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Invariants for Lagrangian tori

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Abstract

We de ne an simple invariant (T) of an embedded nullhomologous Lagrangian torus and use this invariant to show that many symplectic 4{manifolds have in nitely many pairwise symplectically inequivalent nullhomologous Lagrangian tori. We further show that for a large class of examples that (T) is actually a C^1 invariant. In addition, this invariant is used to show that many symplectic 4{manifolds have nontrivial homology classes which are represented by in nitely many pairwise inequivalent Lagrangian tori, a result rst proved by S Vidussi for the homotopy K3{surface obtained from knot surgery using the trefoil knot [19].

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1 Introduction

This paper is concerned with the construction and detection of homologous but inequivalent (under diffeomorphism or symplectomorphism) Lagrangian tori. In recent years there has been considerable progress made on the companion problem for symplectic tori. In fact, it is now known that in a simply connected symplectic 4–manifold with $b^+ > 1$, if T is an embedded symplectic torus, then for each $m > 1$ the homology class of mT contains infinitely many nonisotopic embedded symplectic tori. (See eg [3, 7, 8].) The techniques used for the construction of these examples fail for Lagrangian tori.

The first examples of inequivalent homologous Lagrangian tori were discovered by S Vidussi [19] who presented a technique for constructing infinitely many homologous, but nonisotopic, Lagrangian tori in $E(2)_K$, the result of knot surgery on the K3–surface using the trefoil knot, K. The work in this paper was motivated by an attempt to better understand and distinguish the examples presented in [19]. We found that the key to these examples is the construction of infinite families of nullhomologous Lagrangian tori T in a symplectic 4–manifold X. There is a simple process by which an integer $\lambda(T)$, a Lagrangian framing defect, can be associated to T. In this paper we show that $\lambda(T)$ is an invariant of the symplectomorphism, and in many cases the diffeomorphism, type of $(X;T)$. We then construct infinite families of inequivalent nullhomologous Lagrangian tori distinguished by $\lambda(T)$. Homologically essential examples are created from these by a circle sum process.

Some examples typical of those we which we study can be briefly described: Let X be any symplectic manifold which contains an embedded self-intersection 0 symplectic torus T. For any fibered knot K consider the symplectic manifold X_K constructed by knot surgery [5]. Since X_K is the fiber sum of X and $S^1 \times M_K$ along T and $S^1 \times m$ in $S^1 \times M_K$, where M_K is the result of 0–framed surgery on K and m is a meridian of K, there is a codimension 0 submanifold V in X_K diffeomorphic to $S^1 \times (M_K \setminus m)$. The manifold V is fibered by punctured surfaces Σ', and if γ is any loop on such a surface, the torus $T_\gamma = S^1 \times \gamma$ in X_K is nullhomologous and Lagrangian. We show that $\lambda(T_\gamma)$ is a diffeomorphism invariant. This invariant persists even after circle sums with essential Lagrangian tori, and it distinguishes all our (and Vidussi's) examples.

Here is a more precise summary of our examples:

Theorem 1.1 (a) *Let X be any symplectic manifold with $b_2^+(X) > 1$ which contains an embedded self-intersection 0 symplectic torus with a vanishing cy-*

cle. (See Section 4 for a definition.) Then for each nontrivial fibered knot K in S^3, the result of knot surgery X_K contains infinitely many nullhomologous Lagrangian tori, pairwise inequivalent under orientation-preserving diffeomorphisms.

(b) *Let X_i, $i = 1, 2$, be symplectic 4–manifolds containing embedded self-intersection 0 symplectic tori F_i and assume that F_1 contains a vanishing cycle. Let X be the fiber sum, $X = X_1 \#_{F_1=F_2} X_2$. Then for each nontrivial fibered knot K in S^3, the manifold X_K contains an infinite family of homologically primitive and homologous Lagrangian tori which are pairwise inequivalent.*

In Section 6 we shall also give examples of nullhomologous Lagrangian tori T_i in a symplectic 4–manifold where the (T_i) are mutually distinct, so these tori are inequivalent under symplectomorphisms, but the techniques of this paper, namely relative Seiberg–Witten invariants, fail to distinguish the the T_i. It remains an extremely interesting question whether the these tori are equivalent under diffeomorphisms.

2 Seiberg–Witten invariants for embedded tori

The Seiberg–Witten invariant of a smooth closed oriented 4–manifold X with $b_2^+(X) > 1$ is an integer-valued function SW_X which is defined on the set of *spin^c* structures over X. Corresponding to each *spin^c* structure $\mathfrak{s}$ over X is the bundle of positive spinors $W_{\mathfrak{s}}^+$ over X. Set $c(\mathfrak{s}) \in H_2(X)$ to be the Poincare dual of $c_1(W_{\mathfrak{s}}^+)$. Each $c(\mathfrak{s})$ is a characteristic element of $H_2(X;\mathbf{Z})$ (ie, its Poincare dual $\hat{c}(\mathfrak{s}) = c_1(W_{\mathfrak{s}}^+)$ reduces mod 2 to $w_2(X)$). We shall work with the modified Seiberg–Witten invariant

$$\mathrm{SW}'_X\colon \{k \in H_2(X;\mathbf{Z}) \mid \hat{k} \equiv w_2(TX) \pmod 2\} \to \mathbf{Z}$$

defined by $\mathrm{SW}'_X(k) = \sum_{c(\mathfrak{s})=k} \mathrm{SW}_X(\mathfrak{s})$.

The sign of SW_X depends on a homology orientation of X, that is, an orientation of $H^0(X;\mathbf{R}) \otimes \det H^2_+(X;\mathbf{R}) \otimes \det H^1(X;\mathbf{R})$. If $\mathrm{SW}'_X(\alpha) \neq 0$, then α is called a *basic class* of X. It is a fundamental fact that the set of basic classes is finite. Furthermore, if α is a basic class, then so is $-\alpha$ with $\mathrm{SW}'_X(-\alpha) = (-1)^{(\mathrm{e}+\mathrm{sign})(X)/4}\,\mathrm{SW}'_X(\alpha)$ where $\mathrm{e}(X)$ is the Euler number and $\mathrm{sign}(X)$ is the signature of X. The Seiberg–Witten invariant is an orientation-preserving diffeomorphism invariant of X (together with the choice of a homology orientation).

It is convenient to view the Seiberg–Witten invariant as an element of the integral group ring $\mathbb{Z}H_2(X)$, where for each $\alpha \in H_2(X)$ we let t_α denote the corresponding element in $\mathbb{Z}H_2(X)$. Suppose that $\{\pm\beta_1, \dots, \pm\beta_n\}$ is the set of nonzero basic classes for X. Then the Seiberg–Witten invariant of X is the Laurent polynomial

$$SW_X = \mathrm{SW}'_X(0) + \sum_{j=1}^{n} \mathrm{SW}'_X(\beta_j)\cdot(t_{\beta_j} + (-1)^{(\mathrm{e}+\mathrm{sign})(X)/4}\, t_{\beta_j}^{-1}) \in \mathbb{Z}H_2(X).$$

