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# BPS states of curves in Calabi{Yau 3{folds

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## Abstract

The Gopakumar{Vafa integrality conjecture is de ned and studied for the local geometry of a super-rigid curve in a Calabi{Yau 3{fold. The integrality predicted in Gromov{Witten theory by the Gopakumar{Vafa BPS count is veri ed in a natural series of cases in this local geometry. The method involves Gromov{Witten computations, Möbius inversion, and a combinatorial analysis of the numbers of etale covers of a curve.

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# 1 Introduction and Results

## 1.1 Gromov–Witten and BPS invariants

Let $X$ be a Calabi–Yau 3–fold and let $N^g_\beta(X)$ be the 0–point genus $g$ Gromov–Witten invariant of $X$ in the curve class $\beta \in H_2(X;\mathbb{Z})$. From considerations in M–theory, Gopakumar and Vafa express the invariants $N^g_\beta(X)$ in terms of integer invariants $n^g_\beta(X)$ obtained by BPS state counts [8]. The Gopakumar–Vafa formula may be viewed as providing a definition of the BPS state counts $n^g_\beta(X)$ in terms of the Gromov–Witten invariants.

**Definition 1.1** Define the *Gopakumar–Vafa BPS invariants* $n^r_\beta(X)$ by the formula:

$$\sum_{\beta\neq 0}\sum_{g\geq 0} N^g_\beta(X)t^{2g-2}q^\beta = \sum_{\beta\neq 0}\sum_{g\geq 0} n^g_\beta(X)\sum_{k>0}\frac{1}{k}\left(2\sin\left(\frac{kt}{2}\right)\right)^{2g-2}q^{k\beta}. \qquad (1)$$

Matching the coefficients of the two series yields equations determining $n^g_\beta(X)$ recursively in terms of $N^g_\beta(X)$ (see Proposition 2.1 for an explicit inversion of this formula).

From the above definition, there is no (mathematical) reason to expect $n^g_\beta(X)$ to be an integer. Thus, the physics makes the following prediction.

**Conjecture 1.2** *The BPS invariants are integers:*

$$n^g_\beta(X) \in \mathbb{Z}.$$

*Moreover, for any fixed* $\beta$, $n^g_\beta(X) = 0$ *for* $g \gg 0$.

**Remark 1.3** By the physical arguments of Gopakumar and Vafa, the BPS invariants should be directly defined via the cohomology of the $D$–brane moduli space. First, the $D$–brane moduli space $\hat{M}$ should be defined with a natural morphism $\hat{M} \to M$ to a moduli space $M$ of curves in $X$ in the class $\beta$. The fiber of $\hat{M} \to M$ over each curve $C \in M$ should parameterize flat line bundles on $C$. Furthermore, there should exist an $\mathfrak{sl}_2 \times \mathfrak{sl}_2$ representation on $H^*(\hat{M};\mathbb{C})$ such that the diagonal and right actions are the usual $\mathfrak{sl}_2$ Lefschetz representations on $H^*(\hat{M};\mathbb{C})$ and $H^*(M;\mathbb{C})$ respectively — assuming $\hat{M}$ and $M$ are compact, nonsingular, and Kähler. The BPS state counts $n^g_\beta(X)$ are then the coefficients in the decomposition of the left (fiberwise) $\mathfrak{sl}_2$ representation $H^*(\hat{M};\mathbb{C})$ in the basis given by the cohomologies of the algebraic tori. After these foundations are developed, Equation (1) should be *proven* as the basic result relating Gromov–Witten theory to the BPS invariants.

The correct mathematical definition of the D–brane moduli space is unknown at present, although there has been recent progress in case the curves move in a surface $S \subset X$ (see [12], [13], [14]). The nature of the D–brane moduli space in the case where there are non-reduced curves in the family $M$ is not well understood. The fiber of $\widehat{M} \to M$ over a point corresponding to a non-reduced curve may involve higher rank bundles on the reduction of the curve. It has been recently suggested by Hosono, Saito, and Takahashi [11] that the $\mathfrak{sl}_2 \times \mathfrak{sl}_2$ representation can be constructed in general via intersection cohomology and the Beilinson–Bernstein–Deligne spectral sequence [1].

**Remark 1.4** An extension of formula (1) conjecturally defining integer invariants for arbitrary 3–folds (not necessarily Calabi–Yau) has been found in [16], [17]. Some predictions in the non Calabi–Yau case have been verified in [2]. Though it is not yet known how the relevant physical arguments apply to the non Calabi–Yau geometries, one may hope a mathematical development will provide a unified approach to all 3–folds.

The physical discussion suggests that the BPS invariants will be a sum of integer contributions coming from each component of the D–brane moduli space (whatever space that may be). One obvious source of such components occurs when the curves parameterized by $M$ are rigid or lie in a fixed surface. The moduli space of stable maps has corresponding components given by those maps whose image is the rigid curve or respectively lies in the fixed surface. These give rise to the notion of "local Gromov–Witten invariants" and we expect that the corresponding "local BPS invariants" will be integers.

## 1.2 Local contributions

In this paper we are interested in the contributions of an isolated curve $C \subset X$ to the Gromov–Witten invariants $N^g_{d[C]}(X)$ and the BPS invariants $n^g_{d[C]}(X)$.

To discuss the local contributions of a curve (also often called "multiple cover contributions"), we make the following definitions:

**Definition 1.5** Let $C \subset X$ be a curve and let $M_C \subset \overline{M}_g(X; d[C])$ be the locus of maps whose image is $C$. Suppose that $M_C$ is an open component of $\overline{M}_g(X; d[C])$. Define the *local Gromov–Witten invariant*, $N^g_d(C \subset X) \in \mathbb{Q}$ by the evalution of the well-defined restriction of $[\overline{M}_g(X; d[C])]^{vir}$ to $H_0(M_C; \mathbb{Q})$.

**Definition 1.6** Let $C \subset X$ satisfy the conditions of Definition 1.5. If

$$M_C = \overline{M}_g(C,d)$$

then $C$ is said to be $(d,g)$–*rigid*. If $C$ is $(d,g)$–rigid for all $d$ and $g$, then $C$ is *super-rigid*.

For example, a nonsingular rational curve with normal bundle $O(-1) \oplus O(-1)$ is super-rigid. An elliptic curve $E \subset X$ is super-rigid if and only if $N_{E/X} = L \oplus L^{-1}$ where $L \to E$ is a flat line bundle such that no power of $L$ is trivial (see [16]). An example where $M_C$ is an open component but $M_C \neq \overline{M}_g(C,d)$ is the case where $C \subset X$ is a contractable, smoothly embedded $\mathbf{CP}^1$ with $N_{C/X} = O \oplus O(-2)$. In this case $M_C$ has non-reduced structure coming from the (obstructed) infinitesimal deformations of $C$ in the $O$ direction of $N_{C/X}$ (see [4] for the computation of $N^g_d(C \subset X)$ in this case).

The existence of genus $g$ curves in $X$ with $(d,g+h)$–rigidity is likely to be a subtle question in the algebraic geometry of Calabi–Yau 3–folds. On the other hand, these rigidity issues may be less delicate in the symplectic setting. For a generic almost complex structure on $X$, it is reasonable to hope super-rigidity will hold for any pseudo-holomorphic curve in $X$.

Let $h \geq 0$ and suppose a nonsingular genus $g$ curve $C_g \subset X$ is $(d,g+h)$–rigid. Then $N^{g+h}_d(C_g \subset X)$ can be expressed as the integral of an Euler class of a bundle over $[\overline{M}_{g+h}(C_g,d)]^{vir}$. Let $\pi: U \to \overline{M}_{g+h}(C_g,d)$ be the universal curve and let $f: U \to C_g$ be the universal map. Then

$$N^{g+h}_d(C_g \subset X) = \int_{[\overline{M}_{g+h}(C_g,d)]^{vir}} c(R^1\pi_* f^*(N_{C/X})).$$

In fact, we can rewrite the above integral in the following form:

$$\begin{aligned} \int c(R^1\pi_* f^* N_{C/X}) &= \int c(R\pi_* f^* N_{C/X}[1]) \\ &= \int c(R\pi_* f^*(O_C \oplus \omega_C)[1]) \end{aligned}$$

where all the integrals are over $[\overline{M}_{g+h}(C_g,d)]^{vir}$. The first equality holds because $(d,g+h)$–rigidity implies that $R^0\pi_* f^* N_{C/X}$ is 0. The second equality holds because $N_{C/X}$ deforms to $O_C \oplus \omega_C$, the sum of the trivial sheaf and the canonical sheaf (this follows from an easily generalization of the argument at the top of page 497 in [16]). The last integral depends only upon $g$, $h$, and $d$. We regard this formula as defining the idealized multiple cover contribution of a genus $g$ curve by maps of degree $d$ and genus $g+h$.

We will denote this idealized contribution by the following notation:

$$N_d^h(g) := \int_{[\overline{M}_{g+h}(C_g,d)]^{vir}} c(R^\bullet \pi_* f^*(\mathcal{O}_C \oplus \omega_C)[1]).$$

From the previous discussion, $N_d^h(g) = N_d^{g+h}(C_g)$ for any nonsingular, $(d,g+h)$–rigid, genus $g$ curve $C_g$.

