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Symplectic structures from Lefschetz pencils in high dimensions

Robert E Gompf

Abstract A symplectic structure is canonically constructed on any manifold endowed with a topological linear k–system whose fibers carry suitable symplectic data. As a consequence, the classification theory for Lefschetz pencils in the context of symplectic topology is analogous to the corresponding theory arising in differential topology.

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1 Introduction

There is a classical dichotomy between flexible, topological objects such as smooth manifolds, and rigid, geometric objects such as complex algebraic varieties. Symplectic manifolds lie somewhere between these two extremes, raising the question of whether they should be considered as fundamentally topological or geometric. One approach to this question can be traced back to Lefschetz, who attempted to bridge the gap between topology and algebraic geometry by introducing topological (fibrationlike) structures now called *Lefschetz pencils* on any algebraic variety. These structures and more general *linear systems* can also be defined in the setting of differential topology, where they can be found on many manifolds that do not admit algebraic structures, and provide deep information about the topology of the underlying manifolds. It is now becoming apparent that the appropriate context for studying linear systems is not algebraic geometry, but a larger context that includes all symplectic manifolds. Every closed symplectic manifold (up to deformation) admits linear 1–systems (Lefschetz pencils) [4] and 2–systems [3], and it seems reasonable to expect linear k–systems for all k. Conversely, linear $(n-1)$–systems on smooth $2n$–manifolds determine symplectic structures [5]. In this paper, we show that for any k, a linear k–system, endowed with suitable symplectic data on the fibers, determines a symplectic structure on the underlying manifold (Theorem 2.3). We then apply this to the study of Lefschetz pencils, to provide a framework in which symplectic structures appear much more topological than algebrogeometric. While Lefschetz pencils in the algebrogeometric world carry delicate algebraic structure, topological Lefschetz pencils have a classification theory expressed entirely in terms of embedded spheres and a diffeomorphism group of the fiber. The main conclusion of this article (Theorem 3.3) is that symplectic Lefschetz pencils have an analogous classification theory in terms of Lagrangian spheres and a symplectomorphism group of the fiber. That is, the subtleties of symplectic geometry do not interfere with a topological approach to classification.

To construct a prototypical linear k–system on a smooth algebraic variety $X \subset \mathbb{CP}^N$ of complex dimension n, simply choose a linear subspace $A \subset \mathbb{CP}^N$ of codimension $k+1$, with A transverse to X. The *base locus* $B = X \cap A$ is a complex submanifold of X with codimension $k+1$. The subspace $A \subset \mathbb{CP}^N$ lifts to a codimension–$(k+1)$ linear subspace $\widetilde{A} \subset \mathbb{C}^{N+1}$, and projection to $\mathbb{C}^{N+1}/\widetilde{A} \cong \mathbb{C}^{k+1}$ descends to a holomorphic map $\mathbb{CP}^N - A \to \mathbb{CP}^k$ whose restriction will be denoted $f\colon X - B \to \mathbb{CP}^k$. The *fibers* $F_y = f^{-1}(y) \cup B$ of this linear k–system are the intersections of X with the codimension–k linear subspaces of $\mathbb{CP}^N$ containing A. The transversality hypothesis guarantees that $B \subset X$ has a tubular neighborhood V with a complex vector bundle structure $\pi\colon V \to B$ such that f restricts to projectivization $\mathbb{C}^{k+1} - \{0\} \to \mathbb{CP}^k$ (up to action by $\mathrm{GL}(k+1,\mathbb{C})$) on each fiber.

To generalize this structure to a smooth $2n$–manifold X, we first need to relax the holomorphicity conditions. Recall that an *almost-complex structure* $J\colon TX \to TX$ on X is a complex vector bundle structure on the tangent bundle (with each $J_x\colon T_xX \to T_xX$ representing multiplication by i). This is much weaker than a holomorphic structure on X. For our purposes, it is sufficient to assume J is continuous (rather than smooth). We impose such a structure on X, but rather than requiring $f\colon X - B \to \mathbb{CP}^k$ to be J–holomorphic (complex linear on each tangent space), it suffices to impose a weaker condition. Let ω_{std} denote the standard (Kähler) symplectic structure on $\mathbb{CP}^k$, normalized so that $\int_{\mathbb{CP}^1} \omega_{\mathrm{std}} = 1$. (Recall that a symplectic structure is a closed 2–form that is nondegenerate as a bilinear form on each tangent space.) We require J on X to be $(\omega_{\mathrm{std}}, f)$–tame in the following sense:

Definition 1.1 [5] For a C^1 map $f\colon X \to Y$ and a 2–form ω on Y, an almost-complex structure J on X is *(ω, f)–tame* if $f^*\omega(v, Jv) > 0$ for all $v \in TX - \ker df$.

In the special case $f = \mathrm{id}_X$, this reduces to the standard notion of J being ω–tame. In that case, imposing the additional condition that $\omega(Jv, Jw) = \omega(v, w)$ for all $x \in X$ and $v, w \in T_xX$ gives the notion of ω–compatibility. For example, the standard complex structure on $\mathbb{CP}^k$ is ω_{std}–compatible, so the standard complex structure on our algebraic prototype $X \subset \mathbb{CP}^N$ is $(\omega_{\mathrm{std}}, f)$–tame for $f\colon X - B \to \mathbb{CP}^k$ as above. For $f = \mathrm{id}_X$, the ω–tameness condition (unlike ω–compatibility) is open, ie preserved under small perturbations of ω and J, and a closed ω taming some J is automatically symplectic (since it is nondegenerate: every nonzero $v \in TX$ pairs nontrivially with something, namely Jv). Such pairs ω and J determine the same orientation on X. In general, the (ω, f)–tameness condition is preserved under taking convex combinations of forms ω (for fixed J, f). If J is (ω, f)–tame, then each $\ker df_x \subset T_xX$ is a J–complex subspace (characterized as those $v \in T_xX$ for which $f^*\omega(v, Jv) = 0$), so away from critical points each $f^{-1}(y)$ is a J–complex submanifold of X.

We can now define linear systems on smooth manifolds:

Definition 1.2 For $k \geq 1$, a *linear k–system* (f, J) on a smooth, closed $2n$–manifold X is a closed, codimension–$2(k+1)$ submanifold $B \subset X$, a smooth $f\colon X - B \to \mathbb{CP}^k$, and a continuous almost-complex structure J on X with $J|X - B$ $(\omega_{\mathrm{std}}, f)$–tame, such that B admits a neighborhood V with a (smooth, correctly oriented) complex vector bundle structure $\pi\colon V \to B$ for which f is projectivization on each fiber.

For each $y \in \mathbb{CP}^k$, the fiber $F_y = f^{-1}(y) \cup B$ is a closed subset of X whose intersection with V is a smooth, codimension–$2k$ submanifold. F_y is a J–holomorphic submanifold away from the critical points of f, since J is $(\omega_{\mathrm{std}}, f)$–tame on $X - B$ and continuous at B. The complex orientation of F_y agrees with the preimage orientation induced from the complex orientations of X and $\mathbb{CP}^k$. The base locus $B = F_y \cap F_{y'}$ ($y' \neq y \in \mathbb{CP}^k$) is J–holomorphic. The complex orientation of B, which in the transverse case $k = 1$ is also the intersection orientation of $F_y \cap F_y'$, determines the "correct" orientation for the fibers of π. Later (Lemma 2.1) we will verify that the complex bundle structure on V can be assumed to come from J on $TX|B$ by the Tubular Neighborhood Theorem.

Our first goal is to construct symplectic structures using linear k–systems. This was already achieved in [5] for *hyperpencils*, which are linear $(n-1)$–systems endowed with some additional structure taken from the algebraic prototype. It was shown that every hyperpencil determines a unique symplectic form up to isotopy. (Symplectic forms ω_0 and ω_1 on X are *isotopic* if there is a diffeomorphism $\psi\colon X \to X$ isotopic to id_X with $\psi^*\omega_0 = \omega_1$.) The proof crucially used the fact that fibers of linear $(n-1)$–systems are oriented surfaces (away from the critical points) — note that by Moser's Theorem [9] every closed, connected, oriented surface admits a unique symplectic form (ie area form) up to isotopy and scale. For $k < n-1$, the fibers will have higher dimension, so symplectic forms on them need neither exist nor be unique, and we must hypothesize existence and some compatibility of symplectic structures on the fibers. Similarly, almost-complex structures exist essentially uniquely on oriented surfaces, so the required almost-complex structure on a hyperpencil can be essentially uniquely constructed, given only a local existence hypothesis at the critical points. For higher dimensional fibers, there seems to be no analogous procedure, requiring us to include a global almost-complex structure in the defining data of a linear k–system. (Consider the projection $S^2 \times S^4 \to S^2$ which can be made holomorphic locally, but whose fibers admit no almost-complex structure.) The main result for constructing symplectic forms on linear k–systems is Theorem 2.3. The statement is rather technical, but can be informally summed up as follows:

Principle 1.3 For a linear k–system (f, J) on X, suppose that the fibers admit J–taming symplectic structures (suitably interpreted at the critical points), and that these can be chosen to fit together suitably along B and in cohomology. Then (f, J) determines an isotopy class of symplectic forms on X.

The isotopy class of forms can be explicitly characterized (Addenda 2.4 and 2.6).

Our main application concerns Lefschetz pencils on smooth manifolds. These are structures obtained by generalizing the generic algebraic prototype of linear 1–systems.

