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Monopoles over 4–manifolds containing long necks, I

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Abstract

We study moduli spaces of Seiberg–Witten monopoles over spin^c Riemannian 4–manifolds with long necks and/or tubular ends. This first part discusses compactness, exponential decay, and transversality. As applications we prove two vanishing theorems for Seiberg–Witten invariants.

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1 Introduction

This is the first of two papers devoted to the study of moduli spaces of Seiberg–Witten monopoles over spin^c Riemannian 4–manifolds with long necks and/or tubular ends. Our principal motivation is to provide the analytical foundations for subsequent work on Floer homology. Such homology groups should appear naturally when one attempts to express the Seiberg–Witten invariants of a closed spin^c 4–manifold Z cut along a hypersurface Y, say

$$Z = Z_1 \cup_Y Z_2,$$

in terms of relative invariants of the two pieces Z_1, Z_2. The standard approach, familiar from instanton Floer theory (see [13, 10]), is to construct a 1–parameter family $\{g_T\}$ of Riemannian metrics on Z by stretching along Y so as to obtain a neck $[-T,T]\times Y$, and study the monopole moduli space $M^{(T)}$ over (Z, g_T) for large T. There are different aspects of this problem: compactness, transversality, and gluing. The present paper will focus particularly on compactness, and also establish transversality results sufficient for the construction of Floer homology groups of rational homology 3–spheres. The second paper in this series will be devoted to gluing theory.

Let the monopole equations over the neck $[-T,T]\times Y$ be perturbed by a closed 2–form η on Y, so that temporal gauge solutions to these equations correspond to downward gradient flow lines of the correspondingly perturbed Chern–Simons–Dirac functional ϑ_η over Y. Suppose all critical points of ϑ_η are non-degenerate. Because each moduli space $M^{(T)}$ is compact, one might expect, by analogy with Morse theory, that a sequence $\omega_n \in M^{(T_n)}$ where $T_n \to \infty$ has a subsequence which converges in a suitable sense to a pair of monopoles over the cylindrical-end manifolds associated to Z_1, Z_2 together with a broken gradient line of ϑ_η over $\mathbb{R}\times Y$. The first results in this direction were obtained by Kronheimer–Mrowka [21] (with $\eta = 0$) and Morgan–Szabó–Taubes [28] (in a particular case, with η non-exact). Nicolaescu's book [29] contains some foundational results in the case $\eta = 0$. Marcolli–Wang [26] proved a general compactness theorem for η exact. In this paper we will consider the general case when η may be non-exact. Unfortunately, compactness as stated above may then fail. (A simple class of counter-examples is described after Theorem 1.4 below.) It is then natural to seek topological conditions which ensure that compactness does hold. We will consider two approaches which provide different sufficient conditions. In the first approach, which is a refinement of well-known techniques (see [9, 29]), one first establishes global bounds on what is morally a version of the energy functional (although the energy concept is ambiguous in the presence of perturbations) and then derives local L^2

bounds on the curvature forms. In the second approach, which appears to be new, one begins by placing the connections in Coulomb gauge with respect to a given reference connection and then obtains global bounds on the corresponding connection forms in suitably weighted Sobolev norms, utilizing the *a priori* pointwise bounds on the spinors.

The paper also contains expository sections on configuration spaces and exponential decay, borrowing some ideas from Donaldson [10], to which we also refer for the Fredholm theory. In the transversality theory of moduli spaces we mostly restrict ourselves, for the time being, to the case when all ends of the 4–manifold in question are modelled on rational homology spheres. The perturbations used here are minor modifications of the ones introduced in [15]. It is not clear to us that these perturbations immediately carry over to the case of more general ends, as has apparently been assumed by some authors, although we expect that a modified version may be shown to work with the aid of gluing theory.

In most of this paper we make an assumption on the cohomology class of η which rules out the hardest case in the construction of Floer homology (see Subsection 1.2). This has the advantage that we can use relatively simple perturbations. A comprehensive monopole Floer theory including the hardest case is expected to appear in a forthcoming book by Kronheimer–Mrowka [20]. An outline of their construction (and much more) can be found in [22].

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1.1 Vanishing results for Seiberg–Witten invariants

Before describing our compactness results in more detail we will mention two applications to Seiberg–Witten invariants of closed 4–manifolds.

By a *spin^c manifold* we shall mean an oriented smooth manifold with a spin^c structure. If Z is a spin^c manifold then $-Z$ will refer to the same smooth manifold equipped with the opposite orientation and the corresponding spin^c structure.

If Z is a closed, oriented 4–manifold then by an *homology orientation* of Z we mean an orientation of the real vector space $H^0(Z)^*\oplus H^1(Z)\oplus H^+(Z)^*$, where

$H^+(Z)$ is any maximal positive subspace for the intersection form on $H^2(Z)$. The dimension of H^+ is denoted b_2^+.

In [7] Bauer and Furuta introduced a refined Seiberg–Witten invariant for closed spin^c 4–manifolds Z. This invariant $\widetilde{\mathrm{SW}}(Z)$ lives in a certain equivariant stable cohomotopy group. If Z is connected and $b_2^+(Z) > 1$, and given an homology orientation of Z, then according to [6] there is a natural homomorphism from this stable cohomotopy group to $\mathbb{Z}$ which maps $\widetilde{\mathrm{SW}}(Z)$ to the ordinary Seiberg–Witten invariant $\mathrm{SW}(Z)$ defined by the homology orientation. In [5] Bauer showed that, unlike the ordinary Seiberg–Witten invariant, the refined invariant does not in general vanish for connected sums where both summands have $b_2^+ > 0$. However, $\widetilde{\mathrm{SW}}(Z) = 0$ provided there exists a metric and perturbation 2–form on Z for which the Seiberg–Witten moduli space M_Z is empty (see [6, Remark 2.2] and [18, Proposition 6]).

Theorem 1.1 *Let Z be a closed spin^c 4–manifold and $Y \subset Z$ a closed, orientable 3–dimensional submanifold. Suppose*

(i) *Y admits a Riemannian metric with positive scalar curvature, and*

(ii) *$H^2(Z;\mathbb{Q}) \to H^2(Y;\mathbb{Q})$ is non-zero.*

Then there exists a metric and perturbation 2–form on Z for which M_Z is empty, hence $\widetilde{SW}(Z) = 0$.

This generalizes a result of Fintushel–Stern [12] and Morgan–Szabó–Taubes [28] which concerns the special case when $Y \approx S^1 \times S^2$ is the link of an embedded 2–sphere of self-intersection 0. One can derive Theorem 1.1 from Nicolaescu's proof [29] of their result and the classification of closed orientable 3–manifolds admitting positive scalar curvature metrics (see [23, p 325]). However, we shall give a direct (and much simpler) proof where the main idea is to perturb the monopole equations on Z by a suitable 2–form such that the corresponding perturbed Chern–Simons–Dirac functional on Y has no critical points. One then introduces a long neck $[-T,T] \times Y$. See Section 9 for details.

We now turn to another application, for which we need a little preparation. For any compact spin^c 4–manifold Z whose boundary is a disjoint union of rational homology spheres set

$$d(Z) = \frac{1}{4}\left(c_1(\mathcal{L}_Z)^2 - \sigma(Z)\right) + b_1(Z) - b_2^+(Z).$$

Here $\mathcal{L}_Z$ is the determinant line bundle of the spin^c structure, and $\sigma(Z)$ the signature of Z. If Z is closed then the moduli space M_Z has expected dimension $d(Z) - b_0(Z)$.

In [14] we will assign to every spin^c rational homology 3–sphere Y a rational number $h(Y)$. (A preliminary version of this invariant was introduced in [15].) In Section 9 of the present paper this invariant will be defined in the case when Y admits a metric with positive scalar curvature. It satisfies $h(-Y)=-h(Y)$. In particular, $h(S^3)=0$.

Theorem 1.2 *Let Z be a closed, connected spin^c 4–manifold, and let $W\subset Z$ be a compact, connected, codimension 0 submanifold whose boundary is a disjoint union of rational homology spheres $Y_1,\dots,Y_r$, $r\ge 1$, each of which admits a metric of positive scalar curvature. Suppose $b_2^+(W)>0$ and set $W^c=Z\setminus\operatorname{int}W$. Let each Y_j have the orientation and spin^c structure inherited from W. Then the following hold:*

(i) *If $2\sum_j h(Y_j)\le -d(W)$ then there exists a metric and perturbation 2–form on Z for which M_Z is empty, hence $\widetilde{SW}(Z)=0$.*

(ii) *If $b_2^+(Z)>1$ and $2\sum_j h(Y_j)<d(W^c)$ then $SW(Z)=0$.*

Note that (ii) generalizes the classical theorem (see [30, 29]) which says that $\mathrm{SW}(Z)=0$ if Z is a connected sum where both sides have $b_2^+>0$.

1.2 The Chern–Simons–Dirac functional

Let Y be a closed, connected Riemannian spin^c 3–manifold. We consider the Seiberg–Witten monopole equations over $\mathbb{R}\times Y$, perturbed by adding a 2–form to the curvature part of these equations. This 2–form should be the pull-back of a closed form η on Y. Recall from [21, 28] that in temporal gauge these perturbed monopole equations can be described as the downward gradient flow equation for a perturbed Chern–Simons–Dirac functional, which we will denote by ϑ_η, or just ϑ when no confusion can arise.

For transversality reasons we will add a further small perturbation to the monopole equations over $\mathbb{R}\times Y$, similar to those introduced in [15, Section 2]. This perturbation depends on a parameter $\mathfrak{p}$ (see Subsection 3.3). When $\mathfrak{p}\ne 0$ then the perturbed monopole equations are no longer of gradient flow type. Therefore, $\mathfrak{p}$ has to be kept small in order for the perturbed equations to retain certain properties (see Subsection 4.2).

If S is a configuration over Y (ie a spin connection together with a section of the spin bundle) and $u\colon Y\to \mathrm{U}(1)$ then

$$\vartheta(u(S))-\vartheta(S)=2\pi\int_Y \widetilde{\eta}\wedge[u], \tag{1}$$

where $[u] \in H^1(Y)$ is the pull-back by u of the fundamental class of $\mathrm{U}(1)$, and

$$\widetilde{\eta} = \pi c_1(\mathcal{L}_Y) - [\eta] \in H^2(Y). \tag{2}$$

Here $\mathcal{L}_Y$ is the determinant line bundle of the spin^c structure of Y.

Let $\mathcal{R}_Y$ be the space of (smooth) monopoles over Y (ie critical points of ϑ) modulo all gauge transformations $Y \to \mathrm{U}(1)$, and $\widetilde{\mathcal{R}}_Y$ the space of monopoles over Y modulo null-homotopic gauge transformations.

When no statement is made to the contrary, we will always make the following two assumptions:

(O1) $\widetilde{\eta}$ is a real multiple of some rational cohomology class.

(O2) All critical points of ϑ are non-degenerate.

The second assumption implies that $\mathcal{R}_Y$ is a finite set. This rules out the case when $\widetilde{\eta} = 0$ and $b_1(Y) > 0$, because if $\widetilde{\eta} = 0$ then the subspace of reducible points in $\mathcal{R}_Y$ is homeomorphic to a $b_1(Y)$–dimensional torus. If $\widetilde{\eta} \neq 0$ or $b_1(Y) = 0$ then the non-degeneracy condition can be achieved by perturbing η by an exact form (see Proposition 8.1).

For any $\alpha, \beta \in \widetilde{\mathcal{R}}_Y$ let $M(\alpha, \beta)$ denote the moduli space of monopoles over $\mathbb{R} \times Y$ that are asymptotic to α and β at $-\infty$ and ∞, respectively. Set $\check{M} = M/\mathbb{R}$. By a *broken gradient line* from α to β we mean a sequence $(\omega_1, \dots, \omega_k)$ where $k \geq 0$ and $\omega_j \in \check{M}(\alpha_{j-1}, \alpha_j)$ for some $\alpha_0, \dots, \alpha_k \in \widetilde{\mathcal{R}}_Y$ with $\alpha_0 = \alpha$, $\alpha_k = \beta$, and $\alpha_{j-1} \neq \alpha_j$ for each j. If $\alpha = \beta$ then we allow the empty broken gradient line (with $k = 0$).

1.3 Compactness

Let X be a spin^c Riemannian 4–manifold with tubular ends $\overline{\mathbb{R}}_+ \times Y_j$, $j = 1, \dots, r$, where $r \geq 0$ and each Y_j is a closed, connected Riemannian spin^c 3–manifold. Setting $Y = \cup_j Y_j$ this means that we are given

- an orientation preserving isometric embedding $\iota\colon \overline{\mathbb{R}}_+ \times Y \to X$ such that
$$X_{:t} = X \setminus \iota((t, \infty) \times Y) \tag{3}$$
is compact for any $t \geq 0$,
- an isomorphism between the spin^c structure on $\overline{\mathbb{R}}_+ \times Y$ induced from Y and the one inherited from X via the embedding ι.

Here $\mathbb{R}_+$ is the set of positive real numbers and $\overline{\mathbb{R}}_+ = \mathbb{R}_+ \cup \{0\}$. Usually we will just regard $\overline{\mathbb{R}}_+ \times Y$ as a (closed) submanifold of X.

Let η_j be a closed 2–form on Y_j and define $\widetilde{\eta}_j \in H^2(Y_j)$ in terms of η_j as in (2). We write ϑ instead of ϑ_{η_j} when no confusion is likely to arise. We perturb the curvature part of the monopole equations over X by adding a 2–form μ whose restriction to $\mathbb{R}_+ \times Y_j$ agrees with the pull-back of η_j. In addition we perturb the equations over $\mathbb{R} \times Y_j$ and the corresponding end of X using a perturbation parameter $\mathfrak{p}_j$. If $\vec{\alpha} = (\alpha_1, \dots, \alpha_r)$ with $\alpha_j \in \widetilde{\mathcal{R}}_{Y_j}$ let $M(X;\vec{\alpha})$ denote the moduli space of monopoles over X that are asymptotic to α_j over $\mathbb{R}_+ \times Y_j$.

Let $\lambda_1, \dots, \lambda_r$ be positive constants. We consider the following two equivalent conditions on the spin^c manifold X and $\widetilde{\eta}_j, \lambda_j$:

(A) There exists a class $\widetilde{z} \in H^2(X;\mathbb{R})$ such that $\widetilde{z}|_{Y_j} = \lambda_j \widetilde{\eta}_j$ for $j = 1, \dots, r$.

(A′) For configurations S over $X_{:0}$ the sum $\sum_j \lambda_j \vartheta(S|_{\{0\}\times Y_j})$ depends only on the gauge equivalence class of S.

Note that if $\lambda_j = 1$ for all j then (A) holds precisely when there exists a class $z \in H^2(X;\mathbb{R})$ such that $z|_{Y_j} = [\eta_j]$ for $j = 1, \dots, r$.

Theorem 1.3 *If Condition (A) is satisfied and each $\mathfrak{p}_j$ has sufficiently small C^1 norm then the following holds. For $n = 1, 2, \dots$ let $\omega_n \in M(X;\vec{\alpha}_n)$, where $\vec{\alpha}_n = (\alpha_{n,1}, \dots, \alpha_{n,r})$. If*

$$\inf_n \sum_{j=1}^{r} \lambda_j \vartheta(\alpha_{n,j}) > -\infty \tag{4}$$

then there exists a subsequence of ω_n which chain-converges to an $(r+1)$–tuple $(\omega, \vec{v}_1, \dots, \vec{v}_r)$ where ω is an element of some moduli space $M(X;\vec{\beta})$ and $\vec{v}_j$ is a broken gradient line over $\mathbb{R} \times Y_j$ from β_j to some $\gamma_j \in \widetilde{\mathcal{R}}_{Y_j}$. Moreover, if ω_n chain-converges to $(\omega, \vec{v}_1, \dots, \vec{v}_r)$ then for sufficiently large n there is a gauge transformation $u_n\colon X \to U(1)$ which is translationary invariant over the ends and maps $M(X;\vec{\alpha}_n)$ to $M(X;\vec{\gamma})$

The assumption (4) imposes an "energy bound" over the ends of X, as we will show in Subsection 7.2. The notion of chain-convergence is defined in Subsection 7.1. The limit, if it exists, is unique up to gauge equivalence (see Proposition 7.2 below).

1.4 Compactness and neck-stretching

In this subsection cohomology groups will have real coefficients.

We consider again a spin^c Riemannian 4–manifold X as in the previous subsection, but we now assume that the ends of X are given by orientation preserving isometric embeddings

$$\begin{aligned} \iota'_j\colon\ \overline{\mathbb{R}}_+\times Y'_j &\to X, \qquad j=1,\dots,r',\\ \iota^\pm_j\colon\ \overline{\mathbb{R}}_+\times(\pm Y_j) &\to X, \qquad j=1,\dots,r, \end{aligned}$$

where $r,r'\ge 0$. Here each Y'_j, Y_j should be a closed, connected spin^c Riemannian 3–manifold, and as before there should be the appropriate identifications of spin^c structures. For every $T=(T_1,\dots,T_r)$ with $T_j>0$ for each j, let $X^{(T)}$ denote the manifold obtained from X by gluing, for $j=1,\dots,r$, the two ends $\iota^\pm_j(\overline{\mathbb{R}}_+\times Y_j)$ to form a neck $[-T_j,T_j]\times Y_j$. To be precise, let $X^{\{T\}}\subset X$ be the result of deleting from X the sets $\iota^\pm_j([2T_j,\infty)\times Y_j)$, $j=1,\dots,r$. Set

$$X^{(T)}=X^{\{T\}}/\sim,$$

where we identify

$$\iota^+_j(t,y)\sim\iota^-_j(2T_j-t,y)$$

for all $(t,y)\in(0,2T_j)\times Y_j$ and $j=1,\dots,r$. We regard $[-T_j,T_j]\times Y_j$ as a submanifold of $X^{(T)}$ by means of the isometric embedding $(t,y)\mapsto\pi_T\iota^+_j(t+T_j,y)$, where $\pi_T\colon\ X^{\{T\}}\to X^{(T)}$. Also, we write $\mathbb{R}_+\times(\pm Y_j)$ instead of $\iota^\pm_j(\mathbb{R}_+\times Y_j)$, and similarly for $\mathbb{R}_+\times Y'_j$, if this is not likely to cause any confusion.

Set $X^\#=X^{(T)}$ with $T_j=1$ for all j. The process of constructing $X^\#$ from X (as smooth manifolds) can be described by the unoriented graph γ which has one node for every connected component of X and, for each $j=1,\dots,r$, one edge representing the pair of embeddings $\iota^\pm_j$.

A node in an oriented graph is called a *source* if it has no incoming edges. If e is any node in γ let X_e denote the corresponding component of X. Let $Z_e=(X_e)_{:1}$ be the corresponding truncated manifold as in (3). Let $\gamma\setminus e$ be the graph obtained from γ by deleting the node e and all edges of which e is a boundary point. Given an orientation o of γ let ∂^-Z_e denote the union of all boundary components of Z_e corresponding to incoming edges of (γ,o). Let F_e be the kernel of $H^1(Z_e)\to H^1(\partial^-Z_e)$, and set

$$\Sigma(X,\gamma,o)=\dim\,H^1(X^\#)-\sum_e\dim\,F_e.$$

It will follow from Lemma 5.3 below that $\Sigma(X,\gamma,o)\le 0$ if each connected component of γ is simply-connected.

We will now state a condition on (X,γ) which is recursive with respect to the number of nodes of γ.

(C) If γ has more than one node then it should admit an orientation o such that the following two conditions hold:

- $\Sigma(X,\gamma,o)=0$,
- Condition (C) holds for $(X\setminus X_e,\gamma\setminus e)$ for all sources e of (γ,o).

We are only interested in this condition when each component of γ is simply-connected. If γ is connected and has exactly two nodes e_1,e_2 then (C) holds if and only if $H^1(X^\#)\to H^1(Z_{e_j})$ is surjective for at least one value of j, as is easily seen from the Mayer–Vietoris sequence. See Subsection 5.3 and the proof of Proposition 5.6 for more information about Condition (C).

Let the Chern–Simons–Dirac functionals on Y_j,Y'_j be defined in terms of closed 2–forms η_j,η'_j respectively. Let $\widetilde{\eta}_j$ and $\widetilde{\eta}'_j$ be the corresponding classes as in (2). Let $\lambda_1,\dots,\lambda_r$ and $\lambda'_1,\dots,\lambda'_{r'}$ be positive constants. The following conditions on $X,\widetilde{\eta}_j,\widetilde{\eta}'_j,\lambda_j,\lambda'_j$ will appear in Theorem 1.4 below.

(B1) There exists a class in $H^2(X^\#)$ whose restrictions to Y_j and Y'_j are $[\eta_j]$ and $[\eta'_j]$, respectively, and all the constants λ_j,λ'_j are equal to 1.

(B2) There exists a class in $H^2(X^\#)$ whose restrictions to Y_j and Y'_j are $\lambda_j\widetilde{\eta}_j$ and $\lambda'_j\widetilde{\eta}'_j$, respectively. Moreover, the graph γ is simply-connected, and Condition (C) holds for (X,γ).

Choose a 2–form μ on X whose restriction to each end $\mathbb{R}_+\times(\pm Y_j)$ is the pull-back of η_j, and whose restriction to $\mathbb{R}_+\times Y'_j$ is the pull-back of η'_j. Such a form μ gives rise, in a canonical way, to a form $\mu^{(T)}$ on $X^{(T)}$. We use the forms $\mu,\mu^{(T)}$ to perturb the curvature part of the monopole equations over X, $X^{(T)}$, respectively. We use the perturbation parameter $\mathfrak{p}'_j$ over $\mathbb{R}\times Y'_j$ and the corresponding ends, and $\mathfrak{p}_j$ over $\mathbb{R}\times Y_j$ and the corresponding ends and necks.

Moduli spaces over X will be denoted $M(X;\vec{\alpha}_+,\vec{\alpha}_-,\vec{\alpha}')$, where the j'th component of $\vec{\alpha}_\pm$ specifies the limit over the end $\mathbb{R}_+\times(\pm Y_j)$ and the j'th component of $\vec{\alpha}'$ specifies the limit over $\mathbb{R}_+\times Y'_j$.

Theorem 1.4 *Suppose at least one of the conditions (B1), (B2) holds, and for $n=1,2,\dots$ let $\omega_n\in M(X^{(T(n))};\vec{\alpha}'_n)$, where $\vec{\alpha}'_n=(\alpha'_{n,1},\dots,\alpha'_{n,r'})$ and*

$T_j(n) \to \infty$ for $j = 1, \dots, r$. Suppose also that the perturbation parameters $\mathfrak{p}_j, \mathfrak{p}'_j$ are admissible for each $\vec{\alpha}'_n$, and that

$$\inf_n \sum_{j=1}^{r'} \lambda'_j \vartheta(\alpha'_{n,j}) > -\infty.$$

Then there exists a subsequence of ω_n which chain-converges to an $(r+r'+1)$–tuple $\mathbb{V} = (\omega, \vec{v}_1, \dots, \vec{v}_r, \vec{v}'_1, \dots, \vec{v}'_{r'})$, where

- *ω is an element of some moduli space $M(X; \vec{\alpha}_1, \vec{\alpha}_2, \vec{\beta}')$,*
- *$\vec{v}_j$ is a broken gradient line over $\mathbb{R} \times Y_j$ from α_{1j} to α_{2j},*
- *$\vec{v}'_j$ is a broken gradient line over $\mathbb{R} \times Y'_j$ from β'_j to some $\gamma'_j \in \widetilde{\mathcal{R}}_{Y'_j}$.*

Moreover, if ω_n chain-converges to $\mathbb{V}$ then for sufficiently large n there is a gauge transformation $u_n\colon X^{(T(n))} \to U(1)$ which is translationary invariant over the ends and maps $M(X^{(T(n))}; \vec{\alpha}'_n)$ to $M(X^{(T(n))}; \vec{\gamma}')$.

The notion of chain-convergence is defined in Subsection 7.1. Note that the chain-limit is unique only up to gauge equivalence, see Proposition 7.2.

What it means for the perturbation parameters $\mathfrak{p}_j, \mathfrak{p}'_j$ to be "admissible" is defined in Definition 7.3. As in Theorem 1.3, if (B2) holds and the perturbation parameters have sufficiently small C^1 norm then they are admissible for any $\vec{\alpha}'$, see Proposition 5.6. If (B1) is satisfied but perhaps not (B2) then for any $C_1 < \infty$ there is a $C_2 > 0$ such that if the perturbation parameters have C^1 norm $< C_2$ then they are admissible for all $\vec{\alpha}'$ satisfying $\sum_{j=1}^{r'} \lambda'_j \vartheta(\alpha'_j) > -C_1$, see the remarks after Proposition 4.5.

The conditions (B1), (B2) in the theorem correspond to the two approaches to compactness referred to at the beginning of this introduction: If (B1) is satisfied then one can take the "energy approach", whereas if (B2) holds one can use the "Hodge theory approach"

The conclusion of the theorem does not hold in general when neither (B1) nor (B2) are satisfied. For in that case Theorem 1.1 would hold if instead of (ii) one merely assumed that $b_1(Y) > 0$. Since $\mathbb{R}^4$ contains an embedded $S^1 \times S^2$ this would contradict the fact that there are many spin^c 4–manifolds with $b_2^+ > 1$ and non-zero Seiberg–Witten invariant.

For the moment we will abuse language and say that (B2) holds if it holds for some choice of constants λ_j, λ'_j, and similarly for (B1). Then a simple example where (B1) is satisfied but not (B2) is $X = \mathbb{R} \times Y$, where one glues the two ends to obtain $X^{(T)} = (\mathbb{R}/2T\mathbb{Z}) \times Y$. There are also many examples where (B2) is satisfied but not (B1). For instance, consider the case when X consists of two copies of $\mathbb{R} \times Y$, say $X = \mathbb{R} \times Y \times \{1,2\}$ with Y connected, and one glues $\mathbb{R}_+ \times Y \times \{1\}$ with $\mathbb{R}_- \times Y \times \{2\}$. In this case $r = 1$ and $r' = 2$, so we are given closed 2–forms η_1, η_1', η_2' on Y. Condition (B1) now requires that these three 2–forms represent the same cohomology class, while (B2) holds as long as there are $a_1, a_2 > 0$ such that $[\eta_1] = a_1[\eta_1'] = a_2[\eta_2']$.

1.5 Outline

Here is an outline of the content of the remainder of this paper. In Section 2 we study orbit spaces of configurations over Riemannian n–manifolds with tubular ends. This includes results on the Laplacians on such manifolds, which are also applied later in Subsection 5.1 to the study of the $d^* + d^+$ operator in the case $n = 4$. Section 3 introduces monopoles, perturbations, and moduli spaces. Section 4 establishes local compactness results for monopoles, and technical results on perturbations. In Section 5 the Hodge theory approach to local compactness is presented, which is an alternative to the energy approach of Subsection 4.3. Section 6 is devoted to exponential decay, which is needed for the global compactness results in Section 7, where Theorems 1.3 and 1.4 are proved. Section 8 discusses non-degeneracy of critical points and regularity of moduli spaces. Finally, in Section 9 we prove Theorems 1.1 and 1.2. There are also two appendices. The first of these explains how to patch together sequences of local gauge transformations (an improvement of Uhlenbeck's technique). The second appendix contains a quantitative inverse function theorem which is used in the proof of exponential decay.

2 Configuration spaces

2.1 Configurations and gauge transformations

Let X be a Riemannian n–manifold with tubular ends $\overline{\mathbb{R}}_+ \times Y_j$, $j = 1, \dots, r$, where $n \geq 1$, $r \geq 0$, and each Y_j is a closed, connected Riemannian $(n-1)$–manifold. This means that we are given for each j an isometric embedding

$$\iota_j \colon \overline{\mathbb{R}}_+ \times Y_j \to X;$$

moreover, the images of these embeddings are disjoint and their union have precompact complement. Usually we will just regard $\overline{\mathbb{R}}_+ \times Y_j$ as a submanifold of X. Set $Y=\cup_j Y_j$ and, for $t\ge 0$,

$$X_{:t}=X\setminus(t,\infty)\times Y.$$

Let $\mathbb{S}\to X$ and $\mathbb{S}_j\to Y_j$ be Hermitian complex vector bundles, and $L\to X$ and $L_j\to Y_j$ principal U(1)–bundles. Suppose we are given, for each j, isomorphisms

$$\iota_j^*\mathbb{S}\xrightarrow{\approx}\overline{\mathbb{R}}_+\times\mathbb{S}_j,\quad \iota_j^*L\xrightarrow{\approx}\overline{\mathbb{R}}_+\times L_j.$$

By a *configuration* in $(L,\mathbb{S})$ we shall mean a pair (A,Φ) where A is a connection in L and Φ a section of $\mathbb{S}$. Maps $u\colon X\to \mathrm{U}(1)$ are referred to as *gauge transformations* and these act on configurations in the natural way:

$$u(A,\Phi)=(u(A),u\Phi).$$

The main goal of this section is to prove a "local slice" theorem for certain orbit spaces of configurations modulo gauge transformations.

We begin by setting up suitable function spaces. For $p\ge 1$ and any non-negative integer m let $L^p_m(X)$ be the completion of the space of compactly supported smooth functions on X with respect to the norm

$$\|f\|_{m,p}=\|f\|_{L^p_m}=\left(\sum_{k=0}^m\int_X|\nabla^k f|^p\right)^{1/p}.$$

Here the covariant derivative is computed using some fixed connection in the tangent bundle TX which is translationary invariant over each end. Define the Sobolev space $L^p_k(X;\mathbb{S})$ of sections of $\mathbb{S}$ similarly.

We also need weighted Sobolev spaces. For any smooth function $w\colon X\to\mathbb{R}$ set $L^{p,w}_m(X)=e^{-w}L^p_m(X)$ and

$$\|f\|_{L^{p,w}_k}=\|e^w f\|_{L^p_k}.$$

In practice we require that w have a specific form over the ends, namely

$$w\circ\iota_j(t,y)=\sigma_j t,$$

where the σ_j's are real numbers.

The following Sobolev embeddings (which hold in $\mathbb{R}^n$, hence over X) will be used repeatedly:

$$\begin{aligned} L^p_{m+1}&\subset L^{2p}_m \quad \text{if } p\ge n/2,\ m\ge 0,\\ L^p_2&\subset C^0_B \quad \text{if } p>n/2. \end{aligned}$$

Here C^0_B denotes the Banach space of bounded continuous functions, with the supremum norm. Moreover, if $pm>n$ then multiplication defines a continuous map $L^p_m\times L^p_k\to L^p_k$ for $0\le k\le m$.

For the remainder of this section fix $p > n/2$. Note that this implies $L^p_1 \subset L^2$ over compact n–manifolds.

We will now define an affine space $\mathcal{C}$ of $L^p_{1,\mathrm{loc}}$ configurations in $(L, \mathbb{S})$. Let A_o be a smooth connection in L. Choose a smooth section Φ_o of $\mathbb{S}$ whose restriction to $\mathbb{R}_+ \times Y_j$ is the pull-back of a section ψ_j of $\mathbb{S}_j$. Suppose $\psi_j = 0$ for $j \le r_0$, and $\psi_j \not\equiv 0$ for $j > r_0$, where r_0 is a non-negative integer. Fix a weight function w as above with $\sigma_j \ge 0$ small for all j, and $\sigma_j > 0$ for $j \le r_0$. Set

$$\mathcal{C} = \{(A_o + a, \Phi_o + \phi) \, : \, a, \phi \in L^{p,w}_1\}.$$

We topologize $\mathcal{C}$ using the $L^{p,w}_1$ metric.

We wish to define a Banach Lie group $\mathcal{G}$ of $L^p_{2,\mathrm{loc}}$ gauge transformations over X such that $\mathcal{G}$ acts smoothly on $\mathcal{C}$ and such that if $S, S' \in \mathcal{C}$ and $u(S) = S'$ for some $L^p_{2,\mathrm{loc}}$ gauge transformations u then $u \in \mathcal{G}$. If $u \in \mathcal{G}$ then we must certainly have

$$(-u^{-1}du, (u-1)\Phi_o) = u(A_o, \Phi_o) - (A_o, \Phi_o) \in L^{p,w}_1.$$

Now

$$\|du\|_{L^{p,w}_1} \le \mathrm{const} \cdot (\|u^{-1}du\|_{L^{p,w}_1} + \|u^{-1}du\|^2_{L^{p,w}_1}), \tag{5}$$

and vice versa, $\|du\|_{L^{p,w}_1}$ controls $\|u^{-1}du\|_{L^{p,w}_1}$, so we try

$$\mathcal{G} = \{u \in L^p_{2,\mathrm{loc}}(X; \mathrm{U}(1)) \, : \, du, (u-1)\Phi_o \in L^{p,w}_1\}.$$

By $L^p_{2,\mathrm{loc}}(X; \mathrm{U}(1))$ we mean the set of elements of $L^p_{2,\mathrm{loc}}(X; \mathbb{C})$ that map into $\mathrm{U}(1)$. We will see that $\mathcal{G}$ has a natural smooth structure such that the above criteria are satisfied.

(This approach to the definition of $\mathcal{G}$ was inspired by [10].)

2.2 The Banach algebra

Let $\widetilde{x}$ be a finite subset of X which contains at least one point from every connected component of X where Φ_o vanishes identically.

Definition 2.1 Set

$$\mathcal{E} = \{f \in L^p_{2,\mathrm{loc}}(X; \mathbb{C}) \, : \, df, f\Phi_o \in L^{p,w}_1\},$$

and let $\mathcal{E}$ have the norm

$$\|f\|_{\mathcal{E}} = \|df\|_{L^{p,w}_1} + \|f\Phi_o\|_{L^{p,w}_1} + \sum_{x \in \widetilde{x}} |f(x)|.$$

We will see in a moment that $\mathcal{E}$ is a Banach algebra (without unit if $r_0 < r$). The next lemma shows that the topology on $\mathcal{E}$ is independent of the choice of Φ_o and $\widetilde{x}$.