We define the local BPS invariants in terms of the local Gromov–Witten invariants via the Gopakumar–Vafa formula.

**Definition 1.7** Define the *local BPS invariants* $n_d^h(g)$ in terms of the local Gromov–Witten invariants by the formula

$$\sum_{d\neq 0}\sum_{h\geq 0} N_d^h(g)\, t^{2(g+h-1)} q^d = \sum_{d\neq 0}\sum_{h\geq 0} n_d^h(g) \sum_{k>0} \frac{1}{k}\left(2\sin\left(\frac{kt}{2}\right)\right)^{2(g+h-1)} q^{kd}.$$

The local Gromov–Witten invariants $N_d^h(g)$ are in general difficult to compute. For $g=0$, these integrals were computed in [6]. In terms of local BPS invariants, these calculations yield:

$$n_d^h(0) = \begin{cases} 1 & \text{for } d=1 \text{ and } h=0, \\ 0 & \text{otherwise.} \end{cases}$$

For $g=1$, complete results have also been obtained [16]:

$$n_d^h(1) = \begin{cases} 1 & \text{for } d\geq 1 \text{ and } h=0, \\ 0 & \text{otherwise.} \end{cases}$$

The local invariants of a super-rigid nodal rational curve as well as the local invariants of contractable (non-generic) embedded rational curves were determined in [4].

In this paper we compute certain contributions to the local Gromov–Witten invariants $N_d^h(g)$ for $g>1$ and we determine the corresponding contributions to the BPS invariants $n_d^h(g)$. We prove the integrality of these contributions. In the appendix, we provide tables giving explicit values for $n_d^h(g)$.

## 1.3 Results

The contributions to $N^h_d(g)$ we compute are those that come from maps $[f : D \to C]$ satisfying either of following conditions:

(i) A single component of the domain is an etale cover of $C$ (with any number of auxiliary collapsed components *simply* attached to the etale component).

(ii) The map $f$ has exactly two branch points (and no collapsed components).

The type (i) contributions, the *etale invariants*, correspond to the first level in a natural grading on the set of local Gromov–Witten invariants which will be discussed in Section 2.2. We use an elementary observation to reduce the computation of the etale invariants to the computation of the degree 1 local invariants, ie, $n^h_1(g)$. The computation of the degree 1 invariants was done previously by the second author in [16]. The observation that we use, while elementary, seems useful enough to formalize in a general setting. This we do by the introduction of *primitive Gromov–Witten invariants* in Section 2.2.

The type (ii) contribution we compute by a Grothendieck–Riemann–Roch calculation which is carried out in Section 4.

### 1.3.1 Type (i) contributions (etale contributions)

**Definition 1.8** We define $\overline{M}^{et}_{g+h}(C,d) \subset \overline{M}_{g+h}(C,d)$ to be the union of the moduli components corresponding to stable maps $\pi : D \to C$ satisfying:

(a) $D$ contains a unique component $C'$ of degree $d$, etale over $C$, while all other components are degree 0.

(b) All $\pi$–collapsed components are all simply attached to $C'$ (the vertex in the dual graph of the domain curve corresponding to $C'$ does not contain a cycle).

We define the etale Gromov–Witten invariants by

$$N^h_d(g)^{et} := \int_{[\overline{M}_{g+h}(C,d)^{et}]^{vir}} c(R^1\pi_* f^*(\mathcal{O}_C \oplus \omega_C)[1])$$

and we define the etale BPS invariants $n^h_d(g)^{et}$ in terms of $N^h_d(g)^{et}$ via the Gopakumar–Vafa formula as before.

As we will explain in Section 2, any Gromov–Witten invariant can be written in terms of *primitive* Gromov–Witten invariants. The etale invariants exactly correspond to those that can be expressed in terms of degree 1 primitive invariants.

Our main two Theorems concerning the etale BPS invariants give an explicit formula for $n_d^h(g)^{et}$ and prove they are integers.

**Theorem 1.9** *Let $C_{n;g}$ be number of degree $n$, connected, complete, etale covers of a curve of genus $g$, each counted by the reciprocal of the number of automorphisms of the cover. Let $\mu$ be the Möbius function: $\mu(n) = (-1)^a$ where $a$ is the number of prime factors of $n$ if $n$ is square-free and $\mu(n) = 0$ if $n$ is not square-free. Then the etale BPS invariants are given as the coefficients of the following polynomial:*

$$\sum_{h \geq 0} n_d^h(g)^{et} y^{h+g-1} = \sum_{k|d} k\,\mu(k)\, C_{\frac{d}{k};g}\, P_k(y)^{\frac{d(g-1)}{k}}$$

*where the polynomial $P_k(y)$ is defined*[1] *by*

$$P_k(4\sin^2 t) = 4\sin^2(kt)$$

*which by Lemma P1 is given explicitly by*

$$P_k(y) = \sum_{a=1}^{k} -\frac{k}{a}\binom{a+k-1}{2a-1}(-y)^a.$$

**Theorem 1.10** *The etale BPS invariants are integers: $n_d^h(g)^{et} \in \mathbb{Z}$.*

We note that $C_{n;g}$ is not integral in general, for example $C_{2;g} = (2^{2g}-1)/2$. We also note that the formula given by the Theorem 1.9 shows that for fixed $d$ and $g$, $n_d^h(g)^{et}$ is non-zero only if $0 \leq h \leq (d-1)(g-1)$. See Table 1 for explicit values of $n_d^h(g)^{et}$ for small $d$, $g$, and $h$.

There is a range where the etale contributions are the only contributions to the full local BPS invariant.

**Lemma 1.11** *Let $d_{min}$ be the smallest divisor $d'$ of $d$ that is not 1 and such that $\mu(\frac{d}{d'}) \neq 0$, then*

$$n_d^h(g) = n_d^h(g)^{et} \quad \text{for all } h \geq (d_{min}-1)(g-1).$$

[1] **Warning:** This definition of $P_k(y)$ differs from the one in [3] by a factor of $y$.

**Proof** This follows from Equation 3 (in Section 2) and the simple geometric fact that a degree $d$ stable map $f\colon D_{g+h} \to C_g$ must be of type (i) if $h \leq (d-1)(g-1)$ or if $d=1$. □

**Remark 1.12** *A priori* there is no reason (even physically) to expect that the etale invariants $n_d^h(g)^{et}$ are integers outside of the range where $n_d^h(g)^{et} = n_d^h(g)$. Theorem 1.10 is very suggestive that the D–brane moduli space has a distinguished component (or components) corresponding to these etale contributions. Furthermore, our results suggest that this component has dimension $d(g-1)+1$ and has a product decomposition (at least cohomologically) with one factor a complex torus of dimension $g$.

Theorem 1.9 follows from the computation of $N_d^h(g)^{et}$ by a (reasonably straightforward) inversion of the Gopakumar–Vafa formula that is carried out in Section 2. Theorem 1.10 is proved directly from the formula given in Theorem 1.9 and turns out to be rather involved. It depends on somewhat delicate congruence properties of the polynomials $P_l(y)$ and the number of covers $C_{d,g}$. These are proved in Section 3.

### 1.3.2 Type (ii) contributions

There is another situation where $\overline{M}_{g+h}(C_g, d)$ has a distinguished open component. If

$$h = (d-1)(g-1)+1,$$

then there are exactly two open components, namely the etale component $M^{et}$ and one other $\tilde{M} \subset \overline{M}_{g+h}(C_g, d)$. The generic points of $\tilde{M}$ correspond to maps of nonsingular curves with exactly two simple ramification points. Let $\tilde{N}_g(d)$ be the corresponding contribution to the Gromov–Witten invariants so that

$$N_d^{(d-1)(g-1)+1}(g) = N_d^{(d-1)(g-1)+1}(g)^{et} + \tilde{N}_d(g).$$

The component $\tilde{M}$ admits a finite morphism to $\mathrm{Sym}^2(C_g)$ given by sending a map to its branched locus (see [6] for the existence of such a morphism).

We compute the invariant $\tilde{N}_d(g)$ in Section 4 by a Grothendieck–Riemann–Roch (GRR) computation. The relative Todd class required by GRR is computed using the formula of Mumford [15] adapted to the context of stable maps (see [6] Section 1.1). The intersections in the GRR formula are computed by pushing forward to $\mathrm{Sym}^2(C_g)$. The result of this computation (which is carried out in Section 4) is the following:

**Theorem 1.13**

$$\tilde{N}_d(g) = \int_{\widetilde{M}} c(R \quad f \ (O_{C_g} \quad !_{C_g})[1]) = \frac{g-1}{8} \ (g-1)D_{d;g} - D_{d;g} - \frac{1}{27}D_{d;g} \ :$$

The numbers $D_{d;g}$, $D_{d;g}$, and $D_{d;g}$ are the following Hurwitz numbers of covers of the curve $C_g$.

- $D_{d;g}$ is the number of connected, degree $d$ covers of $C_g$ simply branched over 2 distinct fixed points of $C_g$.
- $D_{d;g}$ is the number of connected, degree $d$, covers of $C_g$ with 1 node lying over a fixed point of $C_g$.
- $D_{d;g}$ is the number of connected, degree $d$ covers of $C_g$ with 1 double ramification point over a fixed point of $C_g$.