Definition 1.4 A *Lefschetz pencil* on a smooth, closed, oriented $2n$–manifold X is a closed, codimension–4 submanifold $B \subset X$ and a smooth $f\colon X - B \to \mathbb{CP}^1$ such that

(1) B admits a neighborhood V with a (smooth, correctly oriented) complex vector bundle structure $\pi\colon V \to B$ for which f is projectivization on each fiber,

(2) for each critical point x of f, there are orientation-preserving coordinate charts about x and $f(x)$ (into $\mathbb{C}^n$ and $\mathbb{C}$, respectively) in which f is given by $f(z_1, \ldots, z_n) = \sum_{i=1}^{n} z_i^2$, and

(3) f is 1–1 on the critical set $K \subset X$.

Condition (2) implies K is finite, so (3) can always be achieved by a perturbation of f. A Lefschetz pencil, together with an (ω_{std}, f)–tame almost-complex structure J, is a linear 1–system (although the latter can have more complicated critical points). Such a Lefschetz 1–system can be constructed as before on any smooth algebraic variety by using a suitably generic linear subspace $A \cong \mathbb{CP}^{N-2} \subset \mathbb{CP}^N$. On the other hand, projection $S^2 \times S^4 \to S^2 = \mathbb{CP}^1$ gives a (trivial) Lefschetz pencil admitting no such J.

The topology of Lefschetz pencils is understood at the most basic level, eg [8] or (in dimension 4) [7]. We first consider the case with $B = \emptyset$, or *Lefschetz fibrations* $f\colon X^{2n} \to S^2$. Choose a collection $A = \bigcup A_j \subset S^2$ of embedded arcs with disjoint interiors, connecting the critical values to a fixed regular value $y_0 \in S^2$. Over a sufficiently small disk $D \subset S^2$ containing y_0, we see the trivial bundle $D \times F_{y_0} \to D$. Expanding D to include an arc A_j adds an n–handle along an $(n-1)$–sphere lying in a fiber. Thus, if we expand D to include A, the result is specified by a cyclically ordered collection of *vanishing cycles*, ie embeddings $S^{n-1} \to F_{y_0}$ with suitable normal data. However, this ordered collection depends on our choice of A. Any change in A can be realized by a sequence of *Hurwitz moves*, moving some arc A_j past its neighbor $A_{j\pm1}$. The effect of a Hurwitz move on the ordered collection of vanishing cycles can be easily described using the monodromy of the fibration around $A_{j\pm1}$, which is an explicitly understood element of π_0 of the diffeomorphism group $\mathcal{D}$ of F_{y_0}. (See Section 3.) Over the remaining disk $S^2 - \text{int } D$, we again have a trivial bundle, so the product of the monodromies of the vanishing cycles must be trivial, and then the Lefschetz fibrations extending fixed data over D are classified by $\pi_1(\mathcal{D})$. The correspondence with $\pi_1(\mathcal{D})$ is determined by fixing an arc from y_0 to ∂D (avoiding A) and a trivialization of f over ∂D. Hurwitz moves involving the new arc will induce additional equivalences. For the case $B \neq \emptyset$, we blow up B to obtain a Lefschetz fibration, then apply the previous analysis. However, extra care is required to preserve the blown up base locus and its normal bundle. We must take $\mathcal{D}$ to be the group of diffeomorphisms of F fixing B and its normal bundle, and the product of monodromies will now be a nontrivial normal twist δ around B. We state the result carefully as Proposition 3.1. For now, we sum up the discussion as follows:

Principle 1.5 To classify Lefschetz pencils with a fixed fiber and base locus, first classify, up to Hurwitz moves, cyclically ordered collections of vanishing cycles for which the product of monodromies is $\delta \in \pi_0(\mathcal{D})$. For any fixed choice of arcs and vanishing cycles, the resulting Lefschetz pencils are classified by $\pi_1(\mathcal{D})$. The final classification results from modding out the effects of Hurwitz moves on the last fiber. (One may also choose to mod out by self-diffeomorphisms of the fiber (F_{y_0}, B).)

Of course, this is an extremely difficult problem in general, but at least we know where to start.

If X is given a symplectic structure ω that is symplectic on the fibers, then the above description can be refined. The vanishing cycles will be *Lagrangian* spheres (ie ω restricts to 0 on them), and the monodromies will be symplectomorphisms (diffeomorphisms preserving ω) [2, 10, 11]. The discussion of arcs and Hurwitz moves proceeds as before, where $\mathcal{D}$ is replaced by a suitable group $\mathcal{D}_{\omega_F}$ of symplectomorphisms of the fiber. However, symplectic forms are *a priori* global analytic objects (satisfying the partial differential equation $d\omega = 0$), so for symplectic forms on X compatible with a given Lefschetz pencil, one might expect both the existence and uniqueness questions to involve delicate analytic invariants. Our main result (Theorem 3.3) is that no such difficulties arise, provided that we choose our definitions with suitable care, for example requiring $[\omega] \in H^2_{\mathrm{dR}}(X)$ to be Poincaré dual to the fibers (as is the case for Donaldson's pencils [4]). We obtain:

Principle 1.6 The classification of (suitably defined) symplectic Lefschetz pencils is purely topological, ie analogous to that of Principle 1.5. More precisely, for a suitable symplectic manifold pair (F, B), let i_* denote the π_1–homomorphism induced by inclusion $\mathcal{D}_{\omega_F} \subset \mathcal{D}$. Then for fixed (suitably symplectic) data over D as preceding Principle 1.5, a given Lefschetz pencil admits a suitably compatible symplectic structure if and only if it is classified by an element of $\operatorname{Im} i_*$. Then such structures are classified up to suitable isotopy by $\pi_2(\mathcal{D}/\mathcal{D}_{\omega_F})$, and by $\ker i_*$ if symplectomorphisms preserving f and fixing $f^{-1}(D)$ are also allowed.

This is the same sort of topological classification one obtains for extending bundle structures over a 2–cell: Given groups $H \subset G$, a space $Y \cup 2$–cell, and a fixed H–bundle over Y (on which we do not allow automorphisms), G–bundle and H–bundle extensions (if they exist) are classified by $\pi_1(G) \cong \pi_2(BG)$ and $\pi_1(H) \cong \pi_2(BH)$, respectively. Inclusion $i\colon H \to G$ induces an exact sequence

$$\pi_2(G/H) \xrightarrow{\partial_*} \pi_1(H) \xrightarrow{i_*} \pi_1(G)$$

with i_* corresponding to the forgetful map from H–structures to G–structures. Thus $\operatorname{Im} i_*$ classifies G–extensions admitting H–reductions, and $\ker i_* = \operatorname{Im} \partial_*$ classifies H–reductions of a fixed G–extension as abstract H–extensions. However, different H–reductions can be abstractly H–isomorphic via a G–bundle automorphism supported over the 2–cell, and if we disallow such equivalences, H–reductions of a fixed G–extension are classified by $\pi_2(G/H)$.

2 Linear systems

In this section, we show how to construct symplectic structures from linear systems with suitable symplectic data along the fibers (Principle 1.3). Our construction is modeled on the corresponding method for hyperpencils [5, Theorem 2.11], but is complicated by the fact that the base locus need no longer be 0–dimensional. We must first gain more control of the normal data along B. Given a linear k–system (f,J) on X as in Definition 1.2, let $\nu\to B$ be any J–complex subbundle of $TX|B$ complementary to TB. (This exists since B is a J–holomorphic submanifold of X.) Then the bundle structure $\pi\colon V\to B$ guaranteed on a neighborhood of B (by Definition 1.2) can be arranged (after precomposing π with an isotopy preserving f) to have its fibers tangent to ν along B.

Lemma 2.1 *For ν and π as above, the complex bundle structure on π (given by Definition 1.2) restricts to J on ν.*

Proof Near B, extend TB to a J–complex subbundle H of TX complementary to the fibers of π and tangent to the fibers F_y of f. Then J induces a complex structure near B on TX/H. The latter bundle is canonically $\mathbb{R}$–isomorphic to the bundle of tangent spaces to the fibers of π; let J' denote the resulting almost-complex structure on the fibers of π. Clearly, $J'=J$ on ν, so it suffices to show that J' agrees with the complex structure of π on ν. This follows immediately from [5, Lemma 4.4(b)], which is restated below. (Note that for $x\notin B$, H_x lies in $\ker df_x$, so J' is $(\omega_{\mathrm{std}},f)$–tame at x since J is.) □

Lemma 2.2 *[5] If $f\colon \mathbb{C}^n-\{0\}\to\mathbb{CP}^{n-1}$ denotes projectivization, $n\geq 2$, and J is a continuous (positively oriented) almost-complex structure on a neighborhood W of 0 in $\mathbb{C}^n$, with $J|W-\{0\}$ $(\omega_{\mathrm{std}},f)$–tame, then $J|T_0\mathbb{C}^n$ is the standard complex structure.*

The main idea of the proof is that $J|T_0\mathbb{C}^n$ has the same complex lines as the standard structure (since the complex lines of $\mathbb{C}^n$ are J–complex by $(\omega_{\mathrm{std}},f)$–tameness), and a linear complex structure is determined by its complex lines.