Lemma 2.1 *Let $Z \subset X$ be a compact, connected codimension 0 submanifold.*

(i) *If $\Phi_o|_Z \not\equiv 0$ then there is a constant C such that*
$$\int_Z |f|^p \le C \int_Z |df|^p + |f\Phi_o|^p \tag{6}$$
for all $f \in L^p_1(Z)$.

(ii) *There are constants C_1, C_2 such that*
$$|f(z_2) - f(z_1)| \le C_1 \|df\|_{L^{2p}(Z)} \le C_2 \|df\|_{L^p_1(Z)}$$
for all $f \in L^p_2(Z)$ and $z_1, z_2 \in Z$.

Proof Part (i) follows from the compactness of the embedding $L^p_1(Z) \to L^p(Z)$. The first inequality in (ii) can either be deduced from the compactness of $L^{2p}_1(Z) \to C^0(Z)$, or one can prove it directly, as a step towards proving the Rellich lemma, by considering the integrals of df along a suitable family of paths from z_1 to z_2. □

Lemma 2.2 *Let Y be a closed Riemannian manifold, and $\sigma > 0$.*

(i) *If $q \ge 1$ and $f\colon \mathbb{R}_+ \times Y \to \mathbb{R}$ is a C^1 function such that $\lim_{t\to\infty} f(t,y) = 0$ for all $y \in Y$ then*
$$\|f\|_{L^{q,\sigma}(\mathbb{R}_+\times Y)} \le \sigma^{-1} \|\partial_1 f\|_{L^{q,\sigma}(\mathbb{R}_+\times Y)}.$$

(ii) *If $q > 1$, $T \ge 1$, and $f\colon [0,T]\times Y \to \mathbb{R}$ is a C^1 function then*
$$\|f\|_{L^q([T-1,T]\times Y)} \le \|f_0\|_{L^q(Y)} + (\sigma r)^{-1/r} \|\partial_1 f\|_{L^{q,\sigma}([0,T]\times Y)},$$
where $f_0(y) = f(0,y)$ and $\frac{1}{q} + \frac{1}{r} = 1$.

Here ∂_1 is the partial derivative in the first variable, ie in the $\mathbb{R}_+$ coordinate.

Proof Part (i):
$$\begin{aligned}
\|f\|_{L^{q,\sigma}(\mathbb{R}_+\times Y)} &= \left(\int_{\mathbb{R}_+\times Y} \left| \int_0^\infty e^{\sigma t} \partial_1 f(s+t,y)\, ds \right|^q dt\, dy \right)^{1/q} \\
&\le \int_0^\infty \left(\int_{\mathbb{R}_+\times Y} |e^{\sigma t}\partial_1 f(s+t,y)|^q dt\, dy \right)^{1/q} ds \\
&\le \left(\int_0^\infty e^{-\sigma s} ds \right) \left(\int_{\mathbb{R}_+\times Y} |e^{\sigma(s+t)}\partial_1 f(s+t,y)|^q dt\, dy \right)^{1/q} \\
&\le \sigma^{-1} \|\partial_1 f\|_{L^{q,\sigma}(\mathbb{R}_+\times Y)}.
\end{aligned}$$

Part (ii) follows by a similar computation. □

Parts (i)–(iv) of the following proposition are essentially due to Donaldson [10].

Proposition 2.1

(i) *There is a constant* C_1 *such that*
$$\|f\|_\infty \le C_1 \|f\|_{\mathcal{E}}$$
for all $f \in \mathcal{E}$.

(ii) *For every* $f \in \mathcal{E}$ *and* $j = 1,\dots,r$ *the restriction* $f|_{\{t\}\times Y_j}$ *converges uniformly to a constant function* $f^{(j)}$ *as* $t \to \infty$, *and* $f^{(j)} = 0$ *for* $j > r_0$.

(iii) *There is a constant* C_2 *such that if* $f \in \mathcal{E}$ *and* $f^{(j)} = 0$ *for all* j *then*
$$\|f\|_{L^{p,w}_2} \le C_2 \|f\|_{\mathcal{E}}.$$

(iv) *There is an exact sequence*
$$0 \to L^{p,w}_2 \xrightarrow{\iota} \mathcal{E} \xrightarrow{e} \mathbb{C}^{r_0} \to 0,$$
where ι *is the inclusion and* $e(f) = (f^{(1)},\dots,f^{(r_0)})$.

(v) $\mathcal{E}$ *is complete, and multiplication defines a continuous map* $\mathcal{E}\times\mathcal{E}\to\mathcal{E}$.

Proof First observe that for any $f \in L^p_{2,\mathrm{loc}}(X)$ and $\epsilon > 0$ there exists a $g \in C^\infty(X)$ such that $\|g - f\|_{L^{p,w}_2} < \epsilon$. Therefore it suffices to prove (i)–(iii) when $f \in \mathcal{E}$ is smooth. Part (i) is then a consequence of Lemma 2.1 and Lemma 2.2 (ii), while Part (ii) for $r_0 < j \le r$ follows from Lemma 2.1.

We will now prove (ii) when $1 \le j \le r_0$. Let $f \in \mathcal{E}$ be smooth. Since $\int_{\mathbb{R}_+\times Y_j} |df| < \infty$ by the Hölder inequality, we have
$$\int_0^\infty |\partial_1 f(t,y)|\,dt < \infty \qquad \text{for almost all } y \in Y_j.$$
For $n \in \mathbb{N}$ set $f_n = f|_{[n-1,n+1]\times Y_j}$, regarded as a function on $B = [n-1,n+1]\times Y_j$. Then $\{f_n\}$ converges a.e., so by Egoroff's theorem $\{f_n\}$ converges uniformly over some subset $T \subset B$ of positive measure. There is then a constant $C > 0$, depending on T, such that for every $g \in L^p_1(B)$ one has
$$\int_B |g|^p \le C(\int_B |dg|^p + \int_T |g|^p).$$
It follows that $\{f_n\}$ converges in L^p_2 over B, hence uniformly over B, to some constant function.

Part (iii) follows from Lemma 2.1 and Lemma 2.2 (i). Part (iv) is an immediate consequence of (ii) and (iii). It is clear from (i) that $\mathcal{E}$ is complete. The multiplication property follows easily from (i) and the fact that smooth functions are dense in $\mathcal{E}$. □

2.3 The infinitesimal action

If $f\colon X \to i\mathbb{R}$ and Φ is a section of $\mathbb{S}$ we define a section of $i\Lambda^1 \oplus \mathbb{S}$ by

$$\mathcal{I}_\Phi f = (-df, f\Phi)$$

whenever the expression on the right makes sense. Here Λ^k denotes the bundle of k–forms (on X, in this case). If $S = (A, \Phi)$ is a configuration then we will sometimes write $\mathcal{I}_S$ instead of $\mathcal{I}_\Phi$. Set

$$\mathcal{I} = \mathcal{I}_{\Phi_o}.$$

If Φ is smooth then the formal adjoint of the operator $\mathcal{I}_\Phi$ is

$$\mathcal{I}_\Phi^*(a, \phi) = -d^*a + i\langle i\Phi, \phi\rangle_{\mathbb{R}},$$

where $\langle\,\cdot\,,\,\cdot\,\rangle_{\mathbb{R}}$ is the real inner product on $\mathbb{S}$. Note that

$$\mathcal{I}_\Phi^*\mathcal{I}_\Phi = \Delta + |\Phi|^2$$

where Δ is the positive Laplacian on X.

Set

$$L\mathcal{G} = \{f \in \mathcal{E} \,:\, f \text{ maps into } i\mathbb{R}\}.$$

From Proposition 2.1 (i) we see that the operators

$$\mathcal{I}_\Phi\colon\ L\mathcal{G} \to L^{p,w}_1, \quad \mathcal{I}^*_\Phi\colon\ L^{p,w}_1 \to L^{p,w}$$

are well-defined and bounded for every $\Phi \in \Phi_o + L^{p,w}_1(X; \mathbb{S})$.

Lemma 2.3 *For every $\Phi \in \Phi_o + L^{p,w}_1(X; \mathbb{S})$, the operators $\mathcal{I}_\Phi^*\mathcal{I}_\Phi$ and $\mathcal{I}_\Phi$ have the same kernel in $L\mathcal{G}$.*

Proof Choose a smooth function $\beta\colon \mathbb{R} \to \mathbb{R}$ such that $\beta(t) = 1$ for $t \le 1$, $\beta(t) = 0$ for $t \ge 2$. For $r > 0$ define a compactly supported function

$$\beta_r\colon\ X \to \mathbb{R}$$

by $\beta_r|_{X:0} = 1$, and $\beta_r(t, y) = \beta(t/r)$ for $(t, y) \in \mathbb{R}_+ \times Y_j$.

Now suppose $f \in L\mathcal{G}$ and $\mathcal{I}_\Phi^*\mathcal{I}_\Phi f = 0$. Lemma 2.1 (i) and elliptic regularity gives $f \in L^p_{2,\mathrm{loc}}$, so we certainly have

$$\mathcal{I}_\Phi f \in L^p_{1,\mathrm{loc}} \subset L^2_{\mathrm{loc}}.$$

Clearly,

$$\|\mathcal{I}_\Phi f\|_2 \le \liminf_{r\to\infty} \|\mathcal{I}_\Phi(\beta_r f)\|_2.$$

Over $\mathbb{R}_+ \times Y_j$ we have

$$\mathcal{I}_\Phi^* \mathcal{I}_\Phi = -\partial_1^2 + \Delta_{Y_j} + |\Phi|^2,$$

where $\partial_1 = \frac{\partial}{\partial t}$ and Δ_{Y_j} is the positive Laplacian on Y_j, so

$$\begin{aligned}
\|\mathcal{I}_\Phi(\beta_r f)\|_2^2 &= \int_X \mathcal{I}_\Phi^* \mathcal{I}_\Phi(\beta_r f) \cdot \beta_r \bar{f} \\
&= -\sum_j \int_{\mathbb{R}_+ \times Y_j} ((\partial_1^2 \beta_r) f + 2(\partial_1 \beta_r)(\partial_1 f)) \cdot \beta_r \bar{f} \\
&\le C_1 \|f\|_\infty^2 \int_0^\infty |r^{-2} \beta''(t/r)|\, dt \\
&\qquad + C_1 \|f\|_\infty \|df\|_p \left(\int_0^\infty |r^{-1} \beta'(t/r)|^q dt \right)^{1/q} \\
&\le C_2 \|f\|_{\mathcal{E}}^2 \left(r^{-1} \int_1^2 |\beta''(u)|\, du + r^{1-q} \int_1^2 |\beta'(u)|^q du \right) \\
&\to 0 \text{ as } r \to \infty,
\end{aligned}$$

where $C_1, C_2 > 0$ are constants and $\frac{1}{p} + \frac{1}{q} = 1$. Hence $\mathcal{I}_\Phi f = 0$. □

Lemma 2.4 *$\mathcal{I}^*\mathcal{I}\colon L_2^{q,w}(X) \to L^{q,w}(X)$ is Fredholm of index $-r_0$, for $1 < q < \infty$.*

Proof Because $\mathcal{I}^*\mathcal{I}$ is elliptic the operator in the Lemma is Fredholm if the operator

$$-\partial_1^2 + \Delta_{Y_j} + |\psi_j|^2\colon L_2^{q,\sigma_j} \to L^{q,\sigma_j}, \tag{7}$$

acting on functions on $\mathbb{R} \times Y_j$, is Fredholm for each j. The proof of [10, Proposition 3.21] (see also [24]) can be generalized to show that (7) is Fredholm if σ_j^2 is not an eigenvalue of $\Delta_{Y_j} + |\psi_j|^2$. Since we are taking $\sigma_j \ge 0$ small, and $\sigma_j > 0$ if $\psi_j = 0$, this establishes the Fredholm property in the Lemma.

We will now compute the index. Set

$$\mathrm{ind}^\pm = \mathrm{index}\{\mathcal{I}^*\mathcal{I}\colon L_2^{q,\pm w}(X) \to L^{q,\pm w}(X)\}.$$

Expressing functions on Y_j in terms of eigenvectors of $\Delta_{Y_j} + |\psi_j|^2$ as in [3, 10] one finds that the kernel of $\mathcal{I}^*\mathcal{I}$ in $L_{k'}^{q',\pm w}$ is the same for all $q' > 1$ and integers $k' \ge 0$. Combining this with the fact that $\mathcal{I}^*\mathcal{I}$ is formally self-adjoint we see that

$$\mathrm{ind}^+ = -\mathrm{ind}^-.$$

Now choose smooth functions $w_j\colon\ \mathbb{R}\to\mathbb{R}$ such that $w_j(t)=\sigma_j|t|$ for $|t|\ge 1$. We will apply the addition formula for the index, see [10, Proposition 3.9]. This is proved only for first order operators in [10] but holds for higher order operators as well, with essentially the same proof. The addition formula gives

$$\begin{aligned}\mathrm{ind}^- &= \mathrm{ind}^+ + \sum_j \mathrm{index}\{\mathcal{I}^*_{\psi_j}\mathcal{I}_{\psi_j}\colon\ L^{q,-w_j}_2(\mathbb{R}\times Y_j)\to L^{q,-w_j}(\mathbb{R}\times Y_j)\}\\ &= \mathrm{ind}^+ + 2\sum_j \dim\ker(\Delta_{Y_j}+|\psi_j|^2)\\ &= \mathrm{ind}^+ + 2r_0.\end{aligned}$$

Therefore, $\mathrm{ind}^+=-r_0$ as claimed. □

Proposition 2.2 *For any $\Phi\in\Phi_o+L^{p,w}_1(X;\mathbb{S})$ the following hold:*

(i) *The operator*
$$\mathcal{I}^*_\Phi\mathcal{I}_\Phi\colon\ L\mathcal{G}\to L^{p,w} \tag{8}$$
is Fredholm of index 0, and it has the same kernel as $\mathcal{I}_\Phi\colon\ L\mathcal{G}\to L^{p,w}_1$ and the same image as $\mathcal{I}^*_\Phi\colon\ L^{p,w}_1\to L^{p,w}$.

(ii) *$\mathcal{I}_\Phi(L\mathcal{G})$ is closed in $L^{p,w}_1$ and*
$$L^{p,w}_1(i\Lambda^1\oplus\mathbb{S})=\mathcal{I}_\Phi(L\mathcal{G})\oplus\ker(\mathcal{I}^*_\Phi). \tag{9}$$

Proof It is easy to deduce Part (ii) from Part (i). We will now prove Part (i). Since
$$\mathcal{I}^*_\Phi\mathcal{I}_\Phi-\mathcal{I}^*\mathcal{I}=|\Phi|^2-|\Phi_o|^2\colon\ L^{p,w}_2\to L^{p,w}$$
is a compact operator, $\mathcal{I}^*_\Phi\mathcal{I}_\Phi$ and $\mathcal{I}^*\mathcal{I}$ have the same index as operators between these Banach spaces. It then follows from Lemma 2.4 and Proposition 2.1 (iv) that the operator (8) is Fredholm of index 0. The statement about the kernels is the same as Lemma 2.3. To prove the statement about the images, we may as well assume X is connected. If $\Phi\ne 0$ then the operator (8) is surjective and there is nothing left to prove. Now suppose $\Phi=0$. Then all the weights σ_j are positive, and the kernel of $\mathcal{I}_\Phi$ in $\mathcal{E}$ consists of the constant functions. Hence the image of (8) has codimension 1. But $\int_X d^*a=0$ for every 1–form $a\in L^{p,w}_1$, so $d^*\colon\ L^{p,w}_1\to L^{p,w}$ is not surjective. □

In the course of the proof of (i) we obtained:

Proposition 2.3 *If X is connected and $r_0=r$ then*
$$d^*d(\mathcal{E})=\left\{g\in L^{p,w}(X;\mathbb{C})\ :\ \int_X g=0\right\}.$$

We conclude this subsection with a result that will be needed in the proofs of Proposition 5.3 and Lemma 5.5 below. Let $1 < q < \infty$ and for any $L^q_{1,\mathrm{loc}}$ function $f\colon X \to \mathbb{R}$ set

$$\delta_j f = \int_{\{0\}\times Y_j} \partial_1 f, \quad j = 1, \dots, r.$$

The integral is well-defined because if n is any positive integer then there is a bounded restriction map $L^q_1(\mathbb{R}^n) \to L^q(\{0\}\times\mathbb{R}^{n-1})$.

Choose a point $x_0 \in X$.

Proposition 2.4 *If X is connected, $1 < q < \infty$, $r \geq 1$, and if $\sigma_j > 0$ is sufficiently small for each j then the operator*

$$\begin{aligned} \beta\colon L^{q,-w}_2(X;\mathbb{R}) &\to L^{q,-w}(X;\mathbb{R}) \oplus \mathbb{R}^r, \\ f &\mapsto (\Delta f, (\delta_1 f, \dots, \delta_{r-1} f, f(x_0))) \end{aligned}$$

is an isomorphism.

Proof By the proof of Lemma 2.4, $\Delta\colon L^{q,-w}_2 \to L^{q,-w}$ has index r, hence $\mathrm{ind}(\beta) = 0$. We will show β is injective. First observe that $\sum_{j=1}^r \delta_j f = 0$ whenever $\Delta f = 0$, so if $\beta f = 0$ then $\delta_j f = 0$ for all j.

Suppose $\beta f = 0$. To simplify notation we will now assume Y is connected. Over $\mathbb{R}_+ \times Y$ we have $\Delta = -\partial_1^2 + \Delta_Y$. Let $\{h_\nu\}_{\nu=0,1,\dots}$ be a maximal orthonormal set of eigenvectors of Δ_Y, with corresponding eigenvalues λ_ν^2, where $0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \cdots$. Then

$$f(t,y) = a + bt + g(t,y),$$

where $a, b \in \mathbb{R}$, and g has the form

$$g(t,y) = \sum_{\nu\geq 1} c_\nu e^{-\lambda_\nu t} h_\nu(y)$$

for some real constants c_ν. Elliptic estimates show that g decays exponentially, or more precisely,

$$|(\nabla^j f)_{(t,y)}| \leq d_j e^{-\lambda_1 t}$$

for $(t,y) \in \mathbb{R}_+ \times Y$ and $j \geq 0$, where $d_j > 0$ is a constant. Now

$$\partial_1 f(t,y) = b - \sum_{\nu\geq 1} c_\nu \lambda_\nu e^{-\lambda_\nu t} h_\nu(y).$$

Since Δ_Y is formally self-adjoint we have $\int_Y h_\nu = 0$ if $\lambda_\nu \neq 0$, hence

$$b\,\mathrm{Vol}(Y) = \int_{\{\tau\}\times Y} \partial_1 f = 0.$$

It follows that f is bounded and df decays exponentially over $\mathbb{R}_+ \times Y$, so

$$0 = \int_X f\Delta f = \int_X |df|^2,$$

hence f is constant. Since $f(x_0) = 0$ we have $f = 0$. □

2.4 Local slices

Fix a finite subset $\mathfrak{b} \subset X$.

Definition 2.2 Set

$$\mathcal{G}_{\mathfrak{b}} = \{u \in 1 + \mathcal{E} \,:\, u \text{ maps into } \mathrm{U}(1) \text{ and } u|_{\mathfrak{b}} \equiv 1\}$$
$$L\mathcal{G}_{\mathfrak{b}} = \{f \in \mathcal{E} \,:\, f \text{ maps into } i\mathbb{R} \text{ and } f|_{\mathfrak{b}} \equiv 0\}$$

and let $\mathcal{G}_{\mathfrak{b}}$ and $L\mathcal{G}_{\mathfrak{b}}$ have the subspace topologies inherited from $1+\mathcal{E} \approx \mathcal{E}$ and $\mathcal{E}$, respectively.

By $1 + \mathcal{E}$ we mean the set of functions on X of the form $1 + f$ where $f \in \mathcal{E}$. If $\mathfrak{b}$ is empty then we write $\mathcal{G}$ instead of $\mathcal{G}_{\mathfrak{b}}$, and similarly for $L\mathcal{G}$.

Proposition 2.5

(i) *$\mathcal{G}_{\mathfrak{b}}$ is a smooth submanifold of $1 + \mathcal{E}$ and a Banach Lie group with Lie algebra $L\mathcal{G}_{\mathfrak{b}}$.*

(ii) *The natural action $\mathcal{G}_{\mathfrak{b}} \times \mathcal{C} \to \mathcal{C}$ is smooth.*

(iii) *If $S \in \mathcal{C}$, $u \in L^p_{2,loc}(X;U(1))$ and $u(S) \in \mathcal{C}$ then $u \in \mathcal{G}$.*

Proof (i) If $r_0 < r$ then $1 \notin \mathcal{E}$, but in any case,

$$f \mapsto \sum_{k=1}^{\infty} \frac{1}{k!} f^k = \exp(f) - 1$$

defines a smooth map $\mathcal{E} \to \mathcal{E}$, by Proposition 2.1 (v). Therefore, the exponential map provides the local parametrization around 1 required for $\mathcal{G}_{\mathfrak{b}}$ to be a submanifold of $1 + \mathcal{E}$. The verification of (ii) and (iii) is left to the reader. □

Let $\mathcal{B}_{\mathfrak{b}} = \mathcal{C}/\mathcal{G}_{\mathfrak{b}}$ have the quotient topology. This topology is Hausdorff because it is stronger than the topology defined by the L^{2p} metric on $\mathcal{B}_{\mathfrak{b}}$ (see [11]). The image in $\mathcal{B}_{\mathfrak{b}}$ of a configuration $S \in \mathcal{C}$ will be denoted $[S]$, and we say S is a *representative* of $[S]$.

Let $\mathcal{C}^*_{\mathfrak{b}}$ be the set of all elements of $\mathcal{C}$ which have trivial stabilizer in $\mathcal{G}_{\mathfrak{b}}$. In other words, $\mathcal{C}^*_{\mathfrak{b}}$ consists of those $(A,\Phi)\in\mathcal{C}$ such that $\mathfrak{b}$ contains at least one point from every component of X where Φ vanishes almost everywhere. Let $\mathcal{B}^*_{\mathfrak{b}}$ be the image of $\mathcal{C}^*_{\mathfrak{b}}\to\mathcal{B}_{\mathfrak{b}}$. It is clear that $\mathcal{B}^*_{\mathfrak{b}}$ is an open subset of $\mathcal{B}_{\mathfrak{b}}$.

If $\mathfrak{b}$ is empty then $\mathcal{C}^*\subset\mathcal{C}$ and $\mathcal{B}^*\subset\mathcal{B}$ are the subspaces of *irreducible* configurations. As usual, a configuration that is not irreducible is called reducible.

We will now give $\mathcal{B}^*_{\mathfrak{b}}$ the structure of a smooth Banach manifold by specifying an atlas of local parametrizations. Let $S=(A,\Phi)\in\mathcal{C}^*_{\mathfrak{b}}$ and set

$$V=\mathcal{I}^*_\Phi(L^{p,w}_1),\quad W=\mathcal{I}^*_\Phi\mathcal{I}_\Phi(L\mathcal{G}_{\mathfrak{b}}).$$

By Proposition 2.2 we have

$$\dim(V/W)=|\mathfrak{b}|-\ell$$

where ℓ is the number of components of X where Φ vanishes almost everywhere Choose a bounded linear map $\rho\colon V\to W$ such that $\rho|_W=I$, and set

$$\mathcal{I}^\#_\Phi=\rho\mathcal{I}^*_\Phi.$$

Then

$$L^{p,w}_1(i\Lambda^1\oplus\mathbb{S})=\mathcal{I}_\Phi(L\mathcal{G}_{\mathfrak{b}})\oplus\ker(\mathcal{I}^\#_\Phi)$$

by Proposition 2.2. Consider the smooth map

$$\Pi\colon\ L\mathcal{G}_{\mathfrak{b}}\times\ker(\mathcal{I}^\#_\Phi)\to\mathcal{C},\quad (f,s)\mapsto\exp(f)(S+s).$$

The derivative of this map at $(0,0)$ is

$$D\Pi(0,0)(f,s)=\mathcal{I}_\Phi f+s,$$

which is an isomorphism by the above remarks. The inverse function theorem then says that Π is a local diffeomorphism at $(0,0)$.

Proposition 2.6 *In the situation above there is an open neighbourhood U of $0\in\ker(\mathcal{I}^\#_\Phi)$ such that the projection $\mathcal{C}\to\mathcal{B}_{\mathfrak{b}}$ restricts to a topological embedding of $S+U$ onto an open subset of $\mathcal{B}^*_{\mathfrak{b}}$.*

It is clear that the collection of such local parametrizations $U\to\mathcal{B}^*_{\mathfrak{b}}$ is a smooth atlas for $\mathcal{B}^*_{\mathfrak{b}}$.

Proof It only remains to prove that $S+U\to\mathcal{B}_{\mathfrak{b}}$ is injective when U is sufficiently small. So suppose (a_k,ϕ_k), (b_k,ψ_k) are two sequences in $\ker(\mathcal{I}^\#_\Phi)$ which both converge to 0 as $k\to\infty$, and such that

$$u_k(A+a_k,\Phi+\phi_k)=(A+b_k,\Phi+\psi_k)$$

for some $u_k \in \mathcal{G}_{\mathfrak{b}}$. We will show that $\|u_k - 1\|_{\mathcal{E}} \to 0$. Since Π is a local diffeomorphism at $(0,0)$, this will imply that $u_k = 1$ for $k \gg 0$.

Written out, the assumption on u_k is that

$$\begin{gathered} u_k^{-1} du_k = a_k - b_k, \\ (u_k - 1)\Phi = \psi_k - u_k \phi_k. \end{gathered}$$

By (5) we have $\|du_k\|_{L^{p,w}_1} \to 0$, which in turn gives $\|u_k\phi_k\|_{L^{p,w}_1} \to 0$, hence

$$\|(u_k - 1)\Phi\|_{L^{p,w}_1} \to 0. \tag{10}$$

Because u_k is bounded and du_k converges to 0 in L^p_1 over compact subsets, we can find a subsequence $\{k_j\}$ such that u_{k_j} converges in L^p_2 over compact subsets to a locally constant function u. Then $u|_{\mathfrak{b}} = 1$ and $u\Phi = \Phi$, hence $u = 1$. Set $f_j = u_{k_j} - 1$ and $\phi = \Phi - \Phi_o \in L^{p,w}_1$. Then $\|df_j \otimes \phi\|_{L^{p,w}} \to 0$. Furthermore, given $\epsilon > 0$ we can find $t > 0$ such that

$$\int_{[t,\infty)\times Y} |e^w \phi|^p < \frac{\epsilon}{4}$$

and N such that

$$\int_{X_{:t}} |e^w f_j \phi|^p < \frac{\epsilon}{2}$$

for $j > N$. Then $\int_X |e^w f_j \phi|^p < \epsilon$ for $j > N$. Thus $\|f_j\phi\|_{L^{p,w}} \to 0$, and similarly $\|f_j \nabla\phi\|_{L^{p,w}} \to 0$. Altogether this shows that $\|f_j\phi\|_{L^{p,w}_1} \to 0$. Combined with (10) this yields

$$\|(u_{k_j} - 1)\Phi_0\|_{L^{p,w}_1} \to 0,$$

hence $\|u_{k_j} - 1\|_{\mathcal{E}} \to 0$. But we can run the above argument starting with any subsequence of $\{u_k\}$, so $\|u_k - 1\|_{\mathcal{E}} \to 0$. $\square$

2.5 Manifolds with boundary

Let Z be a compact, connected, oriented Riemannian n–manifold, perhaps with boundary, and $\mathfrak{b} \subset Z$ a finite subset. Let $\mathbb{S} \to Z$ be a Hermitian vector bundle and $L \to Z$ a principal U(1)–bundle. Fix $p > n/2$ and let $\mathcal{C}$ denote the space of L^p_1 configurations (A, Φ) in $(L, \mathbb{S})$. Let $\mathcal{G}_{\mathfrak{b}}$ be the group of those L^p_2 gauge transformations $Z \to \mathrm{U}(1)$ that restrict to 1 on $\mathfrak{b}$, and $\mathcal{C}^*_{\mathfrak{b}}$ the set of all elements of $\mathcal{C}$ that have trivial stabilizer in $\mathcal{G}_{\mathfrak{b}}$. Then

$$\mathcal{B}^*_{\mathfrak{b}} = \mathcal{C}^*_{\mathfrak{b}} / \mathcal{G}_{\mathfrak{b}}$$

is again a (Hausdorff) smooth Banach manifold. As for orbit spaces of connections (see [11, p 192]) the main ingredient here is the solution to the Neumann problem over Z, according to which the operator

$$\begin{aligned} T_\Phi\colon\ L^p_2(Z) &\to L^p(Z)\oplus\partial L^p_1(\partial Z),\\ f &\mapsto (\Delta f+|\Phi|^2 f,\partial_\nu f) \end{aligned}$$

is a Fredholm operator of index 0 (see [31, Section 5.7] and [16, pages 85–86]). Here ν is the inward-pointing unit normal along ∂Z, and $\partial L^p_1(\partial Z)$ is the space of boundary values of L^p_1 functions on Z. Henceforth we work with imaginary-valued functions, and on ∂Z we identify 3–forms with functions by means of the Hodge $*$–operator. Then $T_\Phi=J_\Phi\mathcal{I}_\Phi$, where

$$J_\Phi(a,\phi)=(\mathcal{I}^*_\Phi(a,\phi),(*a)|_{\partial Z}).$$

Choose a bounded linear map

$$\rho\colon\ L^p(Z)\oplus\partial L^p_1(\partial Z)\to W:=T_\Phi(L\mathcal{G}_{\mathfrak{b}})$$

which restricts to the identity on W, and set $J^\#_\Phi=\rho J_\Phi$. An application of Stokes' theorem shows that

$$\ker(T_\Phi)\subset\ker(\mathcal{I}_\Phi)\quad\text{in } L^p_2(Z),$$

hence

$$T_\Phi=J^\#_\Phi\mathcal{I}_\Phi\colon\ L\mathcal{G}_{\mathfrak{b}}\to W$$

is an isomorphism. In general, if $V_1\xrightarrow{T_1}V_2\xrightarrow{T_2}V_3$ are linear maps between vector spaces such that T_2T_1 is an isomorphism, then $V_2=\operatorname{im}(T_1)\oplus\ker(T_2)$. Therefore, for any $(A,\Phi)\in\mathcal{C}^*_{\mathfrak{b}}$ we have

$$L^p_1(Z;i\Lambda^1\oplus\mathbb{S})=\mathcal{I}_\Phi(L\mathcal{G}_{\mathfrak{b}})\oplus\ker(J^\#_\Phi),$$

where both summands are closed subspaces. Thus we obtain the analogue of Proposition 2.6 with local slices of the form $(A,\Phi)+U$, where U is a small neighbourhood of $0\in\ker(J^\#_\Phi)$.

3 Moduli spaces

3.1 Spin^c structures

It will be convenient to have a definition of spin^c structure that does not refer to Riemannian metrics. So let X be an oriented n–dimensional manifold and P_{GL^+} its bundle of positive linear frames. Let $\widetilde{\mathrm{GL}}^+(n)$ denote the 2–fold universal covering group of the identity component $\mathrm{GL}^+(n)$ of $\mathrm{GL}(n,\mathbb{R})$, and denote by -1 the non-trivial element of the kernel of $\widetilde{\mathrm{GL}}^+(n)\to\mathrm{GL}^+(n)$. Set

$$\mathrm{GL}^c(n)=\widetilde{\mathrm{GL}}^+(n)\underset{\pm(1,1)}{\times}\mathrm{U}(1).$$

Then there is a short exact sequence

$$0\to\mathbb{Z}/2\to\mathrm{GL}^c(n)\to\mathrm{GL}^+(n)\times\mathrm{U}(1)\to 1,$$

and $\mathrm{Spin}^c(n)$ is canonically isomorphic to the preimage of $\mathrm{SO}(n)$ by the projection $\mathrm{GL}^c(n)\to\mathrm{GL}^+(n)$.

Definition 3.1 By a spin^c structure $\mathfrak{s}$ on X we mean a principal $\mathrm{GL}^c(n)$–bundle $P_{\mathrm{GL}^c}\to X$ together with a $\mathrm{GL}^c(n)$ equivariant map $P_{\mathrm{GL}^c}\to P_{\mathrm{GL}^+}$ which covers the identity on X. If $\mathfrak{s}'$ is another spin^c structure on X given by $P'_{\mathrm{GL}^c}\to P_{\mathrm{GL}^+}$ then $\mathfrak{s}$ and $\mathfrak{s}'$ are called *isomorphic* if there is a $\mathrm{U}(1)$ equivariant map $P'_{\mathrm{GL}^c}\to P_{\mathrm{GL}^c}$ which covers the identity on P_{GL^+}.

The natural $\mathrm{U}(1)$–bundle associated to P_{GL^c} is denoted $\mathcal{L}$, and the Chern class $c_1(\mathcal{L})$ is called the *canonical class* of the spin^c structure.

Now suppose X is equipped with a Riemannian metric, and let P_{SO} be its bundle of positive orthonormal frames, which is a principal $\mathrm{SO}(n)$–bundle. Then the preimage P_{Spin^c} of P_{SO} by the projection $P_{\mathrm{GL}^c}\to P_{\mathrm{GL}^+}$ is a principal $\mathrm{Spin}^c(n)$–bundle over X, ie a spin^c structure of X in the sense of [23]. Conversely, P_{GL^c} is isomorphic to $P_{\mathrm{Spin}^c}\underset{\mathrm{Spin}^c n}{\times}\mathrm{GL}^c(n)$. Thus there is a natural 1–1 correspondence between (isomorphism classes of) spin^c structures of the smooth oriented manifold X as defined above, and spin^c structures of the oriented Riemannian manifold X in the sense of [23].

