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Z_2–Systolic-Freedom

Michael H Freedman

Abstract We give the first example of systolic freedom over torsion coefficients. The phenomenon is a bit unexpected (contrary to a conjecture of Gromov's) and more delicate than systolic freedom over the integers.

Dedicated to Rob Kirby, a lover of Mathematics and other wild places. Thank you for your inspiration.

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0 Introduction

For closed Riemannian surfaces, whose topology is different from the 2–sphere,

$$A \quad \frac{2}{\ }L^2 \qquad (0.1)$$

where A is area and L is the length of the shortest essential loop. The boundary case is a round projective plane. See [9] and [5] for a discussion. For closed manifolds of higher dimensions, such "systolic inequalities" have been the focus of much research and many interesting counter-examples exist [1], [6], and [7].

We recall some definitions:

Let M be a closed Riemannian manifold of dimension n and let $0 \quad p; q \quad n;$ $p + q = n$.

$$systole_k(M) = \inf area_k[\ ] \qquad (0.2)$$

where the infimum is taken over all smooth oriented k–cycles with $[\ ] \neq 0 \in H_k(M;Z)$:

$$Z_2 - systole_k(M) = \inf area_k(\ ) \qquad (0.3)$$

where the infimum is taken over unoriented k–cycles , $[\ ] \neq 0 \in H_k(M;Z_2)$:

$$stable - systole_k(M) = \inf\ stable - area_k[\ ] \qquad (0.4)$$

where $[\ ] \neq 0 \; 2 \; H_k(M;Z)$=torsion and

$$stable-area_k \quad = \inf \frac{1}{i} \inf_{i[\ ]} area_k(\ )$$

where $i = 1;2;3;:::$ in the inner infimum is over oriented cycles representing $i[\ ]:$

Gromov proved (see [8] for discussion and generalizations) that \stable systolic rigidity" holds for any product of spheres $S^p \quad S^q =: M^n$, that is there a constant $c(n)$ so that for any Riemannian metric on $M^n = S^p \quad S^q; p+q = n$:

$$vol(M) = stable-systole_n(M) \quad c\ stable-systole_p(M)\ stable-systole_q(M) \qquad (0{:}5)$$

Surprisingly, he also discovered that the corresponding unstable statement is false:

Let $M_r = S^3_r \quad \mathbb{R}=(\ ;t) \quad (\sqrt{r}\quad ;t+1)$, where S^3_r is the 3{sphere of radius r and the identification matches a point with its $\sqrt{r}${rotation along Hopf fibers displaced one unit in the real coordinate. For this r{family of metrics on $S^3 \quad S^1$, we have \(3;1){systolic freedom"

$$\frac{systole_4(M_r)}{systole_3(M_r)\,systole_1(M_r)} = \frac{O(r^3)}{O(r^3)\ O(r^{1=2})}\ !\ 0 \text{ as } r\ !\ 1 \qquad (0{:}6)$$

This original example of systolic freedom has been vastly generalized by several authors (see [1] for an overview and recent advances) to show that \freedom" rather than \ridigidity" predominates for dimension $n \quad 3$.

This left the case of Z_2 coefficients open for $n \quad 3$. This case has a remarkable relevance in quantum information theory, which is the subject of another paper [4]. Classically, there is only one type of error: the \bit flip." In a quantum mechanical context the algebra of possible errors has two generators: \bit flip" and \relative phase." It is possible to map the problem of correcting these (Fourier) dual errors onto the problem of specifying (Poincare) dual cycles in a manifold. Torsion coefficients for the cycles corresponds to finite dimensional quantum state spaces: Z_2{coefficients correspond to expressing quantum states in terms of qubits.

It is reported in [9] that Gromov conjectured Z_2{rigidity, ie, systolic inequalities like $(0{:}1)$ and $(0{:}5)$ would hold in this case of Z_2 coefficients. The ease with which nonoriented cycles can be modified to reduce area, particularly in codimension equal to 1, is well known in geometric measure theory and lends support to the idea that at least $Z_2 - (n-1;1)${rigidity might hold. In fact, the opposite is the case. We will exhibit a family of Riemannian metrics on $S^2 \times S^1$ exhibiting $Z_2-(2,1)$–systolic freedom: the ratios $(Z_2-systole_3/Z_2-systole_2 \cdot Z_2-systole_1)$ approach zero as the parameter approaches infinity. Moreover, from this example, as in [1], quite general Z_2–freedoms can be found.

