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On the characteristic and deformation varieties of a knot

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Abstract The colored Jones function of a knot is a sequence of Laurent polynomials in one variable, whose n th term is the Jones polynomial of the knot colored with the n -dimensional irreducible representation of \mathfrak{sl}_2 . It was recently shown by TTQ Le and the author that the colored Jones function of a knot is q -holonomic, ie, that it satisfies a nontrivial linear recursion relation with appropriate coefficients. Using holonomicity, we introduce a geometric invariant of a knot: the characteristic variety, an affine 1-dimensional variety in \mathbb{C}^2 . We then compare it with the character variety of $SL_2(\mathbb{C})$ representations, viewed from the boundary. The comparison is stated as a conjecture which we verify (by a direct computation) in the case of the trefoil and figure eight knots.

We also propose a geometric relation between the peripheral subgroup of the knot group, and basic operators that act on the colored Jones function. We also define a noncommutative version (the so-called noncommutative A -polynomial) of the characteristic variety of a knot.

Holonomicity works well for higher rank groups and goes beyond hyperbolic geometry, as we explain in the last chapter.

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Dedicated to Andrew Casson on the occasion of his 60th birthday

1 Introduction

1.1 The colored Jones function of a knot

The *colored Jones function*

$$J_K: \mathbb{N} \longrightarrow \mathbb{Z}[q^{\pm}]$$

of a knot K in 3-space is a sequence of Laurent polynomials, whose n th term $J_K(n)$ is the Jones polynomial of a knot colored with the n -dimensional irreducible representation of \mathfrak{sl}_2 ; see [22]. We will normalize it by $J_{\text{unknot}}(n) = 1$ for all n , and (for those who worry about framings), we will assume that K is zero-framed.

The first two terms of the colored Jones function of a knot K are better known. Indeed, $J_K(1) = 1$, and $J_K(2)$ coincides with the *Jones polynomial* of a knot K , defined by Jones in [10]. Although we will not use it, note that the colored Jones function of a knot essentially encodes the Jones polynomial of a knot and its connected parallels.

The starting point for our paper is the key property that the colored Jones function is q -holonomic, as was shown in joint work with TTQ Le; see [6]. Informally, a q -holonomic function is one that satisfies a nontrivial linear recursion relation, with appropriate coefficients. A convenient way to describe recursion relations is the *operator point of view* which we now describe.

1.2 The characteristic variety of a knot

Consider the ring \mathcal{F} of *discrete functions* $f: \mathbb{N} \rightarrow \mathbb{Q}(q)$, and define the linear operators E and Q on \mathcal{F} which act on a discrete function f by:

$$(Qf)(n) = q^n f(n) \quad (Ef)(n) = f(n+1).$$

It is easy to see that $EQ = qQE$, and that E, Q generate a noncommutative *Weyl algebra* (often called a q -Weyl algebra) with presentation

$$\mathcal{A} = \mathbb{Z}[q^\pm]\langle Q, E \rangle / (EQ = qQE).$$

Given a discrete function f , consider the set

$$\mathcal{I}_f = \{P \in \mathcal{A} \mid Pf = 0\}.$$

It is easy to see that \mathcal{I}_f is a left ideal of the Weyl algebra, the so-called *recursion ideal* of f .

If $P \in \mathcal{I}_f$, we may think of the equation $Pf = 0$ as a *linear recursion relation* on f . Thus, the set of linear recursion relations that f satisfies may be identified with the recursion ideal \mathcal{I}_f .

Definition 1.1 We say that f is *q -holonomic* iff $\mathcal{I}_f \neq 0$. In other words, a discrete function is q -holonomic iff it satisfies a nontrivial linear recursion relation.

Consider the quotient $\mathcal{B} = \mathbb{Z}[E, Q]$ of the Weyl algebra and let

$$\epsilon: \mathcal{A} \longrightarrow \mathcal{B} \tag{1}$$

be the *evaluation map* at $q = 1$.

Definition 1.2 If I is a left ideal in \mathcal{A} , we define its *characteristic variety* $\text{ch}(I) \subset (\mathbb{C}^*)^2$ by

$$\text{ch}(I) = \{(x, y) \in (\mathbb{C}^*)^2 \mid P(x, y) = 0 \text{ for all } P \in \epsilon(I)\}.$$

If f is a q -holonomic function, then we define its *characteristic variety* to be $\text{ch}(\mathcal{I}_f)$. Finally, if K is a knot in 3-space, we define its *characteristic variety* $\text{ch}(K)$ to be $\text{ch}(J_K)$.

We will make little distinction between a variety $V \subset (\mathbb{C}^*)^2$ and its closure $\overline{V} \subset \mathbb{C}^2$. For those proficient in holonomic functions, please note that our definition of characteristic variety does *not* agree with the one commonly used in holonomic functions. The latter uses only the symbol (ie the leading E -term) of recursion relations.

1.3 The deformation variety of a knot

The deformation variety of a knot is the character variety of $\text{SL}_2(\mathbb{C})$ representations of the knot complement, viewed from their restriction to the boundary torus. The deformation variety of a knot is of fundamental importance to hyperbolic geometry, and to geometrization, and was studied extensively by Cooper et al and Thurston; see [2] and [21].