By a *spin connection* in P_{Spin^c} we shall mean a connection in P_{Spin^c} that maps to the Levi–Civita connection in P_{SO}. If A is a spin connection in P_{Spin^c} then $\hat{F}_A$ will denote the $i\mathbb{R}$ component of the curvature of A with respect to the isomorphism of Lie algebras

$$\mathrm{spin}(n)\oplus i\mathbb{R}\xrightarrow{\approx}\mathrm{spin}^c(n)$$

defined by the double cover $\mathrm{Spin}(n)\times\mathrm{U}(1)\to\mathrm{Spin}^c(n)$. In terms of the induced connection $\check{A}$ in $\mathcal{L}$ one has

$$\hat{F}_A=\frac{1}{2}F_{\check{A}}.$$

If A,A' are spin connections in P_{Spin^c} then we regard $A-A'$ as an element of $i\Omega^1_X$.

The results of Section 2 carry over to spaces of configurations (A,Φ) where A is a spin connection in P_{Spin^c} and Φ a section of some complex vector bundle $\mathbb{S}\to X$.

When the spin^c structure on X is understood then we will say "spin connection over X" instead of "spin connection in P_{Spin^c}".

If n is even then the complex Clifford algebra $\mathbb{C}\ell(n)$ has up to equivalence exactly one irreducible complex representation. Let $\mathbb{S}$ denote the associated spin bundle over X. Then the eigenspaces of the complex volume element $\omega_{\mathbb{C}}$ in $\mathbb{C}\ell(n)$ defines a splitting $\mathbb{S}=\mathbb{S}^+\oplus\mathbb{S}^-$ (see [23]).

If n is odd then $\mathbb{C}\ell(n)$ has up to equivalence two irreducible complex representations ρ_1,ρ_2. These restrict to equivalent representations of $\mathrm{Spin}^c(n)$, so one gets a well-defined spin bundle $\mathbb{S}$ for any spin^c structure on X [23]. If α is the unique automorphism of $\mathbb{C}\ell(n)$ whose restriction to $\mathbb{R}^n$ is multiplication by -1 then $\rho_1\approx\rho_2\circ\alpha$. Hence if A is any spin connection over X then the sign of the Dirac operator D_A depends on the choice of ρ_j. To remove this ambiguity we decree that Clifford multiplication of TX on $\mathbb{S}$ is to be defined using the representation ρ_j satisfying $\rho_j(\omega_{\mathbb{C}})=1$.

In the case of a Riemannian product $\mathbb{R}\times X$ there is a natural 1–1 correspondence $\mathrm{Spin}^c(\mathbb{R}\times X)=\mathrm{Spin}^c(X)$, and we can identify

$$\mathcal{L}_{\mathbb{R}\times X}=\mathbb{R}\times\mathcal{L}_X.$$

If A is a spin connection over $\mathbb{R}\times X$ then $A|_{\{t\}\times X}$ will denote the spin connection B over X satisfying $\check{A}|_{\{t\}\times X}=\check{B}$.

When n is odd then we can also identify

$$\mathbb{S}^+_{\mathbb{R}\times X}=\mathbb{R}\times\mathbb{S}_X. \tag{11}$$

If e is a tangent vector on X then Clifford multiplication with e on $\mathbb{S}_X$ corresponds to multiplication with e_0e on $\mathbb{S}^+_{\mathbb{R}\times X}$, where e_0 is the positively oriented unit tangent vector on $\mathbb{R}$. Therefore, reversing the orientation of X changes the sign of the Dirac operator on X.

From now on, to avoid confusion we will use ∂_B to denote the Dirac operator over a 3–manifold with spin connection B, while the notation D_A will be reserved for Dirac operators over 4–manifolds.

By a *configuration* over a spin^c 3–manifold Y we shall mean a pair (B,Ψ) where B is a spin connection over Y and Ψ a section of the spin bundle $\mathbb{S}_Y$. By a configuration over a spin^c 4–manifold X we mean a pair (A,Φ) where A is a spin connection over X and Φ a section of the positive spin bundle $\mathbb{S}^+_X$.

3.2 The Chern–Simons–Dirac functional

Let Y be a closed Riemannian spin^c 3–manifold and η a closed 2–form on Y of class C^1. Fix a smooth reference spin connection B_o over Y and for any configuration (B,Ψ) over Y define the Chern–Simons–Dirac functional $\vartheta=\vartheta_\eta$ by

$$\vartheta(B,\Psi)=-\frac{1}{2}\int_Y(\hat{F}_B+\hat{F}_{B_o}+2i\eta)\wedge(B-B_o)-\frac{1}{2}\int_Y\langle\partial_B\Psi,\Psi\rangle.$$

Here and elsewhere $\langle\cdot,\cdot\rangle$ denotes Euclidean inner products, while $\langle\cdot,\cdot\rangle_{\mathbb{C}}$ denotes Hermitian inner products. Note that reversing the orientation of Y changes the sign of ϑ. Let $\mathcal{C}=\mathcal{C}_Y$ denote the space of L^2_1 configurations (B,Ψ). Then ϑ defines a smooth map $\mathcal{C}_Y\to\mathbb{R}$ which has an L^2 gradient

$$\nabla\vartheta_{(B,\Psi)}=(*(\hat{F}_B+i\eta)-\frac{1}{2}\sigma(\Psi,\Psi),-\partial_B\Psi).$$

If $\{a_j\}$ is a local orthonormal basis of imaginary-valued 1–forms on Y then

$$\sigma(\phi,\psi)=\sum_{j=1}^{3}\langle a_j\phi,\psi\rangle a_j.$$

Here and elsewhere the inner products are Euclidean unless otherwise specified. Since $\nabla\vartheta$ is independent of B_o, ϑ is independent of B_o up to additive constants. If $u\colon Y\to\mathrm{U}(1)$ then

$$\vartheta(u(S))-\vartheta(S)=\int_Y(\hat{F}_B+i\eta)\wedge u^{-1}du=2\pi\int_Y\widetilde{\eta}\wedge[u],$$

where $[u]\in H^1(Y)$ is the pull-back by u of the fundamental class of $\mathrm{U}(1)$, and $\widetilde{\eta}$ is as in (2).

The invariance of ϑ under null-homotopic gauge transformations imply

$$\mathcal{I}^*_\Psi\nabla\vartheta_{(B,\Psi)}=0. \tag{12}$$

Let $H_{(B,\Psi)}\colon L^2_1\to L^2$ be the derivative of $\nabla\vartheta\colon\mathcal{C}\to L^2$ at (B,Ψ), ie

$$H_{(B,\Psi)}(b,\psi)=(*db-\sigma(\Psi,\psi),-b\Psi-\partial_B\psi).$$

Note that $H_{(B,\Psi)}$ is formally self-adjoint, and $H_{(B,\Psi)}\mathcal{I}_\Psi=0$ if $\partial_B\Psi=0$. As in [15], a critical point (B,Ψ) of ϑ is called *non-degenerate* if the kernel of $\mathcal{I}^*_\Psi+H_{(B,\Psi)}$ in L^2_1 is zero, or equivalently, if $\mathcal{I}_\Psi+H_{(B,\Psi)}\colon L^2_1\to L^2$ is surjective. Note that if η is smooth then any critical point of ϑ_η has a smooth representative.

Let $\mathcal{G}$ be the Hilbert Lie group of L^2_2 maps $Y \to \mathrm{U}(1)$, and $\mathcal{G}_0 \subset \mathcal{G}$ the subgroup of null-homotopic maps. Set

$$\mathcal{B} = \mathcal{C}/\mathcal{G}, \quad \widetilde{\mathcal{B}} = \mathcal{C}/\mathcal{G}_0.$$

Then ϑ descends to a continuous map $\widetilde{\mathcal{B}} \to \mathbb{R}$ which we also denote by ϑ. If Condition (O1) holds (which we always assume when no statement to the contrary is made) then there is a real number q such that

$$\vartheta(\mathcal{G}S) = \vartheta(S) + q\mathbb{Z}$$

for all configurations S. If (O1) does not hold then $\vartheta(\mathcal{G}S)$ is a dense subset of $\mathbb{R}$.

If S is any smooth configuration over a band $(a,b)\times Y$, with $a<b$, let $\nabla\vartheta_S$ be the section of the bundle $\pi_2^*(\mathbb{S}_Y \oplus i\Lambda^1_Y)$ over $(a,b)\times Y$ such that $\nabla\vartheta_S|_{\{t\}\times Y} = \nabla\vartheta_{S_t}$. Here $\pi_2\colon \mathbb{R}\times Y \to Y$ is the projection. Note that $S \mapsto \nabla\vartheta_S$ extends to a smooth map $L^2_1 \to L^2$.

Although we will normally work with L^2_1 configurations over Y, the following lemma is sometimes useful.

Lemma 3.1 *ϑ extends to a smooth function on the space of $L^2_{1/2}$ configurations over Y.*

Proof The solution to the Dirichlet problem provides bounded operators

$$E\colon L^2_{1/2}(Y) \to L^2_1(\mathbb{R}_+ \times Y)$$

such that, for any $f \in L^2_{1/2}(Y)$, the function Ef restricts to f on $\{0\}\times Y$ and vanishes on $(1,\infty)\times Y$, and Ef is smooth whenever f is smooth. (see [31, p 307]). Similar extension maps can clearly be defined for configurations over Y. The lemma now follows from the observation that if S is any smooth configuration over $[0,1]\times Y$ then

$$\vartheta(S_1) - \vartheta(S_0) = \int_{[0,1]\times Y} \left\langle \nabla\vartheta_S, \frac{\partial S}{\partial t} \right\rangle,$$

and the right hand side extends to a smooth function on the space of L^2_1 configurations S over $[0,1]\times Y$. □

We will now relate the Chern–Simons–Dirac functional to the 4–dimensional monopole equations, cf [21, 28]. Let X be a spin^c Riemannian 4–manifold.

Given a parameter $\mu \in \Omega^2(X)$ there are the following Seiberg–Witten equations for a configuration (A,Φ) over X:

$$\begin{aligned}(\hat{F}_A + i\mu)^+ &= Q(\Phi)\\ D_A\Phi &= 0,\end{aligned} \tag{13}$$

where

$$Q(\Phi) = \frac{1}{4}\sum_{j=1}^{3}\langle\alpha_j\Phi,\Phi\rangle\alpha_j$$

for any local orthonormal basis $\{\alpha_j\}$ of imaginary-valued self-dual 2–forms on X. If Ψ is another section of $\mathbb{S}^+_X$ then one easily shows that

$$Q(\Phi)\Psi = \langle\Psi,\Phi\rangle_{\mathbb{C}}\Phi - \frac{1}{2}|\Phi|^2\Psi.$$

Now let $X = \mathbb{R}\times Y$ and for present and later use recall the standard bundle isomorphisms

$$\begin{aligned}\rho^1\colon\ \pi_2^*(\Lambda^0(Y)\oplus\Lambda^1(Y)) &\to \Lambda^1(\mathbb{R}\times Y), \quad (f,a)\mapsto f\,dt + a,\\ \rho^+\colon\ \pi_2^*(\Lambda^1(Y)) &\to \Lambda^+(\mathbb{R}\times Y), \quad a\mapsto \frac{1}{2}(dt\wedge a + *_Y a).\end{aligned} \tag{14}$$

Here $*_Y$ is the Hodge $*$–operator on Y. Let μ be the pull-back of a 2–form η on Y. Set $\vartheta = \vartheta_\eta$. Let $S = (A,\Phi)$ be any smooth configuration over $\mathbb{R}\times Y$ such that A is in temporal gauge. Under the identification $\mathbb{S}^+_{\mathbb{R}\times Y} = \pi_2^*(\mathbb{S}_Y)$ we have

$$\rho^+(\sigma(\Phi,\Phi)) = 2Q(\Phi).$$

Let $\nabla_1\vartheta$, $\nabla_2\vartheta$ denote the 1–form and spinor parts of $\nabla\vartheta$, respectively. Then

$$\begin{aligned}\rho^+\left(\frac{\partial A}{\partial t} + \nabla_1\vartheta_S\right) &= (\hat{F}_A + i\pi_2^*\eta)^+ - Q(\Phi)\\ \frac{\partial\Phi}{\partial t} + \nabla_2\vartheta_S &= -dt\cdot D_A\Phi,\end{aligned} \tag{15}$$

Thus, the downward gradient flow equation

$$\frac{\partial S}{\partial t} + \nabla\vartheta_S = 0$$

is equivalent to the Seiberg–Witten equations (13).

3.3 Perturbations

For transversality reasons we will, as in [15], add further small perturbations to the Seiberg–Witten equation over $\mathbb{R}\times Y$. The precise shape of these perturbations will depend on the situation considered. At this point we will merely describe a set of properties of these perturbations which will suffice for the Fredholm, compactness, and gluing theory.

To any L^2_1 configuration S over the band $(-\frac{1}{2},\frac{1}{2})\times Y$ there will be associated an element $h(S)\in\mathbb{R}^N$, where $N\ge 1$ will depend on the situation considered. If S is an L^2_1 configuration over $\mathbf{B}^+=(a-\frac{1}{2},b+\frac{1}{2})\times Y$ where $-\infty<a<b<\infty$ then the corresponding function

$$h_S\colon\ [a,b]\to\mathbb{R}^N$$

given by $h_S(t)=h(S|_{(t-1/2,t+1/2)\times Y})$ will be smooth. These functions h_S will have the following properties. Let S_o be a smooth reference configuration over $\mathbf{B}^+$.

(P1) For $0\le k<\infty$ the assignment $s\mapsto h_{S_o+s}$ defines a smooth map $L^2_1\to C^k$ whose image is a bounded set.

(P2) If $S_n\to S$ weakly in L^2_1 then $\|h_{S_n}-h_S\|_{C^k}\to 0$ for every $k\ge 0$.

(P3) h_S is gauge invariant, ie $h_S=h_{u(S)}$ for any smooth gauge transformation u.

We will also choose a compact codimension 0 submanifold $\Xi\subset\mathbb{R}^N$ which does not contain $h(\underline{\alpha})$ for any critical point α, where $\underline{\alpha}$ is the translationary invariant configuration over $\mathbb{R}\times Y$ determined by α. Let $\widetilde{\mathfrak{P}}=\widetilde{\mathfrak{P}}_Y$ denote the space of all (smooth) 2–forms on $\mathbb{R}^N\times Y$ supported in $\Xi\times Y$. For any S as above and any $\mathfrak{p}\in\widetilde{\mathfrak{P}}$ let $h_{S,\mathfrak{p}}\in\Omega^2([a,b]\times Y)$ denote the pull-back of $\mathfrak{p}$ by the map $h_S\times\mathrm{Id}$. It is clear that $h_{S,\mathfrak{p}}(t,y)=0$ if $h_S(t)\notin\Xi$. Moreover,

$$\|h_{S,\mathfrak{p}}\|_{C^k}\le\gamma_k\|\mathfrak{p}\|_{C^k} \tag{16}$$

where the constant γ_k is independent of $S,\mathfrak{p}$.

Now let $-\infty\le a<b\le\infty$ and $\mathbf{B}=(a,b)\times Y$. If $\mathfrak{q}\colon\ \mathbf{B}\to\mathbb{R}$ is a smooth function then by a $(\mathfrak{p},\mathfrak{q})$*–monopole* over $\mathbf{B}$ we shall mean a configuration $S=(A,\Phi)$ over $\mathbf{B}^+=(a-\frac{1}{2},b+\frac{1}{2})\times Y$ (smooth, unless otherwise stated) which satisfies the equations

$$\begin{aligned}(\hat F_A+i\pi_2^*\eta+i\mathfrak{q}h_{S,\mathfrak{p}})^+&=Q(\Phi)\\ D_A\Phi&=0,\end{aligned} \tag{17}$$

over $\mathbf{B}$, where η is as before. If A is in temporal gauge then these equations can also be expressed as

$$\frac{\partial S_t}{\partial t}=-\nabla\vartheta_{S_t}+E_S(t), \tag{18}$$

where the perturbation term $E_S(t)$ depends only on the restriction of S to $(t-\frac{1}{2},t+\frac{1}{2})\times Y$.

To reduce the number of constants later, we will always assume that $\mathfrak{q}$ and its differential $d\mathfrak{q}$ are pointwise bounded (in norm) by 1 everywhere. Note that if $\mathfrak{q}$ is constant then the equations (17) are translationary invariant. A $(\mathfrak{p},\mathfrak{q})$–monopole with $\mathfrak{q}h_{S,\mathfrak{p}}=0$ is called a *genuine* monopole. In expressions like $\|F_A\|_2$ and $\|\Phi\|_\infty$ the norms will usually be taken over $\mathbf{B}$.

For the transversality theory in Section 8 we will need to choose a suitable Banach space $\mathfrak{P}=\mathfrak{P}_Y$ of forms $\mathfrak{p}$ as above (of some given regularity). It will be essential that

$$\mathcal{C}(\mathbf{B}^+)\times\mathfrak{P}\to L^p(\mathbf{B},\Lambda^2),\quad (S,\mathfrak{p})\mapsto h_{S,\mathfrak{p}} \tag{19}$$

be a smooth map when a,b are finite (here $p>2$ is the exponent used in defining the configuration space $\mathcal{C}(\mathbf{B}^+)$). Now, one cannot expect $h_{S,\mathfrak{p}}$ to be smooth in S unless $\mathfrak{p}$ is smooth in the $\mathbb{R}^N$ direction (this point was overlooked in [15]). It seems natural then to look for a suitable space $\mathfrak{P}$ consisting of *smooth* forms $\mathfrak{p}$. Such a $\mathfrak{P}$ will be provided by Lemma 8.2. The topology on $\mathfrak{P}$ will be stronger than the C^∞ topology, ie stronger than the C^k topology for every k. The smoothness of the map (19) is then an easy consequence of property (P1) above and the next lemma.

Lemma 3.2 *Let A be a topological space, U a Banach space, and $K\subset\mathbb{R}^n$ a compact subset. Then the composition map*

$$C_B(A,\mathbb{R}^n)\times C^k(\mathbb{R}^n,U)_K\to C_B(A,U)$$

is of class C^{k-1} for any natural number k. Here $C_B(A,\cdot)$ denotes the supremum-normed space of bounded continuous maps from A into the indicated space, and $C^k(\mathbb{R}^n,U)_K$ is the space of C^k maps $\mathbb{R}^n\to U$ with support in K.

Proof This is a formal exercise in the differential calculus. $\square$

3.4 Moduli spaces

Consider the situation of Subsection 1.3. (We do not assume here that (A) holds.) We will define the moduli space $M(X;\vec{\alpha})$. In addition to the parameter μ this will depend on a choice of perturbation forms $\mathfrak{p}_j\in\widetilde{\mathfrak{P}}_{Y_j}$ and a smooth function $\mathfrak{q}\colon X\to[0,1]$ such that $\|d\mathfrak{q}\|_\infty\le 1$, $\mathfrak{q}^{-1}(0)=X_{:\frac{3}{2}}$, and $\mathfrak{q}=1$ on $[3,\infty)\times Y$.

Choose a smooth reference configuration $S_o=(A_o,\Phi_o)$ over X which is translationary invariant and in temporal gauge over the ends, and such that $S_o|_{\{t\}\times Y_j}$

represents $\alpha_j \in \widetilde{\mathcal{R}}_{Y_j}$. Let $p > 4$ and choose w as in Subsection 2.1. Let $\mathfrak{b}$ be a finite subset of X and define $\mathcal{C}, \mathcal{G}_{\mathfrak{b}}, \mathcal{B}_{\mathfrak{b}}$ as in Subsections 2.1 and 2.4. For clarity we will sometimes write $\mathcal{C}(X;\vec{\alpha})$ etc. Set $\vec{\mathfrak{p}} = (\mathfrak{p}_1, \dots, \mathfrak{p}_r)$ and let

$$M_{\mathfrak{b}}(X;\vec{\alpha}) = M_{\mathfrak{b}}(X;\vec{\alpha};\mu;\vec{\mathfrak{p}}) \subset \mathcal{B}_{\mathfrak{b}}$$

be the subset of gauge equivalence classes of solutions $S = (A, \Phi)$ (which we simply refer to as monopoles) to the equations

$$\begin{aligned} \left(\hat{F}_A + i\mu + i\mathfrak{q}\sum_{j=1}^{r} h_{S,\mathfrak{p}_j}\right)^+ - Q(\Phi) = 0 \\ D_A\Phi = 0. \end{aligned} \tag{20}$$

It is clear that $\mathfrak{q}\sum_j h_{S,\mathfrak{p}_j}$ vanishes outside a compact set in X. If it vanishes everywhere then S is called a *genuine* monopole. If $\mathfrak{b}$ is empty then we write $M = M_{\mathfrak{b}}$.

Note that different choices of S_o give canonically homeomorphic moduli spaces $M_{\mathfrak{b}}(X;\vec{\alpha})$ (and similarly for $\mathcal{B}_{\mathfrak{b}}(X;\vec{\alpha})$).

Unless otherwise stated the forms μ and $\mathfrak{p}_j$ will be smooth. In that case every element of $M_{\mathfrak{b}}(X;\vec{\alpha};\mu;\vec{\mathfrak{p}})$ has a smooth representative, and in notation like $[S] \in M$ we will often implicitly assume that S is smooth.

We define the moduli spaces $M(\alpha,\beta) = M(\alpha,\beta;\mathfrak{p})$ of Subsection 1.2 similarly, except that we here use the equations (17) with $\mathfrak{q} \equiv 1$.

The following estimate will be crucial in compactness arguments later.

Proposition 3.1 *For any element $[A,\Phi] \in M(X;\vec{\alpha})$ one has that either*

$$\Phi = 0 \quad \textit{or} \quad \|\Phi\|_\infty^2 \le -\frac{1}{2}\inf_{x\in X}\mathbf{s}(x) + 4\|\mu\|_\infty + 4\gamma_0 \max_j \|\mathfrak{p}_j\|_\infty,$$

where $\mathbf{s}$ is the scalar curvature of X and the constant γ_0 is as in (16).

Proof Let ψ_j denote the spinor field of $A_o|_{\{t\}\times Y_j}$. If $|\Phi|$ has a global maximum then the conclusion of the proposition holds by the proof of [21, Lemma 2]. Otherwise one must have $\|\Phi\|_\infty = \max_j \|\psi_j\|_\infty$ because of the Sobolev embedding $L^p_1 \subset C^0$ on compact 4–manifolds. But the argument in [21] applied to $\mathbb{R}\times Y$ yields

$$\psi_j = 0 \quad \text{or} \quad \|\psi_j\|_\infty^2 \le -\frac{1}{2}\inf_{x\in X}\mathbf{s}(x) + 4\|\mu\|_\infty$$

for each j, and the proposition follows. ◻

The left hand side of (20) can be regarded as a section $\Theta(S) = \widetilde{\Theta}(S, \mu, \vec{\mathfrak{p}})$ of the bundle $\Lambda^+ \oplus \mathbb{S}^-$ over X. It is clear that Θ defines a smooth map

$$\Theta\colon\, \mathcal{C} \to L^{p,w},$$

which we call the *monopole map*. Let $D\Theta$ denote the derivative of Θ. We claim that

$$\mathcal{I}_\Phi^* + D\Theta(S)\colon\, L_1^{p,w} \to L^{p,w} \tag{21}$$

is a Fredholm operator for every $S = (A, \Phi) \in \mathcal{C}$. Note that the $\mathfrak{p}_j$–perturbations in (20) only contribute a compact operator, so we can take $\mathfrak{p}_j = 0$ for each j. We first consider the case $X = \mathbb{R} \times Y$, with $\mu = \pi_2^*\eta$ as before. By means of the isomorphisms (14), (11) and the isomorphism $\mathbb{S}^+ \to \mathbb{S}^-$, $\phi \mapsto dt \cdot \phi$ we can think of the operator (21) as acting on sections of $\pi_2^*(\Lambda_Y^0 \oplus \Lambda_Y^1 \oplus \mathbb{S}_Y)$. If A is in temporal gauge then a simple computation yields

$$\mathcal{I}_\Phi^* + D\Theta(S) = \frac{d}{dt} + P_{S_t}, \tag{22}$$

where

$$P_{(B,\Psi)} = \begin{pmatrix} 0 & \mathcal{I}_\Psi^* \\ \mathcal{I}_\Psi & H_{(B,\Psi)} \end{pmatrix}$$

for any configuration (B, Ψ) over Y. Note that $P_{(B,\Psi)}$ is elliptic and formally self-adjoint, and if (B, Ψ) is a non-degenerate critical point of ϑ_η then $\ker P_{(B,\Psi)} = \ker \mathcal{I}_\Psi$. Thus, the structure of the linearized equations over a cylinder is analogous to that of the instanton equations studied in [10], and the results of [10] carry over to show that (21) is a Fredholm operator.

The index of (21) is independent of S and is called the *expected dimension* of $M(X; \vec{\alpha})$. If $S \in \mathcal{C}$ is a monopole and $D\Theta(S)\colon\, L_1^{p,w} \to L^{p,w}$ is surjective then $[S]$ is called a *regular* point of $M_{\mathfrak{b}}(X; \vec{\alpha})$. If in addition $S \in \mathcal{C}_{\mathfrak{b}}^*$ then $[S]$ has an open neighbourhood in $M_{\mathfrak{b}}(X; \vec{\alpha})$ which is a smooth submanifold of $\mathcal{B}_{\mathfrak{b}}^*$ of dimension

$$\dim M_{\mathfrak{b}}(X; \vec{\alpha}) = \operatorname{index}(\mathcal{I}_\Phi^* + D\Theta(S)) + |\mathfrak{b}|.$$

4 Local compactness I

This section provides the local compactness results needed for the proof of Theorem 1.4 assuming (B1).

4.1 Compactness under curvature bounds

For the moment let B be an arbitrary compact, oriented Riemannian manifold with boundary, and v the outward unit normal vector field along ∂B. Then

$$\Omega^*(B) \to \Omega^*(B) \oplus \Omega^*(\partial B), \quad \phi \mapsto ((d+d^*)\phi, \iota(v)\phi) \tag{23}$$

is an elliptic boundary system in the sense of [17, 2]. Here $\iota(v)$ is contraction with v. By [17, Theorems 20.1.2, 20.1.8] we then have:

Proposition 4.1 *For $k \geq 1$ the map (23) extends to a Fredholm operator*

$$L^2_k(B, \Lambda^*_B) \to L^2_{k-1}(B, \Lambda^*_B) \oplus L^2_{k-\frac{1}{2}}(\partial B, \Lambda^*_{\partial B})$$

whose kernel consists of C^∞ forms.

Lemma 4.1 *Let X be a spin^c Riemannian 4–manifold and $V_1 \subset V_2 \subset \cdots$ precompact open subsets of X such that $X = \cup_j V_j$. For $n = 1, 2, \ldots$ let μ_n be a 2–form on V_n, and $S_n = (A_n, \Phi_n)$ a smooth solution to the Seiberg–Witten equations (13) over V_n with $\mu = \mu_n$. Let $q > 4$. Then there exist a subsequence $\{n_j\}$ and for each j a smooth $u_j \colon V_j \to U(1)$ with the following significance. If k is any non-negative integer such that*

$$\sup_{n \geq j} \Big(\|\Phi_n\|_{L^q(V_j)} + \|\hat{F}(A_n)\|_{L^2(V_j)} + \|\mu_n\|_{C^k(V_j)} \Big) < \infty \tag{24}$$

for every positive integer j then for every $p \geq 1$ one has that $u_j(S_{n_j})$ converges weakly in L^p_{k+1} and strongly in L^p_k over compact subsets of X as $j \to \infty$.

Before giving the proof, note that the curvature term in (24) cannot be omitted. For if ω is any non-zero, closed, anti-self-dual 2–form over the 4–ball B then there is a sequence A_n of U(1) connections over B such that $F(A_n) = in\omega$. If $S_n = (A_n, 0)$ then there are clearly no gauge transformations u_n such that $u_n(S_n)$ converges (in any reasonable sense) over compact subsets of B.

Proof Let $B \subset X$ be a compact 4–ball. After trivializing $\mathcal{L}$ over B we can write $A_n|_B = \widetilde{d} + a_n$, where $\widetilde{d}$ is the spin connection over B corresponding to the product connection in $\mathcal{L}|_B$. By the solution of the Neumann problem (see [31]) there is a smooth $\xi_n \colon B \to i\mathbb{R}$ such that $b_n = a_n - d\xi$ satisfies

$$d^*b_n = 0; \quad *b_n|_{\partial B} = 0.$$

Using the fact that $H^1(B) = 0$ one easily proves that the map (23) is injective on $\Omega^1(B)$. Hence there is a constant C such that

$$\|b\|_{L^2_1(B)} \leq C(\|(d+d^*)b\|_{L^2(B)} + \|*b|_{\partial B}\|_{L^2_{1/2}(\partial B)})$$

for all $b \in \Omega^1(B)$. This gives

$$\|b_n\|_{L^2_1(B)} \le C\|db_n\|_{L^2(B)} = C\|\hat{F}(A_n)\|_{L^2(B)}.$$

Set $v_n = \exp(\xi_n)$. It is now an exercise in bootstrapping, using the Seiberg–Witten equations for S_n and interior elliptic estimates, to show that, for every $k \ge 0$ for which (24) holds and for every $p \ge 1$, the sequence $v_n(S_n) = (d + b_n, v_n\Phi_n)$ is bounded in L^p_{k+1} over compact subsets of $\mathrm{int}(B)$.

To complete the proof, choose a countable collection of such balls such that the corresponding balls of half the size cover X, and apply Lemma A.1. □

4.2 Small perturbations

If S is any smooth configuration over a band $(a,b)\times Y$ with $a<b$, the *energy* of S is by definition

$$\mathrm{energy}(S) = \int_{[a,b]\times Y} |\nabla\vartheta_S|^2.$$

If S is a genuine monopole then $\partial_t\vartheta(S_t) = -\int_Y |\nabla\vartheta_{S_t}|^2$, and so the energy equals $\vartheta(S_a) - \vartheta(S_b)$. If S is a $(\mathfrak{p},\mathfrak{q})$–monopole then one no longer expects these identities to hold, because the equation (18) is not of gradient flow type. The main object of this subsection is to show that if $\|\mathfrak{p}\|_{C^1}$ is sufficiently small then, under suitable assumptions, the variation of $\vartheta(S_t)$ still controls the energy locally (Proposition 4.2), and there is a monotonicity result for $\vartheta(S_t)$ (Proposition 4.3).

It may be worth mentioning that the somewhat technical Lemma 4.4 and Proposition 4.2 are not needed in the second approach to compactness which is the subject of Section 5.

In this subsection $\mathfrak{q}\colon \mathbb{R}\times Y \to \mathbb{R}$ may be any smooth function satisfying $\|\mathfrak{q}\|_\infty, \|d\mathfrak{q}\|_\infty \le 1$. Constants will be independent of $\mathfrak{q}$. The perturbation forms $\mathfrak{p}$ may be arbitrary elements of $\widetilde{\mathfrak{P}}$.

Lemma 4.2 *There is a constant $C_0 > 0$ such that if $-\infty < a < b < \infty$ and $S = (A,\Phi)$ is any $(\mathfrak{p},\mathfrak{q})$–monopole over $(a,b)\times Y$ then there is a pointwise bound*

$$|\hat{F}(A)| \le 2|\nabla\vartheta_S| + |\eta| + C_0|\Phi|^2 + \gamma_0\|\mathfrak{p}\|_\infty.$$

Proof Note that both sides of the inequality are gauge invariant, and if A is in temporal gauge then

$$F(A) = dt \wedge \frac{\partial A_t}{\partial t} + F_Y(A_t),$$

where F_Y stands for the curvature of a connection over Y. Now use inequalities (18) and (16). $\square$

Lemma 4.3 *There exists a constant $C_1 > 0$ such that for any $\tau > 0$ and any $(\mathfrak{p}, \mathfrak{q})$–monopole S over $(0, \tau) \times Y$ one has*

$$\int_{[0,\tau]\times Y} |\nabla\vartheta_S|^2 \le 2(\vartheta(S_0) - \vartheta(S_\tau)) + C_1^2\tau\|\mathfrak{p}\|_\infty^2.$$

Recall that by convention a $(\mathfrak{p}, \mathfrak{q})$–monopole over $(0, \tau) \times Y$ is actually a configuration over $(-\frac{1}{2}, \tau + \frac{1}{2}) \times Y$, so the lemma makes sense.

Proof We may assume S is in temporal gauge. Then

$$\begin{aligned}\vartheta(S_\tau) - \vartheta(S_0) &= \int_0^\tau \partial_t\vartheta(S_t)\,dt \\ &= \int_{[0,\tau]\times Y} \langle\nabla\vartheta_S, -\nabla\vartheta_S + E_S\rangle\,dt \\ &\le \|\nabla\vartheta_S\|_2(\|E_S\|_2 - \|\nabla\vartheta_S\|_2),\end{aligned}$$

where the norms on the last line are taken over $[0, \tau] \times Y$. If a, b, x are real numbers satisfying $x^2 - bx - a \le 0$ then

$$x^2 \le 2x^2 - 2bx + b^2 \le 2a + b^2.$$

Putting this together we obtain

$$\|\nabla\vartheta_S\|_2^2 \le 2(\vartheta(S_0) - \vartheta(S_\tau)) + \|E_S\|_2^2,$$

and the lemma follows from the estimate (16). $\square$

Lemma 4.4 *For all $C > 0$ there exists an $\epsilon > 0$ with the following significance. Let $\tau \ge 4$, $\mathfrak{p} \in \widetilde{\mathfrak{P}}$ with $\|\mathfrak{p}\|_\infty\tau^{1/2} \le \epsilon$, and let $S = (A, \Phi)$ be a $(\mathfrak{p}, \mathfrak{q})$–monopole over $(0, \tau) \times Y$ satisfying $\|\Phi\|_\infty \le C$. Then at least one of the following two statements must hold:*

(i) $\partial_t\vartheta(S_t) \le 0$ *for* $2 \le t \le \tau - 2$,

(ii) $\vartheta(S_{t_2}) < \vartheta(S_{t_1})$ *for* $0 \le t_1 \le 1$, $\tau - 1 \le t_2 \le \tau$.