In section 3, we discuss the quantification of systolic freedom and note that the present example for Z_2–freedom is measured by a function growing more slowly than log whereas in Gromov's original example freedom grows by a power, and in an example of Pittet [11] freedom grows exponentially. It is now of considerable interest, particularly in connection with quantum information theory, whether the “weakness” of Z_2–freedom is an artifact of the example or inherent.

1 The Example

As raw material, we use a sequence of closed hyperbolic surfaces Σ_g of genus $g \to \infty$ with the following three properties:

(i) $\lambda_1(\Sigma_g) \geq c_1$, λ_1 being the smallest eigenvalue of the Laplacian on functions,

(ii) There exits an isometry $\tau: \Sigma_g \to \Sigma_g$, with order $(\tau) \geq c_2(\log g)^{1/2}$, and

(iii) The map $\Sigma_g \to \Sigma_g/\langle\tau\rangle$ is a covering projection and the base surface $_gS =: \Sigma_g/\langle\tau\rangle$ has injectivity radius $(_gS) \geq c_3(\log g)^{1/2}$

where $c_1, c_2,$ and c_3 are positive constants independent of g.

We will return to the construction of the family $\{\Sigma_g\}$ at the end of this section. Let $M_g = (\Sigma_g \times \mathbb{R})/(x,t) \sim (\tau x, t+1)$ be the Riemannian “mapping torus” of τ. We can also think of $M_g = \Sigma_g \times [0,1]/(x,0) \sim (\tau x, 1)$. By two theorems of Lickorish [10], we may first write τ^{-1} out in the mapping class group of Σ_g as a product of Dehn twists δ_i along simple loops $\gamma_i \subset \Sigma_g$:

$$\tau^{-1} = \delta_{n_g} \circ \cdots \circ \delta_2 \circ \delta_1 \tag{1.1}$$

and second perform Dehn surgeries along pushed-in copies of $\{\gamma_i\}$:

$$\left\{\gamma_1 \times \left(\frac{1}{2}+\frac{1}{3n_g}\right), \gamma_2 \times \left(\frac{1}{2}+\frac{2}{3n_g}\right), \ldots, \gamma_i \times \left(\frac{1}{2}+\frac{i}{3n_g}\right), \ldots, \gamma_{n_g} \times \left(\frac{1}{2}+\frac{1}{3}\right)\right\}$$

to obtain a diffeomorphic copy of $\Sigma_g \times [0,1]$ whose product structure induces $[\tau^{-1}]: \Sigma_g \times 0 \to \Sigma_g \times 1$.

Thus, n_g Dehn surgeries on M_g produce the mapping torus for $^{-1}$, ie $_g$ S^1.

In [4], we will find upper bounds on both n_g and max length (y_i) in order to compute a lower bound on the Z_2{freedom function. To merely establish Z_2{freedom, we do not need these estimates. To convert $_g$ S^1 to S^2 S^1 an additional $2g$ Dehn surgeries are needed: Do half (a \sub kernel") of these surgeries at level $\frac{1}{2} + \frac{1}{6n_g}$ and the dual half at level $\frac{1}{2}$. The result of all $n_g + 2g$ Dehn surgeries is topologically S^2 S^1, and once these surgeries are metrically specified, we obtain a sequence of Riemannian 3{manifolds $(S^2\ \ S^1)_g =: S^2$ S^1_g.

In section 2 where Z_2{freedom is established, four metrical properties of these surgeries will be referenced.