Given a knot K in S^3 , consider the complement $M = S^3 - \text{nb}(K)$ (a 3-manifold with torus boundary $\partial M \cong T^2$), and the set

$$R(M) = \text{Hom}(\pi_1(M), \text{SL}_2(\mathbb{C}))$$

of representations of $\pi_1(M)$ into $\text{SL}_2(\mathbb{C})$. This has the structure of an affine algebraic variety defined over \mathbb{Q} , on which $\text{SL}_2(\mathbb{C})$ acts by conjugation on representations. Let $X(M)$ denote the algebrogeometric quotient. There is a natural restriction map $X(M) \longrightarrow X(\partial M)$, induced by the inclusion $\partial M \subset M$. Notice that $\pi_1(\partial M) \cong \mathbb{Z}^2$, generated by a meridian and longitude of K . Restricting attention to representations of $\pi_1(\partial M)$ which are upper diagonal, we may identify the character variety of ∂M with $(\mathbb{C}^2)^*$, parametrized by L and M , the upper left entry of meridian and longitude. Cooper, Culler, Gillet, Long and Shalen [2] define the *deformation variety* $D(K)$ to be the image of $X(\partial M)$ in $(\mathbb{C}^*)^2$.

1.4 The conjecture

Recall that every affine subvariety V in \mathbb{C}^2 is the disjoint union $V_0 \sqcup V_1 \sqcup V_2$ where V_i is a subvariety of V of *pure dimension* i .

We say that two algebraic subvarieties V and V' of \mathbb{C}^2 are *essentially equal* iff V_1 is equal to V'_1 union some y -lines, where a y -line in $\mathbb{C}^2 = \{(x, y) \mid x, y \in \mathbb{C}\}$ is a line $y = a$ for some a .

Conjecture 1 (The Characteristic equals Deformation Variety Conjecture)
For every knot in S^3 , the characteristic and deformation varieties are essentially equal.

Questions similar to the above conjecture and its polynomial version (Conjecture 2 below) were also raised by Frohman and Gelca who studied the colored Jones function of a knot via *Kauffman bracket skein theory*, [8]. Our approach to recursion relations in [6] and here is via statistical mechanics sums and holonomic functions.

A modest corollary of the above conjecture is the following:

Corollary 1.3 *If a knot has nontrivial deformation variety (eg. the knot is hyperbolic), then it has nontrivial colored Jones function.*

Remark 1.4 Despite our improved understanding of the geometry of 3-manifolds, it is unknown at present whether the deformation variety of a knot complement is positive dimensional. If a knot is hyperbolic or torus, then it is, by above mentioned work of Thurston and Cooper et al. If a knot is a satellite, then it is not known, due to the presence of *forbidden representations*, explained by Cooper–Long in [3, section 9].

As evidence for the conjecture, we will show by a direct calculation that:

Proposition 1.5 *Conjectures 1 and 2 are true for the trefoil and figure–8 knots.*

Let us end this section with three comments:

Remark 1.6 Conjecture 1 may be translated as an equality of two polynomials with two commuting variables and integer coefficients; see Conjecture 2 below. Since these polynomials are computable by elimination, it follows that Conjecture 1 is in principle a decidable question. This is in contrast to the *Hyperbolic Volume Conjecture* (due to Kashaev–Murakami–Murakami; see [11, 13]) which involves the existence and identification of a limit of complex numbers.

Remark 1.7 Both Conjecture 1 and the Hyperbolic Volume Conjecture state a relationship between the colored Jones function of a knot and hyperbolic geometry. Combining both conjectures, it follows that the colored Jones function of a hyperbolic knot determines the volume of the hyperbolic 3-manifolds obtained by Dehn surgery on the knot. Indeed, the variation of the volume function depends on the restriction of a path of $\mathrm{SL}_2(\mathbb{C})$ representations to the boundary of the knot complement. Furthermore, the polynomial that defines the deformation variety can compute the variation of the volume function; see Cooper et al [2, section 4.5] and also Yoshida [25] and Neumann–Zagier [14, equation (47)].

Remark 1.8 Conjecture 1 reveals a close relation between the colored Jones function of a knot and its deformation variety. It does not explain though why we ought to look at characters of $\mathrm{SL}_2(\mathbb{C})$ representations. There is a generalization to higher rank groups, which we present in Section 4. We warn the reader that there is no evidence for this generalization.

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2 A polynomial version of Conjecture 1

2.1 The A -polynomial of a knot

Recall the definition of the deformation variety of a knot from Section 1.3. Since projection of affine algebraic varieties corresponds to elimination in their corresponding ideals (see [4]), it is clear that the deformation variety of a knot can in principle be computed via elimination.

In fact, according to [2], the deformation variety $D(K)$ of a knot K is essentially equal to a complex curve in \mathbb{C}^2 which is defined by the zero-locus of the so-called A -polynomial $A(K)$ of K , where the latter lies in $\mathbb{Z}[L, M^2]$. Here A stands for *affine* and not for Alexander.

2.2 A noncommutative version of the A -polynomial

In this section we define a noncommutative version of the A -polynomial of a knot.

If the Weyl algebra \mathcal{A} were a principal ideal domain, every left ideal (such as the recursion ideal of a discrete function) would be generated by a polynomial in noncommuting variables E and Q . This polynomial would be the noncommutative A -polynomial of an ideal. Applying this to the recursion ideal of J_K would allow us to define the noncommutative A -polynomial of a knot.