Proof Given $C > 0$, suppose that for $n = 1, 2, \ldots$ there exist $\tau_n \ge 4$, $\mathfrak{p}_n \in \widetilde{\mathfrak{P}}$ with $\|\mathfrak{p}_n\|_\infty\tau_n^{1/2} \le 1/n$, and a $(\mathfrak{p}_n, \mathfrak{q}_n)$–monopole $S_n = (A_n, \Phi_n)$ over $(0, \tau_n)\times Y$ satisfying $\|\Phi_n\|_\infty \le C$ such that (i) is violated at some point $t = t_n$ and (ii) also does not hold. By Lemma 4.3 the last assumption implies

$$\|\nabla\vartheta_{S_n}\|_{L^2([1,\tau_n-1]\times Y)} \le C_1\|\mathfrak{p}_n\|_\infty\tau_n^{1/2} \le C_1/n.$$

For $s \in \mathbb{R}$ let $\mathcal{T}_s\colon \mathbb{R}\times Y \to \mathbb{R}\times Y$ be translation by s:

$$\mathcal{T}_s(t,y) = (t+s,y).$$

Given $p>2$ then by Lemmas 4.2 and 4.1 we can find $u_n\colon (-1,1)\times Y \to \mathrm{U}(1)$ in $L^p_{2,\mathrm{loc}}$ such that a subsequence of $u_n(\mathcal{T}^*_{t_n}(S_n))$ converges weakly in L^p_1 over $(-\frac{1}{2},\frac{1}{2})\times Y$ to an L^p_1 solution S' to the equations (13) with $\mu = \pi_2^*\eta$. Then $\nabla\vartheta_{S'} = 0$. After modifying the gauge transformations we can even arrange that S' is smooth and in temporal gauge, in which case there is a critical point α of ϑ such that $S'(t)\equiv\alpha$. After relabelling the subsequence above consecutively we then have

$$h_{S_n}(t_n) \to h_{S'}(0) \notin \Xi.$$

Since Ξ is closed, $h_{S_n}(t_n)\notin\Xi$ for n sufficiently large. Hence, $\partial_t|_{t_n}\vartheta(S_n(t)) = -\|\nabla\vartheta_{S_n(t_n)}\|^2 \le 0$, which is a contradiction. □

Proposition 4.2 *For any constant $C>0$ there exist $C',\delta>0$ such that if $S=(A,\Phi)$ is any $(\mathfrak{p},\mathfrak{q})$–monopole over $(-2,T+4)\times Y$ where $T\ge 2$, $\|\mathfrak{p}\|_\infty \le \delta$, and $\|\Phi\|_\infty \le C$, then for $1\le t\le T-1$ one has*

$$\int_{[t-1,t+1]\times Y} |\nabla\vartheta_S|^2 \le 2\left(\sup_{0\le r\le 1}\vartheta(S_{-r}) - \inf_{0\le r\le 4}\vartheta(S_{T+r})\right) + C'\|\mathfrak{p}\|^2_\infty.$$

Proof Choose $\epsilon>0$ such that the conclusion of Lemma 4.4 holds (with this constant C), and set $\delta = \epsilon/\sqrt{6}$. We construct a sequence $t_0,\dots,t_m$ of real numbers, for some $m\ge 1$, with the following properties:

(i) $-1\le t_0\le 0$ and $T\le t_m\le T+4$,

(ii) For $i=1,\dots,m$ one has $1\le t_i - t_{i-1}\le 5$ and $\vartheta(S_{t_i})\le\vartheta(S_{t_{i-1}})$.

The lemma will then follow from Lemma 4.3. The t_i's will be constructed inductively, and this will involve an auxiliary sequence $t'_0,\dots,t'_{m+1}$. Set $t_{-1} = t'_0 = 0$.

Now suppose t_{i-1},t'_i have been constructed for $0\le i\le j$. If $t'_j\ge T$ then we set $t_j = t'_j$ and $m=j$, and the construction is finished. If $t'_j<T$ then we define t_j,t'_{j+1} as follows:

If $\partial_t\vartheta(S_t)\le 0$ for all $t\in[t'_j,t'_j+2]$ set $t_j=t'_j$ and $t'_{j+1}=t'_j+2$; otherwise set $t_j = t'_j-1$ and $t'_{j+1} = t'_j+4$.

Then (i) and (ii) are satisfied, by Lemma 4.4. □

Proposition 4.3 *For all $C > 0$ there exists a $\delta > 0$ such that if $S = (A, \Phi)$ is any $(\mathfrak{p}, \mathfrak{q})$–monopole in temporal gauge over $(-1, 1) \times Y$ such that $\|\mathfrak{p}\|_{C^1} \leq \delta$, $\|\Phi\|_\infty \leq C$, and $\|\nabla\vartheta_S\|_2 \leq C$ then the following holds: Either $\partial_t|_0\vartheta(S_t) < 0$, or there is a critical point α such that $S_t = \alpha$ for $|t| \leq \frac{1}{2}$.*

Proof First observe that if S is any C^1 configuration over $\mathbb{R} \times Y$ then

$$\vartheta(S_{t_2}) - \vartheta(S_{t_1}) = \int_{t_1}^{t_2} \int_Y \langle \nabla\vartheta_{S_t}, \partial_t S_t \rangle dy\, dt,$$

hence $\vartheta(S_t)$ is a C^1 function of t whose derivative can be expressed in terms of the L^2 gradient of ϑ as usual.

Now suppose there is a $C > 0$ and for $n = 1, 2, \ldots$ a $\mathfrak{p}_n \in \widetilde{\mathfrak{P}}$ and a $(\mathfrak{p}_n, \mathfrak{q}_n)$–monopole $S_n = (A_n, \Phi_n)$ over $(-1, 1) \times Y$ such that $\|\mathfrak{p}_n\|_{C^1} \leq \frac{1}{n}$, $\|\Phi_n\|_\infty \leq C$, $\|\nabla\vartheta_{S_n}\|_2 \leq C$, and $\partial_t|_0\vartheta(S_n(t)) \geq 0$. Let $p > 4$ and $0 < \epsilon < \frac{1}{2}$. After passing to a subsequence and relabelling consecutively we can find $u_n \colon (-1, 1) \times Y \to \mathrm{U}(1)$ in $L^p_{3,\mathrm{loc}}$ such that $\widetilde{S}_n = u_n(S_n)$ converges weakly in L^p_2, and strongly in C^1, over $(-\frac{1}{2} - \epsilon, \frac{1}{2} + \epsilon) \times Y$ to a smooth solution S' of (13) with $\mu = \pi_2^*\eta$. We may arrange that S' is in temporal gauge. Then

$$0 \leq \partial_t|_0\vartheta(\widetilde{S}_n(t)) = \int_Y \langle \nabla\vartheta_{\widetilde{S}_n(0)}, \partial_t|_0\widetilde{S}_n(t) \rangle \to \partial_t|_0\vartheta(S'_t).$$

But S' is a genuine monopole, so $\partial_t\vartheta(S'_t) = -\|\nabla\vartheta_{S'_t}\|_2^2$. It also follows that $\nabla\vartheta_{S'(0)} = 0$, hence $\nabla\vartheta_{S'} = 0$ in $(-\frac{1}{2} - \epsilon, \frac{1}{2} + \epsilon) \times Y$ by unique continuation as in [15, Appendix]. Since $h_{S_n} \to h_{S'}$ uniformly in $[-\epsilon, \epsilon]$, and $h_{S'} \equiv \mathrm{const} \notin \Xi$, the function h_{S_n} maps $[-\epsilon, \epsilon]$ into the complement of Ξ when n is sufficiently large. In that case, S_n restricts to a genuine monopole on $[-\epsilon, \epsilon] \times Y$, and the assumption $\partial_t|_0\vartheta(S_n(t)) \geq 0$ implies that $\nabla\vartheta_{S_n} = 0$ on $[-\epsilon, \epsilon] \times Y$. Since this holds for any $\epsilon \in (0, \frac{1}{2})$, the proposition follows. □

We say a $(\mathfrak{p}, \mathfrak{q})$–monopole S over $\mathbb{R}_+ \times Y$ has *finite energy* if $\inf_{t>0} \vartheta(S_t) > -\infty$. A monopole over a 4–manifold with tubular ends is said to have finite energy if it has finite energy over each end.

Proposition 4.4 *Let C, δ be given such that the conclusion of Proposition 4.3 holds. If $S = (A, \Phi)$ is any finite energy $(\mathfrak{p}, \mathfrak{q})$–monopole over $\mathbb{R}_+ \times Y$ with $\|\mathfrak{p}\|_{C^1} \leq \delta$, $\mathfrak{q} \equiv 1$, $\|\Phi\|_\infty \leq C$, and*

$$\sup_{t \geq 1} \|\nabla\vartheta_S\|_{L^2((t-1, t+1) \times Y)} \leq C$$

then the following hold:

(i) *There is a $t > 0$ such that S restricts to a genuine monopole on $(t,\infty)\times Y$,*

(ii) *$[S_t]$ converges in $\mathcal{B}_Y$ to some critical point as $t\to\infty$.*

Proof Let $p>4$. If $\{t_n\}$ is any sequence with $t_n\to\infty$ as $n\to\infty$ then by Lemmas 4.1 and 4.2 there exist $u_n\in L^p_{3,\mathrm{loc}}(\mathbb{R}\times Y;\mathrm{U}(1))$ such that a subsequence of $u_n(\mathcal{T}^*_{t_n}S)$ converges weakly in L^p_2 (hence strongly in C^1) over compact subsets of $\mathbb{R}\times Y$ to a smooth $(\mathfrak{p},\mathfrak{q})$–monopole S' in temporal gauge. Proposition 4.3 guarantees that $\partial_t\vartheta(S_t)\le 0$ for $t\ge 1$, so the finite energy assumption implies that $\vartheta(S'_t)$ is constant. By Proposition 4.3 there is a critical point α such that $S'_t=\alpha$ for all t. This implies (i) by choice of the set Ξ (see Subsection 3.3). Part (ii) follows by a continuity argument from the facts that $\mathcal{B}_Y$ contains only finitely many critical points, and the topology on $\mathcal{B}_Y$ defined by the L^2–metric is weaker than the usual topology. ◻

The following corollary of Lemma 3.1 shows that elements of the moduli spaces defined in Subsection 3.4 have finite energy.

Lemma 4.5 *Let S be a configuration over $\overline{\mathbb{R}}_+\times Y$ and α a critical point of ϑ such that $S-\underline{\alpha}\in L^p_1$ for some $p\ge 2$. Then*

$$\vartheta(S_t)\to\vartheta(\alpha)\ \text{as}\ t\to\infty.$$

4.3 Neck-stretching I

This subsection contains the crucial step in the proof of Theorem 1.4 assuming (B1), namely what should be thought of as a global energy bound.

Lemma 4.6 *Let X be as in Subsection 1.3 and set $Z=X_{:1}$. We identify $Y=\partial Z$. Let $\mu_1,\mu_2\in\Omega^2(Z)$, where $d\mu_1=0$. Set $\eta=\mu_1|_Y$ and $\mu=\mu_1+\mu_2$. Let A_o be a spin connection over Z, and let the Chern–Simons–Dirac functional ϑ_η over Y be defined in terms of the reference connection $B_o=A_o|_Y$. Then for all configurations $S=(A,\Phi)$ over Z which satisfy the monopole equations (13) one has*

$$\left|2\vartheta_\eta(S|_Y)+\int_Z\left(|\nabla_A\Phi|^2+|\hat{F}_A+i\mu_1|^2\right)\right| \le C\,\mathit{Vol}(Z)\left(1+\|\Phi\|^2_\infty+\|F_{A_o}\|_\infty+\|\mu_1\|_\infty+\|\mu_2\|_\infty+\|\mathbf{s}\|_\infty\right)^2,$$

for some universal constant C, where $\mathbf{s}$ is the scalar curvature of Z.

The upper bound given here is not optimal but suffices for our purposes.

Proof Set $F'_A = \hat{F}_A + i\mu_1$ and define F'_{A_o} similarly. Set $B = A|_Y$. Without the assumption $d\mu_1 = 0$ we have

$$\begin{aligned}\int_Z |F'_A|^2 &= \int_Z (2|(F'_A)^+|^2 + F'_A \wedge F'_A) \\ &= \int_Z \big(2|Q(\Phi) - i\mu_2^+|^2 + F'_{A_o} \wedge F'_{A_o} - 2id\mu_1 \wedge (A - A_o)\big) \\ &\quad + \int_Y (\hat{F}_B + \hat{F}_{B_o} + 2i\eta) \wedge (B - B_o).\end{aligned}$$

Without loss of generality we may assume A is in temporal gauge over the collar $\iota([0,1] \times Y)$. By the Weitzenböck formula we have

$$0 = D_A^2\Phi = \nabla_A^*\nabla_A\Phi + \hat{F}_A^+ + \frac{\mathbf{s}}{4}.$$

This gives

$$\begin{aligned}\int_Z |\nabla_A\Phi|^2 &= \int_Z \langle \nabla_A^*\nabla_A\Phi, \Phi\rangle + \int_Y \langle \partial_t\Phi, \Phi\rangle \\ &= \int_Z \left(-\frac{1}{2}|\Phi|^4 - \frac{\mathbf{s}}{4}|\Phi|^2 + \langle i\mu^+\Phi, \Phi\rangle\right) + \int_Y \langle \partial_B\Phi, \Phi\rangle. \qquad \square\end{aligned}$$

Consider now the situation of Subsection 1.4. If (B1) holds then we can find a closed 2–form μ_1 on X whose restriction to $\mathbb{R}_+ \times (\pm Y_j)$ is the pull-back of η_j, and whose restriction to $\mathbb{R}_+ \times Y'_j$ is the pull-back of η'_j. From Lemma 4.6 we deduce:

Proposition 4.5 *For every constant $C_1 < \infty$ there exists a constant $C_2 < \infty$ with the following significance. Suppose we are given*

- $\tau, C_0 < \infty$ *and an r–tuple T such that $\tau \le T_j$ for each j,*
- *real numbers $\tau_j^\pm$, $1 \le j \le r$ and τ'_j, $1 \le j \le r'$ satisfying $0 \le T_j - \tau_j^\pm \le \tau$ and $0 \le \tau'_j \le \tau$.*

Let Z be the result of deleting from $X^{(T)}$ all the necks $(-\tau_j^-, \tau_j^+) \times Y_j$, $1 \le j \le r$ and all the ends $(\tau'_j, \infty) \times Y'_j$, $1 \le j \le r'$. Then for any configuration $S = (A, \Phi)$ representing an element of a moduli space $M(X^{(T)}; \vec{\alpha}'; \mu; \vec{\mathfrak{p}}, \vec{\mathfrak{p}}')$ where

$\sum_{j=1}^{r'}\vartheta(\alpha'_j) > -C_0$ and $\mathfrak{p}_j,\mathfrak{p}'_j,\mu$ all have L^∞ norm $< C_1$ one has that

$$\int_Z (|\nabla_A\Phi|^2+|\hat{F}_A+i\mu_1|^2)+2\sum_{j=1}^{r}\Big(\vartheta(S|_{\{-\tau_j^-\}\times Y_j})-\vartheta(S|_{\{\tau_j^+\}\times Y_j})\Big) + 2\sum_{j=1}^{r'}\Big(\vartheta(S|_{\{\tau'_j\}\times Y'_j})-\vartheta(\alpha'_j)\Big) < C_2(1+\tau)+2C_0.$$

Thus, if each $\mathfrak{p}_j$, $\mathfrak{p}'_j$ has sufficiently small L^∞ norm then Lemma 4.5 and Proposition 4.2 provides local energy bounds over necks and ends for such monopoles. (To apply Proposition 4.2 one can take $\tau_j^\pm$ to be the point t in a suitable interval where $\pm\vartheta(S|_{\{t\}\times Y_j})$ attains its maximum, and similarly for τ'_j.) Moreover, if these perturbation forms have sufficiently small C^1 norms then we can apply Proposition 4.3 over necks and ends. (How small the C^1 norms have to be depends on C_0.)

5 Local compactness II

This section, which is logically independent from Section 4, provides the local compactness results needed for the proof of Theorem 1.4 assuming (B2).

While the main result of this section, Proposition 5.5, is essentially concerned with local convergence of monopoles, the arguments will, in contrast to those of Section 4, be of a global nature. In particular, function spaces over manifolds with tubular ends will play a central role.

5.1 Hodge theory for the operator $-d^*+d^+$

In this subsection we will study the kernel (in certain function spaces) of the elliptic operator

$$\mathcal{D}=-d^*+d^+:\ \Omega^1(X)\to\Omega^0(X)\oplus\Omega^+(X), \tag{25}$$

where X is an oriented Riemannian 4–manifold with tubular ends. The notation $\ker(\mathcal{D})$ will refer to the kernel of $\mathcal{D}$ in the space $\Omega^1(X)$ of all smooth 1–forms, where X will be understood from the context. The results of this subsection complement those of [10].

We begin with the case of a half-infinite cylinder $X=\mathbb{R}_+\times Y$, where Y is any closed, oriented, connected Riemannian 3–manifold. Under the isomorphisms

(14) there is the identification $\mathcal{D} = \frac{\partial}{\partial t} + P$ over $\mathbb{R}_+ \times Y$, where P is the self-adjoint elliptic operator

$$P = \begin{pmatrix} 0 & -d^* \\ -d & *d \end{pmatrix}$$

acting on sections of $\Lambda^0(Y) \oplus \Lambda^1(Y)$ (cf (22)). Since P^2 is the Hodge Laplacian,

$$\ker(P) = H^0(Y) \oplus H^1(Y).$$

Let $\{h_\nu\}$ be a maximal orthonormal set of eigenvectors of P, say $Ph_\nu = \lambda_\nu h_\nu$.

Given a smooth 1–form a over $\mathbb{R}_+ \times Y$ we can express it as $a = \sum_\nu f_\nu h_\nu$, where $f_\nu \colon \mathbb{R}_+ \to \mathbb{R}$. If $\mathcal{D}a = 0$ then $f_\nu(t) = c_\nu e^{-\lambda_\nu t}$ for some constant c_ν. If in addition $a \in L^p$ for some $p \geq 1$ then $f_\nu \in L^p$ for all ν, hence $f_\nu \equiv 0$ when $\lambda_\nu \leq 0$. Elliptic inequalities for $\mathcal{D}$ then show that a decays exponentially, or more precisely,

$$|(\nabla^j a)_{(t,y)}| \leq \beta_j e^{-\delta t}$$

for $(t,y) \in \mathbb{R}_+ \times Y$ and $j \geq 0$, where β_j is a constant and δ the smallest positive eigenvalue of P.

Now let $\sigma > 0$ be a small constant and $a \in \ker(\mathcal{D}) \cap L^{p,-\sigma}$. Arguing as above we find that

$$a = b + c\,dt + \pi^*\psi, \tag{26}$$

where b is an exponentially decaying form, c a constant, $\pi \colon \mathbb{R}_+ \times Y \to Y$, and $\psi \in \Omega^1(Y)$ harmonic.

We now turn to the case when X is an oriented, connected Riemannian 4–manifold with tubular end $\overline{\mathbb{R}}_+ \times Y$ (so $X \setminus \mathbb{R}_+ \times Y$ is compact). Let $Y_1, \dots, Y_r$ be the connected components of Y and set

$$Y' = \bigcup_{j=1}^{s} Y_j, \qquad Y'' = Y \setminus Y',$$

where $0 \leq s \leq r$. Let $\sigma > 0$ be a small constant and $\kappa \colon X \to \mathbb{R}$ a smooth function such that

$$\kappa = \begin{cases} -\sigma t & \text{on } \mathbb{R}_+ \times Y' \\ \sigma t & \text{on } \mathbb{R}_+ \times Y'', \end{cases}$$

where t is the $\mathbb{R}_+$ coordinate. Our main goal in this subsection is to describe $\ker(\mathcal{D}) \cap L^{p,\kappa}$.

We claim that all elements $a \in \ker(\mathcal{D}) \cap L^{p,\kappa}$ are closed. To see this, note first that the decomposition (26) shows that a is pointwise bounded, and da decays exponentially over the ends. Applying the proof of [11, Proposition 1.1.19] to $X_{:T}=X\setminus(T,\infty)\times Y$ we get

$$\begin{aligned}\int_{X_{:T}}(|d^+a|^2-|d^-a|^2)&=\int_{X_{:T}}da\wedge da=\int_{X_{:T}}d(a\wedge da)\\&=\int_{\partial X_{:T}}a\wedge da\to 0\quad\text{as } T\to\infty.\end{aligned}$$

Since $d^+a=0$, we conclude that $da=0$.

Fix $\tau\geq 0$ and for any $a\in\Omega^1(X)$ and $j=1,\dots,r$ set

$$R_ja=\int_{\{\tau\}\times Y_j}*a. \tag{27}$$

Recall that $d^*=-*d*$ on 1–forms, so if $d^*a=0$ then R_ja is independent of τ. Therefore, if $a\in\ker(\mathcal{D})\cap L^{p,\kappa}$ then $R_ja=0$ for $j>s$, hence

$$\sum_{j=1}^s R_ja=\int_{\partial X_{:\tau}}*a=\int_{X_{:\tau}}d{*a}=0.$$

Set

$$\Xi=\{(z_1,\dots,z_s)\in\mathbb{R}^s\;:\;\sum_j z_j=0\}.$$

Proposition 5.1 *In the situation above the map*

$$\begin{aligned}\alpha\colon\ \ker(\mathcal{D})\cap L^{p,\kappa}&\to\ker(H^1(X)\to H^1(Y''))\oplus\Xi,\\ a&\mapsto([a],(R_1a,\dots,R_sa))\end{aligned}$$

is an isomorphism.

Proof We first prove α is injective. Suppose $\alpha(a)=0$. Then $a=df$ for some function f on X. From the decomposition (26) we see that a decays exponentially over the ends. Hence f is bounded, in which case

$$0=\int_X f\,d^*a=\int_X|a|^2.$$

This shows α is injective.

Next we prove α is surjective. Suppose $b\in\Omega^1(X)$, $db=0$, $[b|_{Y''}]=0$, and $(z_1,\dots,z_s)\in\Xi$. Let $\psi\in\Omega^1(Y)$ be the harmonic form representing $[b|_Y]\in H^1(Y)$. Then

$$b|_{\mathbb{R}_+\times Y}=\pi^*\psi+df,$$

for some $f\colon\ \mathbb{R}_+\times Y\to\mathbb{R}$. Choose a smooth function $\rho\colon\ X\to\mathbb{R}$ which vanishes in a neighbourhood of $X_{:0}$ and satisfies $\rho\equiv 1$ on $[\tau,\infty)\times Y$. Set $z_j=0$ for $j>s$ and let z be the function on Y with $z|_{Y_j}\equiv\mathrm{Vol}(Y_j)^{-1}z_j$. Define

$$\widetilde{b}=b+d(\rho(tz-f)).$$

Then over $[\tau,\infty)\times Y$ we have $\widetilde{b}=\pi^*\psi+z\,dt$, so $d^*\widetilde{b}=0$ in this region, and

$$\int_X d^*\widetilde{b}=-\int_{\partial X_{:\tau}}*\widetilde{b}=-\int_Y z=0.$$

Let $\bar{\kappa}\colon\ X\to\mathbb{R}$ be a smooth function which agrees with $|\kappa|$ outside a compact set. By Proposition 2.3 we can find a smooth $\xi\colon\ X\to\mathbb{R}$ such that $d\xi\in L^{p,\bar{\kappa}}_1$ and

$$d^*(\widetilde{b}+d\xi)=0.$$

Set $a=\widetilde{b}+d\xi$. Then $(d+d^*)a=0$ and $\alpha(a)=([b],(z_1,\dots,z_s))$. □

The following proposition is essentially [10, Proposition 3.14] and is included here only for completeness.

Proposition 5.2 *If $b_1(Y)=0$ and $s=0$ then the operator*

$$\mathcal{D}\colon\ L^{p,\kappa}_1\to L^{p,\kappa}\tag{28}$$

has index $-b_0(X)+b_1(X)-b_2^+(X)$.

Proof By Proposition 5.1 the dimension of the kernel of (28) is $b_1(X)$. From Proposition 2.2 (ii) with $\mathbb{S}=0$ we see that the image of (28) is the sum of $d^*L^{p,\kappa}_1$ and $d^+L^{p,\kappa}_1$. The codimensions of these spaces in $L^{p,\kappa}$ are $b_0(X)$ and $b_2^+(X)$, respectively. □

5.2 The case of a single moduli space

Consider the situation of Subsection 1.3. Initially we do not assume Condition (A).

Proposition 5.3 *Fix $1<q<\infty$. Let $\sigma>0$ be a small constant and $\kappa\colon\ X\to\mathbb{R}$ a smooth function such that $\kappa(t,y)=-\sigma t$ for all $(t,y)\in\mathbb{R}_+\times Y$. Let A_o be a spin connection over X which is translationary invariant over the ends of X. For $n=1,2,\dots$ let $S_n=(A_o+a_n,\Phi_n)$ be a smooth configuration over X which satisfies the monopole equations (20) with $\mu=\mu_n$, $\vec{\mathfrak{p}}=\vec{\mathfrak{p}}_n$.*

Suppose $a_n \in L_1^{q,\kappa}$ *for every* n, *and* $\sup_n \|\Phi_n\|_\infty < \infty$. *Then there exist smooth* $u_n\colon X \to U(1)$ *such that if* k *is any non-negative integer with*

$$\sup_{j,n}\left(\|\mu_n\|_{C^k} + \|\mathfrak{p}_{n,j}\|_{C^k}\right) < \infty \tag{29}$$

then the sequence $u_n(S_n)$ *is bounded in* $L^{p'}_{k+1}$ *over compact subsets of* X *for every* $p' \geq 1$.

Before giving the proof we record the following two elementary lemmas:

Lemma 5.1 *Let* E, F, G *be Banach spaces, and* $E \xrightarrow{S} F$ *and* $E \xrightarrow{T} G$ *bounded linear maps. Set*

$$S + T\colon\ E \to F \oplus G, \quad x \mapsto (Sx, Tx).$$

Suppose S *has finite-dimensional kernel and closed range, and that* $S+T$ *is injective. Then* $S+T$ *has closed range, hence there is a constant* $C > 0$ *such that*

$$\|x\| \leq C(\|Sx\| + \|Tx\|)$$

for all $x \in E$.

Proof Exercise. $\square$

Lemma 5.2 *Let* X *be a smooth, connected manifold and* $x_0 \in X$. *Let* $\mathrm{Map}_0(X, U(1))$ *denote the set of smooth maps* $u\colon X \to U(1)$ *such that* $u(x_0) = 1$, *and let* V *denote the set of all closed* 1–*forms* ϕ *on* X *such that* $[\phi] \in H^1(X;\mathbb{Z})$. *Then*

$$\mathrm{Map}_0(X, U(1)) \to V, \quad u \mapsto \frac{1}{2\pi i}u^{-1}du$$

is an isomorphism of Abelian groups.

Proof If $\phi \in V$ define

$$u(x) = \exp(2\pi i \int_{x_0}^{x} \phi),$$

where $\int_{x_0}^{x}\phi$ denotes the integral of ϕ along any path from x_0 to x. Then $\frac{1}{2\pi i}u^{-1}du = \phi$. The details are left to the reader. $\square$

Proof of Proposition 5.3 We may assume X is connected and that (29) holds at least for $k=0$. Choose closed 3–forms $\omega_1,\dots,\omega_{b_1(X)}$ which are supported in the interior of $X_{:0}$ and represent a basis for $H^3_c(X)$. For any $a\in\Omega^1(X)$ define the coordinates of $Ja\in\mathbb{R}^{b_1(X)}$ by

$$(Ja)_k=\int_X a\wedge\omega_k.$$

Then J induces an isomorphism $H^1(X)\to\mathbb{R}^{b_1(X)}$, by Poincaré duality. By Lemma 5.2 we can find smooth $v_n\colon X\to\mathrm{U}(1)$ such that $J(a_n-v_n^{-1}dv_n)$ is bounded as $n\to\infty$. We can arrange that $v_n(t,y)$ is independent of $t\ge 0$ for every $y\in Y$. Then there are $\xi_n\in L^{q,\kappa}_2(X;i\mathbb{R})$ such that $b_n=a_n-v_n^{-1}dv_n-d\xi_n$ satisfies

$$d^*b_n=0;\qquad R_jb_n=0,\quad j=1,\dots,r-1$$

where R_j is as in (27). If $r\ge 1$ this follows from Proposition 2.4, while if $r=0$ (ie if X is closed) it follows from Proposition 2.3. (By Stokes' theorem we have $R_rb_n=0$ as well, but we don't need this.) Note that ξ_n must be smooth, by elliptic regularity for the Laplacian d^*d. Set $u_n=\exp(\xi_n)v_n$. Then $u_n(A_o+a_n)=A_o+b_n$. By Proposition 5.1 and Lemma 5.1 there is a $C_1>0$ such that

$$\|b\|_{L^{q,\kappa}_1}\le C_1\left(\|(d^*+d^+)b\|_{L^{q,\kappa}}+\sum_{j=1}^{r-1}|R_jb|+\|Jb\|\right)$$

for all $b\in L^{q,\kappa}_1$. From inequality (16) and the curvature part of the Seiberg–Witten equations we find that $\sup_n\|d^+b_n\|_\infty<\infty$, hence

$$\|b_n\|_{L^{q,\kappa}_1}\le C_1(\|d^+b_n\|_{L^{q,\kappa}}+\|Jb_n\|)\le C_2$$

for some constant C_2. We can now complete the proof by bootstrapping over compact subsets of X, using alternately the Dirac and curvature parts of the Seiberg–Witten equation. $\square$

Combining Proposition 5.3 (with $k\ge 1$) and Proposition 3.1 we obtain, for fixed closed 2–forms η_j on Y_j:

Corollary 5.1 *If (A) holds then for every constant $C_0<\infty$ there exists a constant $C_1<\infty$ with the following significance. Suppose $\|\mu\|_{C^1},\|\mathfrak{p}_j\|_{C^1}\le C_0$ for each j. Then for any $\vec\alpha=(\alpha_1,\dots,\alpha_r)$ with $\alpha_j\in\widetilde{\mathcal{R}}_{Y_j}$, and any $[S]\in M(X;\vec\alpha;\mu;\vec{\mathfrak{p}})$ and $t_1,\dots,t_r\in[0,C_0]$ one has*

$$\left|\sum_{j=1}^r\lambda_j\vartheta(S|_{\{t_j\}\times Y_j})\right|\le C_1.$$

Note that if $\sum_j \lambda_j \vartheta(\alpha_j) \geq -C_0$ then this gives

$$\sum_{j=1}^{r} \lambda_j \left(\vartheta(S|_{\{t_j\}\times Y_j}) - \vartheta(\alpha_j) \right) \leq C_0 + C_1. \tag{30}$$

5.3 Condition (C)

Consider the situation in Subsection 1.4 and suppose γ is simply-connected and equipped with an orientation o. Throughout this subsection (and the next) (co)homology groups will have real coefficients, unless otherwise indicated.

We associate to (γ, o) a "height function", namely the unique integer valued function h on the set of nodes of γ whose minimum value is 0 and which satisfies $h(e') = h(e) + 1$ whenever there is an oriented edge from e to e'.

Let Z^k and $Z^{[k}$ denote the union of all subspaces $Z_e \subset X^{\#}$ where e has height k and $\geq k$, respectively. Set

$$\partial^- Z^k = \bigcup_{h(e)=k} \partial^- Z_e.$$

For each node e of γ choose a subspace $G_e \subset H_1(Z_e)$ such that

$$H_1(Z_e) = G_e \oplus \mathrm{im}\left(H_1(\partial^- Z_e) \to H_1(Z_e) \right).$$

Then the natural map $F_e \to G_e^*$ is an isomorphism, where G_e^* is the dual of the vector space G_e.

Lemma 5.3 *The natural map $H^1(X^{\#}) \to \oplus_e G_e^*$ is injective. Therefore, this map is an isomorphism if and only if $\Sigma(X, \gamma, o) = 0$.*

Proof Let N be the maximum value of h and suppose $z \in H^1(X^{\#})$ lies in the kernel of the map in the lemma. It is easy to show, by induction on $k = 0, \ldots, N$, that $H^1(X^{\#}) \to H^1(Z^k)$ maps z to zero for each k. We now invoke the Mayer–Vietoris sequence for the pair of subspaces $(Z^{k-1}, Z^{[k})$ of $X^{\#}$:

$$H^0(Z^{k-1}) \oplus H^0(Z^{[k}) \xrightarrow{a} H^0(\partial^- Z^k) \to H^1(Z^{[k-1}) \xrightarrow{b} H^1(Z^{k-1}) \oplus H^1(Z^{[k}).$$

Using the fact that γ is simply-connected it is not hard to see that a is surjective, hence b is injective. Arguing by induction on $k = N, N-1, \ldots, 0$ we then find that $H^1(X^{\#}) \to H^1(Z^{[k})$ maps z to zero for each k. □

We will now formulate a condition on (X,γ) which is stronger than (C) and perhaps simpler to verify. A connected, oriented graph is called a *tree* if it has a unique node (the *root node*) with no incoming edge, and any other node has a unique incoming edge.

Proposition 5.4 *Suppose there is an orientation o of γ such that (γ,o) is a tree and*

$$H^1(Z^{[k})\to H^1(Z^k)$$

is surjective for all k. Then Condition (C) holds.