They are:

(A) *The core curves for the Dehn surgeries are taken, for convenience, to be geodesics in* $_g$ $[0;1]$ *so that the boundaries* $@T_{i;}$ *of their neighborhoods are Euclidean flat.* (1.2)
Also > 0 *is chosen very small. See (D).*

(B) *The replacement solid tori* $T^0_{i;}$ *have* $@T^0_{i;}$ *isometric to* $@T_{i;}$ *and are defined as twisted products* D^2 $[0;2\ ]=$ *where* ( ) *is a constant slightly larger than so that the meridians in* $T_{i;}$ *have length* 2 *and is an isometric rotation of the disk* D^2 *adjusted to equal the holonomy obtained by traveling orthogonal to the surgery slopes in* $@T_{i;}$ *from* $@D^2$ *pt back to itself.* (1.3)

(C) *The geometry on the disk* D^2 *above is rotationally symmetric and has a product collar on its boundary as long as the boundary itself.* (1.4)

(D) *Finally,* > 0 *is so small that the total volume of all the replacement solid tori,* $[_i T^0_{i;}$ *is* $o(g)$. (1.5)

With specifications: $(A) ::: (D)$, Dehn surgery yields a precise-smooth Riemannian manifold for which all the relevant notions of p{area are defined. We could work in this category but there is no need to do so since perturbing to a smooth metric will not effect the status of (Z_2) systolic freedom versus rigidity.

It is now time to return to the construction of the family $f\ \ _g g$. We follow an approach of [13] and [14] in considering the co-compact torsion free Fuchscian group $\Gamma_{(-1;p)}$, the group of unit norm elements of the type $\frac{-1;p}{Q}$ quaternion algebra where p is prime and p 3 mod 4. The group Γ may be explicitly written as:

$$\Gamma_{(-1,p)} = \left\{ \begin{pmatrix} a+b\sqrt{p} & -c+d\sqrt{p} \\ c+d\sqrt{p} & a-b\sqrt{p} \end{pmatrix} : a,b,c,d \in Z, \det = 1 \right\} \Big/ \mathrm{id}. \qquad (1.6)$$

Analogous to the congruence of $SL(2,R)$, we have for integers $N > 2$ the normal subgroups of $\Gamma_{-1,p}$,

$$\Gamma_{(-1,p)}(N) = \left\{ \begin{pmatrix} 1+N(a+b\sqrt{p}) & N(-c+d\sqrt{p}) \\ N(c+d\sqrt{p}) & 1+N(a-b\sqrt{p}) \end{pmatrix} : a,b,c,d \in Z \right\} \Big/ \mathrm{id}. \qquad (1.7)$$

which are known ([13] and [12]) to satisfy (i).

In Lemma 2 [14] it is proved that:

$$inj.\ rad.(\mathbb{H}^2/\Gamma_{-1,p}(N)) = O(\log N) \qquad (1.8)$$

and in the proof of Theorem 6 that genus $(\mathbb{H}^2/\Gamma_{-1,p}(N)) =: genus\,(\Sigma(N)) =: genus(\Sigma_g) =: genus(N)$ satisfies:

$$O(N^2) \leq genus(N) \leq O(N^3) \qquad (1.9)$$

so

$$inj.\ rad.\ \Sigma_g = O(\log g) \qquad (1.10)$$

Now choose a sequence of h and g to satisfy $\log g = O(\log h)^2$ and so that $N(h)$ divides $N(g)$. Thus, we have a covering projection $\Sigma_g \longrightarrow \Sigma_h$. Let γ be the shortest essential loop in Σ_h, by (1.10) length$(\gamma) = O(\log h)$. Choosing a base point on γ, $[\gamma] \in \Gamma_{(-1,p)}(N(h))/\Gamma_{(-1,p)}(N(g))$ satisfies:

$$order[\gamma] \geq O(\log(h)) = O(\log g)^{1/2} \qquad (1.11)$$

since the translation length of $\gamma = O(\log g)^{1/2}$ must be multiplied by $O(\log g)^{1/2}$ before it reaches length $O(\log g)$, a necessary condition to be an element in the subgroup $\Gamma_{(-1,p)}(N(g))$.

Let τ be the translation determined by $[\gamma]$. We have just checked condition (ii) order $(\tau) > O(\log g)^{1/2}$. Factor the previous covering as:

$$\Sigma_g \longrightarrow \Sigma_g/\langle\tau\rangle \longrightarrow \Sigma_h \qquad (1.12)$$

and set $\Sigma_g/\langle\tau\rangle =: {}_gS.$ Since ${}_gS$ covers Σ_h, we conclude condition (iii):

$$inj.\ rad.({}_gS) \geq inj.\ rad.(\Sigma_h) \geq O(\log h) = O(\log g)^{1/2}. \qquad (1.13)$$

2 Verification of Freedom

We regard the Riemannian manifold $S^2 \times S^1_g$ as essentially specified in section 1. Technically, there is the parameter ϵ to be analyzed in [4] which controls the “thickness” of the Dehn surgeries. On two occasions, we demand this to be sufficiently small (the cost is an increase in the maximum absolute value of the Riemann curvature tensor as a function of g). The first occurrence is in the next proposition.