We can now state the main theorem of this section. By Lemma 2.1, the canonical identification of the vector bundle $\pi\colon V\to B$ with the normal bundle ν

of its 0–section is a J–complex isomorphism. This complex bundle is projectively trivialized by f (in Definition 1.2), so we can reduce the structure group of ν to U(1) (acting diagonally on $\mathbb{C}^{k+1}$) by choosing a suitable Hermitian structure on ν. This Hermitian structure is canonically determined up to a positive scalar function. Let h denote the hyperplane class in $H^2_{\mathrm{dR}}(\mathbb{CP}^k)$ dual to $[\mathbb{CP}^{k-1}]$, and let $c_f \in H^2_{\mathrm{dR}}(X)$ correspond to $f^*h \in H^2_{\mathrm{dR}}(X-B)$ under the obvious isomorphism. (Recall $\operatorname{codim} B \geq 4$.)

Theorem 2.3 *Let (f,J) be a linear k–system on X. Choose a J–complex subbundle $\nu \subset TX|B$ complementary to TB, and a Hermitian form on ν as above. Suppose there is a symplectic form ω_B on B taming $J|B$, with $[\omega_B] = c_f|B \in H^2_{\mathrm{dR}}(B)$. Then ω_B extends to a closed 2–form ζ on X representing c_f, with ν and TB ζ–orthogonal, and ζ agreeing with the given Hermitian form on each 1–dimensional J–complex subspace of ν. Given such an extension ζ, suppose that each F_y, $y \in \mathbb{CP}^k$, has a neighborhood W_y in X with a closed 2–form η_y on W_y taming $J|\ker df_x$ for all $x \in W_y - B$, agreeing with ζ on each $TF_z|B$, $z \in \mathbb{CP}^k$, and with $[\eta_y - \zeta] = 0 \in H^2_{\mathrm{dR}}(W_y, B)$. Then (f,J) determines an isotopy class Ω of symplectic forms on X representing $c_f \in H^2_{\mathrm{dR}}(X)$.*

Each $\ker df_x$ is J–complex, so we define η_y–tameness on it in the obvious way. The class $[\eta_y - \zeta] \in H^2_{\mathrm{dR}}(W_y, B)$ is defined since $\eta_y - \zeta$ vanishes on B by hypothesis. This class vanishes automatically if $[\eta_y] = c_f|W_y$ and the restriction map $H^1_{\mathrm{dR}}(W_y) \to H^1_{\mathrm{dR}}(B)$ is surjective; however surjectivity always fails when (for example) B is a surface of nonzero genus and a generic (4–dimensional) fiber has $b_1 < 2$.

For our subsequent application to Lefschetz pencils, we will need an explicit characterization of Ω and detailed properties of some of its representatives. The characterization below is complicated by our need to perturb J during the proof. A simpler version when no perturbation is necessary will be given as Addendum 2.6 after the required notation is established.

Addendum 2.4 *Fix a metric on X and $\varepsilon > 0$. Let $\mathcal{J}_\varepsilon$ be the C^0–space of continuous almost-complex structures J' on X that are ε–close to J, agree with J on $TX|B$ and outside the ε–neighborhood U of B, and make each $F_y \cap U$ J'–complex. Fix a regular value y_0 of f. Then Ω contains a form ω taming an element of $\mathcal{J}_\varepsilon$ and extending ω_B, such that J is ω–compatible on ν, which is ω–orthogonal to B, and $\omega|F_{y_0}$ is isotopic to $\eta_{y_0}|F_{y_0}$ by an isotopy ψ_s of the pair (F_{y_0}, B) that is symplectic on (B, ω_B). For ε sufficiently small, any two forms representing c_f and taming elements of $\mathcal{J}_\varepsilon$ are isotopic, so these latter conditions uniquely determine Ω.*

Theorem 2.3 was designed for compatibility with [5, Theorem 3.1], which was the main tool for putting symplectic structures on hyperpencils (and domains of locally holomorphic maps [6]). The proof is based on an idea of Thurston [12]. We state and prove a version of the theorem which has been slightly modified, primarily to correct for the failure of H^1–surjectivity observed following Theorem 2.3. We will ultimately apply the theorem to a linear system projection $f\colon X-B \to \mathbb{CP}^k$, working relative to a normal disk bundle C of B (intersected with $X-B$).

Theorem 2.5 *Let $f\colon X\to Y$ be a smooth map between manifolds, and let C be a codimension–0 submanifold (with boundary) that is closed in X, with $X-\operatorname{int} C$ compact. Suppose that ω_Y is a symplectic form on Y, and J is a continuous, (ω_Y, f)–tame almost-complex structure on X. Let ζ be a closed 2–form on X taming J on C. Suppose that for each $y\in Y$, $f^{-1}(y)\cup C$ has a neighborhood W_y in X, with a closed 2–form η_y on W_y agreeing with ζ on C, such that $[\eta_y-\zeta]=0\in H^2_{\mathrm{dR}}(W_y, C)$ and such that η_y tames $J|\ker df_x$ for each $x\in W_y$. Then there is a closed 2–form η on X agreeing with ζ on C, with $[\eta]=[\zeta]\in H^2_{\mathrm{dR}}(X)$, and such that for all sufficiently small $t>0$ the form $\omega_t = t\eta + f^*\omega_Y$ on X tames J (and hence is symplectic). For preassigned $\hat{y}_1,\dots,\hat{y}_m\in Y$, we can assume η agrees with $\eta_{\hat{y}_j}$ near each $f^{-1}(\hat{y}_j)$.*

Proof For each $y\in Y$, $[\eta_y-\zeta]=0\in H^2_{\mathrm{dR}}(W_y,C)$, so we can write $\eta_y = \zeta + d\alpha_y$ for some 1–form α_y on W_y with $\alpha_y|C=0$. Since each $X-W_y$ is compact, each $y\in Y$ has a neighborhood disjoint from $f(X-W_y)$. Thus, we can cover Y by open sets U_i, with each $f^{-1}(U_i)$ contained in some W_y, and each $\hat{y}_j$ lying in only one U_i. Let $\{\rho_i\}$ be a subordinate partition of unity on Y. The corresponding partition of unity $\{\rho_i\circ f\}$ on X can be used to splice the forms α_y; let $\eta = \zeta + d\sum_i(\rho_i\circ f)\alpha_{y_i}$. Clearly, η is closed with $[\eta]=[\zeta]\in H^2_{\mathrm{dR}}(X)$, $\eta=\zeta$ on C, and $\eta=\eta_{\hat{y}_j}$ near $f^{-1}(\hat{y}_j)$, so it suffices to show that ω_t tames J ($t>0$ small). In preparation, perform the differentiation to obtain $\eta = \zeta + \sum_i(\rho_i\circ f)d\alpha_{y_i} + \sum_i(d\rho_i\circ df)\wedge\alpha_{y_i}$. The last term vanishes when applied to a pair of vectors in $\ker df_x$, so on each $\ker df_x$ we have $\eta = \zeta + \sum_i(\rho_i\circ f)d\alpha_{y_i} = \sum_i(\rho_i\circ f)\eta_{y_i}$. By hypothesis, this is a convex combination of taming forms, so we conclude that $J|\ker df_x$ is η–tame for each $x\in X$.

It remains to show that there is a $t_0>0$ for which $\omega_t(v,Jv)>0$ for every $t\in(0,t_0)$ and v in the unit sphere bundle $\Sigma\subset TX$ (for any convenient metric). But

$$\omega_t(v,Jv) = t\eta(v,Jv) + f^*\omega_Y(v,Jv).$$

Since J is (ω_Y,f)–tame, the last term is positive for $v\notin\ker df$ and zero otherwise. Since $J|\ker df$ is η–tame, the continuous function $\eta(v,Jv)$ is positive for all v in some neighborhood U of $\ker df \cap \Sigma$ in Σ. Similarly, for $v \in \Sigma|C$, $\eta(v, Jv) = \zeta(v, Jv) > 0$. Thus, $\omega_t(v, Jv) > 0$ for all $t > 0$ when $v \in U \cup \Sigma|C$. On the compact set $\Sigma|(X - \operatorname{int} C) - U$ containing the rest of Σ, $\eta(v, Jv)$ is bounded and the last displayed term is bounded below by a positive constant, so $\omega_t(v, Jv) > 0$ for $0 < t < t_0$ sufficiently small, as required. □

Proof of Theorem 2.3 and addenda We begin by producing the desired symplectic structure near B, via a local model generalizing the case $\dim B = 0$ from [5]. Assume the fibers of π are tangent to ν. Let $L_0 \to B$ denote the Hermitian line bundle obtained by restricting π to a fixed F_y, so L_0 and ν are associated to the same principal U(1)–bundle $\pi_P\colon P \to B$. Then $c_1(L_0) = c_f|B = [\omega_B]$ (since a generic section of L_0 is obtained by perturbing $B \cup f^{-1}(\mathbb{CP}^{k-1}) \subset X$ and intersecting it with F_y). Let $i\beta_0$ on P be a U(1)–connection form for L_0 with Chern form ω_B, so $-\frac{1}{2\pi}d\beta_0 = \pi_P^*\omega_B$. For $r > 0$, let $S_r \subset V$ denote the sphere bundle of radius r (for the Hermitian metric). The map $(\pi, f)\colon V - B \to B \times \mathbb{CP}^k$ exhibits each S_r as a principal U(1)–bundle. The corresponding line bundle $L \to B \times \mathbb{CP}^k$ restricts to L_0 over B and to the tautological bundle L_{taut} over $\mathbb{CP}^k$. Since $H^2(B \times \mathbb{CP}^k) \cong H^2(B) \oplus (H^0(B) \otimes H^2(\mathbb{CP}^k))$ (over $\mathbb{Z}$), we conclude that $L \cong \pi_1^* L_0 \otimes \pi_2^* L_{\mathrm{taut}}$. Fix this isomorphism, and let $i\beta$ be the U(1)–connection form on S_r induced by $i\beta_0$ on L_0 and the tautological connection on L_{taut}. Then the Chern form of $i\beta$ is given by $-\frac{1}{2\pi}d\beta = \pi^*\omega_B - f^*\omega_{\mathrm{std}}$ (pushed down to $B \times \mathbb{CP}^k$). Define a 2–form ω_V on $V - B$ by

$$\omega_V = (1 - r^2)\pi^*\omega_B + r^2 f^*\omega_{\mathrm{std}} + \frac{1}{2\pi}d(r^2) \wedge \beta.$$