Proof It suffices to verify that $\Sigma(X,\gamma,o)=0$. Set

$$\begin{aligned}F^{[k}&=\ker(H^1(Z^{[k})\to H^1(\partial^- Z^k)),\\ F^k&=\ker(H^1(Z^k)\to H^1(\partial^- Z^k)).\end{aligned}$$

The Mayer–Vietoris sequence for $(Z^k,Z^{[k+1})$ yields an exact sequence

$$0\to F^{[k}\to F^k\oplus H^1(Z^{[k+1})\to H^1(\partial^- Z^{[k+1}).$$

If $H^1(Z^{[k})\to H^1(Z^k)$ is surjective then so is $F^{[k}\to F^k$, hence $\ker(F^{[k}\to F^k)\to F^{[k+1}$ is an isomorphism, in which case

$$\dim\, F^{[k}=\dim\, F^{[k+1}+\dim\, F^k.$$

Therefore,

$$\Sigma(X,\gamma,o)=\dim\, H^1(X^\#)-\sum_k\dim\, F^k=\dim\, H^1(X^\#)-\dim\, F^{[0}=0.\qquad\square$$

5.4 Neck-stretching II

Consider again the situation in Subsection 1.4. The following set-up will be used in the next two lemmas. We assume that γ is simply-connected and that o is an orientation of γ with $\Sigma(X,\gamma,o)=0$. Let $1<p<\infty$.

An end of X that corresponds to an edge of γ is either *incoming* or *outgoing* depending on the orientation o. (These are the ends $\mathbb{R}_+\times(\pm Y_j)$, but the sign here is unrelated to o.) All other ends (ie $\mathbb{R}_+\times Y'_j$, $1\le j\le r'$) are called *neutral*.

Choose subspaces G_e of $H_1(Z_e)\approx H_1(X_e)$ as in the previous subsection, and set $g_e=\dim\, G_e$. For each component X_e of X let $\{q_{ek}\}$ be a collection of closed 3–forms on X_e supported in the interior of $Z'_e=(X_e)_{:0}$ which represents a basis for the image of G_e in $H^3_c(X_e)$ under the Poincaré duality isomorphism. For any $a\in\Omega^1(Z'_e)$ define $J_ea\in\mathbb{R}^{g_e}$ by

$$(J_ea)_k=\int_{X_e}a\wedge q_{ek}.$$

For each e let $\mathbb{R}_+\times Y_{em}$, $m=1,\dots,h_e$ be the outgoing ends of X_e. For any $a\in\Omega^1(Z'_e)$ define $R_ea\in\mathbb{R}^{h_e}$ by

$$(R_ea)_m=\int_{\{0\}\times Y_{em}}*a.$$

Set $n_e=g_e+h_e$ and

$$L_ea=(J_ea,R_ea)\in\mathbb{R}^{n_e}.$$

For any $a\in\Omega^1(X^{(T)})$ let $La\in V=\oplus_e\mathbb{R}^{n_e}$ be the element with components L_ea.

For any tubular end $\mathbb{R}_+\times P$ of X let $t\colon\ \mathbb{R}_+\times P\to\mathbb{R}_+$ be the projection. Choose a small $\sigma>0$ and for each e a smooth function $\kappa_e\colon\ X_e\to\mathbb{R}$ such that

$$\kappa_e=\begin{cases}\sigma t & \text{on incoming ends,}\\ -\sigma t & \text{on outgoing and neutral ends.}\end{cases}$$

Let $X^{\{T\}}\subset X$ be as in Subsection 1.4 and let $\kappa=\kappa_T\colon\ X^{(T)}\to\mathbb{R}$ be a smooth function such that $\kappa_T-\kappa_e$ is constant on $X_e\cap X^{\{T\}}$ for each e. (Such a function exists because γ is simply-connected.) This determines κ_T up to an additive constant.

Fix a point $x_e\in X_e$ and define a norm $\|\cdot\|_T$ on V by

$$\|v\|_T=\sum_e\exp(\kappa_T(x_e))\|v_e\|,$$

where $\|\cdot\|$ is the Euclidean norm on R^{n_e} and $\{v_e\}$ the components of v.

Let $\mathcal{D}$ denote the operator $-d^*+d^+$ on $X^{(T)}$.

Lemma 5.4 *There is a constant C such that for every r–tuple T with $\min_j T_j$ sufficiently large and every $L^{p,\kappa}_1$ 1–form a on $X^{(T)}$ we have*

$$\|a\|_{L^{p,\kappa}_1}\le C(\|\mathcal{D}a\|_{L^{p,\kappa}}+\|La\|_T).$$

Note that adding a constant to κ rescales all norms in the above inequality by the same factor.

Proof Let τ be a function on X which is equal to $2T_j$ on the ends $\mathbb{R}_+\times(\pm Y_j)$ for each j. Choose smooth functions $f_1,f_2\colon\thinspace \mathbb{R}\to\mathbb{R}$ such that $(f_1(t))^2+(f_2(1-t))^2=1$ for all t, and $f_k(t)=1$ for $t\le\frac{1}{3}$, $k=1,2$. For each e define $\beta_e\colon\thinspace X_e\to\mathbb{R}$ by

$$\beta_e=\begin{cases}f_1(t/\tau) & \text{on outgoing ends,}\\ f_2(t/\tau) & \text{on incoming ends,}\\ 1 & \text{elsewhere.}\end{cases}$$

Let $\bar\beta_e$ denote the smooth function on $X^{(T)}$ which agrees with β_e on $X_e\cap X^{\{T\}}$ and is zero elsewhere.

In the following, $C,C_1,C_2,\dots$ will be constants that are independent of T. Set $\check T=\min_j T_j$. Assume $\check T\ge 1$.

Note that $|\nabla\beta_e|\le C_1\check T^{-1}$ everywhere, and similarly for $\bar\beta_e$. Therefore

$$\|\beta_e a\|_{L^{p,\kappa_e}_1}\le C_2\|a\|_{L^{p,\kappa_e}_1}$$

for 1–forms a on X_e.

Let $\mathcal{D}_e$ denote the operator $-d^*+d^+$ on X_e. By Proposition 5.1 the Fredholm operator

$$\mathcal{D}_e\oplus L_e\colon\thinspace L^{p,\kappa_e}_1\to L^{p,\kappa_j}\oplus\mathbb{R}^{n_e}$$

is injective, hence it has a bounded left inverse P_e,

$$P_e(\mathcal{D}_e\oplus L_e)=\mathrm{Id}.$$

If a is a 1–form on $X^{(T)}$ and $v\in V$ set

$$\bar\beta_e(a,v)=(\bar\beta_e a,v_e).$$

Here we regard $\bar\beta_e a$ as a 1–form on X_e. Define

$$P=\sum_e\beta_eP_e\bar\beta_e\colon\thinspace L^{p,\kappa}\oplus V\to L^{p,\kappa}_1.$$

If we use the norm $\|\cdot\|_T$ on V then $\|P\|\le C_3$. Now

$$\begin{aligned}P(\mathcal{D}\oplus L)a&=\sum_e\beta_eP_e(\bar\beta_e\mathcal{D}a,L_ea)\\&=\sum_e\beta_eP_e(\mathcal{D}_e\bar\beta_ea+[\bar\beta_e,\mathcal{D}]a,L_e\bar\beta_ea)\\&=\sum_e(\beta_e\bar\beta_ea+\beta_eP_e([\bar\beta_e,\mathcal{D}]a,0))\\&=a+Ea,\end{aligned}$$

where
$$\|Ea\|_{L^{p,\kappa}_1} \le C_4 \check{T}^{-1} \|a\|_{L^{p,\kappa}}.$$
Therefore,
$$\|P(\mathcal{D} \oplus L) - I\| \le C_4 \check{T}^{-1},$$
so if $\check{T} > C_4$ then $z = P(\mathcal{D} \oplus L)$ will be invertible, with
$$\|z^{-1}\| \le (1 - \|z - I\|)^{-1}.$$
In that case we can define a left inverse of $\mathcal{D} \oplus L$ by
$$Q = (P(\mathcal{D} \oplus L))^{-1} P.$$
If $\check{T} \ge 2C_4$ say, then $\|Q\| \le 2C_3$, whence for any $a \in L^{p,\kappa}_1$ we have
$$\|a\|_{L^{p,\kappa}_1} = \|Q(\mathcal{D}a, La)\|_{L^{p,\kappa}_1} \le C(\|\mathcal{D}a\|_{L^{p,\kappa}} + \|La\|_T). \qquad \square$$

Lemma 5.5 *Let e be a node of γ and for $n = 1, 2, \dots$ let $T(n)$ be an r–tuple and a_n an L^{p,κ_n}_1 1–form on $X^{(T(n))}$, where $\kappa_n = \kappa_{T(n)}$. Suppose*

(i) $\Sigma(X, \gamma, o) = 0$,

(ii) $\min_j T_j(n) \to \infty$ *as* $n \to \infty$,

(iii) *There is a constant $C' < \infty$ such that*
$$\kappa_n(x_{e'}) \le \kappa_n(x_e) + C'$$
for all nodes e' of γ and all n,

(iv) $\sup_n \|d^+ a_n\|_\infty < \infty$.

Then there are smooth $u_n \colon X^{(T(n))} \to U(1)$ such that the sequence $b_n = a_n - u_n^{-1} du_n$ is bounded in L^p_1 over compact subsets of X_e, and $b_n \in L^{p,\kappa_n}_1$ and $d^* b_n = 0$ for every n.

Note that (iii) implies that e must be a source of (γ, o).

Proof Without loss of generality we may assume that $\kappa_n(x_e) = 1$ for all n, in which case
$$\sup_n \|1\|_{L^{p,\kappa_n}} < \infty.$$
By Lemmas 5.2 and 5.3 we can find smooth $v_n \colon X^{(T(n))} \to \mathrm{U}(1)$ such that
$$\sup_n \|J_{e'}(a_n - v_n^{-1} dv_n)\| < \infty$$
for every node e', where $\|\cdot\|$ is the Euclidean norm. (Compare the proof of Proposition 5.3.) Moreover, we can take v_n translationary invariant over each

end of $X^{(T(n))}$. Proposition 2.4 then provides smooth $\xi_n \in L^{p,\kappa_n}_2(X;i\mathbb{R})$ such that

$$b_n = a_n - v_n^{-1}dv_n - d\xi_n \in L^{p,\kappa_n}_1$$

satisfies

$$d^*b_n = 0, \qquad \int_{\{0\}\times Y'_j} *b_n = 0$$

for $j = 1, \dots, r'-1$. Stokes' theorem shows that the integral vanishes for $j = r'$ as well, and since γ is simply-connected we obtain, for $j = 1, \dots, r$,

$$\int_{\{t\}\times Y_j} *b_n = 0 \qquad \text{for } |t| \le T_j(n).$$

In particular, $R_{e'}b_n = 0$ for all nodes e' of γ.

Set $u_n = v_n \exp(\xi_n)$, so that $b_n = a_n - u_n^{-1}du_n$. By Lemma 5.4 we have

$$\begin{aligned}\|b_n\|_{L^{p,\kappa_n}_1} &\le C\left(\|d^+b_n\|_{L^{p,\kappa_n}} + \sum_{e'} \exp(\kappa_n(x_{e'}))\|J_{e'}(b_n)\|\right)\\ &\le C\left(\|d^+a_n\|_{L^{p,\kappa_n}} + \exp(1+C')\sum_{e'}\|J_{e'}(a_n - v_n^{-1}dv_n)\|\right),\end{aligned}$$

which is bounded as $n \to \infty$. ◻

Proposition 5.5 *Suppose γ is simply-connected and that Condition (C) holds for (X,γ). For $n = 1, 2, \dots$ let $[S_n] \in M(X^{(T(n))}; \vec{\alpha}'_n; \mu_n; \vec{\mathfrak{p}}_n; \vec{\mathfrak{p}}'_n)$, where $\min_j T_j(n) \to \infty$. Then there exist smooth maps $w_n \colon X \to U(1)$ such that if k is any positive integer with*

$$\sup_{j,j',n} (\|\mu_n\|_{C^k} + \|\mathfrak{p}_{j,n}\|_{C^k} + \|\mathfrak{p}'_{j',n}\|_{C^k}) < \infty$$

then the sequence $w_n(S_n)$ is bounded in $L^{p'}_{k+1}$ over compact subsets of X for every $p' \ge 1$.

Proof Consider the set-up in the beginning of this subsection where now $p > 4$ is the exponent used in defining configuration spaces, and o is an orientation of γ for which (C) is fulfilled. By passing to a subsequence we can arrange that $\kappa_n(x_e) - \kappa_n(x_{e'})$ converges to a point $\ell(e,e') \in [-\infty,\infty]$ for each pair of nodes e, e' of γ. Define an equivalence relation $\sim$ on the set $\mathcal{N}$ of nodes of γ by declaring that $e \sim e'$ if and only if $\ell(e,e')$ is finite. Then we have a linear ordering on $\mathcal{N}/\sim$ such that $[e] \ge [e']$ if and only if $\ell(e,e') > -\infty$. Here $[e]$ denotes the equivalence class of e.

Choose e such that $[e]$ is the maximum with respect to this linear ordering. Let $S_n = (A_o + a_n, \Phi_n)$. Then all the hypotheses of Lemma 5.5 are satisfied. If u_n is as in that lemma then, as in the proof of Proposition 5.3, $u_n(S_n) = (A_o + b_n, u_n\Phi_n)$ will be bounded in $L^{p'}_{k+1}$ over compact subsets of X_e for every $p' \geq 1$.

For any r–tuple T let $W^{(T)}$ be the result of gluing ends of $X \setminus X_e$ according to the graph $\gamma \setminus e$ and (the relevant part of) the vector T. To simplify notation let us assume that the outgoing ends of X_e are $\mathbb{R}_+ \times (-Y_j)$, $j = 1, \dots, r_1$. Then $\mathbb{R}_+ \times Y_j$ is an end of $W^{T(n)}$ for $j = 1, \dots, r_1$. Let b'_n be the 1–form on $W^{T(n)}$ which away from the ends $\mathbb{R}_+ \times Y_j$, $1 \leq j \leq r_1$ agrees with b_n, and on each of these ends is defined by cutting off b_n:

$$b'_n(t,y) = \begin{cases} f_1(t - 2T_j(n) + 1) \cdot b_n(t,y), & 0 \leq t \leq 2T_j(n), \\ 0, & t \geq 2T_j(n). \end{cases}$$

Here f_1 is as in the proof of Lemma 5.4. Then $\sup_n \|d^+ b'_n\|_\infty < \infty$. After choosing an orientation of $\gamma \setminus e$ for which (C) holds we can apply Lemma 5.5 to each component of $\gamma \setminus e$, with b'_n in place of a_n. Repeating this process proves the proposition. □

Corollary 5.2 *If (B2) holds then for every constant $C_0 < \infty$ there exists a constant $C_1 < \infty$ such that for any element $[S]$ of a moduli space $M(X^{(T)}; \vec{\alpha}'; \mu; \vec{\mathfrak{p}}; \vec{\mathfrak{p}}')$ where $\min_j T_j > C_1$ and $\mu, \mathfrak{p}_j, \mathfrak{p}'_j$ all have C^1–norm $< C_0$ one has*

$$\left| \sum_{j=1}^{r} \lambda_j \left(\vartheta(S|_{\{-T_j\} \times Y_j}) - \vartheta(S|_{\{T_j\} \times Y_j}) \right) + \sum_{j=1}^{r'} \lambda'_j \vartheta(S|_{\{0\} \times Y'_j}) \right| < C_1.$$

The next proposition, which is essentially a corollary of Proposition 5.5, exploits the fact that Condition (C) is preserved under certain natural extensions of (X, γ).

Proposition 5.6 *Suppose γ is simply-connected and (C) holds for (X, γ). Then for every constant $C_0 < \infty$ there is a constant $C_1 < \infty$ such that if $S = (A, \Phi)$ represents an element of a moduli space $M(X^{(T)}; \vec{\alpha}'; \mu; \vec{\mathfrak{p}}; \vec{\mathfrak{p}}')$ where $\min_j T_j > C_1$ and $\mu, \mathfrak{p}_j, \mathfrak{p}'_j$ all have L^∞ norm $< C_0$ then*

$$\|\nabla \vartheta_S\|_{L^2((t-1,t+1) \times Y_j)} < C_1 \quad \text{for } |t| \leq T_j - 1,$$
$$\|\nabla \vartheta_S\|_{L^2((t-1,t+1) \times Y'_j)} < C_1 \quad \text{for } t \geq 1.$$

Proof Given an edge v of γ corresponding to a pair of ends $\mathbb{R}_+ \times (\pm Y_j)$ of X, we can form a new pair $(X_{(j)}, \gamma_{(j)})$ where $X_{(j)} = X \coprod (\mathbb{R} \times Y_j)$, and $\gamma_{(j)}$ is obtained from γ by splitting v into two edges with a common end-point representing the component $\mathbb{R} \times Y_j$ of $X_{(j)}$.

Similarly, if e is a node of γ and $\mathbb{R}_+ \times Y'_j$ an end of X_e then we can form a new pair $(X^{(j)}, \gamma^{(j)})$ where $X^{(j)} = X \coprod (\mathbb{R} \times Y'_j)$, and $\gamma^{(j)}$ is obtained from γ by adding one node e'_j representing the component $\mathbb{R}_+ \times Y'_j$ of $X^{(j)}$ and one edge joining e and e'_j.

One easily shows, by induction on the number of nodes of γ, that if (C) holds for (X, γ) then (C) also holds for each of the new pairs $(X_{(j)}, \gamma_{(j)})$ and $(X^{(j)}, \gamma^{(j)})$. Given this observation, the proposition is a simple consequence of Proposition 5.5. □

6 Exponential decay

In this section we will prove exponential decay results for genuine monopoles over half-cylinders $\mathbb{R}_+ \times Y$ and long bands $[-T, T] \times Y$. The overall scheme of proof will be the same as that for instantons in [10], and Subsections 6.1 and 6.3 follow [10] quite closely. On the other hand, the proof of the main result of Subsection 6.2, Proposition 6.1, is special to monopoles (and is new, as far as we know).

Throughout this section Y will be a closed, connected Riemannian spin^c 3–manifold, and $\eta \in \Omega^2(Y)$ closed. We will study exponential decay towards a non-degenerate critical point α of $\vartheta = \vartheta_\eta$. We make no non-degeneracy assumptions on any other (gauge equivalence classes of) critical points, and we do not assume that (O1) holds, except implicitly in Proposition 6.3. All monopoles will be genuine (ie $\mathfrak{p} = 0$).

Previously, Nicolaescu [29] has proved a slightly weaker exponential decay result in the case $\eta = 0$, using different techniques. Other results (with less complete proofs) were obtained in [28] and [26].

6.1 A differential inequality

We begin by presenting an argument from [10] in a more abstract setting, so that it applies equally well to the Chern–Simons and the Chern–Simons–Dirac functionals.

Let E be a real Banach space with norm $\|\cdot\|$ and E' a real Hilbert space with inner product $\langle\cdot,\cdot\rangle$. Let $E\to E'$ be an injective, bounded operator with dense image. We will identify E as a vector space with its image in E'. Set $|x|=\langle x,x\rangle^{1/2}$ for $x\in E'$.

Let $U\subset E$ be an open set containing 0 and

$$f\colon\ U\to\mathbb{R},\qquad g\colon\ U\to E'$$

smooth maps satisfying $f(0)=0$, $g(0)=0$, and

$$Df(x)y=\langle g(x),y\rangle$$

for all $x\in U$, $y\in E$. Here $Df(x)\colon\ E\to\mathbb{R}$ is the derivative of f at x. Suppose $H=Dg(0)\colon\ E\to E'$ is an isomorphism (of topological vector spaces). Note that H can be thought of as a symmetric operator in E'. Suppose E contains a countable set $\{e_j\}$ of eigenvectors for H which forms an orthonormal basis for E'. Suppose σ,λ are real numbers satisfying $0\le\lambda<\sigma$ and such that H has no positive eigenvalue less than σ.

Lemma 6.1 *In the above situation there is a constant $C>0$ such that for every $x\in U$ with $\|x\|\le C^{-1}$ one has*

$$\begin{aligned}2\sigma f(x)&\le|g(x)|^2+C|g(x)|^3,\\ 2\lambda f(x)&\le|g(x)|^2.\end{aligned}$$

Proof It clearly suffices to establish the first inequality for some C. By Taylor's formula [8, 8.14.3] there is a $C_1>0$ such that for all $x\in U$ with $\|x\|\le C_1^{-1}$ one has

$$\begin{aligned}|f(x)-\frac{1}{2}\langle Hx,x\rangle|&\le C_1\|x\|^3,\\ |g(x)-Hx|&\le C_1\|x\|^2.\end{aligned}$$

Let $He_j=\lambda_je_j$, and set $x_j=\langle x,e_j\rangle e_j$. Then

$$\sigma\langle Hx,x\rangle=\sigma\sum_j\lambda_j|x_j|^2\le\sum_j\lambda_j^2|x_j|^2=|Hx|^2.$$

By assumption, there is a $C_2>0$ such that

$$\|x\|\le C_2|Hx|$$

for all $x\in E$. Putting the above inequalities together we get, for $r=\|x\|\le C_1^{-1}$,

$$\begin{aligned}|Hx|&\le|g(x)|+C_1r^2\\ &\le|g(x)|+rC_1C_2|Hx|.\end{aligned}$$

If $r < (C_1C_2)^{-1}$ this gives
$$|Hx| \leq (1 - rC_1C_2)^{-1}|g(x)|,$$
hence
$$\begin{aligned}2\sigma f(x) &\leq \sigma\langle Hx, x\rangle + 2\sigma C_1 r^3 \\ &\leq |Hx|^2 + 2\sigma r C_1 C_2^2 |Hx|^2 \\ &\leq \frac{1 + 2\sigma r C_1 C_2^2}{(1 - rC_1C_2)^2}|g(x)|^2 \\ &\leq (1 + C_3 r)|g(x)|^2 \\ &\leq |g(x)|^2 + C_4|g(x)|^3\end{aligned}$$
for some constants C_3, C_4. □

We will now apply this to the Chern–Simons–Dirac functional. Let $\alpha = (B_0, \Psi_0)$ be a non-degenerate critical point of ϑ. Set $K_\alpha = \ker(\mathcal{I}_\alpha^*) \subset \Gamma(i\Lambda^1 \oplus \mathbb{S})$ and $\widetilde{H}_\alpha = H_\alpha|_{K_\alpha} \colon K_\alpha \to K_\alpha$. Note that any eigenvalue of $\widetilde{H}_\alpha$ is also an eigenvalue of the self-adjoint elliptic operator
$$\begin{pmatrix} 0 & \mathcal{I}_\alpha^* \\ \mathcal{I}_\alpha & H_\alpha \end{pmatrix}$$
over Y acting on sections of $i\Lambda^0 \oplus i\Lambda^1 \oplus \mathbb{S}$. Let $\lambda^\pm$ be positive real numbers such that $\widetilde{H}_\alpha$ has no eigenvalue in the interval $[-\lambda^-, \lambda^+]$.

In the following lemma, Sobolev norms of sections of the spinor bundle $\mathbb{S}_Y$ over Y will be taken with respect to B_0 and some fixed connection in the tangent bundle TY. This means that the same constant ϵ will work if α is replaced with some monopole gauge equivalent to α.

Lemma 6.2 *In the above situation there exists an $\epsilon > 0$ such that if S is any smooth monopole over the band $(-1,1) \times Y$ satisfying $\|S_0 - \alpha\|_{L^2_1(Y)} \leq \epsilon$ then*
$$\pm 2\lambda^\pm(\vartheta(S_0) - \vartheta(\alpha)) \leq -\partial_t|_0 \vartheta(S_t).$$

Proof Choose a smooth $u\colon (-1,1) \times Y \to \mathrm{U}(1)$ such that $u(S)$ is in temporal gauge. Then
$$\partial_t\vartheta(S_t) = \partial_t\vartheta(u_t(S_t)) = -\|\nabla\vartheta(u_t(S_t))\|_2^2 = -\|\nabla\vartheta(S_t)\|_2^2.$$
If $\epsilon > 0$ is sufficiently small then by the local slice theorem we can find a smooth $v\colon Y \to \mathrm{U}(1)$ which is L^2_2 close to 1 and such that $\mathcal{I}_\alpha^*(v(S_0) - \alpha) = 0$. We now apply Lemma 6.1 with E the kernel of $\mathcal{I}_\alpha^*$ in L^2_1, E' the kernel of $\mathcal{I}_\alpha^*$ in L^2, and $f(x) = \pm(\vartheta(\alpha + x) - \vartheta(\alpha))$. The assumption that α be non-degenerate means that $H = \widetilde{H}_\alpha\colon E \to E'$ is an isomorphism, so the lemma follows. □

6.2 Estimates over $[0,T]\times Y$

Let α be a non-degenerate critical point of ϑ and $\underline{\alpha}=(B,\Psi)$ the monopole over $\mathbb{R}\times Y$ that α defines. Throughout this subsection, the same convention for Sobolev norms of sections of $\mathbb{S}_Y$ will apply as in Lemma 6.2. For Sobolev norms of sections of the spinor bundles over (open subsets of) $\mathbb{R}\times Y$ we will use the connection B.

Throughout this subsection $S=(A,\Phi)$ will be a monopole over a band $\mathbf{B}=[0,T]\times Y$ where $T\geq 1$. Set $s=(a,\phi)=S-\underline{\alpha}$ and

$$\begin{aligned}\delta &= \|s\|_{L^2_2(\mathbf{B})},\\ \nu^2 &= \|\nabla\vartheta_S\|^2_{L^2(\mathbf{B})}=\vartheta(S_0)-\vartheta(S_T).\end{aligned}\tag{31}$$

The main result of this subsection is Proposition 6.1, which asserts in particular that if δ is sufficiently small then S is gauge equivalent to a configuration $\widetilde{S}$ which is in Coulomb gauge with respect to $\underline{\alpha}$ and satisfies $\|\widetilde{S}-\underline{\alpha}\|_{L^2_1(\mathbf{B})}\leq$ const $\cdot\,\nu$.

We will assume $\delta\leq 1$. Let a' denote the contraction of a with the vector field $\partial_1=\frac{\partial}{\partial t}$. Quantities referred to as constants or denoted "const" may depend on $Y,\eta,[\alpha],T$ but not on S. Note that

$$\nu\leq \text{const}\cdot(\|s\|_{1,2}+\|s\|^2_{1,2})\leq\text{const},\tag{32}$$

the last inequality because $\delta\leq 1$.

For real numbers t set

$$i_t\colon\ Y\to\mathbb{R}\times Y,\quad y\mapsto(t,y).$$

If ω is any differential form over $\mathbf{B}$ set $\omega_t=i_t^*\omega$, $0\leq t\leq T$. Similar notation will be used for connections and spinors over $\mathbf{B}$.

Lemma 6.3 *There is a constant $C_0>0$ such that*

$$\|\partial_1\phi\|_2\leq C_0(\nu+\|a'\|_3).$$

Proof We have

$$\partial_1\phi=\partial_1\Phi=\nabla^A_1\Phi-a'\Phi,$$

where ∇^A_1 is the covariant derivative with respect to A in the direction of the vector field $\partial_1=\frac{\partial}{\partial t}$. Now $|\nabla^A_1\Phi|$ depends only on the gauge equivalence class of $S=(A,\Phi)$, and if A is in temporal gauge (ie if $a'=0$) then $(\nabla^A_1\Phi)_t=\partial_{A_t}\Phi_t$. The lemma now follows because $\delta\leq 1$. □

Lemma 6.4 *There is a constant $C_1 > 0$ such that if δ is sufficiently small then the following hold:*

(i) $\|\phi\|_{1,2} \le C_1(\|\mathcal{I}^*_{\underline{\alpha}} s\|_2 + \|a'\|_{1,2} + \nu)$,

(ii) *There is a smooth $\check{f}\colon \mathbf{B} \to i\mathbb{R}$ such that $\check{s} = (\check{a}, \check{\phi}) = \exp(\check{f})(S) - \underline{\alpha}$ satisfies*
$$\|\check{s}_t\|_{1,2} \le C_1 \|\nabla\vartheta_{S_t}\|_2, \quad 0 \le t \le T,$$

(iii) $\|da\|_2 \le C_1 \nu$,

where in (i) and (iii) all norms are taken over $\mathbf{B}$.

Proof The proof will use an elliptic inequality over Y, the local slice theorem for Y, and the gradient flow description of the Seiberg–Witten equations over $\mathbb{R} \times Y$.

(i) Since α is non-degenerate we have
$$\|z\|_{1,2} \le \text{const} \cdot \|(\mathcal{I}^*_\alpha + H_\alpha) z\|_2$$
for L^2_1 sections z of $(i\Lambda \oplus \mathbb{S})_Y$. Recall that
$$\nabla\vartheta_{\alpha+z} = H_\alpha z + z \otimes z$$
where $z \otimes z$ represents a pointwise quadratic function of z. Furthermore, $\|z \otimes z\|_2 \le \text{const} \cdot \|z\|^2_{1,2}$. If $\|z\|_{1,2}$ is sufficiently small then we can rearrange to get
$$\|z\|_{1,2} \le \text{const} \cdot (\|\mathcal{I}^*_\alpha z\|_2 + \|\nabla\vartheta_{\alpha+z}\|_2). \tag{33}$$
By the Sobolev embedding theorem we have
$$\|s_t\|_{L^2_1(Y)} \le \text{const} \cdot \|s\|_{L^2_2(\mathbf{B})}, \quad t \in [0, T],$$
for some constant independent of t, so we can apply inequality (33) with $z = s_t$ when δ is sufficiently small. Because
$$(\mathcal{I}^*_{\underline{\alpha}} s - \partial_1 a')_t = \mathcal{I}^*_\alpha s_t$$
we then obtain
$$\int_0^T \|s_t\|^2_{L^2_1(Y)} dt \le \text{const} \cdot (\|\mathcal{I}^*_{\underline{\alpha}} s\|^2_{L^2(\mathbf{B})} + \|\partial_1 a'\|^2_{L^2(\mathbf{B})} + \nu^2).$$
This together with Lemma 6.3 establishes (i).

(ii) Choose a base-point $y_0 \in Y$. By the local slice theorem there is a constant C such that if δ is sufficiently small then for each $t \in [0, T]$ there is a unique smooth $\check{f}_t\colon Y \to i\mathbb{R}$ such that

- $\|\check{f}_t\|_{2,2} \le C\delta$,
- $\check{f}_t(y_0) = 0$ if α is reducible,
- $\check{s}_t = \exp(\check{f}_t)(S_t) - \alpha$ satisfies $\mathcal{I}^*_\alpha \check{s}_t = 0$.

It is not hard to see that the function $\check{f}\colon \mathbf{B} \to i\mathbb{R}$ given by $\check{f}(t,y) = \check{f}_t(y)$ is smooth. Moreover, $\|\check{s}_t\|_{1,2} \le \text{const} \cdot \|s_t\|_{1,2}$. Part (ii) then follows by taking $z = \check{s}_t$ in (33).

(iii) Choose a smooth $u\colon \mathbf{B} \to \mathrm{U}(1)$ such that $u(S)$ is in temporal gauge, and set $(\underline{a}, \underline{\phi}) = u(S) - \underline{\alpha}$. Then

$$da = d\underline{a} = dt \wedge \partial_1 \underline{a} + d_Y \underline{a} = -dt \wedge \nabla_1 \vartheta_{S_t} + d_Y \check{a}_t,$$

where $\nabla_1 \vartheta$ is the first component of $\nabla\vartheta$. This yields the desired estimate on da. □

Lemma 6.5 *Let $\{v_1, \dots, v_{b_1(Y)}\}$ be a family of closed 2–forms on Y which represents a basis for $H^2(Y;\mathbb{R})$. Then there is a constant C such that*

$$\|b\|_{L^2_1(\mathbf{B})} \le C\left(\|(d^* + d)b\|_{L^2(\mathbf{B})} + \|(*b)|_{\partial\mathbf{B}}\|_{L^2_{1/2}(\partial\mathbf{B})} + \sum_j \left| \int_{\mathbf{B}} dt \wedge \pi^* v_j \wedge b \right| \right) \tag{34}$$

for all L^2_1 1–forms b on $\mathbf{B}$, where $\pi\colon \mathbf{B} \to Y$ is the projection.

Proof Let K denote the kernel of the operator

$$\Omega^1_{\mathbf{B}} \to \Omega^0_{\mathbf{B}} \oplus \Omega^2_{\mathbf{B}} \oplus \Omega^0_{\partial\mathbf{B}}, \quad b \mapsto (d^*b, db, *b|_{\partial\mathbf{B}}).$$

Then we have an isomorphism

$$\rho\colon K \xrightarrow{\approx} H^1(Y;\mathbb{R}), \quad b \mapsto [b_0].$$

For on the one hand, an application of Stokes' theorem shows that ρ is injective. On the other hand, any $c \in H^1(Y;\mathbb{R})$ can be represented by an harmonic 1–form ω, and $\pi^*\omega$ lies in K, hence ρ is surjective.

It follows that every element of K is of the form $\pi^*(\omega)$. Now apply Proposition 4.1 and Lemma 5.1. □

Lemma 6.6 *There is a smooth map $\hat{f}\colon \mathbf{B} \to i\mathbb{R}$, unique up to an additive constant, such that $\hat{a} = a - d\hat{f}$ satisfies*

$$d^*\hat{a} = 0, \quad (*\hat{a})|_{\partial\mathbf{B}} = 0.$$

Given any such $\hat{f}$, if we set $\hat{s} = (\hat{a}, \hat{\phi}) = \exp(\hat{f})(S) - \underline{\alpha}$ then

$$\|\hat{a}\|_{L^2_1(\mathbf{B})} \le C_2\nu, \quad \|\hat{s}\|_{L^2_2(\mathbf{B})} \le C_2\delta$$

for some constant $C_2 > 0$.

Proof The first sentence of the lemma is just the solution to the Neumann problem. If we fix $x_0 \in \mathbf{B}$ then there is a unique $\hat{f}$ as in the lemma such that $\hat{f}(x_0) = 0$, and we have $\|\hat{f}\|_{3,2} \le \text{const} \cdot \|a\|_{2,2}$. Writing

$$\hat{\phi} = \exp(\hat{f})\Phi - \Psi = (\exp(\hat{f}) - 1)\Phi + \phi$$

and recalling that, for functions on $\mathbf{B}$, multiplication is a continuous map $L^2_3 \times L^2_k \to L^2_k$ for $0 \le k \le 3$, we get

$$\begin{aligned}\|\hat{\phi}\|_{2,2} &\le C\|\exp(\hat{f}) - 1\|_{3,2}\|\Phi\|_{2,2} + \|\phi\|_{2,2}\\ &\le C'\|\hat{f}\|_{3,2}\exp(C''\|\hat{f}\|_{3,2}) + \|\phi\|_{2,2}\\ &\le C'''\|s\|_{2,2}\end{aligned}$$

for some constants $C, \dots, C'''$, since we assume $\delta \le 1$. There is clearly a similar L^2_2 bound on $\hat{a}$, so this establishes the L^2_2 bound on $\hat{s}$.