Proposition 2.1 $vol(S^2 \times S^1_g) = \mathbb{Z}_2-systole_3(S^2 \times S^1_g) = O(g)$

Proof $Volume(M_g) = vol(\Sigma_g \times [0,1] = area(\Sigma_g) = 2\pi \chi(\Sigma_g) = O(g))$. By choosing $\epsilon > 0$ small enough as a function of g, the Dehn fillings contribute negligible volume so this property is retained by $S^2 \times S^1_g$. □

The next proposition is more subtle.

Proposition 2.2 $\mathbb{Z}_2 - systole_2(S^2 \times S^1_g) = O(g)$

Proof According to [3] a non-oriented minimizer among all nonzero codimension one cycles always exists and is smooth provided the ambient dimension is at most 7. Let $X_g \subset S^2 \times S^1_g$ be this minimizer. For a contradiction, assume $area(X_g) < O(g)$.

The Dehn surgeries in section 1 were confined to $\Sigma_g \times [\frac{1}{2}, 1]$, so the surfaces $\Sigma_g \times t$, $t \in (0, \frac{1}{2})$ persist as submanifolds of $S^2 \times S^1_g$. By Sard's theorem, for almost all $t \in (0, \frac{1}{2})$, $\Sigma_g \times t$ intersects X_g transversely. Let W_t, $t \in (0, \frac{1}{2})$ denote the intersection. By the co-area formula.

$$O(g) > area(X_g) \geq \int_{t=0}^{1/2} length(W_t)\, dt \tag{2.1}$$

Consequently, for some transverse $t \in (0, \frac{1}{2})$,

$$length(W_t) < O(g) \tag{2.2}$$

Since both $\Sigma_g \times t$ and X_g represent the nonzero element of $H_2(S^2 \times S^1_g; \mathbb{Z}_2)$, the complement $S^2 \times S^1_g \setminus (\Sigma_g \times t \cup X_g)$ can be two colored into black and white regions (change colors when crossing either surface) and the closure B of the black points is a piecewise smooth $\mathbb{Z}_2$–homology between $\Sigma_g \times t$ and X_g.

For homological reasons, the reverse Dehn surgeries $S^2 \quad S^1_g \rightsquigarrow M_g$ have cores with zero (mod 2) intersection with X_g. This means that the tori $\partial T_{i,} = \partial T'_{i,}$ each meet X_g in a null homologous, probably disconnected, 1–manifold $X_g \cap \partial T_{i,} \quad \partial T_{i,}$. Again, if is a sufficiently small function of g, we may “cut off” X_g along these tori to form

$$X'_g = (X_g \setminus \cup_i T_{i,}) \cup \quad {}_{i,}$$

where ${}_i$ denotes a bounding surface for $X_g \cap \partial T_{i,}$ in $\partial T_{i,}$, with negligible increase in area. In particular, we still have:

$$area(X'_g) < O(g) \tag{2.3}$$

More specifically choose ${}_i$ to be the “black” piece of $\partial T_{i,}$, ie ${}_i \quad B$. If we set

$$B' = closure(B \setminus \cup_i T_{i,})$$

and recall

$$\cup_i T_{i,} \cap \quad {}_g \quad t = \emptyset,$$

we see that B' is a Z_2–homology from X'_g to ${}_g \quad t$.