Suppose that T is an embedded (but not necessarily homologically essential) torus of self-intersection 0 in X, and identify a tubular neighborhood of T with $T \times D^2$. Let α, β, γ be simple loops on $\partial(T \times D^2)$ whose homology classes generate $H_1(\partial(T \times D^2))$. Denote by $X_T(p,q,r)$ the result of surgery on T which annihilates the class of $p\alpha + q\beta + r\gamma$; ie,

$$X_T(p,q,r) = (X \setminus T \times D^2) \cup_\varphi T^2 \times D^2 \qquad (1)$$

where $\varphi\colon \partial(X \setminus T \times D^2) \to \partial(T^2 \times D^2)$ is an orientation-reversing diffeomorphism satisfying $\varphi_*[p\alpha + q\beta + r\gamma] = [\partial D^2]$. An important formula for calculating the Seiberg–Witten invariants of surgeries on tori is due to Morgan, Mrowka, and Szabo [13] (see also [12], [17]). Suppose that $b_2^+(X \setminus (T \times D^2)) > 1$. Then each $b_2^+(X_T(p,q,r)) > 1$. Given a class $k \in H_2(X)$:

$$\sum_i \mathrm{SW}'_{X_T(p,q,r)}(k_{(p,q,r)} + i[T]) = p \sum_i \mathrm{SW}'_{X_T(1,0,0)}(k_{(1,0,0)} + i[T]) +$$
$$+\, q \sum_i \mathrm{SW}'_{X_T(0,1,0)}(k_{(0,1,0)} + i[T]) + r \sum_i \mathrm{SW}'_{X_T(0,0,1)}(k_{(0,0,1)} + i[T]) \qquad (2)$$

In this formula, T denotes the torus which is the core $T^2 \times 0 \subset T^2 \times D^2$ in each specific manifold $X_T(a,b,c)$ in the formula, and $k_{(a,b,c)} \in H_2(X_T(a,b,c))$ is any class which agrees with the restriction of k in $H_2(X \setminus T \times D^2, \partial)$ in the diagram:

$$\begin{array}{ccc}
H_2(X_T(a,b,c)) & \longrightarrow & H_2(X_T(a,b,c), T \times D^2) \\
 & & \big\downarrow \cong \\
 & & H_2(X \setminus T \times D^2, \partial) \\
 & & \big\uparrow \cong \\
H_2(X) & \longrightarrow & H_2(X, T \times D^2)
\end{array}$$

Furthermore, in each term of (2), unless the homology class $[T]$ is 2–divisible, each i must be even since the classes $k_{(a,b,c)} + i[T]$ must be characteristic in $H_2(X_T(a,b,c))$.

Let $\psi(a,b,c)\colon H_2(X_T(a,b,c)) \to H_2(X\setminus T\times D^2,\partial)$ be the composition of maps in the above diagram, and $\psi(a,b,c)_*$ the induced map of integral group rings. Since we are interested in invariants of the pair (X,T), we shall work with

$$\overline{SW}_{(X_T(a,b,c),T)} = \psi(a,b,c)_*(SW_{X_T(a,b,c)}) \in \mathbb{Z}H_2(X\setminus T\times D^2,\partial).$$

The indeterminacy in (2) is caused by multiples of $[T]$; so passing to $\overline{SW}$ removes this indeterminacy, and the Morgan–Mrowka–Szabo formula becomes

$$\overline{SW}_{(X_T(p,q,r),T)} = p\,\overline{SW}_{(X_T(1,0,0),T)} + q\,\overline{SW}_{(X_T(0,1,0),T)} + r\,\overline{SW}_{(X_T(0,0,1),T)}. \quad (3)$$

Let T and T' be embedded tori in the oriented 4–manifold X. We shall say that these tori are *C^∞–equivalent* if there is an orientation-preserving diffeomorphism f of X with $f(T)=T'$. Any self-diffeomorphism of X which throws T onto T', takes a loop on the $\partial(T\times D^2)$ to a loop on the boundary of a tubular neighborhood of T'. Set

$$I(X,T) = \{\overline{SW}_{(X_T(a,b,c),T)} \mid a,b,c\in\mathbb{Z}\}.$$

Proposition 2.1 *Let T be an embedded torus of self-intersection* 0 *in the simply connected 4–manifold X with $b_2^+(X\setminus T)>1$. After fixing a homology orientation for X, $I(X,T)$ is an invariant of the pair (X,T) up to C^∞-equivalence.* □

3 The Lagrangian framing invariant

In this section we shall define the invariant $\lambda(T)$ of a nullhomologous Lagrangian torus. To begin, consider a nullhomologous torus T embedded in a smooth 4–manifold X with tubular neighborhood N_T. Let $i\colon \partial N_T \to X\setminus N_T$ be the inclusion.

Definition 3.1 A *framing* of T is a diffeomorphism $\varphi\colon T\times D^2 \to N_T$ such that $\varphi(p)=p$ for all $p\in T$. A framing φ of T is *nullhomologous* if for $x\in\partial D^2$, the homology class $\varphi_*[T\times\{x\}]\in\ker i_*$.

Given a framing $\varphi\colon T\times D^2\to N_T$, there is an associated section $\sigma(\varphi)$ of $\partial N_T\to T$ given by $\sigma(\varphi)(x)=\varphi(x,1)$, and given a pair of framings, φ_0, φ_1 there is a difference class $\delta(\varphi_0,\varphi_1)\in H^1(T;\mathbb{Z})=[T,S^1]$, the homotopy class of the composition

$$T \xrightarrow{\sigma(\varphi_1)} \partial N_T \xrightarrow{\varphi_0^{-1}} T\times\partial D^2 \xrightarrow{\mathrm{pr}_2} \partial D^2 = S^1.$$

Note that if γ is a loop on T, then $\delta(\varphi_0,\varphi_1)[\gamma] = (\varphi_1)_*[\gamma]\cdot(\varphi_0)_*[T]$ using the intersection pairing on ∂N_T, or equivalently,

$$(\varphi_0)_*^{-1}\circ(\varphi_1)_*[\gamma] = [\gamma\times\{1\}] + \delta(\varphi_0,\varphi_1)[\gamma][\partial D^2] \in H_1(T\times\partial D^2;\mathbb{Z}).$$

Proposition 3.2 *A nullhomologous framing of T is unique up to homotopy.*

Proof Since T is nullhomologous, it follows that $H_2(X\setminus N_T)\to H_2(X)$ is onto, and $H_3(X\setminus N_T,\partial N_T) = H_3(X,T) = H_3(X)\oplus H_2(T)$. Then the long exact sequence of $(X\setminus N_T,\partial N_T)$ shows that the kernel of $i_*: H_2(\partial N_T)\to H_2(X\setminus N_T)$ is isomorphic to $H_2(T)=\mathbb{Z}$. So any two nullhomologous framings φ_0, φ_1 give rise to homologous tori $\varphi_i(T)$ in ∂N_T. Thus for any loop γ on T:

$$\delta(\varphi_0,\varphi_1)[\gamma] = (\varphi_1)_*[\gamma]\cdot(\varphi_0)_*[T] = (\varphi_1)_*[\gamma]\cdot(\varphi_1)_*[T] = [\gamma\times\{1\}]\cdot[T\times\{1\}] = 0$$

the last pairing in $T\times\partial D^2$. Hence $\delta(\varphi_0,\varphi_1)=0$. □

We denote by φ_N any such nullhomologous framing of T.

Now suppose that (X,ω) is a symplectic 4–manifold containing an embedded Lagrangian torus T. For any closed oriented Lagrangian surface $\Sigma\subset X$ there is a nondegenerate bilinear pairing

$$(TX|_\Sigma/T\Sigma)\otimes T\Sigma\to\mathbb{R},\quad ([v],u)\mapsto\omega(v,u).$$

Hence, the normal bundle $N_\Sigma\cong T^*\Sigma$, the cotangent bundle; so computing Euler numbers, $2g-2 = -e(\Sigma) = e(T^*(\Sigma)) = e(N_\Sigma) = \Sigma\cdot\Sigma$, for g the genus of Σ. Furthermore, this is true symplectically as well. The Lagrangian neighborhood theorem [20] states that each such Lagrangian surface has a tubular neighborhood which is symplectomorphic to a neighborhood of the zero section of its cotangent bundle with its standard symplectic structure, where the symplectomorphism is the identity on Σ.