We now turn to the L^2_1 bound on $\hat{a}$. Let $\check{a}$ be as in Lemma 6.4. Since $\hat{a} - \check{a}$ is exact we have

$$\left|\int_Y v \wedge \hat{a}_t\right| = \left|\int_Y v \wedge \check{a}_t\right| \le \text{const} \cdot \|v\|_2\|\check{a}_t\|_2$$

for any closed $v \in \Omega^2_Y$. Now take $b = \hat{a}$ in Lemma 6.5 and use Lemma 6.4, remembering that $d\hat{a} = da$. □

Definition 6.1 For any smooth $h\colon Y \to i\mathbb{R}$ define $\underline{h}, P(h)\colon \mathbf{B} \to i\mathbb{R}$ by $\underline{h}(t,y) = h(y)$ and

$$P(h) = \Delta\underline{h} + i\langle i\Psi, \exp(\underline{h})\Phi\rangle,$$

where $\Delta = d^*d$ is the Laplacian over $\mathbb{R} \times Y$. Let $P_t(h)$ be the restriction of $P(h)$ to $\{t\} \times Y$.

Note that

$$\mathcal{I}^*_{\underline{\alpha}}(\exp(\underline{h})(S) - \underline{\alpha}) = -d^*a + P(h).$$

Lemma 6.7 *If α is irreducible then the following hold:*

(i) *There is a $C_3 > 0$ such that if δ is sufficiently small then there exists a unique smooth $h\colon Y \to i\mathbb{R}$ satisfying $\|h\|_{3,2} \le C_3\delta$ and $P_0(h) = 0$.*

(ii) *If $h\colon Y \to i\mathbb{R}$ is any smooth function satisfying $P_0(h) = 0$ then*

$$\|P(h)\|_{L^2(\mathbf{B})} \le \mathit{const} \cdot (\nu + \|a'\|_{L^3(\mathbf{B})}).$$

Proof (i) We will apply Proposition B.1 (ie the inverse function theorem) to the smooth map

$$P_0\colon L^2_3 \to L^2_1, \quad h \mapsto \Delta_Y h + i\langle i\Psi_0, \exp(h)\Phi_0\rangle.$$

The first two derivatives of this map are

$$DP_0(h)k = \Delta_Y k + k\langle \Psi_0, \exp(h)\Phi_0\rangle,$$
$$D^2P_0(h)(k,\ell) = ik\ell\langle i\Psi_0, \exp(h)\Phi_0\rangle.$$

The assumption $\delta \le 1$ gives

$$\|D^2P_0(h)\| \le \mathrm{const}\cdot(1+\|\nabla h\|_3).$$

Set $L = DP_0(0)$. Then

$$(L - \Delta_Y - |\Psi_0|^2)k = k\langle \Psi_0, \phi_0\rangle,$$

hence

$$\|L - \Delta_Y - |\Psi_0|^2\| \le \mathrm{const}\cdot\delta.$$

Thus if δ is sufficiently small then L is invertible and

$$\|L^{-1}\| \le \|(\Delta_Y + |\Psi_0|^2)^{-1}\| + 1.$$

Furthermore, we have $P_0(0) = i\langle i\Psi_0, \phi_0\rangle$, so

$$\|P_0(0)\|_{1,2} \le \mathrm{const}\cdot\delta.$$

By Proposition B.1 (i) there exists a constant $C>0$ such that if δ is sufficiently small then there is a unique $h \in L^2_3$ such that $\|h\|_{3,2} \le C$ and $P_0(h) = 0$ (which implies that h is smooth). Proposition B.1 (ii) then yields

$$\|h\|_{3,2} \le \mathrm{const}\cdot\|P_0(0)\|_{1,2} \le \mathrm{const}\cdot\delta.$$

(ii) Setting $Q = P(h)$ we have, for $0 \le t \le T$,

$$\int_Y |Q(t,y)|^2 dy = \int_Y \left| \int_0^t \partial_1 Q(s,y)\,ds \right|^2 dy \le \mathrm{const}\cdot\int_{\mathbf{B}} |\partial_1 Q|^2.$$

Now, $\partial_1 Q = i\langle i\Psi, \exp(\underline{h})\partial_1\Phi\rangle$, hence

$$\|\partial_1 Q\|_2 \le \mathrm{const}\cdot\|\partial_1\Phi\|_2 \le \mathrm{const}\cdot(\nu + \|a'\|_3)$$

by Lemma 6.3. □

Proposition 6.1 *There is a constant C_4 such that if δ is sufficiently small then there exists a smooth $\widetilde{f}\colon \mathbf{B} \to i\mathbb{R}$ such that $\widetilde{s} = (\widetilde{a}, \widetilde{\phi}) = \exp(\widetilde{f})(S) - \underline{\alpha}$ satisfies*

$$\mathcal{I}^*_{\underline{\alpha}}\widetilde{s} = 0, \quad (*\widetilde{a})|_{\partial\mathbf{B}} = 0, \quad \|\widetilde{s}\|_{L^2_1(\mathbf{B})} \le C_4\nu, \quad \|\widetilde{s}\|_{L^2_2(\mathbf{B})} \le C_4\delta,$$

where δ, ν are as in (31).

This is analogous to Uhlenbeck's theorem [32, Theorem 1.3] (with p=2), except that we assume a bound on δ rather than on ν.

Proof To simplify notation we will write $\mathcal{I} = \mathcal{I}_{\underline{\alpha}}$ in this proof.

Case 1: α reducible In that case the operator $\mathcal{I}^*$ is given by $\mathcal{I}^*(b,\psi) = -d^*b$. Let $\widetilde{f}$ be the $\hat{f}$ provided by Lemma 6.6. Then apply Lemma 6.4 (ii), taking the S of that lemma to be the present $\exp(\widetilde{f})(S)$.

Case 2: α irreducible Let $\hat{f}, \hat{S}$ etc be as in Lemma 6.6. Choose $h\colon\, Y \to i\mathbb{R}$ such that the conclusions of Lemma 6.7 (i) holds with the S of that lemma taken to be the present $\hat{S}$. Set $\acute{S} = (\acute{A}, \acute{\Phi}) = \exp(\underline{h})(\hat{S})$ and $\acute{s} = (\acute{a}, \acute{\phi}) = \acute{S} - \underline{\alpha}$. By Lemmas 6.7 and 6.4 (ii) we have

$$\|\mathcal{I}^*\acute{s}\|_2 \le \text{const}\cdot\nu, \quad \|\acute{s}\|_{2,2} \le \text{const}\cdot\delta, \quad \|\acute{\phi}\|_{1,2} \le \text{const}\cdot\nu.$$

Since $-d^*\acute{a} = \mathcal{I}^*\acute{s} - i\langle i\Psi, \acute{\phi}\rangle$ we also get

$$\|d^*\acute{a}\|_2 \le \text{const}\cdot\nu.$$

Applying Lemma 6.5 as in the proof of Lemma 6.6 we see that

$$\|\acute{a}\|_{1,2} \le \text{const}\cdot\nu.$$

It now only remains to make a small perturbation to $\acute{S}$ so as to fulfil the Coulomb gauge condition. To this end we invoke the local slice theorem for $\mathbf{B}$. This says that there is a $C > 0$ such that if δ is sufficiently small then there exists a unique smooth $f\colon\, \mathbf{B} \to i\mathbb{R}$ such that setting $\widetilde{s} = (\widetilde{a}, \widetilde{\phi}) = \exp(f)(\acute{S}) - \underline{\alpha}$ one has

$$\|f\|_{3,2} \le C\delta, \quad \mathcal{I}^*\widetilde{s} = 0, \quad *\widetilde{a}|_{\partial\mathbf{B}} = 0.$$

We will now estimate first $\|f\|_{2,2}$, then $\|\widetilde{s}\|_{1,2}$ in terms of ν. First note that $*\acute{a}|_{\partial\mathbf{B}} = *\hat{a}|_{\partial\mathbf{B}} = 0$, and

$$\widetilde{a} = \acute{a} - df, \qquad \widetilde{\phi} = \exp(f)\acute{\Phi} - \Psi$$

by definition. Write the imaginary part of $\exp(f)$ as $f + f^3u$. Then f satisfies the equations $(\partial_t f)|_{\partial\mathbf{B}} = 0$ and

$$\begin{aligned}
0 &= -d^*\widetilde{a} + i\langle i\Psi, \widetilde{\phi}\rangle_{\mathbb{R}} \\
&= \Delta f - d^*\acute{a} + i\langle i\Psi, \exp(f)(\acute{\phi} + \Psi)\rangle_{\mathbb{R}} \\
&= \Delta f + |\Psi|^2 f - d^*\acute{a} + i\langle i\Psi, \exp(f)\acute{\phi} + f^3u\Psi\rangle_{\mathbb{R}}.
\end{aligned}$$

By the Sobolev embedding theorem we have

$$\|f\|_\infty \le \text{const}\cdot\|f\|_{3,2} \le \text{const}\cdot\delta,$$

and we assume $\delta \leq 1$, so $\|u\|_\infty \leq \text{const}$. Therefore,

$$\begin{aligned}\|f\|_{2,2} &\leq \text{const} \cdot \|\Delta f + |\Psi|^2 f\|_2 \\ &\leq \nu + \text{const} \cdot \|f^3\|_2 \\ &\leq \nu + \text{const} \cdot \|f\|_{2,2}^3,\end{aligned}$$

cf Subsection 2.5 for the first inequality. If δ is sufficiently small then we can rearrange to get $\|f\|_{2,2} \leq \text{const} \cdot \nu$. Consequently, $\|\widetilde{a}\|_{1,2} \leq \text{const} \cdot \nu$. To estimate $\|\widetilde{\phi}\|_{1,2}$ we write

$$\widetilde{\phi} = g\Psi + \exp(f)\acute{\phi},$$

where $g = \exp(f) - 1$. Then $|dg| = |df|$ and $|g| \leq \text{const} \cdot |f|$. Now

$$\begin{aligned}\|\widetilde{\phi}\|_2 &\leq \text{const} \cdot \|f\|_2 + \|\acute{\phi}\|_2 \leq \text{const} \cdot \nu, \\ \|\nabla\widetilde{\phi}\|_2 &\leq \text{const} \cdot (\|g\|_{1,2} + \|df \otimes \acute{\phi}\|_2 + \|\nabla\acute{\phi}\|_2) \\ &\leq \text{const} \cdot (\nu + \|df\|_4\|\acute{\phi}\|_4) \\ &\leq \text{const} \cdot (\nu + \|f\|_{2,2}\|\acute{\phi}\|_{1,2}) \\ &\leq \text{const} \cdot (\nu + \nu^2) \\ &\leq \text{const} \cdot \nu\end{aligned}$$

by (32). Therefore, $\|\widetilde{\phi}\|_{1,2} \leq \text{const} \cdot \nu$. Thus, the proposition holds with

$$\widetilde{f} = \hat{f} + \underline{h} + f.$$

□

Proposition 6.2 *Let k be a positive integer and $V \Subset \text{int}(\mathbf{B})$ an open subset. Then there are constants $\epsilon_k, C_{k,V}$, where ϵ_k is independent of V, such that if*

$$\mathcal{I}_{\underline{\alpha}}^* s = 0, \quad \|s\|_{L^2_1(\mathbf{B})} \leq \epsilon_k$$

then

$$\|s\|_{L^2_k(V)} \leq C_{k,V}\|s\|_{L^2_1(\mathbf{B})}.$$

Proof The argument in [11, pages 62–63] carries over, if one replaces the operator $d^* + d^+$ with $\mathcal{I}_{\underline{\alpha}}^* + D\Theta_{\underline{\alpha}}$, where $D\Theta_{\underline{\alpha}}$ is the linearization of the monopole map at $\underline{\alpha}$. Note that $\mathcal{I}_{\underline{\alpha}}^* + D\Theta_{\underline{\alpha}}$ is injective over $S^1 \times Y$ because α is nondegenerate, so if $\gamma\colon \mathbf{B} \to \mathbb{R}$ is a smooth function supported in $\text{int}(\mathbf{B})$ then

$$\|\gamma s\|_{k,2} \leq C_k' \|(\mathcal{I}_{\underline{\alpha}}^* + D\Theta_{\underline{\alpha}})(\gamma s)\|_{k-1,2}$$

for some constant C_k'. □

6.3 Decay of monopoles

The two theorems in this subsection are analogues of Propositions 4.3 and 4.4 in [10], respectively.

Let β be a non-degenerate monopole over Y, and $U \subset \mathcal{B}_Y$ an L^2–closed subset which contains no monopoles except perhaps $[\beta]$. Choose $\lambda^{\pm} > 0$ such that $\widetilde{H}_\beta$ has no eigenvalue in the interval $[-\lambda^-, \lambda^+]$, and set $\lambda = \min(\lambda^-, \lambda^+)$. Define

$$\mathbf{B}_t = [t-1, t+1] \times Y.$$

Theorem 6.1 *For any $C > 0$ there are constants $\epsilon, C_0, C_1, \dots$ such that the following holds. Let $S = (A, \Phi)$ be any monopole in temporal gauge over $(-2, \infty) \times Y$ such that $[S_t] \in U$ for some $t \geq 0$. Set*

$$\bar{\nu} = \|\nabla \vartheta_S\|_{L^2((-2,\infty)\times Y)}, \quad \nu(t) = \|\nabla \vartheta_S\|_{L^2(\mathbf{B}_t)}.$$

If $\|\Phi\|_\infty \leq C$ and $\bar{\nu} \leq \epsilon$ then there is a smooth monopole α over Y, gauge equivalent to β, such that if B is the connection part of $\underline{\alpha}$ then for every $t \geq 1$ and non-negative integer k one has

$$\sup_{y \in Y} |\nabla_B^k (S - \underline{\alpha})|_{(t,y)} \leq C_k \sqrt{\nu(0)} e^{-\lambda^+ t}. \tag{35}$$

Proof It follows from the local slice theorem that $\widetilde{\mathcal{B}}_Y \to \mathcal{B}_Y$ is a (topological) principal $H^1(Y;\mathbb{Z})$–bundle. Choose a small open neighbourhood V of $[\beta] \in \mathcal{B}_Y$ which is the image of a convex set in $\mathcal{C}_Y$. We define a continuous function $\bar{f}\colon V \to \mathbb{R}$ by

$$\bar{f}(x) = \vartheta(\sigma(x)) - \vartheta(\sigma([\beta]))$$

where $\sigma\colon V \to \widetilde{\mathcal{B}}_Y$ is any continuous cross-section. It is clear that $\bar{f}$ is independent of σ.

Given $C > 0$, let $S = (A, \Phi)$ be any monopole over $(-2, \infty) \times Y$ such that $\|\Phi\|_\infty \leq C$ and $[S_t] \in U$ for some $t \geq 0$. If $\delta > 0$, and k is any non-negative integer, then provided $\bar{\nu}$ is sufficiently small, our local compactness results (Lemmas 4.2 and 4.1) imply that for every $t \geq 0$ we can find a smooth $u\colon \mathbf{B}_0 \to \mathrm{U}(1)$ such that

$$\|u(S|_{\mathbf{B}_t}) - \underline{\beta}\|_{C^k(\mathbf{B}_0)} < \delta.$$

In particular, if $\bar{\nu}$ is sufficiently small then

$$f\colon \overline{\mathbb{R}}_+ \to \mathbb{R}, \quad t \mapsto \bar{f}([S_t])$$

is a well-defined smooth function. Since $f(t) - \vartheta(S_t)$ is locally constant, and $f(t) \to 0$ as $t \to \infty$, we have

$$f(t) = \vartheta(S_t) - L,$$

where $L = \lim_{t\to\infty} \vartheta(S_t)$. If $\bar\nu$ is sufficiently small then Lemma 6.2 gives $2\lambda^+ f \le -f'$, hence

$$0 \le f(t) \le e^{-2\lambda^+ t} f(0), \quad t \ge 0.$$

This yields

$$\nu(t)^2 = f(t-1) - f(t+1) \le \text{const} \cdot e^{-2\lambda^+ t} f(0), \quad t \ge 1.$$

If $\bar\nu$ is sufficiently small then by Propositions 6.1 and 6.2 we have

$$f(t) \le \text{const} \cdot \nu(t), \quad \sup_{y\in Y} |\nabla_A^k(\nabla\vartheta_S)|_{(t,y)} \le C'_k \nu(t) \tag{36}$$

for every $t \ge 0$ and non-negative integer k, where C'_k is some constant. Here we are using the simple fact that if E, E' are Banach spaces, $W \subset E$ an open neighbourhood of 0, and $h\colon W \to E'$ a differentiable map with $h(0) = 0$ then $\|h(x)\| \le (\|Dh(0)\| + 1)\|x\|$ in some neighbourhood of 0. For instance, to deduce the second inequality in (36) we can apply this to the map

$$h\colon L^2_{k+j+1} \to L^2_j, \quad s = (a, \phi) \mapsto \nabla^k_{B'+a}(\nabla\vartheta_{\underline{\beta}+s})$$

where $j \ge 3$, say, and B' is the connection part of $\underline{\beta}$.

Putting the inequalities above together we get

$$\sup_{y\in Y} |\nabla_A^k(\nabla\vartheta_S)|_{(t,y)} \le C''_k \sqrt{\nu(0)} e^{-\lambda^+ t}, \quad t \ge 1$$

for some constants C''_k. If S is in temporal gauge we deduce, by taking $k = 0$, that S_t converges uniformly to some continuous configuration α. One can now prove by induction on k that α is of class C^k and that (35) holds. ◻

Theorem 6.2 *For any $C > 0$ there are constants $\epsilon, C_0, C_1, \ldots$ such that the following holds for every $T > 1$. Let $S = (A, \Phi)$ be any smooth monopole in temporal gauge over the band $[-T-2, T+2] \times Y$, and suppose $[S_t] \in U$ for some $t \in [-T, T]$. Set*

$$\bar\nu = \|\nabla\vartheta_S\|_{L^2((-T-2,T+2)\times Y)}, \quad \nu(t) = \|\nabla\vartheta_S\|_{L^2(\mathbf{B}_t)}.$$

If $\|\Phi\|_\infty \le C$ and $\bar\nu \le \epsilon$ then there is a smooth monopole α over Y, gauge equivalent to β, such that if B is the connection part of $\underline{\alpha}$ then for $|t| \le T - 1$ and every non-negative integer k one has

$$\sup_{y\in Y} |\nabla_B^k(S - \underline{\alpha})|_{(t,y)} \le C_k(\nu(-T) + \nu(T))^{1/2} e^{-\lambda(T-|t|)}.$$

Proof Given $C>0$, let $S=(A,\Phi)$ be any monopole over $[-T-2,T+2]\times Y$ such that $\|\Phi\|_\infty\le C$ and $[S_t]\in U$ for some $t\in[-T,T]$. If $\bar\nu$ is sufficiently small then we can define the function $f(t)$ for $|t|\le T$ as in the proof of Theorem 6.1, and (36) will hold with $f(t)$ replaced by $|f(t)|$, for $|t|\le T$. Again, $f(t)=\vartheta(S_t)-L$ for some constant L. Lemma 6.2 now gives

$$e^{-2\lambda_-(T-t)}f(T)\le f(t)\le e^{-2\lambda_+(T+t)}f(-T),\quad |t|\le T,$$

which implies

$$|f(t)|\le(|f(-T)|+|f(T)|)e^{-2\lambda(T-|t|)},\quad |t|\le T,$$
$$\nu(t)^2\le\text{const}\cdot(\nu(-T)+\nu(T))e^{-2\lambda(T-|t|)},\quad |t|\le T-1.$$

By Propositions 6.1 and 6.2 there is a critical point α gauge equivalent to β such that

$$\|\nabla^k_B(S_0-\alpha)\|_{L^\infty(Y)}\le C'''_k\nu(0)$$

for some constants C'''_k. It is now easy to complete the proof by induction on k. ◻

6.4 Global convergence

The main result of this subsection is Proposition 6.3, which relates local and global convergence of monopoles over a half-cylinder. First some lemmas.

Lemma 6.8 *Let Z be a compact Riemannian n–manifold (perhaps with boundary), m a non-negative integer, and $q\ge n/2$. Then there is a real polynomial $P_{m,q}(x)$ of degree $m+1$ satisfying $P_{m,q}(0)=0$, such that for any smooth $u\colon Z\to U(1)$ one has*

$$\|du\|_{m,q}\le P_{m,q}(\|u^{-1}du\|_{m,q}).$$

Proof Argue by induction on m and use the Sobolev embedding $L^r_k(Z)\subset L^{2r}_{k-1}(Z)$ for $k\ge1$, $r\ge n/2$. ◻

Lemma 6.9 *Let Z be a compact, connected Riemannian n–manifold (perhaps with boundary), $z\in Z$, m a positive integer, and $q\ge1$. Then there is a $C>0$ such that for any smooth $f\colon Z\to\mathbb{C}$ one has*

(i) $\|f-f_{\mathrm{av}}\|_{m,q}\le C\|df\|_{m-1,q}$,

(ii) $\|f\|_{m,q}\le C(\|df\|_{m-1,q}+|f(z)|)$,

where $f_{\mathrm{av}}=Vol(Z)^{-1}\int_Z f$ is the average of f.

Proof Exercise. □

Lemma 6.10 *Let Z be a compact Riemannian n–manifold (perhaps with boundary), m a positive integer, and q a real number such that $mq > n$. Let Φ be a smooth section of some Hermitian vector bundle $E \to Z$, $\Phi \not\equiv 0$. Then there exists a $C > 0$ with the following significance. Let ϕ_1 be a smooth section of E satisfying $\|\phi_1\|_q \le C^{-1}$ and $w\colon Z \to \mathbb{C}$ a smooth map. Define another section ϕ_2 by*

$$w(\Phi + \phi_1) = \Phi + \phi_2.$$

Then

$$\|w - 1\|_{m,q} \le C(\|dw\|_{m-1,q} + \|\phi_2 - \phi_1\|_q).$$

Proof The equation

$$(w-1)\Phi = \phi_2 - \phi_1 - (w-1)\phi_1$$

gives

$$\begin{aligned}\|w-1\|_{m,q} &\le \text{const} \cdot (\|dw\|_{m-1,q} + \|(w-1)\Phi\|_q) \\ &\le \text{const} \cdot (\|dw\|_{m-1,q} + \|\phi_2 - \phi_1\|_q + \|w-1\|_{m,q}\|\phi_1\|_q).\end{aligned}$$

Here the first inequality is analogous to Lemma 6.9 (ii). If $\|\phi_1\|_q$ is sufficiently small then we can rearrange to get the desired estimate on $\|w-1\|_{m,q}$. □

Now let α be a non-degenerate critical point of ϑ. Note that if $S = (A, \Phi)$ is any finite energy monopole in temporal gauge over $\mathbb{R}_+ \times Y$ such that $\|\Phi\|_\infty < \infty$ and

$$\liminf_{t\to\infty} \int_{[t,t+1]\times Y} |S - \underline{\alpha}|^r = 0$$

for some $r > 1$ then by the results of Section 4 we have $[S_t] \to [\alpha]$ in $\mathcal{B}_Y$, hence $S - \underline{\alpha}$ decays exponentially by Theorem 6.1. In this situation we will simply say that S is *asymptotic to* α.

(Here we used the fact that for any $p > 2$ and $1 < r \le 2p$, say, the L^r metric on the L^p_1 configuration space $\mathcal{B}([0,1] \times Y)$ is well-defined.)

Lemma 6.11 *If $S = (A, \Phi)$ is any smooth monopole over $\mathbb{R}_+ \times Y$ such that $\|\Phi\|_\infty < \infty$ and $S - \underline{\alpha} \in L^p_1$ for some $p > 2$ then there exists a null-homotopic smooth $u\colon \mathbb{R}_+ \times Y \to U(1)$ such that $u(S)$ is in temporal gauge and asymptotic to α.*

Proof By Theorem 6.1 there exists a smooth $u\colon \mathbb{R}_+ \times Y \to \mathrm{U}(1)$ such that $u(S)$ is in temporal gauge and asymptotic to α. Lemma 6.8, Lemma 6.9 (i), and the assumption $S - \underline{\alpha} \in L^p_1$ then gives

$$\|u - u_{\mathrm{av}}\|_{L^\infty([t,t+1]\times Y)} \to 0 \text{ as } t \to \infty,$$

hence u is null-homotopic. □

It follows that all elements of the moduli spaces defined in Subsection 3.4 have smooth representatives that are in temporal gauge over the ends.

Proposition 6.3 *Let $\delta > 0$ and suppose $\vartheta\colon \widetilde{\mathcal{B}}_Y \to \mathbb{R}$ has no critical value in the half-open interval $(\vartheta(\alpha), \vartheta(\alpha) + \delta]$ (this implies Condition (O1)). For $n = 1, 2, \ldots$ let $S_n = (A_n, \Phi_n)$ be a smooth monopole over $\overline{\mathbb{R}}_+ \times Y$ such that*

$$S_n - \underline{\alpha} \in L^p_1, \quad \sup_n \|\Phi_n\|_\infty < \infty, \quad \vartheta(S_n(0)) \le \vartheta(\alpha) + \delta,$$

for some $p > 2$. Let $v_n\colon \overline{\mathbb{R}}_+ \times Y \to U(1)$ be a smooth map such that the sequence $v_n(S_n)$ converges in C^∞ over compact subsets of $\overline{\mathbb{R}}_+ \times Y$ to a configuration S in temporal gauge. Then the following hold:

(i) *S is asymptotic to a critical point α' gauge equivalent to α,*

(ii) *If $\alpha = \alpha'$ then v_n is null-homotopic for all sufficiently large n, and there exist smooth $u_n\colon \overline{\mathbb{R}}_+ \times Y \to U(1)$ with the following significance: For every $t \ge 0$ one has $u_n = 1$ on $[0,t] \times Y$ for all sufficiently large n. Moreover, for any $\sigma < \lambda^+$, $q \ge 1$ and non-negative integer m one has*

$$\|u_n v_n(S_n) - S\|_{L^{q,\sigma}_m} \to 0 \text{ as } n \to \infty.$$

Here λ^+ is as in Subsection 6.1.

Proof It clearly suffices to prove the proposition when $q \ge 2$ and $mq > 4$, which we assume from now on.

By Lemma 4.5 we have

$$\int_{\mathbb{R}_+\times Y} |\nabla\vartheta_{S_n}|^2 = \vartheta(S_n(0)) - \vartheta(\alpha) \le \delta \tag{37}$$

for each n, hence $\int_{\mathbb{R}_+\times Y} |\nabla\vartheta_S|^2 \le \delta$. Part (i) of the proposition is now a consequence of Theorem 6.1 and the following

Claim 6.1 *$[S(t)]$ converges in $\mathcal{B}_Y$ to $[\alpha]$ as $t \to \infty$.*

Proof of claim For $r>0$ let $B_r\subset\mathcal{B}_Y$ denote the open r–ball around $[\alpha]$ in the L^2 metric, and let $\overline{B}_r$ be the corresponding closed ball. Choose $r>0$ such that $\overline{B}_{2r}$ contains no monopole other than $[\alpha]$. Assuming the claim does not hold then by Lemma 4.1 one can find a sequence t'_j such that $t'_j\to\infty$ as $j\to\infty$ and $[S(t'_j)]\notin\overline{B}_{2r}$ for each j. Because of the convergence of $v_n(S_n)$ it follows by a continuity argument that there are sequences n_j,t_j with $t_j,n_j\to\infty$ as $j\to\infty$, such that

$$[S_{n_j}(t_j)]\in\overline{B}_{2r}\setminus B_r$$

for all j. For $s\in\mathbb{R}$ let $\mathcal{T}_s\colon\ \mathbb{R}\times Y\to\mathbb{R}\times Y$ be translation by s:

$$\mathcal{T}_s(t,y)=(t+s,y).$$

Again by Lemma 4.1 there are smooth $\omega_j\colon\ \overline{\mathbb{R}}_+\times Y\to\mathrm{U}(1)$ such that a subsequence of $(\mathcal{T}_{t_j})^*(\omega_j(S_{n_j}))$ converges in C^∞ over compact subsets of $\mathbb{R}\times Y$ to some finite energy monopole S' whose spinor field is pointwise bounded. Moreover, it is clear that $\vartheta\circ\omega_j(0)-\vartheta\in\mathbb{R}$ must be bounded as $j\to\infty$, so by passing to a subsequence and replacing ω_j by $\omega_j\omega_{j_0}^{-1}$ for some fixed j_0 we may arrange that $\vartheta\circ\omega_j(0)=\vartheta$ for all n. Then $\ell=\lim_{t\to-\infty}\vartheta(S'(t))$ must be a critical value of ϑ. Since

$$[S'(0)]\in\overline{B}_{2r}\setminus B_r,$$

$S'(0)$ is not a critical point, whence $\partial_t|_0\vartheta(S'(t))<0$. Therefore,

$$\vartheta(\alpha)+\delta\geq\ell>\vartheta(S'(0))>\vartheta(\alpha),$$

contradicting our assumptions. This proves the claim. □

We will now prove Part (ii). For $\tau\geq 0$ let

$$B^-_\tau=[0,\tau]\times Y,\quad B^+_\tau=[\tau,\infty)\times Y,\quad \mathcal{O}_\tau=[\tau,\tau+1]\times Y.$$

By Lemma 6.11 there is, for every n, a null-homotopic, smooth $\widetilde{v}_n\colon\ \overline{\mathbb{R}}_+\times Y\to\mathrm{U}(1)$ such that $S''_n=\widetilde{v}_n(S_n)$ is in temporal gauge and asymptotic to α.

Note that

$$\lim_{t\to\infty}\limsup_{n\to\infty}\vartheta(S_n(t))=\vartheta(\alpha).$$

For otherwise we could find an $\epsilon>0$ and for every natural number j a pair $t_j,n_j\geq j$ such that

$$\vartheta(S_{n_j}(t_j))\geq\vartheta(\alpha)+\epsilon,$$

and we could then argue as in the proof of Claim 6.1 to produce a critical value of ϑ in the interval $(\alpha,\alpha+\delta]$. Since $|\nabla\vartheta_{S_n}|=|\nabla\vartheta_{S''_n}|$ it follows from (37) and Theorem 6.1 that there exists a $t_1\geq 0$ such that if $\tau\geq t_1$ then

$$\limsup_{n\to\infty}\|S''_n-\underline{\alpha}\|_{L^{q,\sigma}_m(B^+_\tau)}\leq\mathrm{const}\cdot e^{(\sigma-\lambda^+)\tau}$$

where the constant is independent of τ. Then we also have

$$\limsup_{n\to\infty} \|S''_n - S\|_{L^{q,\sigma}_m(B^+_\tau)} \le \text{const} \cdot e^{(\sigma-\lambda^+)\tau}.$$

Set $S'_n = v_n(S_n)$ and $w_n = \widetilde{v}_n v_n^{-1}$. Then we get

$$\limsup_{n\to\infty} \|S''_n - S'_n\|_{L^q_m(\mathcal{O}_\tau)} \le \text{const} \cdot e^{-\lambda^+\tau},$$

which gives

$$\limsup_{n\to\infty} \|dw_n\|_{L^q_m(\mathcal{O}_\tau)} \le \text{const} \cdot e^{-\lambda^+\tau}$$

by Lemma 6.8. In particular, w_n is null-homotopic for all sufficiently large n.

Fix $y_0 \in Y$ and set $x_\tau = (\tau, y_0)$. Choose a sequence τ_n such that $\tau_n \to \infty$ as $n \to \infty$ and

$$\begin{aligned} \|S'_n - S\|_{L^{q,\sigma}_m(B^-_{\tau_n+1})} &\to 0, \\ \|S''_n - S\|_{L^{q,\sigma}_m(B^+_{\tau_n})} &\to 0 \end{aligned} \tag{38}$$

as $n \to \infty$. If α is reducible then by multiplying each $\widetilde{v}_n$ by a constant and redefining w_n, S''_n accordingly we may arrange that $w_n(x_{\tau_n}) = 1$ for all n. (If α is irreducible we keep $\widetilde{v}_n$ as before.) Then (38) still holds. Applying Lemma 6.8 together with Lemma 6.9 (ii) (if α is reducible) or Lemma 6.10 (if α is irreducible) we see that

$$e^{\sigma\tau_n}\|w_n - 1\|_{L^q_{m+1}(\mathcal{O}_{\tau_n})} \to 0$$

as $n \to \infty$.

Let $\beta\colon \mathbb{R} \to \mathbb{R}$ be a smooth function such that $\beta(t) = 0$ for $t \le \frac{1}{3}$ and $\beta(t) = 1$ for $t \ge \frac{2}{3}$. Set $\beta_\tau(t) = \beta(t-\tau)$. Given any function $w\colon \mathcal{O}_\tau \to \mathbb{C} \setminus (-\infty, 0]$ define

$$\mathcal{U}_{w,\tau} = \exp(\beta_\tau \log w)$$

where $\log(\exp(z)) = z$ for complex numbers z with $|\mathrm{Im}(z)| < \pi$. Let m' be any integer such that $m'q > 4$. If $\|w-1\|_{m',q}$ is sufficiently small then

$$\begin{aligned} \|\mathcal{U}_{w,\tau} - 1\|_{m',q} &\le \text{const} \cdot \|w-1\|_{m',q}, \\ \|w^{-1}dw\|_{m'-1,q} &\le \text{const} \cdot \|w-1\|_{m',q}. \end{aligned} \tag{39}$$

To see this recall that for functions on $\mathbb{R}^4$, multiplication defines a continuous map $L^q_{m'} \times L^q_k \to L^q_k$ for $0 \le k \le m'$. Therefore, if V is the set of all functions in $L^q_{m'}(\mathcal{O}_\tau, \mathbb{C})$ that map into some fixed small open ball about $1 \in \mathbb{C}$ then $w \mapsto \mathcal{U}_{w,\tau}$ defines a C^∞ map $V \to L^q_{m'}$. This yields the first inequality in (39), and the proof of the second inequality is similar.