Unfortunately, the algebra \mathcal{A} is not a principal ideal domain. One way to get around this problem is to invert polynomials in Q , as we now explain. Consider the Ore algebra $\mathcal{A}_{\text{loc}} = \mathbb{K}[E, \sigma]$ over the field $\mathbb{K} = \mathbb{Q}(q, Q)$, where σ is the automorphism of \mathbb{K} given by

$$\sigma(f)(q, Q) = f(q, qQ). \tag{2}$$

Additively, we have

$$\mathcal{A}_{\text{loc}} = \left\{ \sum_{k=0}^{\infty} a_k E^k \mid a_k \in \mathbb{K}, a_k = 0 \text{ for } k \gg 0 \right\},$$

where the multiplication of monomials given by $aE^k \cdot bE^l = a\sigma^k(b)E^{k+l}$.

Recall the ring \mathcal{F} of discrete functions $f: \mathbb{N} \rightarrow \mathbb{Q}(q)$, and its quotient ring $\tilde{\mathcal{F}}$ under the equivalence relation $f \sim g$ iff $f(n) = g(n)$ for all but finitely many n . Then, \mathcal{A}_{loc} acts on $\tilde{\mathcal{F}}$. In particular, if f is a discrete function, we may define its recursion ideal, with respect to \mathcal{A}_{loc} . We will call f q -holonomic with respect to \mathcal{A}_{loc} iff its recursion ideal with respect to \mathcal{A}_{loc} does not vanish.

By clearing out denominators, it is easy to see that if f is a discrete function, then it is q -holonomic with respect to \mathcal{A} iff it is q -holonomic with respect to \mathcal{A}_{loc} .

It turns out that every left ideal in \mathcal{A}_{loc} is *principal*; see [5, chapter 2, exercise 4.5]. Given a left ideal I of \mathcal{A}_{loc} , let $A_q(I)$ denote a generator of I , with the following properties:

- $A_q(I)$ has smallest E -degree and lies in \mathcal{A} .
- We can write $A_q(I) = \sum_k a_k E^k$ where $a_k \in \mathbb{Z}[q, Q]$ are coprime (this makes sense since $\mathbb{Z}[q, Q]$ is a unique factorization domain).

These properties uniquely determine $A_q(I)$ up to left multiplication by $\pm q^a Q^b$ for integers a, b .

Definition 2.1 Given a left ideal I in \mathcal{A} , we define its A_q -polynomial $A_q(I) \in \mathcal{A}$ to be $A_q(I)$. Given a knot K in S^3 , we define its A_q -polynomial $A_q(K)$ to be the A_q -polynomial of the \mathcal{A}_{loc} -recursion ideal of J_K .

Recall from Section 2.1 that the A polynomial of a knot lies in the ring $\mathbb{Z}[L, M^2]$ which we will identify with $\mathbb{Z}[E, Q]$ by $L = E$ and $M = Q^{1/2}$. In other words,

Definition 2.2 We identify the geometric pair (L, M^2) of (*meridian, longitude*) of a knot K with the pair (E, Q) of *basic operators* which act on the colored Jones function of K .

Let us comment on this definition. It is not too surprising that the meridian variable M is identified with Q , the multiplication by q^n . This is foreshadowed by the *Euler expansion* of the colored Jones function in terms of powers of q^n and $q - 1$, [7]. The physical meaning of this expansion is, according to Rozansky, a Feynman diagram expansion around a $U(1)$ -connection in the knot complement with holonomy q^n , [18]. Thus, it is not surprising that $M^2 = Q$.

It is more surprising that the longitude variable L corresponds to the shift operator E . This can be explained in the following way. According to Witten (see [24]), the Jones polynomial $J_K(n)$ of a knot K is the average over an infinite dimensional space of connections, of the *trace of the holonomy around K* , where the trace is computed in the n -dimensional representation of \mathfrak{sl}_2 . To a leading order term, computing traces in the n -dimensional representation is equivalent to computing traces of an $(n - 1, 1)$ -connected parallel of the knot in the 2-dimensional representation. Thus, increasing n by 1 corresponds to going once more around the knot. Since holonomy and longitude are synonymous notions, this explains in some sense the relation $E = L$.

Conjecture 2 (The AJ Conjecture)¹ For every knot in S^3 , $A(K)(L, M) = \epsilon A_q(K)(L, M^2)$.

Lemma 2.3 Conjecture 2 implies Conjecture 1.

Proof Consider $f, g \in \mathbb{Z}[E, Q]$. Let us say that f is *essentially equal* to g if their images in $\mathbb{Q}(Q)[E]$ are equal. In other words, f is essentially equal to g iff f/g is a rational function of Q .

¹AJ are the initials of the A -polynomial and the colored Jones polynomial

If $V(f) = \{(x, y) \in \mathbb{C}^2 \mid f(x, y) = 0\}$ denotes the variety of zeros of f , then it is easy to see that if f is essentially equal to g , then $V(f)$ is essentially equal to $V(g)$.

It is easy to see that the characteristic (resp. deformation) variety is essentially equal to $V(\epsilon A_q)$ (resp. $V(A)$). The result follows. \square

Remark 2.4 Conjecture 2 is consistent with the behavior of the colored Jones function and the A -polynomial under mirror image, changing the orientation of the knot, and \mathbb{Z}_2 -symmetry. For the behavior of the A -polynomial under these operations, see Cooper–Long: [3, proposition 4.2]. On the other hand, the colored Jones function satisfies the symmetry $J(n) = J(-n)$. Moreover, J is invariant under the change of orientation of a knot and changes under $q \rightarrow q^{-1}$ under mirror image.