Thus an embedded Lagrangian torus T has self-intersection 0, and small enough tubular neighborhoods N_T have, up to symplectic isotopy, a preferred framing $\varphi_L: T\times D^2\to N_T$ such that for any point $x\in D^2$, the torus $\varphi_L(T\times\{x\})$ is also Lagrangian. We shall call φ_L the *Lagrangian framing* of T.

Thus if T is a nullhomologous Lagrangian torus, we may consider the difference $\delta(\varphi_N,\varphi_L)\in H^1(T;\mathbb{Z}) = [T,S^1]$. It thus induces a well-defined homomorphism $\delta(\varphi_N,\varphi_L)_*: H_1(T;\mathbb{Z})\to H_1(S^1)$.

Definition 3.3 The *Lagrangian framing invariant* of a nullhomologous Lagrangian torus T is the nonnegative integer $\lambda(T)$ such that

$$(\varphi_N;\varphi_L)_*(H^1(T;\mathbb{Z})) = \lambda(T)\,\mathbb{Z} \quad \text{in} \quad H_1(S^1) = \mathbb{Z}.$$

Thus if a and b form a basis for $H_1(T;\mathbb{Z})$ then $\lambda(T)$ is the greatest common divisor of $|(\varphi_N;\varphi_L)(a)|$ and $|(\varphi_N;\varphi_L)(b)|$. Furthermore, if $f\colon X \to Y$ is a symplectomorphism with $f(T) = T'$, then $f\circ\varphi_N$ is a nullhomologous framing of T', and for small enough N_T, $f\circ\varphi_L$ is the Lagrangian framing of T'. Hence:

Theorem 3.4 *Let T be a nullhomologous Lagrangian torus in the symplectic 4–manifold X. Then the Lagrangian framing invariant $\lambda(T)$ is a symplectomorphism invariant of $(X;T)$.* □

Here is an example. Let K be any fibered knot in S^3, and let M_K be the result of 0–surgery on K. Then M_K is a 3–manifold with the same homology as $S^2\times S^1$, and M_K is fibered over the circle. Let γ be any embedded loop which lies on a fiber of the fibration $S^3\setminus K \to S^1$. Ie, γ lies on a Seifert surface Σ of K. The first homology $H_1(M_K) = H_1(S^3\setminus K) = \mathbb{Z}$, and the integer corresponding to a given loop is the linking number of the loop with K. Since γ lies on a Seifert surface, its linking number with K is 0, and so γ is nullhomologous in M_K.

Taking the product with a circle, $S^1\times M_K$ fibers over T^2, and it is a symplectic 4–manifold with a symplectic form which arises from the sum of volume forms in the base and in the fiber. More precisely, one can choose metrics so that the fiber bundle projection, $p\colon M_K \to S^1$ is harmonic. Let α be the volume form on the base S^1, and let β be the volume form on the first S^1 in $S^1\times M_K$. Then $\omega = \beta\wedge p^*(\alpha) + *_3\, p^*(\alpha)$ defines a symplectic form on $S^1\times M_K$. Since γ lies in a fiber, its tangent space at any point is spanned by a vector parallel to the tangent space of the fiber and a vector tangent to S^1. So ω vanishes on $T(S^1\times\gamma)$. (See also [19].) Thus $T_\gamma = S^1\times\gamma$ is a nullhomologous Lagrangian torus in $S^1\times M_K$. If γ' is a pushoff of γ in the Seifert surface Σ, then $S^1\times\gamma'$ is again Lagrangian. This, together with the pushoff of γ onto nearby fibers, describes the Lagrangian framing of T_γ. We shall also say that γ' is the *Lagrangian pushoff* of γ.

Definition 3.5 Let K be a fibered knot in S^3, and let γ be any embedded loop lying on a fiber of the fibration $S^3\setminus K \to S^1$. The *Lagrangian framing defect* $\lambda(\gamma)$ of γ is the linking number of γ with a Lagrangian pushoff of itself.

In Figures 1 and 2, we have $(\gamma_1) = 1$, and $(\gamma_2) = 3$.

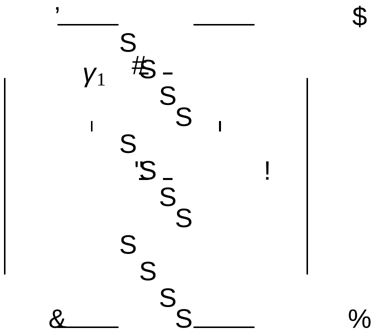
Figure 1

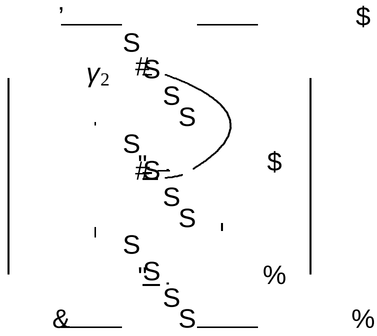
Figure 2

As in the above definition, let γ be an embedded loop lying on a fiber of the fibration $S^3 \setminus K \to S^1$, and let $N(\gamma) = \gamma \times D^2$ be a tubular neighborhood of γ. Further, let $\ell(\gamma) \in H_1(\partial N(\gamma))$ be the (nullhomologous) 0–framing of γ in S^3; that is, the nontrivial primitive class which is sent to 0 by $H_1(\partial(N(\gamma))) \to H_1(S^3 \setminus N(\gamma))$. Then if γ' is a Lagrangian pushoff of γ, in $H_1(\partial(N(\gamma)))$ we have the relation

$$[\gamma'] = \ell(\gamma) + (\gamma)[\partial D^2].$$

In other words, the Lagrangian pushoff corresponds to the framing (γ) with respect to the usual 0–framing $\ell(\gamma)$ in S^3. So, for example, a Lagrangian $1/p$ surgery on the curve γ_1 above corresponds to a $(p+1)/p$ surgery with respect to the usual framing of γ_1 in S^3. More generally, a $1/p$ Lagrangian surgery on a curve γ in the Seifert surface of a fibered knot in S^3 corresponds to a $(p\ (\gamma)+1)/p$ surgery with respect to the usual framing of γ in S^3.

Theorem 3.6 *In $S^1 \times M_K$, the Lagrangian framing invariant of T_γ is $(T_\gamma) = |\ (\gamma)|$.*

Proof As a basis for $H_1(T_\gamma;\mathbf{Z})$ take $[\{1\} \times \gamma]$ and $[S^1 \times \{x\}]$ where $x \in \gamma$. Since the linking number of γ and K is 0, there is a Seifert surface C for γ in S^3 which is disjoint from K. The tubular neighborhood N_T of T_γ is given by $N_T = S^1 \times N(\gamma)$ and $(\varphi_N)(T_\gamma) = (S^1 \times C) \cap \partial N_T = S^1 \times (C \cap \partial N(\gamma)) = S^1 \times \ell(\gamma)$, where we are using '$\ell(\gamma)$' here to denote a curve in the class $\ell(\gamma)$. Thus $(\varphi_N, \varphi_L)[S^1 \times \{x\}] = (\varphi_L)\ [S^1 \times \{x\}]\ [S^1 \times \ell(\gamma)] = [S^1 \times \{pt\}]\ [S^1 \times \ell(\gamma)] = 0$, and

$$\lambda(\varphi_N, \varphi_L)[\{1\} \times \gamma] = \lambda(\varphi_L)\,[\{1\} \times \gamma] \cdot [S^1 \times \ell(\gamma)] =$$
$$[\{1\} \times \gamma'] \cdot [S^1 \times \ell(\gamma)] = (\ell(\gamma) + \lambda(\gamma)[\partial D^2]) \cdot \ell(\gamma) = \lambda(\gamma)$$