The covers are understood to be etale away from the imposed ramification. Also, $D_{d;g}$, $D_{d;g}$, and $D_{d;g}$ are all counts weighted by the reciprocal of the number of automorphisms of the covers.

There is an additional Hurwitz number $D_{g;d}$ which is natural to consider together with the three above:

- $D_{d;g}$ is the number of connected, degree $d$ covers of $C_g$ with 2 distinct ramification points in the domain lying over a fixed point of $C_g$.

However, $D_{d;g}$ is determined from the previous Hurwitz numbers by the degeneration relation:

$$D_{d;g} = D_{d;g} + 3D_{d;g} + 2D_{d;g} \tag{2}$$

(see [10]). Theorem 1.13 therefore involves all of the independent covering numbers which appear in this 2 branch point geometry (see Table 3 for some explicit values of these numbers).

Theorem 1.13 can be used to extend the range where we can compute the full local BPS invariants. Lemma 1.11 generalizes to

**Lemma 1.14** *Let $d_{min}$ be defined as in Lemma 1.11, then*

$$n_d^h(g) = \begin{cases} n_d^h(g)^{et} & \text{for all } h \quad (d_{min}-1)(g-1) \\ n_d^h(g)^{et} + \ \tilde{N}_{d_{min}}(g) & \text{for } h = (d_{min}-1)(g-1)+1 \end{cases}$$

*where is the rational number given by Equation 3, ie,*

$$= (\tfrac{d}{d_{min}})(\tfrac{d}{d_{min}})^{d_{min}(g-1)+2} :$$

For example, if $d$ is prime, then $d_{min} = d$ and  = 1. See Table 2 for explicit values of $n^h_d(g)$ for small $d$, $g$, and $h$.

Since $n^h_d(g)^{et} \in \mathbb{Z}$ by Theorem 1.10, the integrality conjecture predicts that $\tilde{n}_d(g) \in \mathbb{Z}$. In light of our formula in Theorem 1.13, this leads to congruences that are conjecturally satisfied by the Hurwitz numbers $D_{d;g}$, $D_{d;g}$, and $D_{d;g}$.

**Conjecture 1.15** *Let* $_{d;g} = 216\tilde{n}_d(g)$, *that is*

$$_{d;g} = (g-1)\left(27(g-1)D_{d;g} - 27D_{d;g} - D_{d;g}\right):$$

*Suppose that $d$ is not divisible by 4, 6, or 9. Then,*

$$_{d;g} \quad 0 \pmod{216}:$$

Although $D_{d;g}$, $D_{d;g}$, and $D_{d;g}$ are not *a priori* integers, it is proven in [3] that $_{d;g} \in \mathbb{Z}$. It is also proven in [3] that Conjecture 1.15 holds for $d = 2$ and $d = 3$.

**Remark 1.16** Various congruence properties of $C_{d;g}$ (the number of degree $d$ connected etale covers) will also be used in the proof of the integrality of the etale BPS invariants $n^h_d(g)^{et}$ (see Lemma C4). We speculate that these and the above conjecture are the beginning of a series of congruence properties of general Hurwitz numbers that are encoded in the integrality of the local BPS invariants.

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## 2 Inversion of the Gopakumar{Vafa formula

In this section we invert the Gopakumar{Vafa formula in general to give an explicit expression for the BPS invariants in terms of the Gromov{Witten invariants. We then introduce the notion of a primitive Gromov{Witten invariants and show that all Gromov{Witten invariants can be expressed in terms of primitive invariants. In the case of the local invariants of a nonsingular curve, this suggests a natural grading on the set of local Gromov{Witten invariants. We will see that the etale invariants comprise the first level of this grading.

## 2.1 Inversion of the Gopakumar–Vafa formula

Let $\beta \in H_2(X;\mathbb{Z})$ be an indivisible class. Then the Gopakumar–Vafa formula is:

$$\sum_{g\geq 0}\sum_{d>0} N^g_{d\beta}(X)\,\lambda^{2g-2}q^d = \sum_{g\geq 0}\sum_{d>0} n^g_{d\beta}(X)\sum_{k>0}\frac{1}{k}\left(2\sin\left(\frac{k\lambda}{2}\right)\right)^{2g-2}q^{kd}.$$

Fix $n$ and look at the $q^n$ terms on each side:

$$\sum_{g\geq 0} N^g_{n\beta}(X)\,\lambda^{2g-2} = \sum_{g\geq 0}\sum_{d|n} n^g_{d\beta}(X)\frac{d}{n}\left(2\sin\left(\frac{n\lambda}{2d}\right)\right)^{2g-2}.$$

Letting $s = n\lambda$ and multiplying the above equation by $n$ we find

$$\sum_{g\geq 0} N^g_{n\beta}(X)\,n^{3-2g}s^{2g-2} = \sum_{d|n}\sum_{g\geq 0} n^g_{d\beta}(X)\,d\left(2\sin\frac{s}{2d}\right)^{2g-2}.$$

Recall that Möbius inversion says that if $f(n) = \sum_{d|n} g(d)$, then $g(d) = \sum_{k|d}\mu(\frac{d}{k})f(k)$. Applying this to the above equation (more precisely, to the coefficients of each term of the equation separately), we obtain

$$\sum_{g\geq 0} n^g_{d\beta}(X)\,d\left(2\sin\frac{s}{2d}\right)^{2g-2} = \sum_{k|d}\mu(\tfrac{d}{k})\sum_{g\geq 0} N^g_{k\beta}(X)(\tfrac{s}{k})^{2g-2}k.$$

Letting $t = 2\sin\frac{s}{2d}$ and dividing by $d$ we arrive at

$$\sum_{g\geq 0} n^g_{d\beta}(X)\,t^{2g-2} = \sum_{g\geq 0}\sum_{k|d}\mu(\tfrac{d}{k})(\tfrac{d}{k})^{2g-3}N^g_{k\beta}(X)\left(2\arcsin\frac{t}{2}\right)^{2g-2}.$$

By interchanging $k$ and $d/k$ in the sum and restricting to the $t^{2g-2}$ term of the formula we arrive at the following formula for the BPS invariants.

**Proposition 2.1** *Let $\beta \in H_2(X;\mathbb{Z})$ be an indivisible class, then the BPS invariant $n^g_{d\beta}(X)$ is given by the following formula*

$$n^g_{d\beta}(X) = \sum_{g'=0}^{g}\sum_{k|d}\mu(k)k^{2g'-3}\alpha_{g,g'}N^{g'}_{d/k\beta}(X)$$

*where $\alpha_{g,g'}$ is the coefficient of $r^{g-g'}$ in the series*

$$\left(\frac{\arcsin(\sqrt{r}/2)}{\sqrt{r}/2}\right)^{2g'-2}.$$

*In particular, $n^g_{d\beta}(X)$ depends on $N^{g'}_{d'\beta}(X)$ for all $g' \leq g$ and all $d'$ dividing $d$ such that $\mu(\frac{d}{d'}) \neq 0$.*

Note that the local BPS invariants are thus given by

$$n^h_d(g) = \sum_{h'=0}^{h} \sum_{k|d} \mu(k) k^{2(g+h')-3} \sigma_{h+g,h'+g} N^{h'}_{d/k}(g), \tag{3}$$

or in generating function form:

$$\sum_{d>0}\sum_{h\geq 0} n^h_d(g) t^{2(g+h-1)} q^d = \sum_{k,n>0}\sum_{h\geq 0} \mu(k) N^h_n(g) k^{2(g+h)-3} \left(2\arcsin\tfrac{t}{2}\right)^{2(g+h-1)} q^{nk}. \tag{4}$$

## 2.2 Primitive Gromov–Witten invariants

In this subsection, we formalize the observation that certain contributions to the Gromov–Witten invariants of $X$ can be computed in terms of Gromov–Witten invariants of the covering spaces of $X$. We use this to reduce the computation of the etale invariants to the degree 1 invariants (which have been previously computed by the second author [16]).

**Definition 2.2** We say that a stable map $f: C \to X$ is *primitive* if

$$f_*: \pi_1(C) \to \pi_1(X)$$

is surjective. Note that $\mathrm{Im}(f_*) \subset \pi_1(X)$ is locally constant on the moduli space of stable maps. Let $\overline{M}_g(X,\beta)_G$ be the component(s) consisting of maps $f$ with $\mathrm{Im}(f_*) = G \subset \pi_1(X)$. In particular, $\overline{M}_g(X,\beta)_{\pi_1(X)}$ consists of primitive stable maps. Define the *primitive Gromov–Witten invariants*, denoted $\widehat{N}^g_\beta(X)$, to be the invariants obtained by restricting $[\overline{M}_g(X,\beta)]^{vir}$ to the primitive component $\overline{M}_g(X,\beta)_{\pi_1(X)}$.