2.3 Computing the A_q polynomial of a knot

Section 2 defines the A_q polynomial of a knot K . This section explains how to compute the A_q polynomial of a knot. For more details, we refer the reader to [6].

Starting from a generic planar projection of a knot K , it was shown in [6, section 3.2] that the colored Jones function of a knot K can be written as a *multisum*

$$J_K(n) = \sum_{k_1, \dots, k_r=0}^{\infty} F(n, k_1, \dots, k_r) \quad (3)$$

of a *proper q -hypergeometric function* $F(n, k_1, \dots, k_r)$. For a fixed positive n , only finitely many terms are nonzero. Of course, F depends on a planar projection of K . The key property is that F is q -holonomic in all $r + 1$ variables, and that it follows from first principles that multisums of q -holonomic functions are q -holonomic in all remaining free variables.

Working with the Weyl algebra \mathcal{A}_r of $r + 1$ variables, and using the fact that F is q -proper hypergeometric, we may write $EF/F = A/B$ and $E_i F/F = A_i/B_i$ for polynomials $A, B, A_i, B_i \in \mathbb{Q}(q)[q^n, q^{k_1}, \dots, q^{k_r}]$. Replacing q^n by Q and q^{k_i} by Q_i , it follows that the recursion ideal of F in the Weyl algebra \mathcal{A}_{r+1} is generated by $BE - A, B_1 E_1 - A_1, \dots, B_r E_r - A_r$.

The creative telescoping method of Wilf–Zeilberger (the so-called *WZ algorithm*) produces from these generators of F , via noncommutative elimination, operators that annihilate J_K . For a discussion of Wilf–Zeilberger’s algorithm,

see [26, 23, 17] and also [6, section 5]. For an implementation of the algorithm, see [15, 16].

Applying the WZ algorithm to equation (3), we are guaranteed to get an operator $P \in \mathcal{A}_{\text{loc}}$ such that $PJ_K = 0$. It follows that $A_q(K)$ is a right divisor of P . In other words, there exist an operator $P_1 \in \mathcal{A}_{\text{loc}}$ such that $P_1P = A_q(K)$. We caution however that the WZ algorithm does not give in general a minimal order difference operator. For a thorough discussion of this matter, see [17, p164]. In other words, P need not equal to $A_q(K)$.

The problem of computing right factors of an operator has been solved in theory by Petkovšek in [1]. A computer implementation of this solution is not available at present.

In case we are looking for right factors of degree 1 (this is equivalent to deciding whether a discrete function has closed form), there is an algorithm `qHyper` of Petkovšek which decides about this problem in real time; see [17].

In the special examples that we will consider, namely the colored Jones function of 3_1 and 4_1 knots, we can bypass the thorny issue of right factorization of an operator.

3 Proof of the conjecture for the trefoil and figure–8 knots

3.1 The colored Jones function and the A –polynomial of the 3_1 and 4_1 knots

Habiro [9] and Le give the following formula for the colored Jones function of the left-handed trefoil (3_1) and figure–8 (4_1) knots:

$$J_{3_1}(n) = \sum_{k=0}^{\infty} (-1)^k q^{k(k+3)/2} q^{nk} (q^{-n-1}; q^{-1})_k (q^{-n+1}; q)_k \tag{4}$$

$$J_{4_1}(n) = \sum_{k=0}^{\infty} q^{nk} (q^{-n-1}; q^{-1})_k (q^{-n+1}; q)_k. \tag{5}$$

where we define the *rising* and *falling factorials* for $k > 0$ by:

$$\begin{aligned} (a; q)_k &= (1 - a)(1 - aq) \dots (1 - aq^{k-1}) \\ (a; q^{-1})_k &= (1 - a)(1 - aq^{-1}) \dots (1 - aq^{-k+1}) \end{aligned}$$

and $(a; q)_0 = (a; q)_0 = 1$. Notice that the sums in in equations (4) and (5) have compact support, namely for each positive n , only the terms with $k \leq n$ contribute.

These formulas are discussed in detail in Masbaum [12, theorem 5.1], in relation to the cyclotomic expansion of the colored Jones function of twist knots. To compare Masbaum's formula with the one given above, keep in mind that:

$$\begin{aligned} S(n, k) &:= q^{nk}(q^{-n-1}; q^{-1})_k(q^{-n+1}; q)_k \\ &= \frac{\{n-k\}\{n-k+1\}\dots\{n+k\}}{\{n\}} \\ &= \prod_{j=1}^k ((q^{n/2} - q^{-n/2})^2 - (q^{j/2} - q^{-j/2})^2) \end{aligned}$$

where $\{m\} = q^{m/2} - q^{-m/2}$.

On the other hand, Cooper et al [2] compute the A -polynomial of the 3_1 and 4_1 knots, as follows:

$$\begin{aligned} A(3_1) &= (L-1)(L+M^6) & (6) \\ A(4_1) &= (L-1)(-L+LM^2+M^4+2LM^4+L^2M^4+LM^6-LM^8) & (7) \end{aligned}$$

where we include the factor $L-1$ in the A -polynomial which corresponds to the abelian representations of the knot complement.