Thus $\lambda(T_\gamma) = \gcd(|\lambda(\varphi_N, \varphi_L)[\{1\} \times \gamma]|, |\lambda(\varphi_N, \varphi_L)[S^1 \times \{x\}]|) = |\lambda(\gamma)|$. □

Lemma 3.7 *Given any nontrivial fibered knot K in S^3, there is a sequence of embedded loops γ_n contained in a fixed fiber Σ of $S^3 \setminus K \to S^1$ such that $\lim_{n \to \infty} |\lambda(\gamma_n)| = \infty$.*

Proof If we can find any $c \in H_1(\Sigma)$ represented by an embedded loop such that $\lambda(c) \neq 0$, then if $e \in H_1(\Sigma)$ is represented by a loop and is not a multiple of c, $\lambda(e+nc)$ is the linking number of $e+nc$ with $e'+nc'$ (c', e' the Lagrangian pushoffs). Thus

$$\lambda(e + nc) = \lambda(e) + n^2\lambda(c) + n(\mathrm{lk}(c, e') + \mathrm{lk}(e, c'));$$

whose absolute value clearly goes to ∞ as $n \to \infty$. Further, $e+nc$ is represented by an embedded loop for all n for which $e+nc$ is primitive, and this is true for infinitely many n. (To see this, identify $H_1(\Sigma)$ with $\mathbb{Z}^{2g}$. Then since c and e are independent and primitive, we may make a change of coordinates so that in these new coordinates $c = (1, 0, 0, \ldots, 0)$ and $e = (r, s, 0, \ldots, 0)$, for $r, s \in \mathbb{Z}$, $s \neq 0$. Thus $e + nc = (n + r, s, 0, \ldots, 0)$. The first coordinate is prime for infinitely many n, and at most finitely many of these primes can divide s. So these $e + nc$ are primitive.)

To find c with $\lambda(c) \neq 0$, note that $\mathrm{lk}(c, e')$ is the Seifert linking pairing. Since $\Delta_K(t) \neq 1$; this pairing is nontrivial. Let $\{b_i\}$ be a basis for $H_1(\Sigma)$. If all $\lambda(b_i) = 0$, and all $\lambda(b_i + b_j) = 0$ then $\mathrm{lk}(b_j, b_i') = -\mathrm{lk}(b_i, b_j')$ for all $i \neq j$. This means that the Seifert matrix V corresponding to this basis satisfies $V^T = -V$. However, $1 = \Delta_K(1) = \det(V^T - V) = \det(2V^T) = 2^{2g}\det(V^T)$, a contradiction. □

We conclude from Theorem 3.6 and this lemma:

Theorem 3.8 *Let K be any nontrivial fibered knot in S^3. Then in the symplectic manifold $X = S^1 \times M_K$ there are infinitely many nullhomologous Lagrangian tori which are inequivalent under symplectomorphisms of X.* □

The constructions of this section are related to Polterovich's 'linking class' $L \in H^1(T;\mathbb{Z})$ (see [15]) which is defined for Lagrangian tori $T \subset \mathbb{C}^2$, by $L([a]) = \mathrm{lk}(T, a')$, where a' is a pushoff of a representative a of $[a] \in H_1(T;\mathbb{Z})$ in the Lagrangian direction. One quickly sees that L is actually defined for a nullhomologous Lagrangian torus in any symplectic 4–manifold, and $L = \delta(\varphi_N, \varphi_L)$.

The Polterovich linking class is also defined for totally real tori in $\mathbb{C}^2$, and it is shown in [15] that the value of L on totally real tori can be essentially arbitrary, whereas Eliashberg and Polterovich have shown that in $\mathbb{C}^2$ the linking class L vanishes on Lagrangian tori.

The results of this section may be interpreted as saying that this vanishing phenomenon disappears in symplectic 4–manifolds more complicated than $\mathbb{C}^2$.

4 Nullhomologous Lagrangian tori

In this section we shall describe examples of collections of C^∞–inequivalent nullhomologous Lagrangian tori. The key point is that for our examples, the Lagrangian framing invariant is actually a C^∞ invariant.

We begin by describing the symplectic 4–manifolds which contain the examples. Let X be a symplectic 4–manifold with $b_2^+(X) > 1$ which contains an embedded symplectic torus F satisfying

(a) $F \cdot F = 0$

(b) F contains a loop γ, primitive in $\pi_1(F)$, which in $X \setminus F$ bounds an embedded disk Δ of self-intersection -1.

For example, a fiber of a simply connected elliptic surface satisfies this condition. Any torus with a neighborhood symplectically diffeomorphic to a neighborhood of a nodal or cuspidal fiber in an elliptic surface also satisfies the condition, and such tori can be seen to occur in many complex surfaces [4]. Let us describe this situation by saying that X contains an embedded symplectic self-intersection 0 torus *with a vanishing cycle*.

Now consider a genus g fibered knot K in S^3, and let Σ be a fiber of the fibration $M_K \to S^1$ and let m be a meridian of K. Let X be a symplectic 4–manifold with $b_2^+(X) > 1$ and with an embedded symplectic self-intersection 0 torus, F, with a vanishing cycle. Fix tubular neighborhoods $N = S^1 \times m \times D^2$

of the torus $S^1\times m$ in $S^1\times M_K$ and $N_F = F\times D^2$ of F in X, and consider the result of knot surgery

$$X_K = X\#_{F=S^1\times m}(S^1\times M_K)$$

where we require the gluing to take the circle $(S^1\times pt\times pt)$ in ∂N to $(\alpha\times pt)$ in ∂N_F. Then X_K is a symplectic 4–manifold with Seiberg–Witten invariant $SW_{X_K} = \Delta_K^{sym}(t_F^2)\cdot SW_X$ where Δ_K^{sym} is the symmetrized Alexander polynomial of K (see [5]). Fix an embedded loop γ on Σ whose linking number with the chosen meridian m is 0, and let $T_\gamma = S^1\times\gamma$, a Lagrangian torus with tubular neighborhood $N_{T_\gamma} = T_\gamma\times D^2$ in $S^1\times M_K$. Now M_K is a homology $S^1\times S^2$ and $H_1(\Sigma)\to H_1(M_K)$ is the 0–map. Removing the neighborhood N_m of a meridian from M_K does not change H_1. ($N_m\cap\Sigma = D^2$; so ∂D^2 is a meridian to m and it bounds $\Sigma\setminus D^2$.) Thus we have $[\gamma]=0$ in $H_1(M_K\setminus N_m)$, and hence T_γ is nullhomologous in X_K. In fact, since the linking number of γ and K and m is 0, the loop γ bounds an oriented surface $C\subset S^3\setminus(K\cup m)$. Thus $S^1\times C$ provides a nullhomology of T_γ in X_K. Also note that $b_2^+(X_K\setminus T_\gamma) = b_2^+(X) > 1$.

Proposition 4.1 *For loops γ_1, γ_2 in the fiber Σ of $M_K\to S^1$, if the corresponding nullhomologous tori T_{γ_1} and T_{γ_2} in X_K are symplectically equivalent then $\lambda(T_{\gamma_1}) = \lambda(T_{\gamma_1})$.*

Proof Because $S^1\times C$ is a nullhomology of T_g, the invariant $\lambda(T_\gamma)$ is calculated exactly as in Theorem 3.6; so this proposition follows. □

We wish to calculate $I(X_K;T_\gamma)$. First fix a basis for ∂N_{T_γ} which is adapted to the Lagrangian framing of T_γ. This basis is $\{[S^1\times\{y\}],[\gamma'],[\partial D^2]\}$ where γ' is a Lagrangian pushoff of γ in Σ and $y\in\gamma'$. We begin by studying $X_{K,T_\gamma}(1,0,0)$, the manifold obtained from X_K by the surgery on T_γ which kills $S^1\times\{y\}$.