The usual Gromov–Witten invariants can be computed in terms of the primitive invariants using the following observations. Let $\pi: \widetilde{X}_G \to X$ be the covering space of $X$ corresponding to the subgroup $G \subset \pi_1(X)$. Any stable map

$$[f: C \to X] \in \overline{M}_g(X,\beta)_G$$

lifts to a (primitive) stable map $[\tilde{f}: C \to \widetilde{X}_G] \in \overline{M}_g(\widetilde{X}_G,\tilde{\beta})_G$ for some $\tilde{\beta}$ with $\pi_*(\tilde{\beta}) = \beta$. Furthermore, this lift is unique up to automorphisms of the cover $\pi: \widetilde{X}_G \to X$. Conversely, any stable map in $\overline{M}_g(\widetilde{X}_G,\tilde{\beta})_G$ gives rise to a map in $\overline{M}_g(X,\beta)_G$ by composing with $\pi$. Note that the automorphism group of the cover is $\pi_1(X)/N(G)$ where $N(G)$ is the normalizer of $G \subset \pi_1(X)$. If $G$ is finite index in $\pi_1(X)$, then $\widetilde{X}_G$ is compact and the automorphism group of the cover is finite. This discussion leads to:

**Proposition 2.3** *Fix $X$, $g$, and $\beta$. Suppose that for every stable map $[f : C \to X]$ in $\overline{M}_g(X,\beta)$, the index $[\pi_1(X) : f_*(\pi_1(C))]$ is finite. Then*

$$N^g_\beta(X) = \sum_G \sum_{\tilde\beta} \frac{1}{[\pi_1(X) : N(G)]} \tilde N^g_{\tilde\beta}(\tilde X_G)$$

*where the first sum is over $G \subset \pi_1(X)$ and the second sum is over $\tilde\beta \in H_2(\tilde X_G;\mathbb{Z})$ such that $\pi_*(\tilde\beta) = \beta$.*

**Remark 2.4** In the case when $[\pi_1(X) : G] = \infty$, $\tilde X_G$ will not be compact and hence the usual Gromov–Witten invariants are not well-defined. However, this technique sometimes can still be used to compute the invariants (see [4]). This technique originated in [5] where it was used to compute multiple cover contributions of certain nodal curves in surfaces.

This technique is especially well-suited to the case of the local invariants of a nonsingular genus $g$ curve. In this case, the image of the fundamental group under a (non-constant) stable map always has finite index. Furthermore, any degree $k$, complete, etale cover of a nonsingular genus $g$ curve is a nonsingular curve of genus $k(g-1)+1$. Thus the formula in Proposition 2.3 reduces to

$$N^h_n(g) = \sum_{l|n} C_{l,g} \tilde N^{h-(l-1)(g-1)}_{n/l}(l(g-1)+1) \tag{5}$$

where $C_{k,g}$ is the number of degree $k$, connected, complete, etale covers of a nonsingular genus $g$ curve, each counted by the reciprocal of the number of automorphisms. In light of this formula, we can regard the primitive local invariants $\tilde N^h_d(g)$ as the fundamental invariants. We encode these invariants into generating functions as follows:

$$\tilde F_{k,g-1}(\lambda) = \sum_{h\geq 0} \tilde N^h_k(g)\,\lambda^{2(g+h-1)}.$$

Equation 5 can then be written in generating function form as

$$\begin{aligned}\sum_{h\geq 0}\sum_{n>0} N^h_n(g) q^n t^{2(g+h-1)} &= \sum_{h\geq 0}\sum_{k,l>0} C_{l,g}\tilde N^{h-(l-1)(g-1)}_k(l(g-1)+1) q^{kl} t^{2(g+h-1)}\\ &= \sum_{k,l>0} C_{l,g}\tilde F_{k,l(g-1)} q^{kl}.\end{aligned}$$

We re-index and rearrange Equation 4 below

$$\sum_{d>0}\sum_{h\geq 0} n^h_d(g) t^{2(g+h-1)} q^d = \sum_{m>0}\frac{1}{m}\mu(m)\sum_{h\geq 0}\sum_{n>0} N^h_n(g)(q^m)^n\left(2m \arcsin\frac{t}{2}\right)^{2(g+h-1)}$$

and then substitute the previous equation to arrive at the following general equation for the local BPS invariants:

$$\boxed{\sum_{d>0}\sum_{h\geq 0} n^h_d(g)t^{2(g+h-1)}q^d = \sum_{m,k,l>0} \frac{1}{m}\mu(m)C_{l,g}\tilde{P}_{k,l(g-1)}(2m\arcsin\tfrac{t}{2})^{2(g+h-1)}q^{mkl}.}$$

The unknown functions $\tilde{P}_{k,l(g-1)}$ are graded by the two natural numbers $k$ and $l$. The contribution in the above sum corresponding to fixed $l$ and $k$ are from those stable maps that factor into a composition of a degree $k$ primitive stable map and a degree $l$ etale cover of $C_g$. Thus the etale BPS invariants (the type (i) contributions) correspond exactly to restricting $k=1$ in the above sum. Therefore we have

$$\sum_{d>0}\sum_{h\geq 0} n^h_d(g)^{et}t^{2(g+h-1)}q^d = \sum_{m,l>0} \frac{1}{m}\mu(m)C_{l,g}\tilde{P}_{1,l(g-1)}(2m\arcsin\tfrac{t}{2})^{2(g+h-1)}q^{ml}.$$

Since a degree one map onto a nonsingular curve is surjective on the fundamental group, it is primitive. The degree one local invariants were computed in [16], the result can be expressed:

$$\begin{aligned}\tilde{P}_{1,g-1} &= \sum_{h\geq 0} N^h_1(g)\lambda^{2(g+h-1)}\\ &= \left(4\sin^2\frac{\lambda}{2}\right)^{(g-1)}\end{aligned}$$

and so

$$\sum_{d>0}\sum_{h\geq 0} n^h_d(g)^{et}t^{2(g+h-1)}q^d = \sum_{m,l>0} \frac{1}{m}\mu(m)C_{l,g}\left(4\sin^2(m\arcsin\tfrac{t}{2})\right)^{l(g-1)}q^{ml}.$$

By the definition of $P_m$, we have

$$P_m(4\sin^2\theta) = 4\sin^2(m\theta)$$

and so letting $\theta = \arcsin(t/2)$ or equivalently $t = 2\sin\theta$, we get

$$\sum_{d>0}\sum_{h\geq 0} n^h_d(g)^{et}t^{2(g+h-1)}q^d = \sum_{m,l>0} \frac{1}{m}\mu(m)C_{l,g}\left(P_m(t^2)\right)^{l(g-1)}q^{ml}.$$

Finally, by letting $y=t^2$ and re-indexing $m$ by $k$, we get

$$\sum_{d>0}\sum_{h\geq 0} n^h_d(g)^{et}y^hq^d = \sum_{m,l>0} k\,\mu(k)C_{l,g}P_m(y)^{l(g-1)}q^{ml},$$

and so the formula in Theorem 1.9 is proved by comparing the $q^d$ terms. □

# 3 Integrality of the etale BPS invariants

In this section we show how the integrality of the etale BPS invariants (Theorem 1.10) follows from our formula for them (Theorem 1.9) and some properties of the the polynomials $P_l(y)$ and the number of degree $k$ covers $C_{k,g}$.

The facts that we need concerning the polynomials $P_l(y)$ are the following.

**Lemma P1** (Moll) *If $l \in \mathbb{N}$, then $P_l(y)$, defined by $P_l(4\sin^2 t) = 4\sin^2(lt)$, is given explicitly by*

$$P_l(y) = \sum_{a=1}^{l} -\frac{l}{a} \binom{a-1+l}{2a-1} (-y)^a.$$

**Lemma P2** *If $l$ is a positive integer, then $P_l(y)$ is a polynomial with integer coefficients.*

**Lemma P3** *For any and we have*

$$P\ (y) = P\ (P\ (y)).$$

**Lemma P4** *For $p$ a prime number and $b$ a positive integer, we have*

$$P_p(y)^{p^{l-1}b} \quad y^{p^l b} \bmod p^l.$$

We also will need some facts about $C_{k,g}$, the number of connected etale covers.

**Lemma C1** *Let $C$ be a nonsingular curve of genus $g$, let $S_k$ be the symmetric group on $k$ letters, and define*

$$A_{k,g} = \#\,\mathrm{Hom}(\ _1(C), S_k).$$

*Then*

$$a_{k,g} = \frac{A_{k,g}}{k!}$$

*is an integer.*

Note that $A_{k,g}$ is the number of degree $k$ (not necessarily connected) etale covers of $C$ with a marking of one fiber. Thus $a_{k,g}$ is the number of (not necessarily connected) etale covers each counted by the reciprocal of the number of automorphisms. We remark that Lemma C1 was essentially known to Burnsides.

**Lemma C2** *Let $a_{k,g}$ be as above with $a_{0,g} = 1$ by convention, then*

$$\sum_{k=1}^{\infty} C_{k,g} t^k = \log\Big(\sum_{k=0}^{\infty} a_{k,g} t^k\Big).$$

**Lemma C3** *Define $c_{k,g} := kC_{k,g}$. Then $c_{k,g}$ is an integer.*

We remark that in general, $C_{k,g}$ is not an integer (see Table 3).

**Lemma C4** *Let $p$ be a prime number not dividing $k$ and let $l$ be a positive integer. Then*

$$c_{p^l k,g} \equiv c_{p^{l-1}k,g} \bmod p^l.$$

We defer the proof of these lemmas to the subsections to follow and we proceed as follows.