Combining (38) and (39) we conclude that Part (ii) of the proposition holds with

$$u_n = \begin{cases} 1 & \text{in } B^-_{\tau_n}, \\ \mathcal{U}_{w_n,\tau_n} & \text{in } \mathcal{O}_{\tau_n}, \\ w_n & \text{in } B^+_{\tau_n+1}. \end{cases}$$

This completes the proof of Proposition 6.3. □

7 Global compactness

In this section we will prove Theorems 1.3 and 1.4. Given the results of Sections 4 and 5, what remains to be understood is convergence over ends and necks. We will use the following terminology:

$$\text{c-convergence} = C^\infty \text{ convergence over compact subsets.}$$

7.1 Chain-convergence

We first define the notion of chain-convergence. For simplicity we only consider two model cases: first the case of one end and no necks, then the case of one neck and no ends. It should be clear how to extend the notion to the case of multiple ends and/or necks.

Definition 7.1 Let X be a spin^c Riemannian 4–manifold with one tubular end $\mathbb{R}_+\times Y$, where Y is connected. Let $\alpha_1,\alpha_2,\dots$ and $\beta_0,\dots,\beta_k$ be elements of $\widetilde{\mathcal{R}}_Y$, where $k\ge 0$ and $\vartheta(\beta_{j-1})>\vartheta(\beta_j)$ for $j=1,\dots,k$. Let $\omega\in M(X;\beta_0)$ and $\vec{v}=(v_1,\dots,v_k)$, where $v_j\in\check{M}(\beta_{j-1},\beta_j)$. We say a sequence $[S_n]\in M(X;\alpha_n)$ *chain-converges* to $(\omega,\vec{v})$ if there exist, for each n,

- a smooth map $u_n\colon X\to \mathrm{U}(1)$,
- for $j=1,\dots,k$ a smooth map $u_{n,j}\colon \mathbb{R}\times Y\to\mathrm{U}(1)$,
- a sequence $0=t_{n,0}<t_{n,1}<\cdots<t_{n,k}$,

such that

(i) $u_n(S_n)$ c-converges over X to a representative of ω (in the sense of Subsection 2.4),

(ii) $t_{n,j}-t_{n,j-1}\to\infty$ as $n\to\infty$,

(iii) $u_{n,j}(\mathcal{T}^*_{t_{n,j}}S_n)$ c-converges over $\mathbb{R}\times Y$ to a representative of v_j,

(iv) $\limsup_{n\to\infty}\,[\vartheta(S_n(t_{n,j-1}+\tau))-\vartheta(S_n(t_{n,j}-\tau))]\to 0$ as $\tau\to\infty$,

(v) $\limsup_{n\to\infty}\,[\vartheta(S_n(t_{n,k}+\tau))-\vartheta(\alpha_n)]\to 0$ as $\tau\to\infty$,

where (ii), (iii) and (iv) should hold for $j=1,\dots,k$.

Conditions (iv) and (v) mean, in familiar language, that "no energy is lost in the limit". As before, $\mathcal{T}_s$ denotes translation by s, ie $\mathcal{T}_s(t,y)=(t+s,y)$.

We now turn to the case of one neck and no ends.

Definition 7.2 In the situation of Subsection 1.4, suppose $r=1$ and $r'=0$. Let $\beta_0,\dots,\beta_k\in\widetilde{\mathcal{R}}_Y$, where $k\geq 0$ and $\vartheta(\beta_{j-1})>\vartheta(\beta_j)$, $j=1,\dots,k$. Let $\omega\in M(X;\beta_0,\beta_k)$ and $\vec{v}=(v_1,\dots,v_k)$, where $v_j\in\check{M}(\beta_{j-1},\beta_j)$. Let $T(n)\to\infty$ as $n\to\infty$. We say a sequence $[S_n]\in M(X^{(T(n))})$ *chain-converges* to $(\omega,\vec{v})$ if there exist, for every n,

- a smooth map $u_n\colon\ X^{(T(n))}\to\mathrm{U}(1)$,
- for $j=1,\dots,k$ a smooth map $u_{n,j}\colon\ \mathbb{R}\times Y\to\mathrm{U}(1)$,
- a sequence $-T(n)=t_{n,0}<t_{n,1}<\cdots<t_{n,k+1}=T(n)$,

such that (i)–(iv) of Definition 7.1 hold for the values of j for which they are defined (in other words, (ii) and (iv) should hold for $1\leq j\leq k+1$ and (iii) for $1\leq j\leq k$).

In the notation of Subsection 1.2, if $J\subset\mathbb{R}$ is an interval with non-empty interior then a smooth configuration S over $J\times Y$ is called *regular* (with respect to ϑ) if either $\partial_t\vartheta(S_t)<0$ for every $t\in J$, or S is gauge equivalent to the translationary invariant configuration $\underline{\alpha}$ determined by some critical point α of ϑ. Proposition 4.3 guarantees the regularity of certain $(\mathfrak{p},\mathfrak{q})$–monopoles when $\mathfrak{p}$ is sufficiently small. In particular, genuine monopoles are always regular.

Consider now the situation of Subsection 1.4 (without assuming (B1) or (B2)), and let the 2–form μ on X be fixed.

Definition 7.3 A set of perturbation parameters $\vec{\mathfrak{p}},\vec{\mathfrak{p}}'$ is *admissible* for a vector $\vec{\alpha}'$ of critical points if for some $t_0\geq 1$ the following holds. Let $\mathcal{M}$ be the disjoint union of all moduli spaces $M(X^{(T)};\vec{\alpha}';\vec{\mathfrak{p}};\vec{\mathfrak{p}}')$ with $\min_j T_j\geq t_0$. Then we require, for all j,k, that

(i) If $\widetilde{S}$ is any configuration over $[-1,1]\times Y_j$ which is a C^∞ limit of configurations of the form $S|_{[t-1,t+1]\times Y_j}$ with $S\in\mathcal{M}$ and $|t|\leq T_j-1$, then $\widetilde{S}$ is regular.

(ii) If $\widetilde{S}$ is any configuration over $[-1,1]\times Y'_k$ which is a C^∞ limit of configurations of the form $S|_{[t-1,t+1]\times Y'_k}$ with $S\in\mathcal{M}$ and $t\ge 1$, then $\widetilde{S}$ is regular.

In particular, the zero perturbation parameters are always admissible.

The next two propositions describe some properties of chain-convergence.

Proposition 7.1 *In the notation of Theorem 1.3, suppose ω_n chain-converges to $(\omega,\vec{v}_1,\dots,\vec{v}_r)$, where $\vec{\alpha}_n=\vec{\beta}$ for all n, each $\vec{v}_j$ is empty, and $\vec{\mathfrak{p}}$ is admissible for $\vec{\beta}$. Then $\omega_n\to\omega$ in $M(X;\vec{\beta})$ with its usual topology.*

Proof This follows from Proposition 6.3. □

In other words, if a sequence ω_n in a moduli space M chain-converges to an element $\omega\in M$, then $\omega_n\to\omega$ in M provided the perturbations are admissible.

Proposition 7.2 *In the notation of Theorem 1.4, suppose that the sequence $\omega_n\in M(X^{(T(n))};\vec{\alpha}'_n)$ chain-converges to $\mathbb{V}=(\omega,\vec{v}_1,\dots,\vec{v}_r,\vec{v}'_1,\dots,\vec{v}'_{r'})$, where $\min_j T_j(n)\to\infty$. Suppose also that the perturbation parameters $\vec{\mathfrak{p}},\vec{\mathfrak{p}}'$ are admissible for each $\vec{\alpha}'_n$. Then the following hold:*

(i) *For sufficiently large n there is a smooth map $u_n\colon X^{(T(n))}\to U(1)$ such that $v_{n,j}=u_n|_{\{0\}\times Y'_j}$ satisfies $v_{n,j}(\alpha'_{n,j})=\gamma'_j$, $j=1,\dots,r'$.*

(ii) *The chain limit is unique up to gauge equivalence, ie if $\mathbb{V}$, $\mathbb{V}'$ are two chain limits of ω_n then there exists a smooth $u\colon X^\#\to U(1)$ which is translationary invariant over the ends of $X^\#$, and such that $u(\mathbb{V})=\mathbb{V}'$.*

In (i), recall that moduli spaces are labelled by critical points modulo null-homotopic gauge transformations. Note that we can arrange that the maps u_n are translationary invariant over the ends. This allows us to identify the moduli spaces $M(X;\vec{\alpha}'_n)$ and $M(X;\vec{\gamma}')$, so that we obtain a sequence $u_n(\omega_n)$, $n\gg 0$ in a *fixed* moduli space.

In (ii) we define $u(\mathbb{V})$ as follows. Let $w_j\colon \mathbb{R}\times Y_j\to \mathrm{U}(1)$ and $w'_j\colon \mathbb{R}\times Y'_j\to \mathrm{U}(1)$ be the translationary invariant maps which agree with u on $\{0\}\times Y_j$ and $\mathbb{R}_+\times Y'_j$, respectively. Let $w\colon X\to \mathrm{U}(1)$ be the map which is translationary invariant over each end and agrees with u on $X_{:1}$. Then $u(\mathbb{V})$ is the result of applying the appropriate maps w,w_j,w'_j to the various components of $u(\mathbb{V})$.

Proof (i) For simplicity we only discuss the case of one end and no necks, ie the situation of Definition 7.1. The proof in the general case is similar.

Using Condition (v) of Definition 7.1 and a simple compactness argument it is easy to see that α_n is gauge equivalent to β_k for all sufficiently large n. Moreover, Conditions (iv) and (v) of Definition 7.1 ensure that there exist $\tau, n' > 0$ such that if $n > n'$ then ω_n restricts to a genuine monopole on $(t_{n,k}+\tau,\infty)\times Y$ and on $(t_{n,j-1}+\tau, t_{n,j}-\tau)\times Y$ for $j=1,\dots,k$. It then follows from Proposition 6.3 that $v_n = u_{n,k}|_{\{0\}\times Y}$ satisfies $v_n(\alpha_n)=\beta_k$ for $n\gg 0$. (Recall again that $\alpha_n,\beta_k\in\widetilde{\mathcal{R}}_Y$ are critical points modulo null-homotopic gauge transformations, so $v_n(\alpha_n)$ depends only on the homotopy class of v_n.) Similarly, it follows from Theorems 6.1, 6.2 that $u_{n,j-1}|_{\{0\}\times Y}$ is homotopic to $u_{n,j}|_{\{0\}\times Y}$ for $j=1,\dots,r$ and $n\gg 0$, where $u_{n,0}=u_n$. Therefore, v_n extends over $X_{:0}$.

(ii) This is a simple exercise. □

7.2 Proof of Theorem 1.3

By Propositions 3.1,5.6, and 4.3, if each $\mathfrak{p}_j$ has sufficiently small C^1 norm then $\vec{\mathfrak{p}}$ will be admissible for all $\vec{\alpha}$. Choose $\vec{\mathfrak{p}}$ so that this is the case. Set

$$C_0 = -\inf_n \sum_j \lambda_j \vartheta(\alpha_{n,j}) < \infty.$$

Let S_n be a smooth representative for ω_n. The energy assumption on the asymptotic limits of S_n is unaffected if we replace S_n by $u_n(S_n)$ for some smooth $u_n\colon X\to \mathrm{U}(1)$ which is translationary invariant on $(t_n,\infty)\times Y$ for some $t_n>0$. After passing to a subsequence we can therefore, by Proposition 5.3, assume that S_n c-converges over X to some monopole S' which is in temporal gauge over the ends. Because $\vec{\mathfrak{p}}$ is admissible we have that

$$\partial_t \vartheta(S_n|_{\{t\}\times Y_j}) \le 0$$

for all j,n and $t\ge 0$. From the energy bound (30) we then see that S' must have finite energy. Let γ_j denote the asymptotic limit of S' over the end $\mathbb{R}_+\times Y_j$ as guaranteed by Proposition 4.4. Then

$$\limsup_n \vartheta(\alpha_{n,j}) \le \vartheta(\gamma_j)$$

for each j. Hence there is a constant $C_2<\infty$ such that for $h=1,\dots,r$ and all n one has

$$C_2 + \lambda_h\vartheta(\alpha_{n,h}) \ge \sum_j \lambda_j\vartheta(\alpha_{n,j}) \ge -C_0.$$

Consequently,

$$\sup_{n,j} |\vartheta(\alpha_{n,j})| < \infty.$$

For the remainder of this proof we fix j and focus on one end $\mathbb{R}\times Y_j$. For simplicity we drop j from notation and write Y, α_n instead of $Y_j, \alpha_{n,j}$ etc.

After passing to a subsequence we may arrange that $\vartheta(\alpha_n)$ has the same value L for all n (here we use Condition (O1)). If $\vartheta(\gamma) = L$ then we set $k = 0$ and the proof is complete. Now suppose $\vartheta(\gamma) > L$. Then there is an n' such that $\partial_t\vartheta(S_n(t)) < 0$ for all $n \geq n'$, $t \geq 0$. Set

$$\delta = \frac{1}{2}\min\{|x-y| \, : \, x, y \text{ are distinct critical values of } \vartheta\colon \widetilde{\mathcal{B}}_Y \to \mathbb{R}\}.$$

The minimum exists by (O1). For sufficiently large n we define $t_{n,1} \gg 0$ implicitly by

$$\vartheta(S_n(t_{n,1})) = \vartheta(\gamma) - \delta.$$

It is clear that $t_{n,1} \to \infty$ as $n \to \infty$. Moreover, Definition 7.1 (iv) must hold for $j = 1$. For otherwise we can find $\epsilon > 0$ and sequences τ_ℓ, n_ℓ with $\tau_\ell, n_\ell \to \infty$ as $\ell \to \infty$, such that

$$\vartheta(S_{n_\ell}(\tau_\ell)) - \vartheta(S_{n_\ell}(t_{n_\ell,1} - \tau_\ell)) > \epsilon \tag{40}$$

for every ℓ. As in the proof of Claim 6.1 there are smooth $\widetilde{u}_\ell\colon \mathbb{R}\times Y \to \mathrm{U}(1)$ satisfying $\vartheta \circ \widetilde{u}_\ell(0) = \vartheta$ such that a subsequence of $\widetilde{u}_\ell(\mathcal{T}^*_{t_{n_\ell,1}} S_{n_\ell})$ c-converges over $\mathbb{R}\times Y$ to a finite energy monopole $\widetilde{S}$ in temporal gauge. The asymptotic limit $\widetilde{\gamma}$ of $\widetilde{S}$ at $-\infty$ must satisfy

$$\epsilon \leq \vartheta(\gamma) - \vartheta(\widetilde{\gamma}) < \delta,$$

where the first inequality follows from (40). This contradicts the choice of δ. Therefore, Definition 7.1 (iv) holds for $j = 1$ as claimed.

After passing to a subsequence we can find $u_{n,1}\colon \mathbb{R}\times Y \to \mathrm{U}(1)$ such that $u_{n,1}(\mathcal{T}^*_{t_{n,1}} S_n)$ c-converges over $\mathbb{R}\times Y$ to some finite energy monopole S_1' in temporal gauge. Let $\beta_1^\pm$ denote the limit of S_1' at $\pm\infty$. A simple compactness argument shows that γ and β_1^- are gauge equivalent, so we can arrange that $\gamma = \beta_1^-$ by modifying the $u_{n,1}$ by a fixed gauge transformation $\mathbb{R}\times Y \to \mathrm{U}(1)$. As in the proof of Proposition 7.2 (i) we see that $u_{n,1}$ must be null-homotopic for all sufficiently large n. Hence $\vartheta(\beta_1^+) \geq L$. If $\vartheta(\beta_1^+) = L$ then we set $k = 1$ and the proof is finished. If on the other hand $\vartheta(\beta_1^+) > L$ then we continue the above process. The process ends when, after passing successively to subsequences and choosing $u_{n,j}, t_{n,j}, \beta_j^\pm$ for $j = 1, \dots, k$ (where $\beta_{j-1}^+ = \beta_j^-$, and $u_{n,j}$ is null-homotopic for $n \gg 0$) we have $\vartheta(\beta_k^+) = L$. This must occur after finitely many steps; in fact $k \leq (2\delta)^{-1}(\vartheta(\gamma) - L)$. ◻

7.3 Proof of Theorem 1.4

For simplicity we first consider the case when there is exactly one neck (ie $r=1$), and we write $Y=Y_1$ etc. We will make repeated use of the local compactness results proved earlier.

Let S_n be a smooth representative of ω_n. After passing to a subsequence we can find smooth maps

$$u_n\colon\ X^{(T(n))}\setminus(\{0\}\times Y)\to \mathrm{U}(1)$$

such that $\widetilde{S}_n=u_n(S_n)$ c-converges over X to some finite energy monopole S' which is in temporal gauge over the ends. Introduce the temporary notation $S_n(t)=S_n|_{\{t\}\times Y}$, and similarly for $\widetilde{S}_n$ and u_n. For $0\le\tau<T(n)$ set

$$\Theta_{\tau,n}=\vartheta(S_n(-T(n)+\tau))-\vartheta(S_n(T(n)-\tau)).$$

Let $u_n^\pm=u_n(\pm T(n))$ and

$$I_n^\pm=2\pi\int_Y\widetilde{\eta}_j\wedge[u_n^\pm],$$

cf Equation (1). Since $\Theta_{0,n}$ is bounded as $n\to\infty$, it follows that $I_n^+-I_n^-$ is bounded as $n\to\infty$. By Condition (O1) there is a $q>0$ such that $qI_n^\pm$ is integral for all n. Hence we can arrange, by passing to a subsequence, that $I_n^+-I_n^-$ is constant. In particular,

$$I_n^+-I_1^+=I_n^--I_1^-.$$

Choose a smooth map $w\colon\ X\to\mathrm{U}(1)$ which is translationary invariant over the ends, and homotopic to u_1^{-1} over $X_{:0}$. After replacing u_n by wu_n for every n we then obtain $I_n^+=I_n^-$. Set $I_n=I_n^\pm$. We now have

$$\Theta_{\tau,n}=\vartheta(\widetilde{S}_n(-T(n)))-\vartheta(\widetilde{S}_n(T(n))).$$

Let β_0,β' denote the asymptotic limits of S' over the ends $\iota^+(\mathbb{R}_+\times Y)$ and $\iota^-(\mathbb{R}_+\times Y)$, respectively. Set

$$L=\lim_{\tau\to\infty}\lim_{n\to\infty}\Theta_{\tau,n}=\vartheta(\beta_0)-\vartheta(\beta').$$

Since $\Theta_{\tau,n}\ge0$ for $\tau\ge0$ we have $L\ge0$.

Suppose $L=0$. Then a simple compactness argument shows that there is a smooth $v\colon\ Y\to\mathrm{U}(1)$ such that $v(\beta_0)=\beta'$. Moreover, there is an n_0 such that $v\,u_n^-\sim u_n^+$ for $n\ge n_0$, where $\sim$ means 'homotopic'. Therefore, we can find a smooth $z\colon\ X\to\mathrm{U}(1)$ which is translationary invariant over the ends and homotopic to $u_{n_0}^{-1}$ over $X_{:0}$, such that after replacing u_n by zu_n for every

n we have that $\beta_0 = \beta'$ and $u_n^+ \sim u_n^-$. In that case we can in fact assume that u_n is a smooth map $X^{(T(n))} \to \mathrm{U}(1)$. The remainder of the proof when $L = 0$ (dealing with convergence over the ends) is now a repetition of the proof of Theorem 1.3.

We now turn to the case $L > 0$. For large n we must then have $\partial_t S_n(t) < 0$ for $|t| \le T(n)$. Let δ be as in the proof of Theorem 1.3. We define $t_{n,1} \in (-T(n), T(n))$ implicitly for large n by

$$\vartheta(\beta_0) = \vartheta(S_n(t_{n,1})) + I_n + \delta.$$

Then $|t_{n,1} \pm T(n)| \to \infty$ as $n \to \infty$. As in the proof of Theorem 1.3 one sees that

$$\limsup_{n\to\infty} [\vartheta(S_n(-T(n)+\tau)) - \vartheta(S_n(t_{n,1}-\tau))] \to 0$$

as $\tau \to \infty$, and after passing to a subsequence we can find smooth $u_{n,1}\colon \mathbb{R} \times Y \to \mathrm{U}(1)$ such that $u_{n,1}(\mathcal{T}^*_{t_{n,1}} S_n)$ c-converges over $\mathbb{R} \times Y$ to a finite energy monopole S'_1 in temporal gauge whose asymptotic limit at $-\infty$ is β_0. Let β_1 denote the asymptotic limit of S'_1 at ∞. We now repeat the above process. The process ends when, after passing successively to subsequences and choosing $u_{n,j}, t_{n,j}, \beta_j$ for $j = 1, \dots, k$ one has that

$$\limsup_{n\to\infty} [\vartheta(S_n(t_{n,k}+\tau)) - \vartheta(S_n(T(n)-\tau))] \to 0$$

as $\tau \to \infty$. As in the case $L = 0$ one sees that β_k, β' must be gauge equivalent, and after modifying $u_n, u_{n,j}$ one can arrange that $\beta_k = \beta'$. This establishes chain-convergence over the neck. As in the case $L = 0$ we can in fact assume that u_n is a smooth map $X^{(T(n))} \to \mathrm{U}(1)$, and the rest of the proof when $L > 0$ is again a repetition of the proof of Theorem 1.3.

In the case of multiple necks one applies the above argument successively to each neck. In this case, too, after passing to a subsequence one ends up with smooth maps $u_n\colon X^{(T(n))} \to \mathrm{U}(1)$ such that $u_n(S_n)$ c-converges over X. One can then deal with convergence over the ends as before. □

8 Transversality

We will address two kinds of transversality problems: non-degeneracy of critical points of the Chern–Simons–Dirac functional, and regularity of moduli spaces over 4–manifolds.

In this section we do not assume Condition (O1).

Recall that a subset of a topological space Z is called *residual* if it contains a countable intersection of dense open subsets of Z.

8.1 Non-degeneracy of critical points

Lemma 8.1 *Let Y be a closed, connected, Riemannian spin^c 3–manifold and η any closed (smooth) 2–form on Y. Let G^* be the set of all $\nu \in \Omega^1(Y)$ such that all irreducible critical points of $\vartheta_{\eta+d\nu}$ are non-degenerate. Then $G^* \subset \Omega^1(Y)$ is residual, hence dense (with respect to the C^∞ topology).*

Proof The proof is a slight modification of the argument in [15]. For $2 \le k \le \infty$ and $\delta > 0$ let $W_{k,\delta}$ be the space of all 1–forms ν on Y of class C^k which satisfy $\|d\nu\|_{C^1} < \delta$. Let $W_{k,\delta}$ have the C^k topology. For $1 \le k < \infty$ we define a $\mathcal{G}$–equivariant smooth map

$$\begin{aligned}\Upsilon_k\colon\ \mathcal{C}^* \times L^2_1(Y;i\mathbb{R}) \times W_{k,\delta} &\to L^2(Y;i\Lambda^1 \oplus \mathbb{S}),\\ (B,\Psi,\xi,\nu) &\mapsto \mathcal{I}_\Psi\xi + \nabla\vartheta_{\eta+d\nu}(B,\Psi),\end{aligned}$$

where $\mathcal{G}$ acts trivially on forms, and by multiplication on spinors. Now if $\Upsilon_k(B,\Psi,\xi,\nu) = 0$ then

$$\|\mathcal{I}_\Psi\xi\|_2^2 = -\int_Y \langle\nabla\vartheta_{\eta+d\nu}(B,\Psi),\mathcal{I}_\Psi\xi\rangle = 0$$

by (12), which implies $\xi = 0$ since $\Psi \ne 0$. The derivative of Υ_k at a point $x = (B,\Psi,0,\nu)$ is

$$D\Upsilon_k(x)(b,\psi,f,v) = H_{(B,\Psi)}(b,\psi) + \mathcal{I}_\Psi f + (i{*}dv,0). \tag{41}$$

Let (B,Ψ) be any irreducible critical point of ϑ_η. We will show that $P = D\Upsilon_k(B,\Psi,0,0)$ is surjective. Note that altering (B,Ψ) by an L^2_2 gauge transformation u has the effect of replacing P by uPu^{-1}. We may therefore assume that (B,Ψ) is smooth. Since $P_1 = \mathcal{I}_\Psi + H_{(B,\Psi)}$ has surjective symbol, the image of the induced operator $L^2_1 \to L^2$ is closed and has finite codimension. The same must then hold for $\mathrm{im}(P)$. Suppose $(b,\psi) \in L^2$ is orthogonal to $\mathrm{im}(P)$, ie $db = 0$ and $P_1^*(b,\psi) = 0$. The second equation implies that b and ψ are smooth, by elliptic regularity. Writing out the equations we find as in [15] that on the complement of $\Psi^{-1}(0)$ we have $-b = idr$ for some smooth function $r\colon Y \setminus \Psi^{-1}(0) \to \mathbb{R}$. We now invoke a result of Bär [4] which says that, because B is smooth and $\Psi \not\equiv 0$, the equation $\partial_B\Psi = 0$ implies that the zero-set of Ψ is contained in a countable union of smooth 1–dimensional submanifolds of Y. In particular, any smooth loop in Y can be deformed slightly so that it misses $\Psi^{-1}(0)$. Hence b is exact. From Bär's theorem (or unique continuation for ∂_B, which holds when B is of class C^1, see [19]) we also deduce that the complement of $\Psi^{-1}(0)$ is dense and connected. Therefore, f has a smooth extension to all of Y, and as in [15] this gives $(b,\psi) = 0$. Hence P is surjective.

Consider now the vector bundle

$$E=(\mathcal{C}^*\underset{\mathcal{G}}{\times}L^2(Y;i\Lambda^1\oplus\mathbb{S}))\times L^2_1(Y;i\mathbb{R})\to\mathcal{B}^*\times L^2_1(Y;i\mathbb{R}).$$

For $1\le k<\infty$ the map Υ_k defines a smooth section $\sigma_{k,\delta}$ of the bundle

$$E\times W_{k,\delta}\to\mathcal{B}^*\times L^2_1(Y;i\mathbb{R})\times W_{k,\delta}.$$

By the local slice theorem, a zero of Υ_k is a regular point of Υ_k if and only if the corresponding zero of $\sigma_{k,\delta}$ is regular. Since surjectivity is an open property for bounded operators between Banach spaces, a simple compactness argument shows that the zero-set of $\sigma_{2,\delta}$ is regular when $\delta>0$ is sufficiently small. Fix such a δ. Observe that the question of whether the operator (41) is surjective for a given x is independent of k. Therefore, the zero-set $M_{k,\delta}$ of $\sigma_{k,\delta}$ is regular for $2\le k<\infty$. In the remainder of the proof assume $k\ge 2$.

For any $\rho>0$ let $\mathcal{B}_\rho$ be the set of elements $[B,\Psi]\in\mathcal{B}$ satisfying

$$\int_Y|\Psi|\ge\rho.$$

Define $M_{k,\delta,\rho}\subset M_{k,\delta}$ similarly. For any given ν, the formula for Υ_k defines a Fredholm section of E which we denote by σ_ν. Let $G_{k,\delta,\rho}$ be the set of those $\nu\in W_{k,\delta}$ such that σ_ν has only regular zeros in $\mathcal{B}_\rho\times\{0\}$. For $k<\infty$ let

$$\pi\colon\ M_{k,\delta}\to W_{k,\delta}$$

be the projection, and $\Sigma\subset M_{k,\delta}$ the closed subset consisting of all singular points of π. A compactness argument shows that π restricts to a closed map on $M_{k,\delta,\rho}$, hence

$$G_{k,\delta,\rho}=W_{k,\delta}\setminus\pi(M_{k,\delta,\rho}\cap\Sigma)$$

is open in $W_{k,\delta}$. On the other hand, applying the Sard–Smale theorem as in [11, Section 4.3] we see that $G_{k,\delta,\rho}$ is residual (hence dense) in $W_{k,\delta}$. Because $W_{\infty,\delta}$ is dense in $W_{k,\delta}$, we deduce that $G_{\infty,\delta,\rho}$ is open and dense in $W_{\infty,\delta}$. But then

$$\bigcap_{n\in\mathbb{N}}G_{\infty,\delta,\frac{1}{n}}$$

is residual in $W_{\infty,\delta}$, and this is the set of all $\nu\in W_{\infty,\delta}$ such that σ_ν has only regular zeros.

An irreducible critical point of $\vartheta_{\eta+d\nu}$ is non-degenerate if and only if the corresponding zero of σ_ν is regular. Thus we have proved that among all smooth 1–forms ν with $\|d\nu\|_{C^1}<\delta$, those ν for which all irreducible critical points of $\vartheta_{\eta+d\nu}$ are non-degenerate make up a residual subset in the C^∞ topology. The same must hold if η is replaced with $\eta + d\nu$ for any $\nu \in \Omega^1(Y)$, so we conclude that G^* is locally residual in $\Omega^1(Y)$, ie any point in G^* has a neighbourhood V such that $G^* \cap V$ is residual in V. Hence G^* is residual in $\Omega^1(Y)$. (This last implication holds if $\Omega^1(Y)$ is replaced with any second countable, regular space.) □

Proposition 8.1 *Let Y be a closed, connected, Riemannian spin^c 3–manifold and η any closed 2–form on Y such that either $b_1(Y) = 0$ or $\widetilde{\eta} \neq 0$. Let G be the set of all $\nu \in \Omega^1(Y)$ such that all critical points of $\vartheta_{\eta+d\nu}$ are non-degenerate. Then G is open and dense in $\Omega^1(Y)$ with respect to the C^∞ topology.*

Proof A compactness argument shows that G is open. If $b_1(Y) > 0$ then $\vartheta_{\eta+d\nu}$ has no reducible critical points and the result follows from Lemma 8.1.

Now suppose $b_1(Y) = 0$. Then we may assume $\eta = 0$. For $0 \le k \le \infty$ let W_k be the space of 1–forms on Y of class C^k, with the C^k topology. If $\nu \in W_1$ then $\vartheta_{d\nu}$ has up to gauge equivalence a unique reducible critical point, represented by $(B - i\nu, 0)$ for any smooth spin connection B over Y with $\check{B}$ flat. This critical point is non-degenerate precisely when

$$\ker(\partial_{B-i\nu}) = 0 \quad \text{in } L^2_1. \tag{42}$$

Let G'_k be the set of all $\nu \in W_k$ such that (42) holds. This is clearly an open subset of W_k. The last part of the proof of [15, Proposition 3] shows that G'_k is residual (hence dense) in W_k for $2 \le k < \infty$. Hence G'_∞ is open and dense in $\Omega^1(Y) = W_\infty$. Now apply Lemma 8.1. □

Marcolli [25] proved a weaker result in the case $b_1(Y) > 0$, allowing η to vary freely among the closed 2–forms.

8.2 Regularity of moduli spaces

The following lemma will provide us with suitable Banach spaces of perturbation forms.

Lemma 8.2 *Let X be a smooth n–manifold, $K \subset X$ a compact, codimension 0 submanifold, and $E \to X$ a vector bundle. Then there exists a separable Banach space W consisting of smooth sections of E supported in K, such that the following hold:*

(i) *The natural map $W \to \Gamma(E|_K)$ is continuous with respect to the C^∞ topology on $\Gamma(E|_K)$.*

(ii) *For every point $x \in \mathrm{int}(K)$ and every $v \in E_x$ there exists a section $s \in \Gamma(E)$ with $s(x) = v$ and a smooth embedding $g\colon \mathbb{R}^n \to X$ with $g(0) = x$ such that for arbitrarily small $\epsilon > 0$ there are elements of W of the form fs where $f\colon X \to [0,1]$ is a smooth function which vanishes outside $g(\mathbb{R}^n)$ and satisfies*

$$f(g(z)) = \begin{cases} 0, & |z| \geq 2\epsilon, \\ 1, & |z| \leq \epsilon. \end{cases}$$

Proof Fix connections in E and TX, and a Euclidean metric on E. For any sequence $a = (a_0, a_1, \dots)$ of positive real numbers and any $s \in \Gamma(E)$ set

$$\|s\|_a = \sum_{k=0}^{\infty} a_k \|\nabla^k s\|_\infty$$

and

$$W_a = \{s \in \Gamma(E) \,:\, \mathrm{supp}(s) \subset K,\ \|s\|_a < \infty\}.$$

Then $W = W_a$, equipped with the norm $\|\cdot\|_a$, clearly satisfies (i) for any a. We claim that one can choose a such that (ii) also holds. To see this, first observe that there is a finite dimensional subspace $V \subset \Gamma(E)$ such that

$$V \to E_x, \quad s \mapsto s(x)$$

is surjective for every $x \in K$. Fix a smooth function $b\colon \mathbb{R} \to [0,1]$ satisfying

$$b(t) = \begin{cases} 1, & t \leq 1, \\ 0, & t \geq 4. \end{cases}$$

We use functions f that in local coordinates have the form

$$f_r(z) = b(r|z|^2),$$

where $r \gg 0$. Note that for each k there is a bound $\|f_r\|_{C^k} \leq \mathrm{const} \cdot r^k$ where the constant is independent of $r \geq 1$. It is now easy to see that a suitable sequence a can be found. ◻

In the next two propositions, $X, \vec{\alpha}, \mu$ will be as in Subsection 1.3. Let $K \subset X$ be any non-empty compact codimension 0 submanifold. Let W be a Banach space of smooth self-dual 2–forms on X supported in K, as provided by Lemma 8.2. The following proposition will be used in the proof of Theorem 1.2.