It is time to use property (i): W_t separates ${}_g \quad t$ into two subsurfaces meeting along their boundaries: One subsurface sees black on the positive side, the other on its negative side. An inequality of Buser’s [2], a converse to the Cheeger’ isoperimetric inequality, states that $area > constant \quad length$, in the presence of bounded sectional curvatures, yields an upper bound on ${}_1$. Thus, the smaller of these two subsurfaces, call it $Y \quad {}_g \quad t$ must satisfy:

$$area(Y) \quad c_4 \; length(W_t) \tag{2.4}$$

where c_4 is independent of g. Combining with line (2.2), we have:

$$area(Y) \quad O(\log g) \tag{2.5}$$

Now modify X'_g to Z by cutting along W_t and inserting two parallel copies of Y. This may be done so that the result is disjoint from ${}_g \quad t$ but bordant to it by a slight modification B'' of B', with B'' still disjoint from ${}_g \quad t$. See Figure 2.1 and Figure 2.2.

combining (2.3) and (2.5):

$$area(Z) \quad 3 \quad O(\log g) = O(\log g) \tag{2.6}$$

Now reverse the Dehn surgeries and consider:

$$(B''; \; {}_g \quad t; Z) \quad M_g \setminus \; {}_g \quad (t \quad) \quad M_g. \tag{2.7}$$

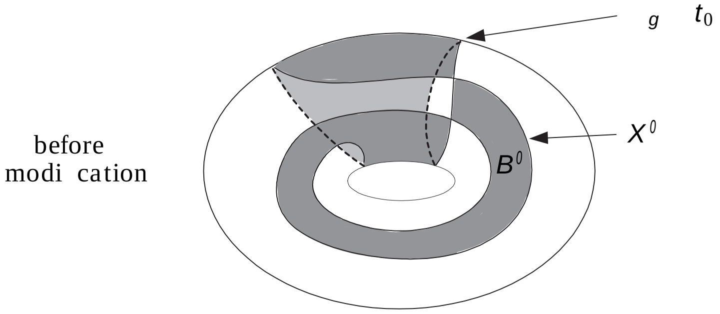

Figure 2.1

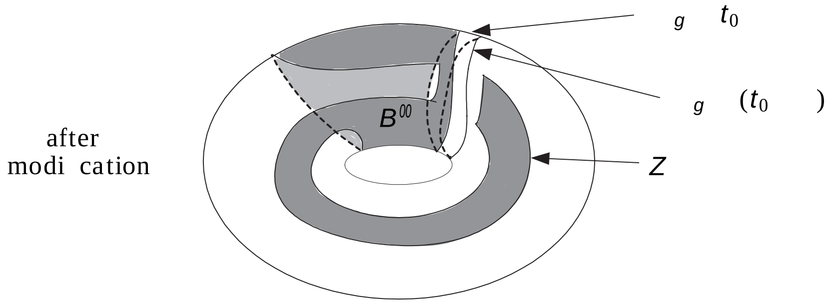

Figure 2.2

The middle term of line (2.7) is diffeomorphic to $_g$ $\mathbb{R}$, which is a codimension 0 submanifold of $\mathbb{R}^3$. This proves that B^{00} and in particular Z is orientable. But this looks absurd. Apparently, we have constructed an oriented surface Z oriented-homologous to the fiber $_g$ t of M_g of smaller area (compare line (2.6) with the first line in the proof of proposition 1.1).

Let $\frac{@}{@t}$ be the divergenceless flow in the interval direction on M_g. Lift Z to $\hat{Z}$ in the infinite cyclic cover $_g$ $\mathbb{R}$ and consider the flow through the lift $\hat{B}^{00}$, the lift of B^{00}. The divergence theorem states that the flux through $\hat{Z}$ is equal to the flux through $_g$ t. Since $\frac{@}{@t}$ is orthogonal to $_g$ t,

$$area(\ _g\ \ t\ )\quad area(\hat{Z}) = area(Z) \qquad (2{:}8)$$

completing the contradiction. □

Proposition 2.3 $Z_2 - systole_1(S^2\ \ S^1_g)\quad O(\log g)^{1=2}$

Proof We actually show that any homotopically essential loop obeys this estimate. The long collar condition C (section 1) implies that any arc in $T^0_{i;}$ with end points on $@T^0_{i;}$ can be replaced with a shorter arc with the same end points lying entirely within $@T^0_{i;}$. It follows that any essential loop in $S^2 \quad S^1_g$ can be homotoped to a shorter loop lying in the complement of the Dehn surgeries.