An easy calculation shows that $d\omega_V = 0$, and it is routine to verify [5] that ω_V restricts to the given Hermitian form on each fiber of π (up to a constant factor of π, arising from our choice of normalization of ω_{std}, which can be eliminated by a constant rescaling of r). Let H be the smooth distribution on V consisting of TB on B together with its β–horizontal lifts to each S_r. Clearly, H is tangent to each S_r and F_y, so it is ω_V–orthogonal to the fibers of π. Since $\omega_V|H = (1-r^2)\pi^*\omega_B$ extends smoothly over B, as does ω_V on the π–fibers, ω_V extends smoothly to all of V, with $\omega_V|B = \omega_B$. If J_V denotes the almost-complex structure on V obtained by lifting $J|B$ to H and summing with the complex bundle structure on the fibers of π, then J_V is ω_V–tame for $r < 1$. (Check this separately on the π–fibers and their ω_V–orthogonal complements H.) Note that $J_V = J$ on $TX|B$ (Lemma 2.1).

We can now state the remaining addendum:

Addendum 2.6 *If J agrees with J_V near B for some choice of $\pi\colon V \to B$ and β_0 as above, then Ω has the simpler characterization that it contains forms ω taming J with $[\omega] = c_f$. In fact, there is a J–taming form $\omega \in \Omega$ satisfying all the conclusions of Addendum 2.4 with ν induced by π, and such that the given forms $\psi_s^* \eta_{y_0}$ on F_{y_0} between η_{y_0} and ω all tame J.*

To construct the required form ζ, choose a form ζ_0 representing $c_f \in H^2_{\mathrm{dR}}(X)$. Then $[\omega_V - \zeta_0] = 0 \in H^2_{\mathrm{dR}}(V)$, so there is a 1–form α on V with $d\alpha = \omega_V - \zeta_0$. Let $\zeta = \zeta_0 + d(\rho\alpha)$, where $\rho\colon X \to \mathbb{R}$ has support in V and $\rho = 1$ near B. Then $\zeta = \omega_V$ near B, so ζ satisfies the required conditions for the theorem. If ζ_0 was already the hypothesized extension of ω_B, satisfying these conditions and suitably compatible with forms η_y, then $\zeta_0 = \omega_V = \zeta$ on each $TF_z|B \cong TB \oplus L_0$, so ζ still agrees with each η_y as required along B. We also could have arranged $\alpha|B = 0$ since $H^2_{\mathrm{dR}}(V, B) = 0$, so that we still have $[\eta_y - \zeta] = 0 \in H^2_{\mathrm{dR}}(W_y, B)$. Thus, we can assume the given ζ agrees with ω_V near B.

Since we must perturb J near B, we verify that for sufficiently small ε, every $J' \in \mathcal{J}_\varepsilon$ as in Addendum 2.4 is $(\omega_{\mathrm{std}}, f)$–tame on $X - B$. Choose ε so that the ε–neighborhood U of B in X (in the given metric) has closure in V, and let $\Sigma \subset TX$ be the compact subset consisting of unit vectors over $\mathrm{cl}(U)$ that are ω_V–orthogonal to fibers F_y. For $J' \in \mathcal{J}_\varepsilon$, each $\ker df_x = T_x F_{f(x)}$ over $U - B$ is J'–complex, so it suffices to show that $f^*\omega_{\mathrm{std}}(v, J'v) > 0$ for $v \in \Sigma \cap T(U - B)$. We replace $f^*\omega_{\mathrm{std}}$ by ω_V, since these agree on such vectors v (which are tangent to the π–fibers and S_r) up to the scale factor $r^2 > 0$. But $\omega_V(v, Jv) > 0$ for $v \in \Sigma$ (since J equals J_V on $TX|B$ and J is $(\omega_{\mathrm{std}}, f)$–tame elsewhere), so the corresponding inequality holds for all $J' \in \mathcal{J}_\varepsilon$ for ε sufficiently small, by compactness of Σ and openness of the taming condition.

We must also modify the pairs (W_y, η_y) so that for all sufficiently small ε, every $J' \in \mathcal{J}_\varepsilon$ is η_y–tame on $\ker df_x$ for each $y \in \mathbb{CP}^k$ and $x \in W_y - B$. Shrink each W_y so that η_y is defined on $\mathrm{cl}(W_y)$. Each W_y contains $f^{-1}(U_y)$ for some neighborhood U_y of y (cf proof of Theorem 2.5). After passing to a finite subcover of $\{U_y\}$, we can assume $\{W_y\}$ is finite, so the pairs (W_y, η_y) for all $y \in \mathbb{CP}^k$ are taken from a finite set, and $\eta_{y_0}|F_{y_0}$ is preserved. Now for each η_y, $\eta_y(v, Jv) > 0$ on the compact space of unit tangent vectors to fibers F_y in $\mathrm{cl}(W_y \cap V)$. (Note that on $TX|B$, ζ tames J.) Thus, each $J' \in \mathcal{J}_\varepsilon$ has the required η_y–taming for ε sufficiently small.

Next we splice our local model ω_V and J_V into each η_y and J. For $y \in \mathbb{CP}^k$, η_y equals ζ on $TF_y|B$, so it tames J there and hence is symplectic on F_y near B. Thus, Weinstein's symplectic tubular neighborhood theorem

[13] on F_y produces an isotopy of F_y fixing B (pointwise) and supported in a preassigned neighborhood of B, sending $\eta_y|F_y$ to a form η'_y agreeing with $\zeta = \omega_V$ near B on F_y. To extend η'_y to a neighborhood of F_y in X, first extend it as ω_V near B and as η_y farther away, leaving a gap in between (inside V). Let $r\colon W_y \to W_y$ be a smooth map agreeing with id_{W_y} away from the gap and on F_y, collapsing W_y onto F_y near the gap. Then $r^*\eta'_y$ is a closed form near F_y extending η'_y (cf [5]). Now recall that the vector field generating Weinstein's isotopy vanishes to second order on B. (It is symplectically dual to the 1–form $-\int_0^1 \pi_t^*(X_t \lrcorner (\eta_y - \zeta))\,dt$, where π_t is fiberwise multiplication by t, and the radial vector field $X_t = \frac{d}{dt}\pi_t$ vanishes to first order on B, as does $\eta_y - \zeta$.) Thus we can assume our isotopy is arbitrarily C^1–small (by working in a sufficiently small neighborhood of B), so we can replace η_y on W_y by $r^*\eta'_y$ on a sufficiently small neighborhood of F_y without disturbing our original hypotheses. In particular, we can assume J is still η_y–tame on each $\ker df_x$ (or similarly for all J' in a preassigned compact subset of $\mathcal{J}_\varepsilon$ with ε as in the previous paragraph). Since we have shrunk the sets W_y, the set $\{W_y\}$ may again be infinite, but we can reduce to a finite subcollection as before. Then there is a single neighborhood W of B in X, contained in $\bigcap W_y$, on which each η_y agrees with ω_V and ζ. Since $\eta_{y0}|F_{y0}$ has only been changed by a C^1–small isotopy fixing B, its use in the addenda is unaffected.

To complete splicing the local model, we perturb J to J' agreeing with J_V near B. Under the hypothesis of Addendum 2.6, we simply set $J' = J$. Otherwise, we invoke [5, Corollary 4.2], which was adapted from [1, page 100].

Lemma 2.7 *[5] For any finite dimensional, real vector space V, there is a canonical retraction $j(A) = A(-A^2)^{-1/2}$ from the open subset of operators in $\mathrm{Aut}(V)$ without real eigenvalues to the set of linear complex structures on V. For any linear $T\colon V \to W$ with $TA = BT$, we have $Tj(A) = j(B)T$ (when both sides are defined).*

Since $J = J_V$ on $TX|B$, $J_t = j((1-t)J + tJ_V)$ is well-defined for $0 \le t \le 1$ near B, and each F_y is J_t–complex there (as seen by letting T be inclusion $T_xF_y \to T_xX$). For any $\varepsilon > 0$, we can thus define $J' \in \mathcal{J}_\varepsilon$ to be J_ρ, for $\rho\colon X \to I$ supported sufficiently close to B and with $\rho \equiv 1$ near B, extended as J away from $\operatorname{supp}\rho$. Then for ε sufficiently small, the preceding three paragraphs show that (f, J') is a linear k–system satisfying the hypotheses of Theorem 2.3 with J', ζ and each η_y agreeing with the standard model on a suitably reduced W.