Proposition 4.2 $SW_{X_{K,T_\gamma}(1,0,0)} = 0$.

Proof Let δ be a path in Σ from y to the point x at which m intersects Σ. By construction, $S^1\times\{x\}$ is identified with $\alpha\times pt\in\partial N_F$. This means that $S^1\times\{x\}$ is the boundary of a disk Δ of self-intersection -1 in $X\setminus N_F$. The surgery curve, $S^1\times\{y\}$, bounds a disk D of self-intersection 0 in $X_{K,T_\gamma}(1,0,0)$ (disjoint from $X_K\setminus T_\gamma\times D^2$); so the surgered manifold $X_{K,T_\gamma}(1,0,0)$ contains the sphere $C = \Delta\cup(S^1\times\delta)\cup D$ of self-intersection -1.

The rim torus $R = m\times\partial D^2\subset\partial N_F$ intersects the sphere C in a single positive intersection point, but this is impossible if $SW_{X_{K,T_\gamma}(1,0,0)}\neq 0$. For, if $SW_{X_{K;T_y}(1;0;0)} \neq 0$, then blowing down C, we obtain a 4–manifold Z (with $b_2^+ > 1$) which contains a torus R' of self-intersection $+1$, and the Seiberg–Witten invariant of Z is nontrivial. However, the adjunction inequality states that for any basic class of Z we have $0 \quad 1 + |\quad R'|$, an obvious contradiction. □

Before proceeding further, note that since T_y is nullhomologous in X_K,

$$j : H_2(X_K) \to H_2(X_K; T_y)$$

is an injection. Thus we may identify $\overline{SW}_{(X_K;T_y)} = j\,(SW_{X_K})$ with SW_{X_K}. We shall make use of an important result due to Meng and Taubes concerning the Seiberg–Witten invariant of a closed 3–manifold M [12]:

$$\begin{cases} SW_M = \ {}^{sym}_{M}(t^2) \ (t - t^{-1})^{-2}; & b_1(M) = 1 \\ SW_M = \ {}^{sym}_{M}; & b_1(M) > 1 \end{cases} \tag{4}$$

where ${}^{sym}_{M}$ is the symmetrized Alexander polynomial of M, and if $b_1(M) = 1$ then $t \in \mathbf{Z}H_1(M;\mathbf{R})$ corresponds to the generator of $H_1(M;\mathbf{R})$.

Since $X_{K;T_y}(0;0;1)$ is the result of the surgery which kills ∂D^2, it is X_K again, and we know that $SW_{X_K} = \ {}^{sym}_{K}(t_F^2) \ SW_X$. This also means that $\overline{SW}_{(X_K;T_y)} = \ {}^{sym}_{K}(t_F^2) \ SW_X$. Thus to calculate $I(X_K;T_y)$, it remains only to calculate the Seiberg–Witten invariant of $X_{K;T_y}(0;1;0)$, the manifold obtained by the surgery on T_y which makes y' bound a disk.

Let $M_K(y)$ denote the result of surgery on y in M_K with the Lagrangian framing. In terms of the usual nullhomologous framing, this is the result of surgery on the link $K \cup y$ in S^3 with framings 0 on K and (y) on y. In case $(y) \neq 0$, we have $b_1(M_K(y)) = 1$ and if $(y) = 0$ then $b_1(M_K(y)) = 2$. In this case, the extra generator of $H_1(M_K(y);\mathbf{R})$ is given by a meridian to y in S^3. Accordingly, the Seiberg–Witten invariant of $M_K(y)$ (equivalently, the Seiberg–Witten invariant of $S^1 \quad M_K(y)$) is given by

$$SW_{M_K(y)} = \begin{cases} {}^{sym}_{M_K(y)}(t^2) \ (t - t^{-1})^{-2}; & (y) \neq 0 \\ {}^{sym}_{M_K(y)}(t^2; s^2); & (y) = 0 \end{cases} \tag{5}$$

where t corresponds to the meridian of K and s to the meridian of y.

Proposition 4.3 *Suppose that* $(y) \neq 0$, *then* $|\ _{M_K(y)}(1)| = |\ (y)|$.

Proof We have $H_1(M_K(y)) = \mathbf{Z} \quad \mathbf{Z}_{|\ (y)|}$. It is a well-known fact [18] that for 3–manifolds with $b_1 = 1$, the sum of the coe cients of the Alexander polynomial is, up to sign, the order of the torsion of H_1. □

Let Z denote the 3–manifold obtained from M_K by doing $+1$ surgery on T_γ with respect to the Lagrangian framing. This is the surgery that kills the class $[\gamma'] + [\partial D^2]$ on the boundary of a tubular neighborhood $\gamma \times D^2$ of γ. Then $H_1(Z) = \mathbb{Z} \oplus \mathbb{Z}_b$ where $b = |\kappa(\gamma) + 1|$. Since M_K is fibered over the circle, the manifold Z is also fibered over the circle with the same fiber Σ. (This is true for any $(1/p)$–Lagrangian-framed surgery. The effect of such a surgery on the monodromy is to compose it with the pth power of a Dehn twist about γ. See [16, 1].)

If $\kappa(\gamma) \neq -1$, then $b_1(Z) = 1$, and its symmetrized Alexander polynomial $\Delta_Z^{sym}(t)$ is a function of one variable. If $\kappa(\gamma) = -1$, we have $H_1(Z) = \mathbb{Z} \oplus \mathbb{Z}$. In this case, the Alexander polynomial of Z is a 2–variable polynomial $\Delta_Z^{sym}(t, s)$ where s corresponds to the meridian of γ. Let $\Delta_Z^{sym}(t) = \Delta_Z^{sym}(t, 1)$.

Write $\Delta_K^{sym}(t) = a_0 + (t^g + t^{-g}) + \sum_{i=1}^{g-1} a_i(t^i + t^{-i})$. (This is equal to $\Delta_{M_K}^{sym}(t)$.)

Lemma 4.4 *The symmetrized Alexander polynomial of Z is given by*

$$\Delta_Z^{sym}(t) = b_0 + (t^g + t^{-g}) + \sum_{i=1}^{g-1} b_i(t^i + t^{-i}), \quad \kappa(\gamma) \neq -1$$

$$\Delta_Z^{sym}(t) = \left(b_0 + (t^g + t^{-g}) + \sum_{i=1}^{g-1} b_i(t^i + t^{-i})\right)(t^{1/2} - t^{-1/2})^{-2}, \quad \kappa(\gamma) = -1$$

for some choice of coefficients b_i.

Proof For a 3–manifold with $b_1 = 1$ which is fibered over the circle, the Alexander polynomial is the characteristic polynomial of the (homology) monodromy. (Compare [18, VII.5.d].) This is a monic symmetric polynomial of degree $2g$, as claimed.