In light of Lemmas P2 and C3, we see from the formula in Theorem 1.9 that $n^h_d(g)^{et} \in \mathbb{Z}$ if and only if $\Phi_{d,g} \equiv 0 \bmod d$, where

$$\Phi_{d,g} = \sum_{k|d} \mu(k) c_{\frac{d}{k},g} P_k(y)^{\frac{d(g-1)}{k}}.$$

Suppose that $p^l$ divides $d$ and that $p^{l+1}$ does not divide $d$ for some prime number $p$. For notational clarity, we will suppress the second subscript of $c$ (which is always $g$) in the following calculation. Let $a = d/p^l$; then we get

$$\begin{aligned}
\Phi_{d,g} &= \sum_{k|a} \sum_{i=0}^{l} \mu(p^i k)\, c_{\frac{d}{p^i k}}\, P_{p^i k}(y)^{\frac{d(g-1)}{p^i k}} \\
&= \sum_{k|a} \mu(k)\, c_{\frac{p^l a}{k}}\, P_k(y)^{\frac{p^l a(g-1)}{k}} - \mu(k)\, c_{\frac{p^{l-1} a}{k}}\, P_{pk}(y)^{\frac{p^{l-1} a(g-1)}{k}}.
\end{aligned}$$

Let $\Psi = P_k(y)$. Then by Lemma P3 we have $P_{pk}(y) = P_p(\Psi)$ and so

$$\Phi_{d,g} = \sum_{k|a} \mu(k) \left( c_{\frac{p^l a}{k}} \Psi^{\frac{p^l a(g-1)}{k}} - c_{\frac{p^{l-1} a}{k}} P_p(\Psi)^{\frac{p^{l-1} a(g-1)}{k}} \right).$$

Then by Lemmas P4 and C4 we have

$$\begin{aligned}
\Phi_{d,g} &\equiv \sum_{k|a} \mu(k) \left( c_{\frac{p^l a}{k}} \Psi^{\frac{p^l a(g-1)}{k}} - c_{\frac{p^l a}{k}} \Psi^{\frac{p^l a(g-1)}{k}} \right) \bmod p^l \\
&\equiv 0 \bmod p^l
\end{aligned}$$

and so $\Phi_{d,g} \equiv 0 \bmod d$ and thus $n^h_d(g)^{et} \in \mathbb{Z}$. □

### 3.1 Properties of the polynomials $P_l(y)$: the proofs of Lemmas P1–P4

This subsection is independent of the rest of the paper. We prove various properties of the following family of power series:

**Definition 3.1** Let $\alpha \in \mathbb{R}$, we define the formal power series $P_\alpha(y)$ by

$$P_\alpha(y) = 4\sin^2(\alpha t)$$

where

$$y = 4\sin^2 t.$$

Note that $P_\alpha(y) \in \mathbb{R}[[y]]$ since $\sin^2(\alpha t)$ is a power series in $t^2$ and $y(t) = 4\sin^2 t = 4t^2 - \frac{4}{3!}t^4 + \dots$ is an invertible power series in $t^2$. (**Warning**: This definition differs from the one in [3] by a power of $y$.)

**Proof of Lemma P3** This is immediate from the definition. □

**Proof of Lemma P1** We prove the formula for $P_l(y)$ with $l \in \mathbb{N}$. This formula and its proof was discovered by Victor Moll; we are grateful to him for allowing us to use it.

From [19] page 170 we can express $\sin^2(lt)/\sin^2 t$ in terms of $\cos(2jt)$ for $1 \leq j \leq l-1$ and from [9] 1.332.3 we can in turn express $\cos(2jt)$ in terms of $\sin^2 t$. Substituting, rearranging, and simplifying we arrive a formula for the coefficients of $P_l$. Let $P_l(y) = \sum_{n=1}^{l} -p_{n;l}(-y)^n$, then $p_{1;l} = l^2$ and for $l > 1$,

$$p_{n;l} = \frac{1}{n-1}\sum_{j=n}(l-j+1)(j-1)\binom{j+n-3}{j-n}. \tag{6}$$

By standard recursion methods (see, for example, the book "$A = B$" [18]) one can derive the identity for the binomial sum that transforms the above expression for $p_{n;l}$ into the one asserted by the Lemma:

$$p_{n;l} = \frac{l}{n}\binom{l+n-1}{2n-1}. \tag{7}$$

□

**Proof of Lemma P2** We need to show that $p_{n;l} \in \mathbb{Z}$. By Equation 7, we have that $np_{n;l} \in \mathbb{Z}$ and by Equation 6, we have that $(n-1)p_{n;l} \in \mathbb{Z}$. Thus $np_{n;l} - (n-1)p_{n;l} = p_{n;l} \in \mathbb{Z}$. □

Note that $-P_l(-y)$ has all positive integral coefficients.

**Proof of Lemma P4** To prove the lemma, clearly it suffices to prove that

$$P_p(y)^{p^{l-1}} \equiv y^{p^l} \bmod p^l$$

for $p$ prime and $l \in \mathbb{N}$.

For $n < p$, we have that $p$ divides $p_{n;p}$ since

$$p_{n;p} = \frac{p}{n}\binom{p+n-1}{2n-1}$$

and $n$ does not divide $p$ (except $n = 1$). Noting that $p_{p;p} = 1$ we have

$$P_p(y) = y^p + pyf(y)$$

for $f \in \mathbb{Z}[y]$. This proves the lemma for $l = 1$. Proceeding by induction on $l$, we assume the lemma for $l - 1$ so that we can write

$$P_p(y)^{p^{l-1}} = y^{p^{l-1}} + p^{l-1}g(y)$$

where $g(y) \in \mathbb{Z}[y]$. But then

$$\begin{aligned} P_p(y)^{p^l} &= \left(y^{p^{l-1}} + p^{l-1}g(y)\right)^p \\ &= y^{p^l} + \text{terms that } p^l \text{ divides} \end{aligned}$$

and so the lemma is proved. □

## 3.2 Properties of the number of covers: the proofs of Lemmas C1–C4

In this subsection we prove the properties concerning the numbers $A_{k;g}$, $a_{k;g}$, $C_{k;g}$, and $c_{k;g}$ that were asserted by the Lemmas.

We begin with a proposition from group theory due to M. Aschbacher:

**Proposition 3.2** (Aschbacher) *Let $G$ be a finite group with conjugacy classes $C_i$, $1 \le i \le r$. Pick a representative $g_i \in C_i$; define*

$$b_{i,j,k} = |\{(g,h) \in C_i \times C_j : gh = g_k\}|$$

*and*

$$\gamma_i = |\{(g,h) \in G \times G : [g,h] \in C_i\}|.$$

*Then*

$$\gamma_k = |G| \sum_{i=1}^{r} b_{i,k,i}$$

*so, in particular, $|G|$ divides $\gamma_k$.*

**Proof** We use the notation $h^{-g} := g^{-1}h^{-1}g$. For $(g,h) \in G \times G$,

$$[g,h] = g^{-1}h^{-1}gh = h^{-g}h \in (h^{-1})^G h^G.$$

Furthermore, $[x,h] = [y,h]$ if and only if $h^{-x} = h^{-y}$ if and only if $xh^{-1} \in C_G(h)$, so

$$\alpha_{j,k} = |\{(g,h) \in G \times C_j : [g,h] = g_k\}| = |C_G(g_j)|\, b_{j',j,k}$$

where $C_{j'}$ is the conjugacy class of $g_j^{-1}$. Of course

$$\gamma_k = |C_k| \sum_{j=1}^{r} \alpha_{j,k}$$

so

$$\gamma_k = |C_k| \sum_{j=1}^{r} |C_G(g_j)|\, b_{j',j,k}.$$

Let

$$\Omega_{j,k} = \{(g,h) \in C_{j'} \times C_j : gh \in C_k\}.$$

Then

$$\begin{aligned} |\Omega_{j,k}| &= |C_j|\, |\{g \in C_{j'} : gg_j \in C_k\}| \\ &= |C_j|\, |\{(g,h) \in C_{j'} \times C_k : g_j = g^{-1}hg\}| \\ &= |C_j|\, b_{j,k,j} \end{aligned}$$

and similarly

$$\begin{aligned} |\Omega_{j,k}| &= |C_k|\, |\{(g,h) \in C_{j'} \times C_j : gh = g_k\}| \\ &= |C_k|\, b_{j',j,k} \end{aligned}$$

so $|C_k| b_{j',j,k} = |C_j| b_{j,k,j'}$. Therefore

$$
\begin{aligned}
\chi_k &= \sum_{j=1} |C_G(g_j)| b_{j',j,k} |C_k| \\
&= \sum_{j=1} |C_G(g_j)| \, |C_j| b_{j,k,j'} \\
&= |G| \sum_{j=1} b_{j,k,j'}
\end{aligned}
$$

which proves the proposition. □

**Proof of Lemma C1** Recall that the lemma asserts that $d!$ divides

$$A_{k,g} = \#\operatorname{Hom}(\pi_1(C_g), S_d).$$

For $x \in S_d$ let $c(x)$ denote the conjugacy class of $x$. We will prove, by induction on $g$, that $d!$ divides the number of solutions $(x_1, \ldots, x_g, y_1, \ldots, y_g)$ to

$$\prod_{i=1}^{g} [x_i, y_i] \in c(z) \tag{8}$$

where $z$ is fixed. The lemma is then the special case where $z$ is the identity.