We now construct a symplectic form ω on X as in [5]. First we apply Theorem 2.5 to $f\colon X - B \to \mathbb{CP}^k$ and J', with $C \subset W$ a normal disk bundle to B

(intersected with $X - B$). Note that $[\eta_y - \zeta] \in H^2_{\mathrm{dR}}(W_y - B, C) \cong H^2_{\mathrm{dR}}(W_y, B)$ vanishes as required. We obtain a closed 2–form η on $X - B$ agreeing with ω_V on C (hence extending over X), with $[\eta] = c_f \in H^2_{\mathrm{dR}}(X)$ and $\eta = \eta_{y_0}$ on F_{y_0}, such that $\omega_t = t\eta + f^*\omega_{\mathrm{std}}$ tames J' on $X - B$ for $t > 0$ chosen sufficiently small. On C, the symplectic form ω_t is given by

$$\omega_t(r) = t(1 - r^2)\pi^*\omega_B + (1 + tr^2)f^*\omega_{\mathrm{std}} + \frac{t}{2\pi}d(r^2) \wedge \beta.$$

Unfortunately, this is singular at B. (Compare the middle term with that of $\omega_V \neq 0$.) However, we can desingularize by a dilation in the manner of [5]: The radial change of variables $R^2 = \frac{1+tr^2}{1+t}$ shows that $\omega_V(R) = \frac{1}{1+t}\omega_t(r)$, so there is a radial symplectic embedding $\varphi\colon (C, \frac{1}{1+t}\omega_t) \to (V, \omega_V)$ onto a collar surrounding the bundle $R^2 \leq \frac{1}{1+t}$. Let $\varphi_0\colon V \to V$ be a radially symmetric diffeomorphism covering id_B and agreeing with φ near ∂C. Let ω be $\varphi_0^*\omega_V$ on $C \cup B$ and $\frac{1}{1+t}\omega_t$ elsewhere. These pieces fit together to define a symplectic form on X, since φ is a symplectic embedding. (This construction is equivalent to blowing up B, applying Theorem 2.5 with $C = \emptyset$ to the resulting singular fibration, and then blowing back down, but it avoids technical difficulties associated with taming on the blown up base locus.)

The form ω satisfies the properties required by Theorem 2.3 and its addenda: To compute the cohomology class $[\omega] \in H^2_{\mathrm{dR}}(X)$, it suffices to work outside C. Then $[\omega] = \frac{1}{1+t}[\omega_t] = \frac{1}{1+t}(tc_f + f^*[\omega_{\mathrm{std}}]) = c_f$ as required, since $[\omega_{\mathrm{std}}] = h \in H^2_{\mathrm{dR}}(\mathbb{CP}^k)$. For Addendum 2.4, note that ω obviously extends ω_B and is compatible with J on ν, which is ω–orthogonal to B. Outside C, we already know that $\omega = \frac{1}{1+t}\omega_t$ tames $J' \in \mathcal{J}_\varepsilon$, so taming need only be checked for $J' = J_V$ on $C \cup B$ with $\omega = \varphi_0^*\omega_V$, and this is easy on $TX|B = TB \oplus \nu$. For C, consider the ω_V–orthogonal, J_V–complex splitting $T(V - B) = P \oplus P^\perp \oplus H$, where P and $P^\perp$ are tangent and normal, respectively, to the complex lines through B in the bundle structure π. The radial map φ_0 preserves the splitting but scales each summand by a different positive function. (Although the fibers of P are scaled differently along their two axes, φ_0^* only rescales $\omega_V|P$ since it is an area form.) Now J_V is ω–tame on C since it is ω_V–tame on each summand. To verify that $\omega|F_{y_0}$ is pairwise isotopic to $\eta_{y_0}|F_{y_0}$, recall that $\eta|F_{y_0} = \eta_{y_0}|F_{y_0}$, so $\omega|F_{y_0} = \frac{t}{1+t}\eta_{y_0}|F_{y_0}$ outside C. When $t \to \infty$ we have $R \to r$ and $\varphi_0 \to \mathrm{id}_V$, so $\omega \to \eta$. Note that η and ω (for all $t > 0$) are symplectic on F_{y_0}, although not necessarily on X (unless t is small). The required isotopy now follows from Moser's method [9] applied pairwise to (F_{y_0}, B): Starting from ω as constructed above with t sufficiently small, let $\widetilde{\omega}_s$, $s = \frac{1}{t} \in [0, a]$, be the corresponding family of cohomologous symplectic forms on F_{y_0} obtained by letting $t \to \infty$ (so $\widetilde{\omega}_a = \omega|F_{y_0}$ and $\widetilde{\omega}_0 = \eta_{y_0}|F_{y_0}$). Moser gives a family α_s of 1–forms on F_{y_0}

with $d\alpha_s = \frac{d}{ds}\widetilde{\omega}_s$, then flows by the vector field Y_s for which $\widetilde{\omega}_s(Y_s,\cdot) = --\alpha_s$ to obtain an isotopy with $\psi_s^*\eta_{y_0} = \widetilde{\omega}_s$. If we first subtract dg_s from α_s, where $g_s\colon F_{y_0} \to \mathbb{R}$ is obtained by pushing $\alpha_s\colon TF_{y_0} \to \mathbb{R}$ from $TB^{\perp_{\widetilde{\omega}_s}}$ down to a tubular neighborhood of B and tapering to 0 away from B, then we can assume $\alpha_s|TB^{\perp_{\widetilde{\omega}_s}} = 0$. Thus Y_s is $\widetilde{\omega}_s$–orthogonal to $TB^{\perp_{\widetilde{\omega}_s}}$, so Y_s is tangent to B, and its flow ψ_s preserves B as required, completing verification of the conditions of Addendum 2.4. (The isotopy restricts to symplectomorphisms on B since each $\widetilde{\omega}_s|B = \omega_B$.) Addendum 2.6 now follows immediately from the observation that the forms $\widetilde{\omega}_s = \psi_s^*\eta_{y_0}$ on F_{y_0} all tame $J = J'$ in this case. (For the characterization of Ω, note that any two cohomologous forms taming a fixed J are isotopic by convexity of the taming condition and Moser's Theorem.)

To complete the proof of Theorem 2.3 and Addendum 2.4, we show that for sufficiently small δ, any two forms ω_u, $u = 0,1$, taming structures $J_u \in \mathcal{J}_\delta$ and representing $c_f \in H^2_{\mathrm{dR}}(X)$, are isotopic, implying that Ω is canonically defined. (Note that for $0 < \delta < \varepsilon$ we have $J \in \mathcal{J}_\delta \subset \mathcal{J}_\varepsilon$, so Ω is then independent of sufficiently small $\varepsilon > 0$ and agrees with its usage in Addendum 2.6. Metric independence follows, since for metrics g, g' on X and $\varepsilon > 0$ there is a $\delta > 0$ with $\mathcal{J}_\delta(g') \subset \mathcal{J}_\varepsilon(g)$.) Let $J_u = j((1-u)J_0 + uJ_1)$, $0 \le u \le 1$. For δ sufficiently small, this is a well-defined path from J_0 to J_1, and each J_u satisfies the defining conditions for $\mathcal{J}_\delta$ except possibly for δ–closeness to J. For δ sufficiently small, there is a compact subset K of the bundle $\mathrm{Aut}(TX) \to X$ lying in the domain of j, containing a δ–neighborhood of the image of the section J. By uniform continuity of $j|K$, we can choose $\delta \in (0,\varepsilon)$ such that J_u must be a path in $\mathcal{J}_\varepsilon$, with ε small enough to satisfy all of the previous requirements. Now for fixed $J_0, J_1 \in \mathcal{J}_\delta$, we can assume the forms η_y were constructed as above to agree with ω_V on W and tame each $J_u|\ker df_x$, $0 \le u \le 1$. Perturb the entire family as before to $J'_u \in \mathcal{J}_\varepsilon$, $0 \le u \le 1$, with each J'_u agreeing with J_V on a fixed W. For a small enough perturbation, J'_u will be ω_u–tame, $u = 0,1$. For $0 < u < 1$, the previous argument produces symplectic forms ω_u taming J'_u. The family ω_u, $0 \le u \le 1$, need not be continuous. However, each ω_u tames J'_v for v in a neighborhood of u, so splicing by a partition of unity on the interval I produces (by convexity of taming) a smooth family ω'_u taming J'_u, $0 \le u \le 1$, with $\omega'_u = \omega_u$ for $u = 0,1$. Applying Moser's Theorem to this family of cohomologous symplectic forms gives the required isotopy. □

3 Lefschetz pencils

We now return to the investigation of Lefschetz pencils (Definition 1.4) and complete the discussion of their classification theory (Proposition 3.1, cf Principle 1.5). We then apply the results of the previous section on linear 1–systems, to show that a similar topological classification theory applies in the symplectic setting (Theorem 3.3, cf Principle 1.6).