3.2 Computer calculations

The colored Jones function of the 3_1 and 4_1 knots given in Equations (4) and (5) has no *closed form*. However, it is *guaranteed* to obey nontrivial recursion relations. Moreover, these relations can be found by computer. There are various programs that can compute the recursion relations for multisums. In maple, one may use `qEKHAD` developed by Zeilberger [17]. In Mathematica, one may use `qZeil.m` developed by Paule and Riese [15, 16]. We will give explicit examples in Mathematica, using Paule and Riese's `qZeil.m` package.

We start in computer talk by loading the packages:

```
Mathematica 5.0 for Sun Solaris
Copyright 1988-2000 Wolfram Research, Inc.
-- Motif graphics initialized --
In[1]:= << qZeil.m
q-Zeilberger Package by Axel Riese -- ©RISC Linz -- V 2.35 (04/29/03)
In[2]:= << qMultiSum.m
```

qMultiSum Package by Axel Riese -- ©RISC Linz -- V 2.45 (04/02/03)

Let us type the colored Jones function J_{3_1} from Equation (4):

```
In[3]:= summandtrefoil = (-1)^k q^(k(k+3)/2) q^(n-k) qfac[q^(-n-1),
      q^(-1), k] qfac[q^(-n+1), q, k]
Out[3]= (-1)^k q^(k(k+3)/2+k*n) qPochhammer[q^(-1-n), 1/q, k]
> qPochhammer[q^(1-n), q, k]
```

We now ask for a recursion relation for J_{3_1} :

```
In[4]:= qZeil[summandtrefoil, {k, 0, Infinity}, n, 1]
qZeil::natbounds: Assuming appropriate convergence.
Out[4]= SUM[n] == (q^(-2+n) (-q + q^(2n)) / (-1 + q^n) - q^(-1+3n) (1 - q^(-1+n)) / (1 - q^n)) SUM[-1+n]
```

In other words, for $J(n) = J_{3_1}(n)$ we have:

$$J(n) = q^{-2+n} \frac{-q + q^{2n}}{-1 + q^n} - q^{-1+3n} \frac{1 - q^{-1+n}}{1 - q^n} J(n-1),$$

The above relation is a first order inhomogeneous recursion relation. We may convert it into a second order homogeneous recursion relation as follows:

```
In[5]:= rec31 = MakeHomRec[%, SUM[n]]
Out[5]= (q^(-1+2n) (q^2 - q^n) SUM[-2+n]) / (q^3 - q^(2n)) +
> ((q - q^n) (q + q^n) (q^4 + q^(4n) - q^(3+n) + q^(2+2n) - q^(3+2n) -
> q^(1+3n)) SUM[-1+n]) / (q^n (q - q^(2n)) (q^3 - q^(2n))) +
> (q^(2-n) (-1 + q^n) SUM[n]) / (q - q^(2n)) == 0
```

Perhaps the reader is displeased to see the above recursion relation written in *backwards shifts*, ie, $SUM[-k+n]$ where $k \geq 0$. This can be converted into a recursion relation using *forward shifts* by:

```
In[6]:= ForwardShifts[%]
Out[6]= (q^(3+2n) (q^2 - q^(2+n)) SUM[n]) / (q^3 - q^(4+2n)) +
> (q^(-2-n) (q - q^(2+n)) (q + q^(2+n))) / (q^4 - q^(5+n) - q^(6+2n) - q^(7+2n) - q^(7+3n) - q^(8+4n))
```

$$\begin{aligned}
 &> \quad (q^4 - q^2 + q^2 - q^2 - q^2 + q^2) \text{SUM}[1 + n] \\
 &> \quad / ((q^4 - q^2)^2 (q^3 - q^2)) + \frac{(-1 + q^{2+n}) \text{SUM}[2 + n]}{q^n (q^4 - q^2)} == 0
 \end{aligned}$$

The next command converts the recursion relation `rec31` into an operator, where (due to Mathematica annoyance), we use the symbol X to denote the shift E :

```

In[7]:= ToqHyper[rec31[[1]] - rec31[[2]]] /. {SUM[N] -> 1, SUM[N q^c_.] :> X^c}
/. N -> Q
Out[7]= 
$$\frac{q^2 (-1 + Q)}{Q (q - Q)^2} + \frac{(q^2 - Q) Q^2}{q (q - Q)^2 X^2} + \frac{(q - Q) (q + Q) (q^4 - q^3 Q + q^2 Q^2 - q^3 Q^2 - q Q^3 + Q^4)}{Q (q - Q)^2 (q^3 - Q^2) X}$$


```

This operator right-divides the A_q polynomial of the 3_1 knot. Let us assume for now that it equals to the A_q polynomial, after clearing denominators. Setting $q = 1$, and replacing X by L and Q by M^2 , and obtain:

```

In[8]:= Factor[ToqHyper[rec31[[1]] - rec31[[2]]] /. {SUM[N] -> 1,
SUM[N q^c_.] :> X^c} /. {N -> Q, q -> 1}] /. {Q -> M^2, X -> L}
Out[8]= 
$$-\frac{(-1 + L) (L + M^6)}{L^2 M^2 (1 + M^2)}$$


```

The result agrees, up to multiplication by a rational function of M and a power of E , with the A -polynomial of 3_1 from (6).