In case $\kappa(\gamma) = -1$, one can either apply the theorem of Turaev *op.cit.* or apply the work of Hutchings and Lee. According to [10] together with Mark [11], after appropriate symmetrization, the zeta invariant of the monodromy, namely the characteristic (Laurent) polynomial of the homology monodromy times the term $(t^{1/2} - t^{-1/2})^{-2}$ is equal to a (Laurent) polynomial in t, whose coefficient of t^n is the sum over m of the coefficients of all terms of $\Delta_Z^{sym}(t, s)$ of the form $a_{n,m}t^n s^m$. In other words, $\Delta_Z^{sym}(t) = \Delta_Z^{sym}(t, 1)$ is this Laurent polynomial. This proves the second statement of the lemma. □

Lemma 4.5 *The Seiberg–Witten invariant of $X_{K,T_\gamma}(0, 1, 0)$ is*

$$\overline{SW}_{(X_{K,T_\gamma}(0,1,0),T_\gamma)} = \left(\Delta_Z^{sym}(t_F^2) - \Delta_K^{sym}(t_F^2)\right) \cdot SW_X$$

if $\lambda(\gamma) \neq -1$, *and if* $\lambda(\gamma) = -1$:

$$\overline{SW}_{(X_{K;T_\gamma}(0,1,0);T_\gamma)} = \left(\Delta^{sym}_Z(t_F^2)\cdot(t_F - t_F^{-1})^2 - \Delta^{sym}_K(t_F^2)\right)\cdot SW_X;$$

Proof The result of (+1)–Lagrangian-framed surgery on T_γ in X_K is the fiber sum $X\#_{F=S^1\times m}(S^1\times Z)$. If $\lambda(\gamma)\neq -1$, it follows from (4) and the usual gluing formulas that this manifold has Seiberg–Witten invariant equal to $\Delta^{sym}_Z(t_F^2)\cdot SW_X$. Applying the surgery formula,

$$\overline{SW}_{(X\#_{F=S^1\times m}(S^1\times Z);T_\gamma)} = \overline{SW}_{(X_{K;T_\gamma}(0,1,0);T_\gamma)} + \overline{SW}_{(X_K;T_\gamma)}$$

or

$$\Delta^{sym}_Z(t_F^2)\cdot SW_X = \overline{SW}_{(X_{K;T_\gamma}(0,1,0);T_\gamma)} + \Delta^{sym}_K(t_F^2)\cdot SW_X$$

and the lemma follows.

If $\lambda(\gamma) = -1$, then

$$\begin{aligned}\overline{SW}_{(X\#_{F=S^1\times m}(S^1\times Z);T_\gamma)} &= SW_X\cdot \overline{SW}_{(S^1\times Z;T_\gamma)}\cdot(t_F - t_F^{-1})^2\\ &= SW_X\cdot\Delta^{sym}_Z(t_F^2)\cdot(t_F - t_F^{-1})^2\end{aligned}$$

and the result follows as above. □

Theorem 4.6 *Let* X *be a symplectic* 4–*manifold with* $b^+>1$ *containing an embedded self-intersection* 0 *torus* F *with a vanishing cycle. Let* K *be a nontrivial fibered knot, and let* γ *be an embedded loop on a fiber of* $S^3\setminus K \to S^1$. *Then the Lagrangian framing invariant* $\lambda(T_\gamma)$ *is an orientation-preserving diffeomorphism invariant of the pair* $(X_K;T_\gamma)$.

Proof Using the notation above and Lemmas 4.4 and 4.5,

$$\overline{SW}_{(X_{K;T_\gamma}(0,1,0);T_\gamma)} = \left((b_0-a_0)+\sum_{i=1}^{g-1}(b_i-a_i)(t_F^{2i}+t_F^{-2i})\right)\cdot SW_X \qquad (6)$$

It follows from (3) that

$$\begin{aligned}\overline{SW}_{(X_{K;T_\gamma(0,p,q)};T_\gamma)} &= p\left((b_0-a_0)+\sum_{i=1}^{g-1}(b_i-a_i)(t_F^{2i}+t_F^{-2i})\right)\cdot SW_X\\ &\quad + q\left(a_0+(t_F^{2g}+t_F^{-2g})+\sum_{i=1}^{g-1}a_i(t_F^{2i}+t_F^{-2i})\right)\cdot SW_X \qquad (7)\end{aligned}$$

Since ${}^{sym}_{K}(1) = \;\; 1$, we have $2 + a_0 + 2\sum_{1}^{g-1} a_i = " = \;\; 1$. Furthermore,

$$\overline{SW}_{(X_{K;T_\gamma}(0;1;0);T_\gamma)} = \begin{cases} SW_X \;\; {}^{sym}_{M_K(\gamma)}(t_F^2); & (\gamma) \neq 0; \\ SW_X \;\; {}^{sym}_{M_K(\gamma)}(t_F^2) \;\; (t_F - t_F^{-1})^2; & (\gamma) = 0: \end{cases} \tag{8}$$

(See (5).) Thus if $(\gamma) \neq 0$,

$${}^{sym}_{M_K(\gamma)}(t^2) = (b_0 - a_0) + \sum_{i=1}^{g-1} (b_i - a_i)(t^{2i} + t^{-2i})$$

and by Proposition 4.3,

$$(b_0 - a_0) + 2\sum_{1}^{g-1} (b_i - a_i) = \;\; (\gamma) = \;\; (\gamma) \tag{9}$$

If $(\gamma) = 0$, it follows from equations (6) and (8) that

$${}^{sym}_{M_K(\gamma)}(t^2) \;\; (t - t^{-1})^2 = (b_0 - a_0) + \sum_{i=1}^{g-1} (b_i - a_i)(t^{2i} + t^{-2i})$$

so $(b_0 - a_0) + 2\sum_{1}^{g-1} (b_i - a_i) = 0$ in this case, and we see that (9) holds in general.

Let $(p;q)$ be the sum of all coe cients of $\overline{SW}_{(X_{K;T_\gamma}(0;p;q);T_\gamma)}{=}SW_X$ from terms of degree not equal to $2g$. Then it follows from (7) that $(p;q) = p \;\; (\gamma) + q("-2)$. Let $(p;q)$ be the coe cient of t_F^{2g} in $\overline{SW}_{X_{T_\gamma}(0;p;q)}{=}SW_X$; so $(p;q) = q$.

We have seen that

$$I(X_K;T_\gamma) = \{\overline{SW}_{(X_{K;T_\gamma}(a;p;q);T_\gamma)} | a;p;q \in \mathbb{Z}\} = \{\overline{SW}_{(X_{K;T_\gamma}(0;p;q);T_\gamma)} | p;q \in \mathbb{Z}\}$$

(the last equality by Proposition 4.2) is an orientation-preserving di eomorphism invariant of the pair $(X_K;T_\gamma)$. From $I(X_K;T_\gamma)$ we can extract the invariant

$$\gcd\{| \;\; (p;q) + (2 - ") \;\; (p;q)| \;\; p;q \in \mathbb{Z}\} = \gcd\{|p \;\; (\gamma)| \;\; p \in \mathbb{Z}\} = | \;\; (\gamma)| = \;\; (T_\gamma):$$

□

Theorem 4.7 *Let X be a symplectic 4–manifold with $b^+ > 1$ containing an embedded self-intersection 0 torus F with a vanishing cycle, and let K be a nontrivial bered knot. Then in X_K there is an in nite sequence of pairwise inequivalent nullhomologous Lagrangian tori T_{γ_n}.*

Proof Choose a sequence of loops γ_n as in the statement of Lemma 3.7, then it is clear that the elements γ_n give inequivalent T_{γ_n}. □

5 Circle sums and essential Lagrangian tori

We next discuss a technique for utilizing families of inequivalent nullhomologous Lagrangian tori to build families of inequivalent essential homologous Lagrangian tori. For a fibered knot K in S^3, let m_1, m_2 be meridians, and consider a meridian γ_1 of m_1 in M_K as shown in Figure 3.