To analyze the topology of a Lefschetz critical point (eg [8]), recall the local model $f\colon \mathbb{C}^n \to \mathbb{C}$, $f(z) = \sum_{i=1}^n z_i^2$, given in Definition 1.4(2). To see that a regular neighborhood of the singular fiber is obtained from that of a regular fiber by adding an n–handle, note that the core of the n–handle appears in the local model as the ε–disk D_ε in $\mathbb{R}^n \subset \mathbb{C}^n$. Thus, the handle is attached to the fiber F_{ε^2} along an embedding $S^{n-1} \hookrightarrow F_{\varepsilon^2} - B$ whose normal bundle $\nu S^{n-1} = -iTS^{n-1}$ in the complex bundle TF is identified with T^*S^{n-1} (by contraction with $\omega_{\mathbb{C}^n}$). We will call such an embedding, together with its isomorphism $\nu S^{n-1} \cong T^*S^{n-1}$, a *vanishing cycle*. Regular fibers intersect the local model in manifolds diffeomorphic to T^*S^{n-1}, and the singular fiber is obtained by collapsing the 0–section (vanishing cycle) to a point. (The latter assertion can be seen explicitly by writing the real and imaginary parts of the equation $\sum z_i^2 = 0$ as $\|x\| = \|y\|$, $x \cdot y = 0$.) The monodromy around the singular fiber is obtained from the geodesic flow on $T^*S^{n-1} \cong TS^{n-1}$, renormalized to be 2π–periodic near the 0–section (on which the flow is undefined), and tapered to have compact support [2, 11]. At time π, the resulting diffeomorphism extends over the 0–section as the antipodal map, defining the monodromy, which is called a (positive) Dehn twist. (To verify this description, note that multiplication by $e^{i\theta}$ acts as the 2π–periodic geodesic flow on the singular fiber, and makes f equivariant with respect to $e^{2i\theta}$ on the base. Thus the sphere ∂D_ε is transported around the singular fiber by $e^{i\theta}$, returning to its original position when $\theta = \pi$, with antipodal monodromy. Away from ∂D_ε, the monodromy is obtained via projection to the singular fiber, where it can be tapered from the geodesic flow near 0 to the identity outside a compact set by an isotopy.) Given arcs $A = \bigcup A_j$ in $\mathbb{CP}^1$ as in the introduction, connecting each critical value of a Lefschetz pencil to a fixed regular value, say [1:0], we may interpret all vanishing cycles and monodromies as occurring on the single fiber $F_{[1:0]}$. The disk D_ε at each critical point extends to a disk D_j with $f(D_j) = A_j$ and ∂D_j the vanishing cycle in $F_{[1:0]}$. Following Lefschetz, we will call such a disk a *thimble*, but we also require that each $f|D_j\colon D_j \to A_j$ has a nondegenerate, unique critical point, and that there is a local trivialization of f near $F_{[1:0]}$ in which each D_j is horizontal.

All of the above structure on the local model of a critical point is compatible with suitable symplectic forms. (See [11].) For the standard Kähler form $\omega_{\mathbb{C}^n}$, the sphere ∂D_ε in $\mathbb{C}^n$ is Lagrangian in the symplectic submanifold F_{ε^2}, so by Weinstein's theorem [13] it has a neighborhood symplectomorphic to a neighborhood of the 0–section in T^*S^{n-1}. This allows Dehn twists to be defined symplectically by a Hamiltonian flow in $T^*S^{n-1} - (0\text{–section})$ [2, 11], determining the monodromy around the singular fiber up to symplectic (Hamiltonian) isotopy. The Lagrangian embedding $S^{n-1} \hookrightarrow F_{\varepsilon^2} - B$ determines a vanishing cycle, and will be called the *Lagrangian vanishing cycle* for the critical point. If ω is an arbitrary Kähler form near 0 on $\mathbb{C}^n$, any small arc A_j from $0 \in \mathbb{C}$ (such as $[0, \varepsilon^2]$ above) still determines a smooth Lagrangian thimble and vanishing cycle, by a trick of Donaldson [11, Lemma 1.13]. The disk consists of the trajectories under symplectic parallel transport (ie the flow over A_j ω–normal to the fibers) that limit to the critical point. If ω is only given to be compatible with i at 0, this structure still exists. (In fact, ω agrees at 0 with some Kähler form; after rescaling the coordinates, we may assume the two forms are arbitrarily close, as are the resulting disks and vanishing cycles. The case of an arbitrary taming ω is less clear.) For a given Lefschetz pencil $f\colon X - B \to \mathbb{CP}^1$, arcs $A \subset \mathbb{CP}^1$, and symplectic form ω on X that is symplectic on each $F_y - K$ (where $K \subset X - B$ is the critical set as in Definition 1.4(3)), any such disk at $x \in K$ is uniquely determined and uniquely extends to a Lagrangian thimble, by symplectic parallel transport.

For a closed, oriented manifold pair $B \subset F$ of dimensions $2n-4$ and $2n-2$, respectively, let $\mathcal{D} = \mathcal{D}(F, B)$ denote the group of orientation-preserving self-diffeomorphisms of F fixing (pointwise) B and $TF|B$. If ω_F is a symplectic form on F whose restriction to B is symplectic, let $\mathcal{D}_{\omega_F} = \mathcal{D}_{\omega_F}(F, B) \subset \mathcal{D}$ be the subgroup of symplectomorphisms of F fixing B and $TF|B$. Let $\delta \in \pi_0(\mathcal{D})$ be the element obtained by a 2π counterclockwise rotation of the normal fibers of B, extended in the obvious way (by tapering to id_F) to a diffeomorphism of F. In the symplectic case, δ canonically pulls back to $\delta_{\omega_F} \in \pi_0(\mathcal{D}_{\omega_F})$, determined by the Hamiltonian flow of a suitable radial function on a tubular neighborhood of B. Any vanishing cycle in $F - B$ determines a Dehn twist in $\pi_0(\mathcal{D})$ as described above. In the symplectic case, a Lagrangian embedding $S^{n-1} \hookrightarrow F - B$ determines a symplectic Dehn twist in $\pi_0(\mathcal{D}_{\omega_F})$, whose image in $\pi_0(\mathcal{D})$ is generated by the corresponding vanishing cycle. If ω' is obtained from ω_F by a pairwise diffeomorphism of (F, B), there is an induced isomorphism $\mathcal{D}_{\omega'} \cong \mathcal{D}_{\omega_F}$ sending $\delta_{\omega'}$ to δ_{ω_F} and inducing an obvious correspondence of symplectic Dehn twists.

Proposition 3.1 *Let $w = (t_1, \ldots, t_m)$ be a word in positive Dehn twists*

$t_j \in \pi_0(\mathcal{D})$, whose product $\prod_{j=1}^m t_j$ equals δ. Then there is a manifold X with a Lefschetz pencil f whose fiber over $[1{:}0] \in \mathbb{CP}^1$ is $F \subset X$, whose base locus is B, and whose monodromy around the singular fibers (with respect to fixed arcs $A \subset \mathbb{CP}^1$) is given by w. For a fixed choice of A and vanishing cycles determining the Dehn twists, such Lefschetz pencils are classified by $\pi_1(\mathcal{D})$.

Proof Choose $0 < \theta_1 < \cdots < \theta_m < 2\pi$. For each j, attach an n–handle to $D^2 \times F$ using the given vanishing cycle for t_j in $\{e^{i\theta_j}\} \times F$. We obtain a singular fibration over D^2, with m singularities as described above, and monodromy given by w. Since $\prod_{j=1}^m t_j = \delta$ is isotopic to id_F fixing B (but rotating its normal bundle clockwise), the fibration over ∂D^2 can be identified with $\partial D^2 \times (F, B)$, and the freedom to choose this identification (without losing control of $TF|B$) is given by $\pi_1(\mathcal{D})$. For any such identification, we can glue on a copy of $D^2 \times F$ to obtain a Lefschetz fibration $\widetilde{f}\colon\ \widetilde{X} \to \mathbb{CP}^1$. This $\widetilde{X}$ contains a canonical copy of $\mathbb{CP}^1 \times B$, on which $\widetilde{f}$ restricts to the obvious projection. The twist defining δ forces the normal bundle of $\mathbb{CP}^1 \times B$ to restrict to the tautological bundle on each $\mathbb{CP}^1 \times \{b\}$, so we can blow down the submanifold to obtain the required Lefschetz pencil. □

To completely determine the above correspondence between Lefschetz pencils and $\pi_1(\mathcal{D})$, we must make a choice determining which pencil corresponds to $0 \in \pi_1(\mathcal{D})$. For a fixed Lefschetz pencil, arcs A and vanishing cycles, assume the disk $D \subset \mathbb{CP}^1$ containing A is embedded so that $1 \in \partial D$ maps to the central vertex $[1{:}0]$ of A. The monodromy around ∂D is given to us as a product of Dehn twists, each of which is well-defined up to isotopies supported near its vanishing cycle. We choose an arc γ in $\mathcal{D}$ from this product to the rotation determining δ. (We are given that such arcs exist.) Since the rotation untwists to id_F by a canonical isotopy of F fixing B, we have now fixed an identification of the fibration over ∂D with $\partial D \times (F, B)$, determining the correspondence with $\pi_1(\mathcal{D})$. Note that the freedom to change γ is essentially $\pi_1(\mathcal{D})$, so unless we fix γ, the correspondence is only determined up to translations in $\pi_1(\mathcal{D})$. In the symplectic setting, we choose γ in $\mathcal{D}_{\omega_F}$ similarly, to fix a correspondence with $\pi_1(\mathcal{D}_{\omega_F})$. In this case, passing back to the smooth setting results in a correspondence between pencils and $\pi_1(\mathcal{D})$ that changes with our choice of γ in $\mathcal{D}_{\omega_F}$ only through translation by elements of $\mathrm{Im}\, i_*$, where $i_*\colon\ \pi_1(\mathcal{D}_{\omega_F}) \to \pi_1(\mathcal{D})$ is induced by inclusion. In particular, ω_F picks out a subcollection of pencils corresponding to $\mathrm{Im}\, i_*$ that is independent of our choice of γ in $\mathcal{D}_{\omega_F}$. We will see that these are precisely the pencils admitting symplectic structures suitably compatible with ω_F. For our symplectic classification, we wish to allow some flexibility in the form over the model fiber $F = F_{[1:0]}$, so we only require it to be suitably isotopic to ω_F. However, the subtlety in specifying the correspondence with $\pi_1(\mathcal{D}_{\omega_F})$ forces us to keep track of a preassigned isotopy on F. Thus, we classify pairs consisting of a suitable symplectic form ω on X and a suitable isotopy from $\omega|F_{[1:0]}$ to ω_F, up to deformations of such pairs.