The case of $g = 1$ is Proposition 3.2 where $G = S_d$. For each fixed $r \in S_d$, the number of solutions to (8) with

$$\prod_{i=1}^{g-1} [x_i, y_i] = r$$

is the number of solutions to

$$w[x_g, y_g]^{-1} = r \tag{9}$$

as $w$ varies over $c(z)$ and $x_g$ and $y_g$ each vary over $S_d$. The number of solutions to (9) depends only on the conjugacy class of $r$ since if $q = srs^{-1}$, then (9) holds if and only if

$$sws^{-1}[sx_gs^{-1}, sy_gs^{-1}]^{-1} = q$$

holds. Thus the number of solutions to (8) can be counted by summing up over $\{C_i\}$, the set of conjugacy classes of $S_d$, the product of

$$|\{(x_g, y_g, w) \in S_d \times S_d \times c(z) : w[x_g, y_g]^{-1} \in C_i\}|$$

with

$$|\{f(x_1,\dots,x_{g-1},y_1,\dots,y_{g-1}) \in (S_d)^{2g-2} : \prod_{i=1}^{g-1}[x_i,y_i] \in C_i\}|.$$

By the induction hypothesis, this latter term is always divisible by $d!$, thus the sum is also divisible by $d!$. □

**Proof of Lemma C2** $A_{k,g}$ is the number of $k$–fold (not necessarily connected), complete étale covers of a nonsingular genus $g$ curve $C_g$ with a fixed labeling of one fiber (the bijection is given by monodromy). Thus $a_{k,g}$ is the number of such covers (without the label), each counted by the reciprocal of the number of its automorphisms.

The relationship between $a_{k,g}$, the total number of $k$–covers, and $C_{k,g}$, the number of connected covers, is given by

$$a_{k,g} = \sum_{\alpha=(1^{\alpha_1}2^{\alpha_2}\dots)\in P(k)} \frac{1}{\prod_{i\geq 1}\alpha_i!}\prod_{i\geq 1} C_{i,g}^{\alpha_i}$$

where $P(k)$ is the set of partitions of $k$ ($\alpha_i$ is the number of $i$'s in the partition so $k=\sum i\alpha_i$). This formula is easily obtained by considering how each cover breaks into a union of connected covers (keeping track of the number of automorphisms).

Thus we have

$$\begin{aligned}
\sum_{k=0}^{\infty} a_{k,g}t^k &= \sum_{\alpha_1\geq 0}\sum_{\alpha_2\geq 0}\cdots\prod_{i=1}^{\infty}\frac{1}{\alpha_i!}C_{i,g}^{\alpha_i}t^{i\alpha_i} \\
&= \sum_{\alpha\geq 0}\frac{1}{\alpha!}\left(\sum_{i=1}^{\infty} C_{i,g}t^i\right)^{\alpha} \\
&= \exp\Big(\sum_{i=1}^{\infty} C_{i,g}t^i\Big)
\end{aligned}$$

and so

$$\sum_{k=1}^{\infty} C_{k,g}t^k = \log\Big(\sum_{k=0}^{\infty} a_{k,g}t^k\Big)$$

which proves the lemma. □

**Proof of Lemma C3** Recall that the lemma asserts that $c_{k;g} := kC_{k;g}$ is an integer. From the previous lemma we have:

$$t\frac{d}{dt}\left(\sum_{k=1}^{\infty} C_{k;g}t^k\right) = t\frac{d}{dt}\log\left(\sum_{k=0}^{\infty} a_{k;g}t^k\right)$$

therefore

$$\sum_{k=1}^{\infty} c_{k;g}t^k = \frac{\sum_{k=1}^{\infty} ka_{k;g}t^k}{\sum_{k=0}^{\infty} a_{k;g}t^k}$$

which implies

$$la_{l;g} = \sum_{n=0}^{l-1} a_{n;g}c_{l-n;g}.$$

Now $a_{0;g} = 1$ and so we can obtain the $c$'s recursively from the $a$'s and then induction immediately implies that $c_{l;g} \in \mathbb{Z}$. □

**Proof of Lemma C4** We want to prove that if $p$ is a prime number not dividing $k$ and $l$ a positive integer, then

$$c_{p^l k;g} \equiv c_{p^{l-1}k;g} \mod p^l.$$

We begin with two sublemmas:

**Lemma 3.3** *Let $p$ be a prime, $l$ a positive number, and $x$ and $y$ variables, then*

$$(y+x)^{p^l} \equiv (y^p+x^p)^{p^{l-1}} \mod p^l.$$

**Proof** We use induction on $l$; the case $l = 1$ is well known. By induction, we may assume that there exists $\alpha \in \mathbb{Z}[x;y]$ such that

$$(y+x)^{p^{l-1}} = (y^p+x^p)^{p^{l-2}} + \alpha p^{l-1}.$$

Thus

$$\begin{aligned}(y+x)^{p^l} &= \left((y^p+x^p)^{p^{l-2}} + \alpha p^{l-1}\right)^p \\ &= (y^p+x^p)^{p^{l-1}} + \text{terms divisible by } p^l\end{aligned}$$

which proves the sublemma.

**Lemma 3.4** *Let $1 \le k \le l-1$ and let $p$ be prime. Then $p^{l-k}$ divides $\binom{p^{l-1}}{k}$.*

**Proof** Recall Legendre's formula for $v_p(m!)$, the number of $p$'s in the prime decomposition of $m!$:

$$v_p(m!) = \frac{m - S_p(m)}{p-1}$$

where $S_p(m)$ is the sum of the digits in the base $p$ expansion of $m$.

Let $(a_{l-2}, \ldots, a_0)$ and $(b_{l-2}, \ldots, b_0)$ be base $p$ expansions of $k$ and $p^{l-1} - k$ respectively, then a simple calculation yields:

$$v_p\left(\binom{p^{l-1}}{k}\right) = \frac{1}{p-1}\left(\sum a_i + \sum b_i - 1\right).$$

Let $n = v_p(k)$ so that $a_n$ is the first non-zero digit of $k = (a_{l-2}, \ldots, a_0)$. Now addition in base $p$ gives $(a_{l-2}, \ldots a_0) + (b_{l-2}, \ldots, b_0) = (1, 0, \ldots, 0)$ so we have that $b_0 = b_1 = \cdots = b_{n-1} = 0$, $b_n = p - a_n$, and $b_i = p - 1 - a_i$ for $n+1 \leq i \leq l-2$. Thus we see that

$$\sum a_i + \sum b_i - 1 = (l-1-n)(p-1)$$

and so, observing that $k \geq n+1$, we have

$$v_p\left(\binom{p^{l-1}}{k}\right) = l-1-n \geq l-k$$

which proves the sublemma.

Now let $a(t) = \sum_{i=1}^{\infty} a_{i,g} t^i$ so that Lemma C2 can be written

$$\sum_{i=1}^{\infty} c_{i,g} \frac{t^i}{i} = \log(1 + a(t)).$$

Thus we have

$$c_{p^l k,g} = p^l k \operatorname{Coeff}_{t^{p^l k}} \{\log(1 + a(t))\}$$
$$c_{p^{l-1} k,g} = p^{l-1} k \operatorname{Coeff}_{t^{p^l k}} \{\log(1 + a(t^p))\}$$

and so

$$c_{p^l k,g} - c_{p^{l-1} k,g} = k \operatorname{Coeff}_{t^{p^l k}} \left\{ \log\left( \frac{(1+a(t))^{p^l}}{(1+a(t^p))^{p^{l-1}}} \right) \right\}$$
$$= k \operatorname{Coeff}_{t^{p^l k}} \left\{ Q(t) + \frac{Q^2(t)}{2} + \frac{Q^3(t)}{3} + \cdots \right\}$$

where $Q \in t\mathbb{Z}[[t]]$ is defined by

$$\frac{(1+a(t))^p}{1+a(t^p)} = 1 - Q(t).$$

To prove Lemma C4 it suffices to prove that $Q(t) \equiv 0 \mod p^l$ since then $Q + Q^2/2 + Q^3/3 + \cdots \in \mathbb{Z}_{(p)}[[t]]$ and $Q + Q^2/2 + Q^3/3 + \cdots \equiv 0 \mod p^l$ which then proves that $c_{p^l k,g} - c_{p^{l-1}k,g} \equiv 0 \mod p^l$.