Thus, it is sufficient to show that any homotopically essential loop γ in M_g has $length\,\gamma \quad O(\log g)^{1=2}$. For a contradiction, suppose the opposite. Since the bundle projection $: M_g \; ! \; [0;1]=0 \quad 1$ is length nonincreasing, degree $\gamma < O(\log g)^{1=2}$. Lift $\gamma n pt.$ to an arc $\tilde{\gamma}$ in $_g \quad R$. The lift $\tilde{\gamma}$ joins some point $(p;t)$ to $(\, ^d p; t+d)$ where $d =$ degree γ. Since $d < O(\log g)^{1=2}$ and since condition (ii) requires order$(\;)$ $O(\log g)^{1=2}$, we see that p and $^d p$ differ by a non-trivial covering translation of the cover $_g \; ! \; _g S$. Nevertheless, any non-trivial covering translation moves each point of the total space at least twice the injectivity radius of the base, a quantity guaranteed by (iii) to be $O(\log g)^{1=2}$. Now using that the projection $_g \quad R \; ! \quad _g$ is also length nonincreasing, we see that $length\,(\tilde{\gamma}) \quad O(\log g)^{1=2}$. Since $length\,(\tilde{\gamma}) = length\,(\gamma)$, the same estimate applies to γ. □

Theorem 2.4 *The family* $fS^2 \quad S^1_g g$ *exhibits* Z_2{*systolic freedom.*

Proof From propositions 2.1, 2.2, and 2.3, we have:

$$\frac{Z_2 - systole_3(S^2 \quad S^1_g)}{Z_2 - systole_2(S^2 \quad S^1_g) \;\; Z_2 - systole_1(S^2 \quad S^1_g)} \quad \frac{O(g)}{O(g) \;\; O(\log g)^{1=2}} \; ! \; 0:$$

□

Many further examples in higher dimensions can now be generated. It is easy to check that if C is a circle of radius $\frac{O(g)}{O(\log g)^{1=2}}$ then $(S^2 \quad S^1_g) \quad C$ has $Z_2-(2;2)${freedom. As in [1], two further 1{surgeries give a family of metrics on $S^2 \quad S^2$ with $Z_2 - (2;2)${freedom. Curiously, the homotopy theoretic methods in [1] do not resolve whether CP^2 has Z_2{freedom. The difficulty is that a crucial \meromorphic map" $CP^2 \; ! \; S^2 \quad S^2$ has even degree. Whether CP^2 admits a metric of volume = in which every surface, orientable or not, of area 1 is null homotopic is an open question. I would like to thank M. Katz for his explanation of this difficulty, and for orienting me within the literatures on systolic inequalities.

3 Curvature Normalization

The precise arithmetic of both the theorem and Gromov's example (See introduction.) suggests that the amount of systolic freedom exhibited in a parameter family should be quantified. The natural way to do this is to homothetically rescale each metric in the family (say g is the parameter) to make the spaces as small as possible while keeping all sectional curvatures bounded between -1 and $+1$.

Given a family exhibiting $(p;q)$–freedom, for some choice of coefficients, first rescale the members of the family to obtain bounded curvature and then write the “$denominator$” $= systole_p(g)\ systole_q(g)$ as a function of the rescaled “$numerator$” $= volume = systole_n(g)$. The function $F(n) = \frac{d(n)}{n}$ measures the “freeness” of the family.

In the constructions of Gromov and Babenko–Katz, $F(n)$ grows like a positive power of n. Pittet [11] replaced a Nil geometry construction of [1] with an analogous Solv geometry construction to realize what our definition interprets as an exponentially growing $F(n)$. When properly rescaled the growth function for the examples in this paper will be considerably slower than root log (to be estimated in [4]). Perhaps the most interesting question to arise from our example is whether manifolds are “nearly” Z_2–rigid, ie, do their Z_2–freeness functions even when maximized over all families of metrics grow with extreme slowness. A negative answer would be very interesting both within geometry and for the implication for quantum codes. A positive answer would require a new technical idea: eg, translating some as yet unproved upper bound on the efficiency of quantum codes into differential geometry.

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Microsoft Corporation, One Microsoft Way
Redmond, WA 98052-6399, USA

Email: `michaelf@microsoft.com`

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