To state the theorem we need one further fact. It is natural to study symplectic forms on X that are symplectic on the fibers of f, but this condition makes no sense on the critical set K. For that we show that f determines a complex structure J^* on $TX|K$, and require our forms to be compatible with J^*. We also require similar compatibility normal to B.

Lemma 3.2 *A Lefschetz pencil canonically determines a complex structure J^* on $TX|K$ (for $n \neq 1$) and on any subbundle ν of $TX|B$ complementary to TB. J^* is obtained by restricting any local (ω_{std}, f)–tame almost-complex structure J defined near a point of K or B, provided ν is J–complex in the latter case.*

Proof First check that in a standard chart at $x \in K$, each hyperplane through 0 is a limit of tangent spaces to regular fibers, so it is J–complex for any (ω_{std}, f)–tame local J. Any 1–dimensional complex subspace at x is an intersection of such hyperplanes, so it is also J–complex for any such J. But J_x is uniquely determined by its complex lines for $n \neq 1$ ([5, Lemma 4.4(a)], cf also Lemma 2.2). For $x \in B$, we obtain a suitable J from the complex bundle structure π of Definition 1.4(1), by perturbing the latter to have fibers tangent to ν as preceding Lemma 2.1. That lemma (which only requires J locally) then gives uniqueness on ν. □

Theorem 3.3 *Let ω_F be a symplectic form on (F, B) as preceding Proposition 3.1, with $[\omega_F] \in H^2_{\text{dR}}(F)$ Poincaré dual to B. Let $S_1, \dots, S_m$ be Lagrangian embeddings $S^{n-1} \hookrightarrow F - B$ determining a word $w = (t_1, \dots, t_m)$ in positive symplectic Dehn twists with $\prod_{j=1}^m t_j = \delta_{\omega_F} \in \pi_0(\mathcal{D}_{\omega_F})$. If $n = 2$, assume each component of each $F - S_j$ intersects B. Then the corresponding symplectic Lefschetz pencils are classified by $\pi_1(\mathcal{D}_{\omega_F})$. More precisely, a Lefschetz pencil $f\colon X - B \to \mathbb{CP}^1$ obtained from $S_1, \dots, S_m$ as in Proposition 3.1, with a fixed choice of thimbles D_j bounded by S_j and covering the given arcs A, corresponds to an element of $\operatorname{Im} i_*$ if and only if X admits a symplectic structure ω that*

(1) *on $(F_{[1:0]}, B)$ comes with a pairwise isotopy to ω_F, defining a deformation of forms that is fixed on B and $S_1, \dots, S_m$,*

(2) *is symplectic on each* $F_y - K$ *and (for* $n \neq 2$*) Lagrangian on each* D_j,

(3) *is compatible with* J^* *on* $TX|K$ *(for* $n \geq 2$*) and on the* ω*–normal bundle* ν *of* B, *and*

(4) *satisfies* $[\omega] = c_f \in H^2_{\mathrm{dR}}(X)$.

For fixed f *and* $D_1, \ldots, D_m$, *such forms are classified up to deformation through such forms by* $\pi_2(\mathcal{D}/\mathcal{D}_{\omega_F})$, *and classified by* $\ker i_*$ *if symplectomorphisms preserving* f *and fixing* $f^{-1}(A)$ *are also allowed.*

Note that in this classification, a single ω with different isotopies as in (1) could represent distinct equivalence classes. Since a deformation with $[\omega]$ fixed determines an isotopy, the isotopy classes of forms on X as above (for fixed f) are classified by the quotient of $\pi_2(\mathcal{D}/\mathcal{D}_{\omega_F})$ by some equivalence relation. In our case, $[\omega] = c_f$ is Poincaré dual to the fiber class $[F_{[1:0]}]$, the same condition that arises in Donaldson's construction of Lefschetz pencils on symplectic manifolds [4].

Proof First we assume $n \geq 3$ and prepare to apply Theorem 2.3 by constructing a smooth family σ of symplectic structures on the fibers of a fixed f. As in that proof, the cohomology class of $\omega_B = \omega_F|B$ equals the normal Chern class of B in F, so we can define the model symplectic form ω_V and almost-complex structure J_V on $V \subset X$ as before (starting from any fixed choice of $\pi\colon V \to B$ as in Definition 1.4(1) and any ω_B–tame J_B on B). By Weinstein's theorem, we can assume ω_V agrees with ω_F near B on $F = F_{[1:0]}$. At each critical point x_j, choose a standard chart for f (necessarily inducing J^* on $T_{x_j}X$). Then D_j is not tangent to any complex curve at x_j (since $f|D_j$ is nondegenerate and f is constant or locally surjective on any complex curve). Thus there is a complex isomorphism $(T_{x_j}X, T_{x_j}D_j) \cong (\mathbb{C}^n, \mathbb{R}^n)$, and $\omega_{\mathbb{C}^n}$ pushes down to a symplectic form ω_j near x_j, compatible with J^* at x_j, and Lagrangian on a disk $\Delta_j \subset D_j$ containing x_j. Since D_j is a thimble for f, we can identify the fibers over $\operatorname{int} A_j$ with F by an isotopy in X fixing B and preserving D_j, so that S_j matches with $\partial\Delta_j$ (with the correct normal correspondence) and their tubular neighborhoods in the fiber correspond symplectically (by Weinstein) relative to ω_F and ω_j, respectively. We can assume (by U(2)–invariance of ω_V) that the isotopy is ω_V–symplectic near B, and that it agrees near K with symplectic parallel transport in the local model. Thus it maps the tubular neighborhood of $\partial\Delta_j$ in its fiber to a neighborhood of the singularity in the singular fiber, by a map that is a symplectomorphism except on $\partial\Delta_j$, which collapses to the singular point (cf [11]). Pulling ω_F back by the isotopy now gives a family σ of symplectic structures on the fibers over A, agreeing with the local models ω_V and ω_j near B and K. Extend σ over a disk $D\subset\mathbb{CP}^1$ whose interior contains $A-[1{:}0]$. Since $\prod_{j=1}^m t_j=\delta_{\omega_F}$ is symplectically isotopic to id_F fixing B (rotating its normal bundle), we can fix a path γ in $\mathcal{D}_{\omega_F}$ as above and identify all fibers over ∂D with (F,B) so that σ is constant. The set of all such choices of identification (agreeing with the given one on $F_{[1:0]}$ and $TF|B$, up to fiberwise symplectic isotopy fixing $TF|B$) is then given by $\pi_1(\mathcal{D}_{\omega_F})$. Passing to $\pi_1(\mathcal{D})$ classifies Lefschetz pencils f as in Proposition 3.1, and σ has a constant extension over the remaining fibers of any f coming from $\operatorname{Im} i_*\subset\pi_1(\mathcal{D})$ (for any choice of element in the corresponding coset of $\ker i_*$).

Next we apply Theorem 2.3. By contractibility of the space of σ–tame complex structures on each $T_xF_{f(x)}$, we obtain a σ–tame family of almost-complex structures on the fibers. After declaring a suitable horizontal distribution to be complex, we obtain a fiberwise σ–tame complex structure J on X, which we may assume agrees with J_V near B and with the structures on the chosen standard charts near the critical points. Now (f,J) is a linear 1–system as required. Set $\nu=(\ker d\pi)|B$. For each F_y, define η_y on a neighborhood W_y by pulling back $\sigma|F_y$ by a map $r\colon W_y\to W_y$ collapsing W_y onto F_y away from $B\cup K$ (cf proof of Theorem 2.3). Then each η_y agrees with ω_V on a fixed neighborhood of B, and with ω_j on a neighborhood of the critical point if F_y is singular. We can assume each $\eta_y|D_j\cap W_y=0$. Let ζ be any form on X representing c_f, agreeing with $\eta_{[1:0]}$ near $F_{[1:0]}$, and vanishing on each thimble. (Note $c_f|F_{[1:0]}=[\omega_F]=[\eta_{[1:0]}]|F_{[1:0]}$ as required, and the thimbles add no 2–homology to $W_{[1:0]}$ since $n\ge 3$.) The condition $[\eta_y-\zeta]=0\in H^2_{\mathrm{dR}}(W_y,B)$ is trivially true for $y=[1{:}0]$. The case of any regular value y then follows since F_y comes with an isotopy rel B in X to $F_{[1:0]}$, sending $\eta_y|F_y$ to ω_F. For a critical value, we can assume W_y is obtained from a tubular neighborhood of a regular fiber by adding an n–handle. Since $n\ge 3$, the handle adds no 2–homology, so the condition holds for all y. Now Theorem 2.3 and Addendum 2.6 provide a unique isotopy class of symplectic forms ω on X taming J, with $[\omega]=c_f$ and $\omega|F_{[1:0]}$ pairwise isotopic to ω_F. This ω can be assumed to satisfy the required conditions for Theorem 3.3: Compatibility of ω with $J^*=J$ is given on ν. It follows on K since ω is made from $\omega_t=t\eta+f^*\omega_{\mathrm{std}}$; the second term vanishes on K, and the first agrees with each $t\omega_j$ if we set $\{\hat{y}_1,\dots,\hat{y}_m\}=f(K)\cup\{[1{:}0]\}$ when applying Theorem 2.5. Similarly, the thimbles D_j are Lagrangian, since $f^*\omega_{\mathrm{std}}$ vanishes on them, as does η if the forms α_y arising from Theorem 2.5 are chosen to vanish there. (This can be arranged since $H^1_{\mathrm{dR}}(S^{n-1})=0$ for $n\ge 3$, and $\alpha_{[1:0]}=0$.) The forms $\widetilde{\omega}_s=\psi_s^*\eta_{y_0}$ in the deformation constructed for (1) also restrict to scalar multiples of $\eta=0$ on each S_j as required.