It remains to prove that `rec31:=Out[7]` coincides with $A_q(3_1)$, after clearing denominators. Notice that `rec31 = P Aq(31)` for some operator P and $\text{ord}_E(\text{rec31}) = 2$, where $\text{ord}_E(P)$ denotes the E -order of an operator E . Thus $\text{ord}_E(A_q(3_1))$ is 1 or 2. If $\text{ord}_E(A_q(3_1)) = 1$, then J_{3_1} would have a closed form. This problem can be decided by computer using `qHyper` (see [17]), which indeed confirms that J_{3_1} does not have closed form. Thus $\text{ord}_E(A_q(3_1)) = 2 = \text{ord}_E(\text{rec31})$. It follows that (up to left multiplication by units), $A_q(3_1)$ equals to `rec31`. This completes the proof in the case of the trefoil.

Now, let us repeat the process for the colored Jones function of the figure 8 knot, given in Equation (5).

```

In[9]:= summandfigure8 = q^(n - k) qfac[q^(-n - 1), q^(-1), k] qfac[q^(-n + 1),
q, k]

```

```

Out[9]= qk n qPochhammer[q-1 - n,  $\frac{1}{q}$ , k] qPochhammer[q1 - n, q, k]
In[10]:= qZeil[summandfigure8, {k, 0, Infinity}, n, 2]
qZeil::natbounds: Assuming appropriate convergence.
Out[10]= SUM[n] ==  $\frac{q^{-1-n} (q+q^n) (-q+q^{2n})}{(1-q^{-2+n}) (1-q^{-1+2n}) \text{SUM}[-2+n] + \frac{(1-q^n) (1-q^{-3+2n})}{(q^{-2-2n} (1-q^{-1+n})^2 (1+q^{-1+n}))} + (q^4 + q^{4n} - q^{3+n} - q^{1+2n} - q^{3+2n} - q^{1+3n}) \text{SUM}[-1+n] / ((1-q^n) (1-q^{-3+2n}))}$ 

```

gives a second-order inhomogeneous recursion relation, which we convert into a third-order homogeneous recursion relation:

```

In[11]:= rec41 = MakeHomRec[%, SUM[n]]
Out[11]=  $\frac{q^{2+n} (-q+q^n) \text{SUM}[-3+n]}{(q+q^2) (-q+q^5) \text{SUM}[-2+n] + (q^{-2-n} (q^2 - q^n) (q^8 + q^{4n} - 2q^{6+n} + q^{7+n} - q^{3+2n} + q^{4+2n} - q^{5+2n} + q^{1+3n} - 2q^{2+3n}) \text{SUM}[-2+n] / ((q+q^n) (q^5 - q^{2n})) + (q^{-1-n} (-q+q^n) (q^4 + q^{4n} + q^{2+n} - 2q^{3+n} - q^{1+2n} + q^{2+2n} - q^{3+2n} - 2q^{1+3n} + q^{2+3n}) \text{SUM}[-1+n] / ((q+q^2) (-q+q^3)) + \frac{q^{1+n} (-1+q^n) \text{SUM}[n]}{(q+q^2) (-q+q^2)}} = 0$ 

```

In forward shifts, we have:

```

In[12]:= ForwardShifts[%]
Out[12]=  $\frac{q^{5+n} (-q+q^3) \text{SUM}[n]}{(q+q^2) (-q+q^5) \text{SUM}[n] + (q^{-5-n} (q^2 - q^{3+n}) (q^8 - 2q^{9+n} + q^{10+n} - q^{9+2n} + q^{10+2n} - q^{11+2n} + q^{10+3n} - 2q^{11+3n} + q^{12+4n}) \text{SUM}[1+n] / ((q+q^3) (q^5 - q^{6+2n})) + (q^{-4-n} (-q+q^3) (q^4 + q^{5+n} - 2q^{6+n} - q^{7+2n} + q^{8+2n} -$ 

```

$$\begin{aligned}
 &> \frac{q^{9+2n} - 2q^{10+3n} + q^{11+3n} + q^{12+4n}}{q} \text{SUM}[2+n] / \\
 &> \frac{((q^2 + q^{3+n})(-q + q^{6+2n})) + \frac{q^{4+n}(-1 + q^{3+n}) \text{SUM}[3+n]}{(q + q^{3+n})(q - q^{6+2n})}}{q} = 0
 \end{aligned}$$

In operator form, `rec41` becomes:

```
In[13]:= ToqHyper[rec41[[1]] - rec41[[2]]] /. {SUM[N -> 1, SUM[N q^c_.] :> X^c} /. N -> Q
```

$$\begin{aligned}
 \text{Out[13]} = & \frac{q(-1+Q)Q}{(q+Q)(q-Q)^2} + \frac{q^2 Q(-Q+Q)^3}{(q^2+Q)(-q+Q)^2 X^3} - \\
 & \frac{((q^2-Q)(q^8-2q^6Q+q^7Q-q^3Q^2+q^4Q^2-q^5Q^2+q^3Q^3-2q^2Q^3+Q^4)) / (q^2Q(q+Q)(q^5-Q^2)X^2)}{(((-q+Q)^4(q^4+q^2Q-2q^3Q-q^2Q^2+q^2Q^2-q^3Q^2-2qQ^3+q^2Q^3+Q^4)) / (qQ(q^2+Q)(-q+Q^2)X)}
 \end{aligned}$$

where $X = E$. Let us assume that this coincides with $A_q(4_1)$, after we clear denominators. Setting $q = 1$, and replacing X by L and Q by M^2 , and obtain:

```
In[14]:= Factor[ToqHyper[rec41[[1]] - rec41[[2]]] /. {SUM[N -> 1, SUM[N q^c_.] :> X^c} /. {N -> Q, q -> 1} /. {Q -> M^2, X -> L}
```

$$\text{Out[14]} = \frac{(-1+L)(L-LM^2-M^4-2LM^4-L^2M^4-LM^6+LM^8)}{L^3M^2(1+M^2)^2}$$