Thus we just need to show that

$$(1 + a(t))^{p^l} \equiv (1 + a(t^p))^{p^{l-1}} \mod p^l.$$

From Lemma 3.3 we have

$$\begin{aligned}(1 + a(t))^{p^l} &\equiv (1 + a(t)^p)^{p^{l-1}} \mod p^l \\ &\equiv (1 + a(t^p) + pf(t))^{p^{l-1}} \mod p^l \\ &\equiv (1 + a(t^p))^{p^{l-1}} + \sum_{k=1}^{p^{l-1}} \binom{p^{l-1}}{k} p^k f(t)^k (1 + a(t^p))^{p^{l-1}-k} \mod p^l.\end{aligned}$$

By Lemma 3.4, $p^l$ divides all the terms in the sum and thus Lemma C4 is proved. □

## 4 The Grothendieck–Riemann–Roch calculation

In order to prove Theorem 1.13, we will apply the Grothendieck–Riemann–Roch formula to the morphism $\pi : U \to \widetilde{M}$ of nonsingular stacks. Here $U$ is the universal curve; see Subsection 1.3.2 for the definition of $\widetilde{M}$. The first step is to compute the relative Todd class of the morphism $\pi$ — that is:

$$Td(\pi) = Td(T_U)/Td(\pi^*(T_{\widetilde{M}})).$$

As the singularities of the morphism $\pi$ occur exactly at the nodes of the universal curve (and the deformations of the 1–nodal map surject onto the versal deformation space of the node), we may use a formula derived by D. Mumford for the relative Todd class [15] (c.f. [6] Section 1.1).

Let $S \subset U$ denote the (nonsingular) substack of nodes. $S$ is of pure codimension 2. There is canonical double cover of $S$,

$$\iota : Z \to S$$

obtained by ordering the branches of the node. $Z$ carries two natural line bundles: the cotangent lines on the first and second branches. Let $\psi_+$, $\psi_-$ denote the Chern classes of these line bundles in $H^2(Z;\mathbb{Q})$. Let $K = c_1(\omega_\pi) \in H^2(U;\mathbb{Q})$. Mumford's formula is:

$$Td(\pi) = \frac{K}{e^K - 1} + \frac{1}{2}\iota_* \sum_{l=1}^{\infty} \frac{B_{2l}}{(2l)!} \frac{\psi_+^{2l-1} + \psi_-^{2l-1}}{\psi_+ + \psi_-}.$$

Since $U$ is a threefold and $S$ is a curve, we find:

$$Td(\pi) = 1 - \frac{K}{2} + \frac{K^2}{12} + \frac{[S]}{12}. \tag{10}$$

Let $\gamma_0 + \gamma_1 + \gamma_2 \in H^*(\widetilde{M};\mathbb{Q})$ denote the cohomological $\pi$ push-forward of $Td(\pi)$. The evaluations:

$$\gamma_0 = -g - (d-1)(g-1); \quad \gamma_1 = \pi_*\left(\frac{K^2 + [S]}{12}\right); \quad \gamma_2 = 0; \tag{11}$$

follow from equation (10).

The Grothendieck–Riemann–Roch formula determines the Chern character of the $\pi$ push-forward:

$$ch(R\pi_* f^*(\mathcal{O}_C)) = \pi_*(ch(f^*(\mathcal{O}_C)) \cdot Td(\pi)).$$

The right side is just $\gamma_0 + \gamma_1 + \gamma_2$. By GRR again,

$$ch(R\pi_* f^*(\omega_C)) = \pi_*(ch(f^*(\omega_C)) \cdot Td(\pi)). \tag{12}$$

We may express the right side as

$$\tilde{\gamma}_0 + \tilde{\gamma}_1 + \tilde{\gamma}_2 \in H^*(\widetilde{M};\mathbb{Q})$$

by the following formulas:

$$\tilde{\gamma}_0 = g - 2 - (d-1)(g-1); \quad \tilde{\gamma}_1 = \pi_*\left(\frac{K^2 + [S]}{12} - \frac{K \cdot W}{2}\right); \tag{13}$$

$$\tilde{\gamma}_2 = \pi_*\left(\frac{K^2 \cdot W + [S] \cdot W}{12}\right).$$

These equations are obtained by simply expanding (12) where we use the notation:

$$W := f^*(c_1(\omega_C)).$$

The Chern characters of $R\pi_* f^*(\mathcal{O}_C \oplus \omega_C)$ determine the classes of the expression:

$$c(R\pi_* f^*(\mathcal{O}_C \oplus \omega_C)[1]).$$

A direct calculation shows:

$$\int_{\widetilde{M}} c(R\pi_* f^*(\mathcal{O}_C \oplus \omega_C)[1]) = \int_{\widetilde{M}} \tilde{\gamma}_2 + \frac{\gamma_1^2 + \tilde{\gamma}_1^2}{2} + \gamma_1\tilde{\gamma}_1. \tag{14}$$

Therefore, our next step is to compute the intersections of the $\gamma$ and $\tilde{\gamma}$ classes in $H^4(\widetilde{M};\mathbb{Q})$.

## 4.1 $\mathrm{Sym}^2(C)$

Let $\mathrm{Sym}^2(C)$ be the symmetric product of $C$. $\mathrm{Sym}^2(C)$ is a nonsingular scheme. There is a canonical branch morphism

$$\rho : \overline{M} \to \mathrm{Sym}^2(C)$$

which associates the branch divisor to each point $[f : D \to C] \in \overline{M}$ (see [7]). The degree of the morphism $\rho$ is $C_{d,g}$. We will relate the required intersections in $\overline{M}$ to the simpler intersection theory of $\mathrm{Sym}^2(C)$.

Let $L \in H^2(\mathrm{Sym}^2(C);\mathbb{Q})$ denote the divisor class corresponding to the subvariety:

$$L_p = \{(p,q) \mid q \in C\}.$$

Let $\Delta$ denote the diagonal divisor class of $\mathrm{Sym}^2(C)$. It is easy to compute the products:

$$L^2 = 1, \quad L\cdot\Delta = 2, \quad \Delta^2 = 4 - 4g$$

in $\mathrm{Sym}^2(C)$.

## 4.2 $R$, $S$, and $T$

An analysis of the ramification of the universal map $f : U \to C$ is required to relate the integrals (14) over $\overline{M}$ to the intersection theory of $\mathrm{Sym}^2(C)$. Consider first the maps:

$$U \xrightarrow{\iota} \overline{M} \times C \xrightarrow{\pi} \overline{M}$$

where $\iota = (\pi, f)$ and $\pi$ is the projection onto the first factor. Let

$$R \subset U,$$
$$B \subset \overline{M} \times C,$$

denote universal ramification and branch loci respectively. Certainly,

$$\iota_*([R]) = [B] \tag{15}$$

as the $\iota$ restricts to a birational morphism from $R$ to $B$. By the Riemann–Hurwitz correspondence, we find:

$$K = W + [R] \in H^2(U;\mathbb{Q}). \tag{16}$$

After taking the square of this equation and pushing forward via $\iota$, we find the equation

$$\iota_*(K^2) = 2[B]\cdot c_1(\omega_C) + \iota_*([R]^2) \tag{17}$$

holds on $\widetilde{M} \quad C$.

The term $\quad ([R]^2)$ in (17) may be determined by the following considerations. The line bundle $\omega \, |_R$ is naturally isomorphic to $\mathcal{O}_U(R)|_R$ at each point of $R$ not contained in the locus of nodes $S$ or the locus of double rami cation points $T$. We will use local calculations to show that the coe cients of $[S]$ and $[T]$ are 1 in the following equation:

$$[R]^2 = -K \quad [R] + [S] + [T]. \tag{18}$$

To compute the coe cient of $[S]$ it su ces to study the local family $\quad : U_{loc} \to \mathbb{C}$ given by $U_{loc} = \{f(x,y,t) \in \mathbb{C}^3 : xy = t\}$ with the maps $f(x,y,t) = x+y$ and $\quad (x,y,t) = t$. For the coe cient of $[T]$, we note $\omega \quad$ is the $\quad$–vertical tangent bundle of $U$ on $T$. Near $T$, $R$ is a double cover of $\widetilde{M}$ with simple rami cation at $T$. Hence, the natural map on $R$ near $T$:

$$\omega \, |_R \to \mathcal{O}_U(R)|_R$$

has a zero of order 1 along T. The coe cient of $[T]$ in (18) is thus 1. We may rewrite (18) using (16)

$$[R]^2 = \frac{-W \quad [R] + [S] + [T]}{2},$$

which will be substituted in (17).

The nal equation for $\quad (K^2)$ using the above results is:

$$(K^2) = \frac{3}{2}[B] \quad c_1(\omega_C) + \quad \frac{[S] + [T]}{2} \quad . \tag{19}$$

Note the branch divisor $B$ is simply the $\quad$ pull-back of the universal family

$$B_S \quad \mathrm{Sym}^2(C) \quad C.$$

Let $\quad_S$ denote the projection of $\mathrm{Sym}^2(C) \quad C$ to the rst factor. Applying to (19) and using the $\quad$ pull-back structure of $B$, we nd:

$$(K^2) = \quad (K^2) = \quad_S \quad \frac{3}{2}[B_S] \quad c_1(\omega_C)) + \quad \frac{[S] + [T]}{2} \quad .$$

A simple calculation in $\mathrm{Sym}^2(C) \quad C$ then yields:

$$_S \, ([B_S] \quad c_1(\omega_C)) = (2g-2)L,$$

We nally arrive at the central equation:

$$(K^2) = \quad \frac{3}{2}(2g-2)L \quad + \quad \frac{[S] + [T]}{2} \quad . \tag{20}$$

Equation (20) will be used to transfer intersections on $\widetilde{M}$ to $\mathrm{Sym}^2(C)$.