For fixed X and f, we wish to compare two arbitrary symplectic structures

ω_u, $u = 0, 1$, satisfying the conclusions of the theorem. We adapt the previous procedure to 1–parameter families, beginning with the construction of σ. For $u = 0, 1$, choose $\pi_u\colon V_u \to B$ with fibers tangent to the ω_u–normal bundle $\nu_u \to B$, and construct structures $\omega_{V,u}$ and $J_{V,u}$ as before, using the Hermitian form $\omega_u|\nu_u$. By Weinstein (cf proof of Theorem 2.3), we can assume $\omega_u = \omega_{V,u}$ near B after an arbitrarily C^1–small isotopy. (First isotope π_u to get equality on $F_{[1:0]}$, preserving the fibers of f, then isotope ω fixing $F_{[1:0]}$.) Let σ_u be the family obtained by restricting ω_u to the fibers. Symplectic parallel transport gives a fiber-preserving map $\varphi_u\colon A \times F_{[1:0]} \to X$ for which $\varphi_u^*\sigma_u$ is constant, and the Lagrangian thimbles D_j are horizontal. Now smoothly extend π_u, $\omega_{V,u}$, $J_{V,u}$ and φ_u for $0 \le u \le 1$. Also extend ω_u near K by linear interpolation, so that it is J^*–compatible on K and has Lagrangian disks Δ_j, $0 \le u \le 1$. Condition (1) gives a pairwise isotopy from ω_0 through ω_F to ω_1 on $(F_{[1:0]}, B)$. Use this to extend σ_u for $0 \le u \le 1$ to $F_{[1:0]}$ with $\sigma_{1/2} = \omega_F$, and then extend as before to the fibers over A and D, agreeing with $\omega_{V,u}$ near B and ω_u near K, and with $\varphi_u^*\sigma_u$ constant for each u.

To complete the construction of σ_u, $u \in I = [0,1]$, over the fibers of $\mathrm{id}_I \times f$ on $I \times X$, we attempt to fill the hole over $(0,1) \times (\mathbb{CP}^1 - D)$, encountering obstructions. As before, we can smoothly identify all fibers over $I \times \partial D$ with $F_{[1:0]}$ so that the family σ_u is constant for each u. To fix this identification τ_u for each u, pull the preassigned path γ back from $\mathcal{D}_{\omega_F}$ to $\mathcal{D}_{\sigma_u|F_{[1:0]}}$ by the given isotopy. This moves the spheres S_j, but the same isotopy shows how to restore them to their original position through families of Lagrangian spheres (by the last part of (1)), yielding a canonically induced path γ_u from the required representative of $\prod t_j$ to δ_{σ_u} in $\mathcal{D}_{\sigma_u|F_{[1:0]}}$, which determines τ_u. Now let $\widetilde{\tau}_0$ be the identification of all fibers over $\{0\} \times (\mathbb{CP}^1 - \operatorname{int} D)$ with $F_{[1:0]}$ by ω_0–symplectic parallel transport along straight lines to [1:0] in $\mathbb{CP}^1 - \operatorname{int} D$. Comparing $\widetilde{\tau}_0|\partial D$ with τ_0, we obtain the element of $\pi_1(\mathcal{D})$ classifying f, and see that this must lie in $\operatorname{Im} i_*$ (being explicitly represented by a loop in $\mathcal{D}_{\sigma_0|F_{[1:0]}} \simeq \mathcal{D}_{\omega_F}$). The corresponding construction of $\widetilde{\tau}_1$ from ω_1 also gives f, so $\widetilde{\tau}_0|\partial D$ and $\widetilde{\tau}_1|\partial D$ differ by an element β of $\ker i_*$ (after we extend each $\widetilde{\tau}_u$ over $I \times \partial D$ to $u = 1/2$ so that we can work with the fixed symplectic form ω_F). Similarly, a direct comparison of $\widetilde{\tau}_0$ and $\widetilde{\tau}_1$ yields an element $\alpha \in \pi_2(\mathcal{D}, \mathcal{D}_{\omega_F}) \cong \pi_2(\mathcal{D}/\mathcal{D}_{\omega_F})$ with $\partial_*\alpha = \beta$. If $\alpha = 0$, we can extend σ_u over X for $0 \le u \le 1$. If only $\beta = 0$, then we can perturb $\widetilde{\tau}_1$ so that $\alpha \in \pi_2(\mathcal{D})$, and α provides a self-diffeomorphism of X preserving f and fixing $f^{-1}(D)$, after which σ_u extends. These vanishing conditions are also necessary for the deformation and symplectomorphism, respectively, specified by the theorem, since any allowable deformation ω_u, $0 \le u \le 1$, determines a family $\widetilde{\tau}_u$ as above interpolating between $\widetilde{\tau}_0$ and $\widetilde{\tau}_1$, showing that $\alpha = 0$. (Note that the family ω_u comes with a smooth family of isotopies from $\omega_u|F_{[1:0]}$ to ω_F as in (1), allowing us to continuously pull back γ to each $\mathcal{D}_{\sigma_u|F_{[1:0]}}$ as before, to define the required interpolation τ_u between τ_0 and τ_1 in the absence of the condition $\sigma_{1/2}|F_{[1:0]} = \omega_F$.)

To complete the proof for $n \geq 3$, it suffices to construct the required deformation between ω_0 and ω_1 from the completed family σ_u on $I \times X$. First find a continuous family J_u of fiberwise σ_u–tame almost-complex structures on X as before, using a horizontal distribution that is ω_u–orthogonal to the fibers when $u = 0, 1$. Then J_u is ω_u–tame, $u = 0, 1$. For $0 < u < 1$, the previous argument now produces suitable symplectic structures ω_u on X, with σ_u replacing ω_F in (1). The family ω_u, $0 \leq u \leq 1$, need not be continuous (particularly at 0,1), but we can smooth it as for Theorem 2.3, with a partition of unity on I, obtaining the required deformation ω'_u: First pull each ω_u back to a neighborhood I_u of $u \in I$, by a map preserving the bundles ν_u, forms ω_B and disks D_j. For I_u sufficiently small, the required conditions (1–4) are preserved, where the isotopy in (1) from ω_u to σ_u on $F_{[1:0]}$ at each $v \in I_u$ is through J_v–taming forms (by Addendum 2.6) and is defined to be constant for $u = 0, 1$. Splicing by a partition of unity on I preserves (2–4), producing the required family ω'_u and a 2–parameter family of symplectic structures on $F_{[1:0]}$, interpolating between ω'_u and a convex combination σ'_u of nearby forms σ_v for each u. If the intervals I_u were sufficiently small, we can extend by $(1-s)\sigma'_u + s\sigma_u$ to obtain a 2–parameter family of symplectic structures from ω'_u to σ_u, all agreeing with ω_F on B and each S_j, and constant for $u = 0, 1$. Moser's technique, parametrized by u, now produces a 2–parameter family of diffeomorphisms of (F, B), which can be reinterpreted as a smooth family of isotopies as in (1) from $\omega'_u|F_{[1:0]}$ to $\omega_F = \sigma_{1/2}|F_{[1:0]}$, interpolating between the given ones for ω_0, ω_1.

For the remaining case $n \leq 2$, inclusion $\mathcal{D}_{\omega_F} \subset \mathcal{D}$ is a homotopy equivalence, so we must show each f has a unique deformation class of structures ω as specified. The case $n = 1$ ($f\colon X \to \mathbb{CP}^1$ a simple branched covering) is trivial, so we assume $n = 2$. Then f is a hyperpencil, so [5, Theorem 2.11(b)] gives the required form ω, provided we arrange J^*–compatibility as before. Uniqueness of the deformation class follows the method of proof of [6, Theorem 1.4] with $m = 0$ and K replaced by $K \cup B$: Given two forms ω_u, $u = 0, 1$, as specified in Theorem 3.3, we can find an $(\omega_{\mathrm{std}}, f)$–tame, ω_u–tame almost-complex structure J_u for each u (cf also [5, Lemma 2.10]). We interpolate to a family J_u, $0 \leq u \leq 1$ (eg by contractibility in [5, Theorem 2.11(a)]), and construct the required deformation ω_u, $0 \leq u \leq 1$, using a partition of unity on I as before. Conditions (1–4) for each ω_u are easily verified, with only (1) requiring comment: We are given isotopies rel B from $\omega_u|F_{[1:0]}$ to ω_F, $u=0,1$. Moser provides an isotopy rel B from ω_0 to ω_1 on $F_{[1:0]}$. Combining these three isotopies gives a path representing an element of $\pi_1(\mathcal{D}', \mathcal{D}'_{\omega_F})$, where the prime indicates rotations of $TF|B$ are allowed. This group vanishes, allowing us to extend our path into the required 2–parameter family of diffeomorphisms. □

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Department of Mathematics, The University of Texas at Austin
1 University Station C1200, Austin, TX 78712–0257, USA

Email: gompf@math.utexas.edu

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