The result agrees, up to multiplication by a rational function of M and a power of E , with the A -polynomial of 4_1 from (7).

It remains to prove that `rec41:=Out[13]` is equal, up to units, to $A_q(4_1)$. Notice that `rec41 = PAq(41)` for some operator P and $\text{ord}_E(\text{rec41}) = 3$. Thus $\text{ord}_E(A_q(4_1))$ is 1 or 2 or 3.

If $\text{ord}_E(A_q(4_1)) = 1$, then J_{4_1} would have a closed form. This problem can be decided by computer using `qHyper` (see [17]), which indeed confirms that J_{3_1} does not have closed form.

If $\text{ord}_E(A_q(4_1)) = 2$, recall the map ϵ which evaluates at $q = 1$. We have: $\epsilon \text{rec41} = \epsilon P \epsilon A_q(4_1)$. Since $\text{ord}_E(\epsilon \text{rec41}) = 3$, it follows that we must have $\text{ord}_E(\epsilon A_q(4_1)) = 2$.

Furthermore, the computer calculation above shows that $\epsilon A_q(4_1)$ divides $A(4_1)$. The latter, given by Equation (7) can be factored as a product of two irreducible polynomials of E -degree 1 and 2.

On the other hand, Lemma 3.1 below implies that $E - 1$ divides $(\epsilon A_q(4_1))|_{Q=1}$. Combining these facts, it follows that $\epsilon A_q(4_1) = A(4_1)$ (and therefore, also $A_q(4_1)$) is of E -degree 3, a contradiction to our hypothesis.

Thus, it follows that $\text{ord}_E(A_q(4_1)) = 3 = \text{ord}_E(\text{rec}4_1)$. This implies that, up to left multiplication by units, $A_q(4_1)$ coincides with $\text{rec}4_1$. This concludes the proof in the case of the figure-8 knot.

Lemma 3.1 For every knot K , $\epsilon A_q(K)(1, 1) = 0$.

Proof Recall that the colored Jones function of a knot K is given by a multisum formula of a q -proper hypergeometric function. Consider the evaluation of the colored Jones function ϵJ_K at $q = 1$. This is a discrete function which is given by a multisum of a proper hypergeometric function. Applying the WZ algorithm, it follows that $\epsilon_Q \epsilon A_q(K)$ annihilates ϵJ_K , where ϵ_Q is the evaluation at $Q = 1$. However, $\epsilon J_K(n) = 1$ for all n ; see [6]. Thus $E - 1$ divides $\epsilon_Q \epsilon A_q(K)$. The result follows. \square

4 Higher rank groups

The purpose of this section is to formulate a generalization of the characteristic and deformation varieties of a knot to higher rank groups.

Consider a *simple* simply connected compact Lie group G with Lie algebra \mathfrak{g} and complexified group $G_{\mathbb{C}}$. Let $\Lambda \cong \mathbb{Z}^r$ denote its weight lattice, which is a free abelian group of rank r , the rank of G , and let $\Lambda_+ \cong \mathbb{N}^r$ denote the cone of positive dominant weights.

One can define the \mathfrak{g} -colored Jones function

$$J_{\mathfrak{g}}: \mathbb{N}^r \longrightarrow \mathbb{Z}[q^{\pm}].$$

In [6], we showed that $J_{\mathfrak{g}}$ is q -holonomic, with respect to the Weyl algebra of r variables:

$$\mathcal{A}_r = \frac{\mathbb{Z}[q^{\pm}] \langle Q_1, \dots, Q_r, E_1, \dots, E_r \rangle}{(\text{Rel}_q)}$$

where the relations are given by:

$$\begin{aligned} Q_i Q_j &= Q_j Q_i & E_i E_j &= E_j E_i \\ Q_i E_j &= E_j Q_i \text{ for } i \neq j & E_i Q_i &= q Q_i E_i \end{aligned} \tag{Rel}_q$$

Loosely speaking, holonomicity of a discrete function of r variables means that it satisfies r independent linear recursion relations.

A precise definition in several equivalent forms was given in [6, section 2]. For the benefit of the reader, we recall here the definition in its form most useful for our purposes.