Proposition 8.2 *In the above situation, let G be set of all $\nu \in W$ such that all irreducible points of the moduli space $M(X; \vec{\alpha}; \mu + \nu; 0)$ are regular (here $\mathfrak{p}_j = 0$ for each j). Then $G \subset W$ is residual, hence dense.*

There is another version of this proposition where W is replaced with the Fréchet space of all smooth self-dual 2–forms on X supported in K, at least if one assumes that (O1) holds for each pair Y_j, η_j and that (A) holds for $X, \widetilde{\eta}_j, \lambda_j$. The reason for the extra assumptions is that the proof then seems to require global compactness results (cf the proof of Lemma 8.1).

Proof We may assume X is connected. Let $\widetilde{\Theta}$ be as in Subsection 3.4. Then

$$(S,\nu) \mapsto \widetilde{\Theta}(S, \mu+\nu, 0)$$

defines a smooth map

$$f\colon\, \mathcal{C}^* \times W \to L^{p,w}(X; i\Lambda^+ \oplus \mathbb{S}^-),$$

where $\mathcal{C}^* = \mathcal{C}^*(X;\vec{\alpha})$. We will show that 0 is a regular value of f. Suppose $f(S,\nu)=0$ and write $S=(A,\Phi)$. We must show that the derivative $P = Df(S,\nu)$ is surjective. Because of the gauge equivariance of f we may assume that S is smooth. Let P_1 denote the derivative of $f(\,\cdot\,,\nu)$ at S. Since the image of P_1 in $L^{p,w}$ is closed and has finite codimension, the same holds for the image of P. Let p' be the exponent conjugate to p and suppose $(z,\psi) \in L^{p',-w}(X; i\Lambda^+ \oplus \mathbb{S}^-)$ is L^2 orthogonal to the image of P, ie

$$\int_X \langle P(a,\phi,\nu'), (z,\psi)\rangle = 0$$

for all $(a,\phi) \in L^{p,w}_1$ and $\nu' \in W$. Taking $\nu'=0$ we see that $P_1^*(z,\psi)=0$. Since P_1^* has injective symbol, z,ψ must be smooth. On the other hand, taking $a,\phi=0$ and varying ν' we find that $z|_K = 0$ by choice of W. By assumption, Φ is not identically zero. Since $D_A\Phi = 0$, the unique continuation theorem in [19] applied to D_A^2 says that Φ cannot vanish in any non-empty open set. Hence Φ must be non-zero at some point x in the interior of K. Varying a alone near x one sees that ψ vanishes in some neighbourhood of x. But $P_1P_1^*$ has the same symbol as $D_A^2 \oplus d^+(d^+)^*$, so another application of the same unique continuation theorem shows that $(z,\psi)=0$. Hence P is surjective.

Consider now the vector bundle

$$E = \mathcal{C}^* \times_{\mathcal{G}} L^{p,w}(X; i\Lambda^+ \oplus \mathbb{S}^-)$$

over $\mathcal{B}^*$. The map f defines a smooth section σ of the bundle

$$E \times W \to \mathcal{B}^* \times W.$$

Because of the local slice theorem and the gauge equivariance of f, the fact that 0 is a regular value of f means precisely that σ is transverse to the zero-section. Since $\sigma(\,\cdot\,,\nu)$ is a Fredholm section of E for any ν, the proposition follows by another application of the Sard–Smale theorem. $\square$

We will now establish transversality results for moduli spaces of the form $M(X,\vec{\alpha})$ or $M(\alpha,\beta)$ involving perturbations of the kind discussed in Subsection 3.3. For the time being we limit ourselves to the case where the 3–manifolds Y, Y_j are all rational homology spheres. We will use functions h_S that are a small modification of those in [15]. To define these, let Y be a closed Riemannian spin^c 3–manifold satisfying $b_1(Y)=0$, and ϑ the Chern–Simons–Dirac functional on Y defined by some closed 2–form η. Choose a smooth, nonnegative function $\chi\colon \mathbb{R}\to\mathbb{R}$ which is supported in the interval $(-\frac{1}{4},\frac{1}{4})$ and satisfies $\int\chi=1$. If S is any L^2_1 configuration over a band $(a-\frac{1}{4},b+\frac{1}{4})$ where $a\le b$ define the smooth function $\widetilde{\vartheta}_S\colon [a,b]\to\mathbb{R}$ by

$$\widetilde{\vartheta}_S(T)=\int_{\mathbb{R}}\chi(T-t)\vartheta(S_t)\,dt,$$

where we interpret the right hand side as an integral over $\mathbb{R}\times Y$. A simple exercise, using the Sobolev embedding theorem, shows that if $S_n\to S$ weakly in L^2_1 over $(a-\frac{1}{4},b+\frac{1}{4})\times Y$ then $\widetilde{\vartheta}_{S_n}\to\widetilde{\vartheta}_S$ in C^∞ over $[a,b]$.

Choose a smooth function $c\colon \mathbb{R}\to\mathbb{R}$ with the following properties:

- $c'>0$,
- c and all its derivatives are bounded,
- $c(t)=t$ for all critical values t of ϑ,

where c' is the derivative of c. The last condition is added only for convenience.

For any L^2_1 configuration S over $(a-\frac{1}{2},b+\frac{1}{2})\times Y$ with $a\le b$ define

$$h_S(t)=\int_{\mathbb{R}}\chi(t_1)c(\widetilde{\vartheta}_S(t-t_1))dt_1.$$

It is easy to verify that h_S satisfies the properties (P1)–(P3).

It remains to choose Ξ and $\mathfrak{P}$. Choose one compact subinterval (with nonempty interior) of each bounded connected component of $\mathbb{R}\setminus\mathrm{crit}(\vartheta)$, where $\mathrm{crit}(\vartheta)$ is the set of critical values of ϑ. Let Ξ be the union of these compact subintervals. Let $\mathfrak{P}=\mathfrak{P}_Y$ be a Banach space of 2–forms on $\mathbb{R}\times Y$ supported in $\Xi\times Y$ as provided by Lemma 8.2.

We now return to the situation described in the paragraph preceding Proposition 8.2. Let $W'\subset W$ be the open subset consisting of those elements ν that satisfy $\|\nu\|_{C^1}<1$. Let Π_δ denote the set of all $\vec{\mathfrak{p}}=(\mathfrak{p}_1,\dots,\mathfrak{p}_r)$ where $\mathfrak{p}_j\in\mathfrak{P}_{Y_j}$ and $\|\mathfrak{p}_j\|_{C^1}<\delta$ for each j.

Proposition 8.3 *Suppose each Y_j is a rational homology sphere and $K \subset X_{:0}$. Then there exists a $\delta > 0$ such that the following holds. Let G be the set of all $(\nu, \vec{\mathfrak{p}}) \in W' \times \Pi_\delta$ such that every irreducible point of the moduli space $M(X; \vec{\alpha}; \mu + \nu; \vec{\mathfrak{p}})$ is regular. Then $G \subset W' \times \Pi_\delta$ is residual, hence dense.*

It seems necessary here to let $\vec{\mathfrak{p}}$ vary as well, because if any of the $\mathfrak{p}_j$ is non-zero then the linearization of the monopole map is no longer a differential operator, and it is not clear whether one can appeal to unique continuation as in the proof of Proposition 8.2.

Proof To simplify notation assume $r = 1$ and set $Y = Y_1$, $\alpha = \alpha_1$ etc. (The proof in the general case is similar.) Note that (A) is trivially satisfied, since each Y_j is a rational homology sphere. Therefore, by Propositions 4.3 and 5.6, if $\delta > 0$ is sufficiently small then for any $(\nu, \mathfrak{p}) \in W' \times \Pi_\delta$ and $[S] \in M(X; \alpha; \mu + \nu; \mathfrak{p})$ one has that either

(i) $[S_t] = \alpha$ for $t \geq 0$, or

(ii) $\partial_t \vartheta(S_t) < 0$ for $t \geq 0$.

As in the proof of Proposition 8.2 it suffices to prove that 0 is a regular value of the smooth map

$$\begin{aligned} \widetilde{f}\colon \mathcal{C}^* \times W' \times \Pi_\delta &\to L^{p,w}, \\ (S, \nu, \mathfrak{p}) &\mapsto \widetilde{\Theta}(S, \mu + \nu, \mathfrak{p}). \end{aligned}$$

The smoothness of the perturbation term $g(S, \mathfrak{p}) = \mathfrak{q} h_{S,\mathfrak{p}}$ follows from the smoothness of the map (19), for by (P1) there exist a t_0 and a neighbourhood $U \subset \mathcal{C}$ of S such that $h_{S'}(t) \notin \Xi$ for all $t > t_0$ and $S' \in U$.

Now suppose $\widetilde{f}(S, \nu, \mathfrak{p}) = 0$ and $(z, \psi) \in L^{p', -w}$ is orthogonal to the image of $D\widetilde{f}(S, \nu, \mathfrak{p})$. We will show that z is orthogonal to the image of $T = Dg(S, \mathfrak{p})$, or equivalently, that (z, ψ) is orthogonal to the image of $Df(S, \nu)$, where $f = \widetilde{f} - g$ as before. The latter implies $(z, \psi) = 0$ by the proof of Proposition 8.2.

Let $h_S\colon [\frac{1}{2}, \infty) \to \mathbb{R}$ be defined in terms of the restriction of S to $\mathbb{R}_+ \times Y$. If $h_S(J) \subset \mathbb{R} \setminus \Xi$ for some compact interval J then by (P1) one has that $h_{S'}(J) \subset \mathbb{R} \setminus \Xi$ for all S' in some neighbourhood of S in $\mathcal{C}$. Therefore, all elements of $\mathrm{im}(T)$ vanish on $h_S^{-1}(\mathbb{R} \setminus \Xi) \times Y$.

We now digress to recall that if u is any locally integrable function on $\mathbb{R}^n$ then the complement of the Lebesgue set of u has measure zero, and if v is any continuous function on $\mathbb{R}^n$ then any Lebesgue point of u is also a Lebesgue point of uv. The notion of Lebesgue set also makes sense for sections τ of a vector bundle of finite rank over a finite dimensional smooth manifold M. In that case a point $x \in M$ is called a Lebesgue point of τ if it is a Lebesgue point in the usual sense for some (hence any) choice of local coordinates and local trivialization of the bundle around x.

Returning to our main discussion, there are now two cases: If (i) above holds then $h_S(t) = \vartheta(\alpha) \notin \Xi$ for $t \geq \frac{1}{2}$, whence $T = 0$ and we are done (recall the overall assumption $\mathfrak{q}^{-1}(0) = X_{:\frac{3}{2}}$ made in Subsection 3.4). Otherwise (ii) must hold. In that case we have $\partial_t c(\widetilde{\vartheta}(t)) < 0$ for $t \geq \frac{1}{4}$ and $\partial_t h_S(t) < 0$ for $t \geq \frac{1}{2}$. Since z is orthogonal to $\mathfrak{q} h_{S,\mathfrak{p}'}$ for all $\mathfrak{p}' \in \mathfrak{P}_Y$ we conclude that $z(t,y) = 0$ for every Lebesgue point (t,y) of z with $t > \frac{3}{2}$ and $h_S(t) \in \operatorname{int}(\Xi)$. Since $h_S^{-1}(\partial\Xi) \cap (\frac{3}{2},\infty)$ is a finite set, z must vanish almost everywhere in $[h_S^{-1}(\Xi) \cap (\frac{3}{2},\infty)] \times Y$. Combining this with our earlier result we deduce that z is orthogonal to $\operatorname{im}(T)$. □

In the next proposition (which is similar to [15, Proposition 5]) let Π_δ be as above with $r = 1$, and set $Y = Y_1$.

Proposition 8.4 *In the situation of Subsection 1.2, suppose Y is a rational homology sphere and $\alpha, \beta \in \mathcal{R}_Y = \widetilde{\mathcal{R}}_Y$. Then there exists a $\delta > 0$ such that the following holds. Let G be the set of all $\mathfrak{p} \in \Pi_\delta$ such that every point in $M(\alpha,\beta;\mathfrak{p})$ is regular. Then $G \subset \Pi_\delta$ is residual, hence dense.*

Proof If $\alpha = \beta$ then an application of Proposition 4.3 shows that if $\|\mathfrak{p}\|_{C^1}$ is sufficiently small then $M(\alpha,\beta;\mathfrak{p})$ consists of a single point represented by $\underline{\alpha}$, which is regular because α is non-degenerate.

If $\alpha \neq \beta$ and $\|\mathfrak{p}\|_{C^1}$ is sufficiently small then for any $[S] \in M(\alpha,\beta;\mathfrak{p})$ one has $\partial_t \vartheta(S_t) < 0$ for all t. Moreover, the moduli space contains no reducibles, since α, β cannot both be reducible. The proof now runs along the same lines as that of Proposition 8.3. Note that the choice of Ξ is now essential: it ensures that $\operatorname{im}(h_S) = (\vartheta(\alpha), \vartheta(\beta))$ contains interior points of Ξ. □

9 Proof of Theorems 1.1 and 1.2

In these proofs we will only use genuine monopoles.

Proof of Theorem 1.1 We may assume Y is connected. Let η be a closed non-exact 2–form on Y which is the restriction of a closed form on Z. Let Y have a metric of positive scalar curvature. If $s \neq 0$ is a small real number then $\vartheta_{s\eta}$ will have no irreducible critical points, by the *a priori* estimate on the spinor fields and the positive scalar curvature assumption. If in addition $s[\eta]-\pi c_1(\mathcal{L}_Y)\neq 0$ then $\vartheta_{s\eta}$ will have no reducible critical points either.

Choose a spin^c Riemannian 4–manifold X as in Subsection 1.4, with $r=1$, $r'=0$, such that there exists a diffeomorphism $X^\# \to Z$ which maps $\{0\}\times Y_1$ isometrically onto Y. Let η_1 be the pull-back of $s\eta$. Then (B1) is satisfied (but perhaps not (B2)), so it follows from Theorem 1.4 that $M(X^{(T)})$ is empty for $T\gg 0$. $\square$

We will now define an invariant h for closed spin^c 3–manifolds Y that satisfy $b_1(Y)=0$ and admit metrics with positive scalar curvature. Let g be such a metric on Y. Recall that for the unperturbed Chern–Simons–Dirac functional ϑ the space $\mathcal{R}_Y$ of critical points modulo gauge equivalence consists of a single point θ, which is reducible. Let $(B,0)$ be a representative for θ. Let $Y_1,\dots,Y_r$ be the connected components of Y and choose a spin^c Riemannian 4–manifold X with tubular ends $\mathbb{R}_+\times Y_j$, $j=1,\dots,r$ (in the sense of Subsection 1.3) and a smooth spin connection A over X such that the restriction of $\check{A}$ to $\mathbb{R}_+\times Y$ is equal to the pull-back of $\check{B}$. (The notation here is explained in Subsection 3.1.) Define

$$\begin{aligned} h(Y,g) &= \mathrm{ind}_{\mathbb{C}}(D_A)-\frac{1}{8}(c_1(\mathcal{L}_X)^2-\sigma(X)) \\ &= \frac{1}{2}(\dim\, M(X;\theta)-d(X)+b_0(X)), \end{aligned}$$

where $D_A\colon\ L^2_1\to L^2$, 'dim' is the expected dimension, and $d(X)$ is the quantity defined in Subsection 1.1. Since $\mathrm{ind}_{\mathbb{C}}(D_A)=\frac{1}{8}(c_1(\mathcal{L}_X)^2-\sigma(X))$ when X is closed, it follows easily from the addition formula for the index (see [10, Proposition 3.9]) that $h(Y,g)$ is independent of X and that

$$h(-Y,g)=-h(Y,g).$$

Clearly,

$$h(Y,g)=\sum_j h(Y_j,g_j),$$

where g_j is the restriction of g to Y_j. To show that $h(Y,g)$ is independent of g we may therefore assume Y is connected. Suppose g' is another positive scalar curvature metric on Y and consider the spin^c Riemannian manifold $X=\mathbb{R}\times Y$ where the metric agrees with $1\times g$ on $(-\infty,-1]\times Y$ and with $1\times g'$ on $[1,\infty)\times Y$. Theorem 1.3 and Proposition 7.2 (ii) say that $M(X;\theta,\theta)$ is compact. This moduli space contains one reducible point, and arguing as in the proof of [15, Theorem 6] one sees that the moduli space must have non-positive (odd) dimension. Thus,

$$h(Y,g') + h(-Y,g) = \frac{1}{2}(\dim M(X;\theta,\theta)+1) \le 0.$$

This shows $h(Y) = h(Y,g)$ is independent of g.

Proof of Theorem 1.2 Let each Y_j have a positive scalar curvature metric. Choose a spin^c Riemannian 4–manifold X as in Subsection 1.4, with $r' = 0$ and with the same r, such that there exists a diffeomorphism $f\colon X^{\#} \to Z$ which maps $\{0\}\times Y_j$ isometrically onto Y_j. Then (B1) is satisfied (but perhaps not (B2)). Let X_0 be the component of X such that $W = f((X_0)_{:1})$. For each j set $\eta_j = 0$ and let $\alpha_j \in \mathcal{R}_{Y_j}$ be the unique (reducible) critical point. Choose a reference connection A_o as in Subsection 3.4 and set $A_0 = A_o|_{X_0}$. Since each α_j has representatives of the form $(B,0)$ where $\check{B}$ is flat it follows that $\hat{F}(A_o)$ is compactly supported. In the following, μ will denote the (compactly supported) perturbation 2–form on X and μ_0 its restriction to X_0.

Let $\mathcal{H}^+$ be the space of self-dual closed L^2 2–forms on X_0. Then $\dim \mathcal{H}^+ = b_2^+(X_0) > 0$, so $\mathcal{H}^+$ contains a non-zero element z. By unique continuation for harmonic forms we can find a smooth 2–form μ_0 on X_0, supported in any given small ball, such that $\hat{F}^+(A_0) + i\mu_0^+$ is not L^2 orthogonal to z. (Here $\hat{F}^+$ is the self-dual part of $\hat{F}$.) Then

$$\hat{F}^+(A_0) + i\mu_0^+ \notin \operatorname{im}(d^+\colon\ L^{p,w}_1 \to L^{p,w}),$$

where w is the weight function used in the definition of the configuration space. Hence $M(X_0;\vec{\alpha})$ contains no reducible monopoles. After perturbing μ_0 in a small ball we can arrange that $M(X_0;\vec{\alpha})$ is transversally cut out as well, by Proposition 8.2.

To prove (i), recall that

$$\dim M(X_0;\vec{\alpha}) = d(W) - 1 + 2\sum_j h(Y_j), \tag{43}$$

so the inequality in (i) simply says that

$$\dim M(X_0;\vec{\alpha}) < 0,$$

hence $M(X_0;\vec{\alpha})$ is empty. Since there are no other moduli spaces over X_0, it follows from Theorem 1.4 that $M(X^{(T)})$ is empty when $\min_j T_j \gg 0$.

We will now prove (ii). If $M(X^{(T)})$ has odd or negative dimension then there is nothing to prove, so suppose this dimension is $2m \ge 0$. Since $M(X_0;\vec{\alpha})$ contains no reducibles we deduce from Theorem 1.4 that $M(X^{(T)})$ is also free of reducibles when $\min_j T_j$ is sufficiently large. Let $\mathbf{B} \subset X_0$ be a compact 4–ball and $\mathcal{B}^*(\mathbf{B})$ the Banach manifold of irreducible L^p_1 configurations over $\mathbf{B}$ modulo L^p_2 gauge transformations. Here $p > 4$ should be an even integer to ensure the existence of smooth partitions of unity. Let $\mathbb{L} \to \mathcal{B}^*(\mathbf{B})$ be the natural complex line bundle associated to some base-point in $\mathbf{B}$, and s a generic section of the m–fold direct sum $m\mathbb{L}$. For $\min_j T_j \gg 0$ let

$$S^{(T)} \subset M(X^{(T)}), \qquad S_0 \subset M(X_0;\vec{\alpha})$$

be the subsets consisting of those elements ω that satisfy $s(\omega|_{\mathbf{B}}) = 0$. By assumption, S_0 is a submanifold of codimension $2m$. For any T for which $S^{(T)}$ is transversely cut out the Seiberg–Witten invariant of Z is equal to the number of points in $S^{(T)}$ counted with sign. Now, the inequality in (ii) is equivalent to

$$d(W) + 2\sum_j h(Y_j) < d(Z) = 2m+1,$$

which by (43) gives

$$\dim\, S_0 = \dim\, M(X_0;\vec{\alpha}) - 2m < 0.$$

Therefore, S_0 is empty. By Theorem 1.4, $S^{(T)}$ is empty too when $\min_j T_j \gg 0$, hence $\mathrm{SW}(Z) = 0$. □

A Patching together local gauge transformations

In the proof of Lemma 4.1 we encounter sequences S_n of configurations such that for any point x in the base-manifold there is a sequence v_n of gauge transformations defined in a neighbourhood of x such that $v_n(S_n)$ converges (in some Sobolev norm) in a (perhaps smaller) neighbourhood of x. The problem then is to find a sequence u_n of global gauge transformations such that $u_n(S_n)$ converges globally. If v_n, w_n are two such sequences of local gauge transformations then $v_n w_n^{-1}$ will be bounded in the appropriate Sobolev norm, so the problem reduces to the lemma below.

This issue was discussed by Uhlenbeck in [32, Section 3]. Our approach has the advantage that it does not involve any "limiting bundles".

Lemma A.1 *Let X be a Riemannian manifold and $P \to X$ a principal G–bundle, where G is a compact subgroup of some matrix algebra $M_r(\mathbb{R})$. Let $M_r(\mathbb{R})$ be equipped with an Ad_G–invariant inner product, and fix a connection in the Euclidean vector bundle $E = P \times_{\mathrm{Ad}_G} M_r(\mathbb{R})$ (which we use to define*

Sobolev norms of automorphisms of E). Let $\{U_i\}_{i=1}^\infty$, $\{V_i\}_{i=1}^\infty$ be open covers of X such that $U_i \Subset V_i$ for each i. We also assume that each V_i is the interior of a compact codimension 0 submanifold of X, and that ∂V_i and ∂V_j intersect transversally for all $i \neq j$. For each i and $n = 1, 2, \dots$ let $v_{i,n}$ be a continuous automorphism of $P|_{V_i}$. Suppose $v_{i,n}v_{j,n}^{-1}$ converges uniformly over $V_i \cap V_j$ for each i, j (as maps into E). Then there exist

- *a sequence of positive integers $n_1 \le n_2 \le \cdots$,*
- *for each positive integer k an open subset $W_k \subset X$ with*
$$\bigcup_{i=1}^{k} U_i \Subset W_k \subset \bigcup_{i=1}^{k} V_i,$$
- *for each k and $n \ge n_k$ a continuous automorphism $w_{k,n}$ of $P|_{W_k}$,*

such that

(i) *If $1 \le j \le k$ and $n \ge n_k$ then $w_{j,n} = w_{k,n}$ on $\bigcup_{i=1}^{j} U_i$,*

(ii) *For each i, k the sequence $w_{k,n}v_{i,n}^{-1}$ converges uniformly over $W_k \cap V_i$,*

(iii) *If $1 \le p < \infty$, and $m > \frac{n}{p}$ is an integer such that $v_{i,n} \in L^p_{m,loc}$ for all i, n then $w_{k,n} \in L^p_{m,loc}$ for all k and $n \ge n_k$. If in addition*
$$\sup_n \|v_{i,n}v_{j,n}^{-1}\|_{L^p_m(V_i \cap V_j)} < \infty \qquad \text{for all } i, j$$
then
$$\sup_{n \ge n_k} \|w_{k,n}v_{i,n}^{-1}\|_{L^p_m(W_k \cap V_i)} < \infty \qquad \text{for all } k, i.$$

The transversality condition ensures that the Sobolev embedding theorem holds for $V_i \cap V_j$ (see [1]). Note that this condition can always be achieved by shrinking the V_i's a little.

Proof Let $N' \subset LG$ be a small Ad_G invariant open neighbourhood of 0. Then $\exp\colon LG \to G$ maps N' diffeomorphically onto an open neighbourhood N of 1. Let $f\colon N \to N'$ denote the inverse map. Let $\mathrm{Aut}(P)$ the bundle of fibre automorphisms of P and $\mathfrak{g}_P$ the corresponding bundle of Lie algebras. Set $\mathbf{N} = P \times_{\mathrm{Ad}_G} N \subset \mathfrak{g}_P$ and let $\exp^{-1}\colon \mathbf{N} \to \mathrm{Aut}(P)$ be the map defined by f.

Set $w_{1,n} = v_{1,n}$ and $W_1 = V_1$. Now suppose $w_{k,n}$, W_k have been chosen for $1 \le k < \ell$, where $\ell \ge 2$, such that (i)–(iii) hold for these values of k. Set $z_n = w_{\ell-1,n}(v_{\ell,n})^{-1}$ on $W_{\ell-1} \cap V_\ell$. According to the induction hypothesis the sequence z_n converges uniformly over $W_{\ell-1}\cap V_\ell$, hence there exists an integer $n_\ell\ge n_{\ell-1}$ such that $y_n=(z_{n_\ell})^{-1}z_n$ takes values in $\mathbf{N}$ for $n\ge n_\ell$.

Choose an open subset $\mathcal{W}\subset X$ which is the interior of a compact codimension 0 submanifold of X, and which satisfies

$$\bigcup_{i=1}^{\ell-1}U_i\Subset\mathcal{W}\Subset W_{\ell-1}.$$

We also require that $\partial\mathcal{W}$ intersect $\partial V_i\cap\partial V_j$ transversally for all i,j. (For instance, one can take $\mathcal{W}=\alpha^{-1}([0,\epsilon])$ for suitable ϵ, where $\alpha\colon X\to[0,1]$ is any smooth function with $\alpha=0$ on $\cup_{i=1}^{\ell-1}U_i$ and $\alpha=1$ on $W_{\ell-1}$.) Choose also a smooth, compactly supported function $\phi\colon W_{\ell-1}\to\mathbb{R}$ with $\phi|_{\mathcal{W}}=1$. Set $W_\ell=\mathcal{W}\cup V_\ell$ and for $n\ge n_\ell$ define an automorphism $w_{\ell,n}$ of $P|_{W_\ell}$ by

$$w_{\ell,n}=\begin{cases}w_{\ell-1,n} & \text{on } \mathcal{W},\\ z_{n_\ell}\exp(\phi\exp^{-1}y_n)v_{\ell,n} & \text{on } W_{\ell-1}\cap V_\ell,\\ z_{n_\ell}v_{\ell,n} & \text{on } V_\ell\setminus\operatorname{supp}(\phi).\end{cases}$$

Then (i)–(iii) hold for $k=\ell$ as well. To see that (iii) holds, note that our transversality assumptions guarantee that the Sobolev embedding theorem holds for $W_{\ell-1}\cap V_\ell$ and for all $V_i\cap V_j$. Since $mp>n$, L^p_m is therefore a Banach algebra for these spaces (see [1]). Recalling the proof of this fact, and the behaviour of L^p_m under composition with smooth maps on the left (see [27, p 184]), one obtains (iii). □

B A quantitative inverse function theorem

In this section E,E' will be Banach spaces. We denote by $\mathcal{B}(E,E')$ the Banach space of bounded operators from E to E'. If $T\in\mathcal{B}(E,E')$ then $\|T\|=\sup_{\|x\|\le1}\|Tx\|$. If $U\subset E$ is open and $f\colon U\to E'$ smooth then $Df(x)\in\mathcal{B}(E,E')$ is the derivative of f at $x\in U$. The second derivative $D(Df)(x)\in\mathcal{B}(E,\mathcal{B}(E,E'))$ is usually written $D^2f(x)$ and can be identified with the symmetric bilinear map $E\times E\to E'$ given by

$$D^2f(x)(y,z)=\left.\frac{\partial^2}{\partial s\partial t}\right|_{(0,0)}f(x+sy+tz).$$

The norm of the second derivative is

$$\|D^2f(x)\|=\sup_{\|y\|,\|z\|\le1}\|D^2f(x)(y,z)\|.$$

For $r > 0$ let
$$B_r = \{x \in E \,:\, \|x\| < r\}.$$

Lemma B.1 *Let $\epsilon, M > 0$ be positive real numbers such that $\epsilon M < 1$, and suppose $f\colon B_\epsilon \to E$ is a smooth map satisfying*
$$f(0) = 0; \quad Df(0) = I; \quad \|D^2 f(x)\| \le M \quad \text{for } x \in B_\epsilon.$$
Then f restricts to a diffeomorphism $f^{-1}B_{\frac{\epsilon}{2}} \xrightarrow{\approx} B_{\frac{\epsilon}{2}}$.

The conclusion of the lemma holds even when $\epsilon M = 1$, see Proposition B.1 below.

Proof The estimate on $D^2 f$ gives
$$\|Df(x) - I\| = \|Df(x) - Df(0)\| \le M\|x\|. \tag{44}$$
Therefore the map
$$h(x) = f(x) - x = \int_0^1 (Df(tx) - I)x\,dt$$
satisfies
$$\|h(x)\| \le \frac{M}{2}\|x\|^2,$$
$$\|h(x_2) - h(x_1)\| \le \epsilon M \|x_2 - x_1\|$$
for all $x, x_1, x_2 \in B_\epsilon$. Hence for every $y \in B_{\frac{\epsilon}{2}}$ the assignment $x \mapsto y - h(x)$ defines a map $B_\epsilon \to B_\epsilon$ which has a unique fix-point. In other words, f maps $f^{-1}B_{\frac{\epsilon}{2}}$ bijectively onto $B_{\frac{\epsilon}{2}}$. Moreover, $Df(x)$ is an isomorphism for every $x \in B_\epsilon$, by (44). Applying the contraction mapping argument above to f around an arbitrary point in B_ϵ then shows that f is an open map. It is then a simple exercise to prove that the inverse $g\colon B_{\frac{\epsilon}{2}} \to f^{-1}B_{\frac{\epsilon}{2}}$ is differentiable and $Dg(y) = (Df(g(y)))^{-1}$ (see [8, 8.2.3]). Repeated application of the chain rule then shows that g is smooth. □

For $r > 0$ let $B_r \subset E$ be as above, and define $B'_r \subset E'$ similarly.

Proposition B.1 *Let ϵ, M be positive real numbers and $f\colon B_\epsilon \to E'$ a smooth map such that $f(0) = 0$, $L = Df(0)$ is invertible, and*
$$\|D^2 f(x)\| \le M \quad \text{for all } x \in B_\epsilon.$$
Set $\kappa = \|L^{-1}\|^{-1} - \epsilon M$ and $\epsilon' = \epsilon\|L^{-1}\|^{-1}$. Then the following hold:

(i) *If $\kappa \geq 0$ then f is a diffeomorphism onto an open subset of E' containing $B'_{\epsilon'/2}$.*

(ii) *If $\kappa > 0$ and $g\colon B'_{\epsilon'/2} \to B_\epsilon$ is the smooth map satisfying $f \circ g = I$ then for all $x \in B_\epsilon$ and $y \in B'_{\epsilon'/2}$ one has*

$$\|Df(x)^{-1}\|, \|Dg(y)\| < \kappa^{-1}, \quad \|D^2g(y)\| < M\kappa^{-3}.$$

The reader may wish to look at some simple example (such as a quadratic polynomial) to understand the various ways in which these results are optimal.

Proof (i) For every $x \in B_\epsilon$ we have

$$\|Df(x)L^{-1} - I\| \leq \|Df(x) - L\| \cdot \|L^{-1}\| < \epsilon M \|L^{-1}\| \leq 1,$$

hence $Df(x)$ is invertible. Thus f is a local diffeomorphism by Lemma B.1. Set $h(x) = f(x) - Lx$. If $x_1, x_2 \in B_\epsilon$ and $x_1 \neq x_2$ then

$$\|f(x_2) - f(x_1)\| \geq \|L(x_2 - x_1)\| - \|h(x_2) - h(x_1)\| > \kappa \|x_2 - x_1\|,$$

hence f is injective. By choice of ϵ' the map

$$\widetilde{f} = f \circ L^{-1}\colon B'_{\epsilon'/2} \to E'$$

is well defined, and for every $y \in B'_{\epsilon'/2}$ one has

$$\|D^2\widetilde{f}(y)\| \leq M\|L^{-1}\|^2.$$

Because

$$\epsilon' M \|L^{-1}\|^2 = \epsilon M \|L^{-1}\| \leq 1,$$

Lemma B.1 says that the image of $\widetilde{f}$ contains every ball $B'_{\delta/2}$ with $0 < \delta < \epsilon'$, hence also $B'_{\epsilon'/2}$.

(ii) Set $c = I - Df(x)L^{-1}$. Then

$$Df(x)^{-1} = L^{-1} \sum_{n=0}^{\infty} c^n,$$

hence

$$\|Df(x)^{-1}\| \leq \frac{\|L^{-1}\|}{1 - \|c\|} < \frac{\|L^{-1}\|}{1 - \epsilon M \|L^{-1}\|} = \kappa^{-1}.$$

This also gives the desired bound on $Dg(y) = Df(g(y))^{-1}$.

To estimate D^2g, let $\mathrm{Iso}(E, E') \subset \mathcal{B}(E, E')$ be the open subset of invertible operators, and let $\iota\colon \mathrm{Iso}(E, E') \to \mathcal{B}(E', E)$ be the inversion map: $\iota(a) = a^{-1}$. Then ι is smooth, and its derivative is given by

$$D\iota(a)b = -a^{-1}ba^{-1},$$

see [8]. The chain rule says that

$$Dg = \iota \circ Df \circ g,$$
$$D(Dg)(y) = D\iota(Df(g(y))) \circ D(Df)(g(y)) \circ Dg(y).$$

This gives

$$\|D(Dg)(y)\| \le \|Df(g(y))^{-1}\|^2 \cdot \|D(Df)(g(y))\| \cdot \|Dg(y)\| < \kappa^{-2} \cdot M \cdot \kappa^{-1}. \qquad \square$$

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