Let X_i, $i = 1,2$ be symplectic 4–manifolds, containing embedded symplectic tori F_i of self-intersection 0, and suppose that F_1 has a vanishing cycle. Denote by X the fiber sum $X = X_1 \#_{F_1=F_2} X_2$. Then

$$X_K = (X_1 \#_{F_1=F_2} X_2)_K = X_1 \#_{F_1=S^1\times m_1} S^1 \times M_K \#_{S^1\times m_2=F_2} X_2$$

As in the previous section, we insist that the gluing map from the boundary of a tubular neighborhood of $S^1\times m_1$ to the boundary of a tubular neighborhood of F_1 should take $S^1\times \mathrm{pt}\times \mathrm{pt}$ to the loop $\gamma\times\mathrm{pt}$ representing the vanishing cycle. Because $T = S^1\times\gamma_1$ is a rim torus to F_1 it follows easily that T is a Lagrangian torus which represents an essential, in fact primitive, class in $H_2(X_K)$.

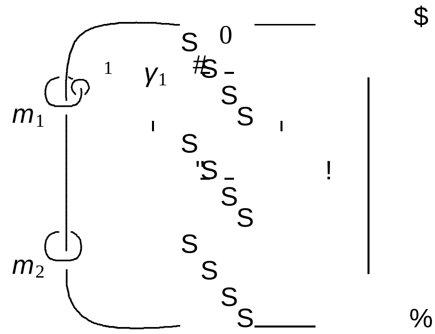

Figure 3

Proposition 5.1 $I(X_K,T) = \{q\,\overline{SW}_{(X_K,T)} \mid q \in \mathbb{Z}\}$.

Proof If we choose a tubular neighborhood $T\times D^2$ with the Lagrangian framing, then we may use the basis $\{[S^1\times\{y\}],[\gamma_1'],[\partial D^2]\}$ for $H_1(\partial(T\times D^2))$, and as in Proposition 4.2, $SW_{X_{K,T}(1,0,0)} = 0$. Of course, $SW_{X_{K,T}(0,0,1)} = SW_{X_K}$. It remains to calculate $SW_{X_{K,T}(0,1,0)}$. This can be done via Kirby calculus. Let Y denote the result of surgery on γ_1 in M_K with respect to the Lagrangian framing. This is shown in Figure 4.

Slide the 0–surgered handle corresponding to K over the 0–surgered $\;_1$, to get Figure 5. Thus $Y = M_K \# (S^1 \quad S^2)$. Hence

$$X_{K;T}(0;1;0) = (X_1 \#_{F_1 = S^1 \quad m_1} T^2 \quad S^2) \setminus (D^3 \quad S^1) \cup_{S^2 \quad S^1} (X_2 \#_{F_2 = S^1 \quad m_2} S^1 \quad M_K) \setminus (D^3 \quad S^1)$$

Each of the two sides of the union has $b^+ \quad 1$, and $S^2 \quad S^1$ admits a metric of positive scalar curvature. This means that $SW_{X_{K;T}(0;1;0)} = 0$, and the proposition follows. □

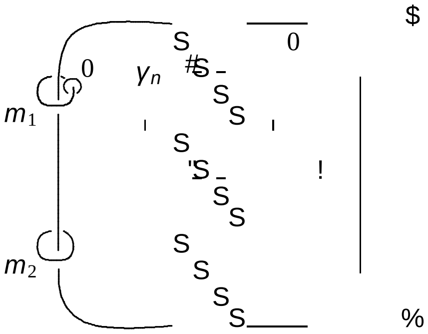

Figure 4

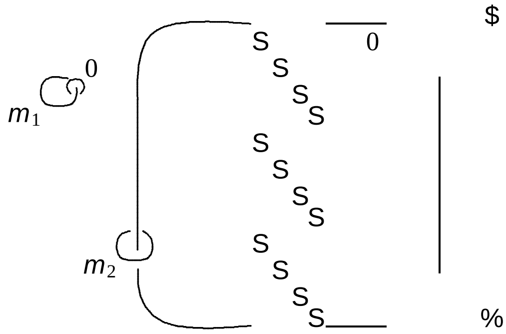

Figure 5

Let $\{T_{y_n}\}$ be the family of nullhomologous Lagrangian tori in X_K given by Theorem 4.7 (thinking of $m = m_1$). For any T_{y_n} in the family, we can form the circle sum of T_{y_n} with T . This is done by xing a path in the ber running from a point of y_n to a point in $\;_1$ and taking the connected sum y_n' of y_n with $\;_1$ along this path. (See Figures 4 and 6.) The resulting torus $T_{y_n'} = S^1 \quad y_n'$ in X is Lagrangian and homologous to T . Note that $(y_n') = (y_n)$; in fact, y_n' and y_n are isotopic in M_K.

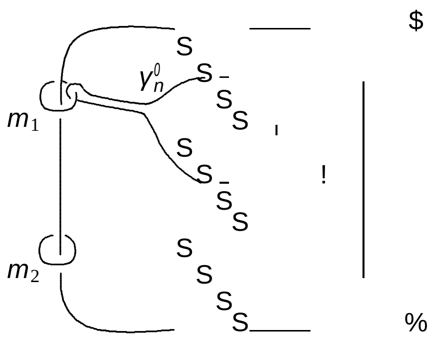

Figure 6

We wish to calculate $I(X_K;T_{y_n^0})$. Using, as in the previous section, the basis adapted to the Lagrangian framing of $T_{y_n^0}$, because $SW_{X_{K;T_{y_n^0}}(1;0;0)} = 0$ and $SW_{X_{K;T_{y_n^0}}(0;0;1)} = SW_{X_K}$, we only need to calculate $\overline{SW}_{(X_{K;T_{y_n^0}}(0;1;0);T_{y^0})}$.

Consider $M_K(y_n^0)$ the 3–manifold obtained by surgery on y_n^0 in M_K using the Lagrangian framing. Since y_n^0 and y_n are isotopic in M_K, the manifolds $M_K(y_n^0)$ and $M_K(y_n)$ are diffeomorphic. We may as well assume that $(y) \neq 0$. We have $H_1(M_K(y_n^0)) = \mathbb{Z} \quad \mathbb{Z}_{j \ (y_n)j}$ where the infinite cyclic summand is generated by the class of m_2 and the finite cyclic summand by the class of , a meridian to y_n. Note that $[m_1] = [m_2] + [\ ]$.

To obtain $M_K(y_n^0)$, one does surgery which kills the curve $y_n^0 + \ (y_n)$. Thus $[y_n^0] = - \ (y_n)[\ ]$ in $H_1(M_K(y_n^0) \setminus (m_1 \cup m_2))$. Furthermore, in the manifold with boundary $M_K(y_n^0) \setminus (m_1 \cup m_2)$, the core C of the surgery solid torus is homologous to $(1/\ (y_n))\, y_n^0$. We are interested in

$$X_{K;T_{y_n^0}}(0;1;0) = X_1 \#_{F_1 = S^1 \ m_1} S^1 \quad M_K(y_n^0) \#_{S^1 \ m_2 = F_2} X_2$$

In $H_2(X_{K;T_{y_n^0}}(0;1;0))$ we have

$$[F_1] = [F_2] + [S^1 \quad] = [F_2] - (1/\ (y_n))[S^1 \quad y_n^0] = [F_2] - [S^1 \quad C]$$

Let $T = S^1 \quad C$. We need to calculate $\overline{SW}_{(X_{K;T_{y_n^0}}(0;1;0);T_{y_n^0})}$ which is an element of

$$\mathbb{Z}H_2(X_{K;T_{y_n^0}}(0;1;0) \setminus T \quad D^2;\partial) = \mathbb{Z}H_2(X_{K;T_{y_n^0}}(0;1;0);T)$$