Given a discrete function $f: \mathbb{N}^r \rightarrow \mathbb{Q}(q)$, we define the *recursion ideal* \mathcal{I}_f and the *q -Weyl module* M_f by:

$$\mathcal{I}_f = \{P \in \mathcal{A}_r \mid Pf = 0\} \quad M_f := \mathcal{A}_r f \cong \mathcal{A}_r / \mathcal{I}_f.$$

M_f is a cyclic left \mathcal{A}_r module. Every finitely generated left \mathcal{A}_r module has a Hilbert dimension. In case $M = \mathcal{A}_r / I$ is cyclic, its *Hilbert dimension* $d(M)$ is defined as follows. Let F_m be the sub-space of \mathcal{A}_r spanned by polynomials in Q_i, E_i of total degree $\leq m$. Then the module \mathcal{A}_r / I can be approximated by the sequence $F_m / (F_m \cap I), m = 1, 2, \dots$. It turns out that, for $m \gg 1$, the dimension of the vector space $F_m / (F_m \cap I) \otimes_{\mathbb{Z}[q^\pm]} \mathbb{Q}(q)$ (over the field $\mathbb{Q}(q)$) is a polynomial in m of degree equal (by definition) to $d(M)$.

Bernstein's *famous inequality* (proved by Sabbah in the q -case, [19]) states that $d(M) \geq r$, if $M \neq 0$ and M has *no monomial torsions*, ie, any non-trivial element of M cannot be annihilated by a monomial in Q_i, E_i . Note that the left \mathcal{A}_r module $M_f := \mathcal{A}_r \cdot f \cong \mathcal{A}_r / \mathcal{I}_f$ does not have monomial torsion.

Definition 4.1 We say that a discrete function f is q -holonomic if $d(M_f) \leq r$.

Note that if $d(M_f) \leq r$, then by Bernstein's inequality, either $M_f = 0$ or $d(M_f) = r$. The former can happen only if $f = 0$. Of course, for $r = 1$, definitions 1.1 and 4.1 agree.

Let us now define the characteristic variety of a cyclic \mathcal{A}_r module $M = \mathcal{A}_r / I$. Let

$$\mathcal{B}_r = \mathbb{Z}[Q_1, \dots, Q_r, E_1, \dots, E_r]$$

and $\epsilon: \mathcal{A}_r \rightarrow \mathcal{B}_r$ denote the evaluation map at $q = 1$.

Definition 4.2 The *characteristic variety* $\text{ch}(M)$ of M is defined by

$$\text{ch}(M) = \{(x, y) \in (\mathbb{C}^*)^{2r} \mid P(x, y) = 0 \text{ for all } P \in \epsilon(I \cap \mathcal{A}_r)\}$$

This definition may be extended to define the characteristic variety of finitely generated left \mathcal{A}_r modules. As before, we will make little distinction between the characteristic variety and its closure in \mathbb{C}^{2r} .

Lemma 4.3 *If M is a q -holonomic \mathcal{A}_r module, then $\dim_{\mathbb{C}} \text{ch}(M) \geq r$.*

Proof Since M is q -holonomic, it follows that the Hilbert dimension of $(\mathcal{A}_r \otimes \mathbb{Q}(q))/I$ is r , and from this it follows that the Hilbert dimension of $(\mathcal{A}_r \otimes \mathbb{Q}(q))/I$ for generic $q \in \mathbb{C}$ is r . Since dimension is upper semicontinuous and it coincides with the Hilbert dimension at the generic point [20], the result follows. \square

Definition 4.4 If K is a knot in S^3 , and G as above, we define its G -characteristic variety $V_G(K) \subset \mathbb{C}^{2r}$ to be the characteristic variety of its \mathfrak{g} -colored Jones function.

Similarly to the case of $\text{SL}_2(\mathbb{C})$, given a knot K in S^3 , consider the complement $M = S^3 - \text{nbnd}(K)$ and the set $R_{G_{\mathbb{C}}}(M)$ of representations of $\pi_1(M)$ into $G_{\mathbb{C}}$. This has the structure of an affine algebraic variety, on which $G_{\mathbb{C}}$ acts by conjugation on representations. Let $X_{G_{\mathbb{C}}}(M)$ denote the algebrogeometric quotient. There is a natural restriction map $X_{G_{\mathbb{C}}}(M) \rightarrow X_{G_{\mathbb{C}}}(\partial M)$. Notice that $\pi_1(\partial M) \cong \mathbb{Z}^2$, generated by the meridian and longitude of K . Restricting attention to representations of $\pi_1(\partial M)$ which are upper diagonal with respect to a Borel decomposition, we may identify the character variety $X_{G_{\mathbb{C}}}(\partial M)$ with T^2 where T is a maximal torus in $G_{\mathbb{C}}$.

Definition 4.5 The $G_{\mathbb{C}}$ -deformation variety $D_{G_{\mathbb{C}}}(K)$ of K is the image of $X_{G_{\mathbb{C}}}(\partial M)$ in T^2 .

Notice that the maximal torus T of $G_{\mathbb{C}}$ can be identified with $(\mathbb{C}^*)^r$, once we choose fundamental weights λ_i . This allows us to identify the values of meridian and longitude with T^2 . Notice further that the deformation variety of a knot contains an r -dimensional component which corresponds to abelian representations.

Let us say that two varieties V and V' in $\mathbb{C}^{2r} = \{(x, y) \mid x, y \in \mathbb{C}^r\}$ are *essentially equal* if the pure r -dimensional part of V equals to that of V' union some r -dimensional varieties of the form $f(y) = 0$.

Question 1 Is it true that for every G as above and for every knot K , the characteristic and deformation varieties $V_G(K)$ and $D_{G_{\mathbb{C}}}(K)$ are essentially equal?

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