Thus we may assume that $[F_1] = [F_2] = [F]$, say, for the purpose of this calculation. Now precisely the same proof as that of Theorem 4.6 (replacing X by $X_1 \#_{F_1 = F_2} X_2$) gives:

Theorem 5.2 *Let X_i, $i = 1;2$ be symplectic 4–manifolds containing embedded symplectic tori F_i of self-intersection 0. Suppose also that F_1 has a vanishing cycle. Set $X = X_1 \#_{F_1 = F_2} X_2$. Let K be a nontrivial fibered knot and let y be an embedded loop on a fiber of $S^3 \setminus K \to S^1$. Let m_1 and m_2 be meridians to K which do not link y, a meridian to m_1 which lies on , and y^0 the connected sum of y and in . Then $T_{y^0} = S^1 \quad y^0$ is a Lagrangian torus in X_K and represents a primitive homology class.*

If is another loop on which has linking number 0 with m_1 and m_2 and ^{0} is the connected sum of and in with corresponding Lagrangian torus $T_{\ 0} = S^1 \quad ^0$, then $T_{\ 0}$ is homologous to T_{y^0} and if T_{y^0} and $T_{\ 0}$ are equivalent in X_K, it follows that $(T\ ) = (T_y)$. □

This completes the proof of Theorem 1.1. Since the fiber F of $E(1)$ has a vanishing cycle and since $(E(1)\#_F X)_K = X_K\#_F E(1)$ we have, for example, the following corollary:

Corollary 5.3 *Let X be any symplectic 4–manifold containing an embedded symplectic torus of self-intersection 0. Let K be any nontrivial fibered knot. Then in $X_K\#_F E(1)$ there is an infinite sequence of essential Lagrangian tori $T_{\gamma'_n}$ which are pairwise homologous but no two of which are equivalent.* □

6 Symplectically inequivalent Lagrangian tori which are not distinguished by relative Seiberg–Witten invariants

These examples live in simple versions of the manifolds constructed in [6]. It would be very easy to give much larger classes of examples, but we shall content ourselves with those below.

Let X be the K3–surface with elliptic fiber F, and let K be a nontrivial fibered knot. Let γ_1, γ_2 be embedded loops on a fiber of $S^3\setminus K\to S^1$ with different $\phi(\gamma_i)>0$, and consider the nullhomologous Lagrangian tori $T_{\gamma_i}\subset X_K$. Theorem 4.6 implies that these tori are C^∞–inequivalent. Suppose that the knot K has genus g. In the construction of X_K we have replaced a 2–disk in a section of the elliptic fibration of K3 with a punctured surface of genus g, a fiber of $S^3\setminus K\to S^1$. Thus X_K contains a symplectic genus g surface S of self-intersection -2 and $[S]\cdot[F]=1$. Consider another smooth fiber F' of the elliptic fibration of $(X\setminus N(T))\subset X_K$. Then $F'+S$ is a singular surface with one double point, which can be smoothed to give a symplectically embedded surface S' of genus $g+1$ representing the homology class $[S']=[F]+[S]$. Thus $[S']^2=0$ and $[S']\cdot[F]=1$.

Next, let K' denote the trefoil knot in S^3. Since K' is a fibered genus 1 knot, the 4–manifold $S^1\times M_{K'}$ is a smooth T^2–fiber bundle over T^2. We obtain a symplectic manifold Y by forming the fiber sum of $g+1$ copies of $S^1\times M_{K'}$ along the tori T where m_i is a meridian of K' in the ith copy of $M_{K'}$. There is a fiber bundle

$$\begin{array}{ccl} T^2 \longrightarrow & Y = & S^1\times M_{K'}\#_{T^2}\cdots\#_{T^2}S^1\times M_{K'} \\ & \Big\downarrow & \\ & C_0 & \end{array}$$

where C_0 is a genus $g+1$ surface. There is a symplectic section $C \quad Y$ given by the connected sum of the individual sections $S^1 \quad m_i$. Now form the symplectic manifold $Z_K = X_K \#_{S^0 = C} Y$. We perform the ber sum so that Z_K is spin [9]. In [6] it is shown that Z_K is simply connected and that $SW_{Z_K} = t \quad + (-1)^g t^{-1}$ where is the canonical class of Z_K.

Since $S^0 \quad X_K$ can be chosen disjoint from a nullhomology of T_{γ_i}, we still have $(T_{\gamma_i}) = (\gamma_i)$ for the Lagrangian framing invariants of the nullhomologous Lagrangian tori $T_{\gamma_i} \quad Z_K$. Thus T_{γ_1} and T_{γ_2} are symplectically inequivalent in Z_K.

We next compute $I(Z_K; T_{\gamma_i})$, using the same basis that we used in Section 4. As in Proposition 4.2, $SW_{Z_{K;T_\gamma}(1;0;0)} = 0$. The point is that

$$Z_{K;T_\gamma}(1;0;0) = X_{K;T_\gamma}(1;0;0) \#_{S^0 = C} Y$$

and the sphere of self-intersection -1 and its dual torus of square 0 found in Proposition 4.2 both live in the complement of the surface S^0.

Next consider $Z_{K;T_\gamma}(0;1;0)$. For any manifold X obtained from X by surgery on T_γ, $H_2(X) = H_2(X)$ if the surgery curve is homologically nontrivial in $X \, n \, T_\gamma$ and $H_2(X) = H_2(X) \quad U$ where U has rank 2 if the surgery curve is homologically trivial in $X \, n \, T\gamma$. Since $(\gamma_i) > 0$, the surgery curve γ^0 for the surgery giving $Z_{K;T_\gamma}(0;1;0)$ is not nullhomologous. This means that for

$$Z_{K;T_\gamma}(0;1;0) = X_{K;T_\gamma}(0;1;0) \#_{S^0 = C} Y$$

$H_2(Z_{K;T_\gamma}(0;1;0)) = H_2(Z_K)$, and the arguments of [6] using the adjunction inequality and [14] to show that $SW_{Z_K} = t \quad + (-1)^g t^{-1}$, will again show that $SW_{Z_{K;T_\gamma}(0;1;0)} = t \quad + (-1)^g t^{-1}$. Hence $I(Z_K; T_{\gamma_i}) = fp(t \quad + (-1)^g t^{-1}) + q(t \quad + (-1)^g t^{-1}) jp; q \; 2 \; Zg = fr(t \quad + (-1)^g t^{-1}) jr \; 2 \; Zg$, independent of γ_i; so relative Seiberg{Witten invariants don't detect whether or not the T_{γ_i} are C^1{equivalent. It would be extremely interesting if they were not.

7 Discussion

As we have already mentioned in Section 1, the rst examples like those of Corollary 5.3 were recently discovered by S. Vidussi [19]. The examples of [19] live in $E(2)_K$ and are of the type described in Theorem 5.2, thus they can be distinguished by using the Lagrangian framing invariant. Again view $E(2)_K$ as a double ber sum, $E(1) \#_{F = S^1 \quad m_1} S^1 \quad M_K \#_{S^1 \quad m_2 = F} E(1)$. Vidussi points out that for a torus $T = S^1 \quad ,$ a loop in whose linking number with m_2

is 0, the homology class of T in $E(2)_K$ is determined by the linking number of with m_1. We have restricted ourselves to the case where the linking number is 1, but this is completely unnecessary, and the Lagrangian framing invariant gives an invariant for the general situation.

One need not restrict to genus one Lagrangian submanifolds in order to take advantage of the technique of Lagrangian circle sums with nullhomologous Lagrangian tori. However the authors have not yet been able to nd invariants for higher genus Lagrangian surfaces which are as simple to calculate as those in this paper.

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