## 4.3 Proof of Theorem 1.13

We will calculate all terms on the right side of integral equation:

$$\int_{\widetilde M} c(R \quad f\ (O_C \quad !_C)[1]) = \int_{\widetilde M} \tilde\gamma_2 + \frac{\gamma_1^2 + \tilde\gamma_1^2}{2} + \gamma_1\tilde\gamma_1 . \tag{21}$$

Consider first the class $\tilde\gamma_2$. By equation (13),

$$\int_{\widetilde M} \tilde\gamma_2 = \int_{\widetilde M} \frac{(K^2 \cdot W) + [S]\cdot W}{12} . \tag{22}$$

The first summand on the right may be computed from the relation:

$$(K^2 \cdot W) = \frac{[S]\cdot W + [T]\cdot W}{2} .$$

The definition of the Hurwitz numbers $D_{d,g}$ and $D_{d,g}$ imply:

$$[S]\cdot W = D_{d,g}(2g-2),$$
$$[T]\cdot W = D_{d,g}(2g-2).$$

Using the above formulas, we find:

$$\int_{\widetilde M} \tilde\gamma_2 = \frac{g-1}{12}\left(3D_{d,g} + D_{d,g}\right) .$$

For the quadratic terms involving $\gamma_1$ and $\tilde\gamma_1$ in equation (21), we will need to compute several integrals. The first two integrals are:

$$\int_{\widetilde M} ([S])^2 = (4-4g)D_{g,d}, \qquad \int_{\widetilde M} ([T])^2 = \frac{4-4g}{3}D_{g,d}.$$

Both equations require a study of the local geometry of the morphism . As is etale at the points of $(S)$, the self-intersection of the curve $(S)$ is simply $^2$ $D_{d,g}$. As has double ramification at the points of $(T)$, the self-intersection of the curve $(T)$ is one third of $^2$ $D_{d,g}$ (see [10]). The integral :

$$\int_{\widetilde M} (K^2)^2 = \frac{9}{4}(2g-2)^2 D_{d,g} + (5g-5)D_{d,g} + \frac{17g-17}{3}D_{d,g}.$$

then follows easily from (20).

Next, the integral

$$\int_{\widetilde M} (K^2) \cdot ([S]) = (4g-4)D_{g,d}$$

follows from the intersection theory of $\mathrm{Sym}^2(C)$ and the definition of the Hurwitz numbers.

Finally, as $(K \; W) = ((2g-2)L)$, the remaining integrals:

$$\int_{\widetilde{M}} (K^2) \quad (K \; W) = \frac{3}{2}(2g-2)^2 D_{d;g} + (2g-2)(D_{d;g} + D_{d;g});$$

$$\int_{\widetilde{M}} ([S]) \quad (K \; W) = (4g-4)D_{g;d};$$

$$\int_{\widetilde{M}} (K \; W)^2 = (2g-2)^2 D_{d;g};$$

are easily obtained.

The final formula for Theorem 1.13 is now obtained from the above integral equations together with (11), (13), and (21):

$$\int_{\widetilde{M}} c(R \quad f \; (O_C \quad \omega_C)[1]) = \frac{g-1}{8} \; (g-1)D_{d;g} - D_{d;g} - \frac{1}{27}D_{d;g} \; :$$

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# A Appendix: Tables of numbers

The tables in this appendix list the values of the invariants studied in the paper in the rst few cases:

(1) The etale BPS invariants $n_d^h(g)^{et}$ (for small values of $d$, $g$, and $h$) as given by Theorem 1.9.

(2) The full local BPS invariants $n_d^h(g)$ (again for small values of $d$, $g$, and $h$) in the range where they are known as given by Lemma 1.14 | question marks where they are unknown.

(3) The various Hurwitz numbers that arise.

The Hurwitz numbers were computed from rst principles and recursion when possible (see [3] for example), and by a naive computer program elsewhere. The Hurwitz numbers that were beyond either of these methods are left as variables in the tables. Note that by Lemma C2, the rational numbers $C_{d;g}$ can be expressed in terms of the integers $a_{d;g}$; it is easy to write a computer program that computes the $a_{d;g}$'s (albeit slowly).

If the etale BPS invariants do indeed arise from corresponding component(s) in the D{brane moduli space (see Remark 1.12) then the horizontal rows of the tables for the etale invariants should be the coe cients of the $\mathfrak{sl}_2 \quad \mathfrak{sl}_2$ decomposition of the cohomology of that space. So for example, the zeros in the beginning of the $n_4^h(g)^{et}$ table suggest that this space factors o (cohomologically) a torus of dimension $2g-1$ (as oppose to the torus factor of dimension $g$ for the other cases).

| $n_2^h(g)^{et}$ | $h=0$ | $h=1$ | $h=2$ | $h=3$ | $h=4$ | $h=5$ | $h=6$ | $h=7$ | $h=8$ | $h=9$ |
|---|---|---|---|---|---|---|---|---|---|---|
| $g=2$ | $-2$ | 8 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $g=3$ | $-8$ | 4 | 31 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $g=4$ | $-32$ | 24 | $-6$ | 128 | 0 | 0 | 0 | 0 | 0 | 0 |
| $g=5$ | $-128$ | 128 | $-48$ | 8 | 511 | 0 | 0 | 0 | 0 | 0 |
| $g=6$ | $-512$ | 640 | $-320$ | 80 | $-10$ | 2048 | 0 | 0 | 0 | 0 |
| $g=7$ | $-2048$ | 3072 | $-1920$ | 640 | $-120$ | 12 | 8191 | 0 | 0 | 0 |

| $n_3^h(g)^{et}$ | $h=0$ | $h=1$ | $h=2$ | $h=3$ | $h=4$ | $h=5$ | $h=6$ | $h=7$ |
|---|---|---|---|---|---|---|---|---|
| $g=2$ | $-3$ | 2 | 73 | 0 | 0 | 0 | 0 | 0 |
| $g=3$ | $-27$ | 36 | $-18$ | 4 | 2641 | 0 | 0 | 0 |
| $g=4$ | $-243$ | 486 | $-405$ | 180 | $-45$ | 6 | 93913 | 0 |
| $g=5$ | $-2187$ | 5832 | $-6804$ | 4536 | $-1890$ | 504 | $-84$ | 8 |
| $g=6$ | $-19683$ | 65610 | $-98415$ | 87480 | $-51030$ | 20412 | $-5670$ | 1080 |
| $g=7$ | $-177147$ | 708588 | $-1299078$ | 1443420 | $-1082565$ | 577368 | $-224532$ | 64152 |

| $n_4^h(g)^{et}$ | $h=0$ | $h=1$ | $h=2$ | $h=3$ | $h=4$ | $h=5$ | $h=6$ | $h=7$ |
|---|---|---|---|---|---|---|---|---|
| $g=2$ | 0 | $-60$ | 30 | 1315 | 0 | 0 | 0 | 0 |
| $g=3$ | 0 | 0 | $-4032$ | 4032 | $-1512$ | 252 | 689311 | 0 |
| $g=4$ | 0 | 0 | 0 | $-261120$ | 391680 | $-244800$ | 81600 | $-15300$ |
| $g=5$ | 0 | 0 | 0 | 0 | $-16760832$ | 33521664 | $-29331456$ | 14665728 |

| $n_5^h(g)^{et}$ | $h=0$ | $h=1$ | $h=2$ | $h=3$ | $h=4$ | $h=5$ | $h=6$ | $h=7$ |
|---|---|---|---|---|---|---|---|---|
| $g=2$ | $-5$ | 10 | $-7$ | 2 | $-1935+a_{5;2}$ | 0 | 0 | 0 |
| $g=3$ | $-125$ | 500 | $-850$ | 800 | $-455$ | 160 | $-34$ | 4 |
| $g=4$ | $-3125$ | 18750 | $-50625$ | 81250 | $-86250$ | 63750 | $-33625$ | 12750 |
| $g=5$ | $-78125$ | 625000 | $-2312500$ | 5250000 | $-8181250$ | 9275000 | $-7910000$ | 5175000 |

Table 1: The etale BPS invariants $n_d^h(g)^{et}$ for small $d$, $g$, and $h$.

| $n_2^h(g)$ | $h=0$ | $h=1$ | $h=2$ | $h=3$ | $h=4$ | $h=5$ | $h=6$ | $h=7$ | $h=8$ | $h=9$ |
|---|---|---|---|---|---|---|---|---|---|---|
| $g=2$ | $-2$ | 8 | 0 | ? | ? | ? | ? | ? | ? | ? |
| $g=3$ | $-8$ | 4 | 31 | 8 | ? | ? | ? | ? | ? | ? |
| $g=4$ | $-32$ | 24 | $-6$ | 128 | 96 | ? | ? | ? | ? | ? |
| $g=5$ | $-128$ | 128 | $-48$ | 8 | 511 | 768 | ? | ? | ? | ? |
| $g=6$ | $-512$ | 640 | $-320$ | 80 | $-10$ | 2048 | 5120 | ? | ? | ? |
| $g=7$ | $-2048$ | 3072 | $-1920$ | 640 | $-120$ | 12 | 8191 | 30720 | ? | ? |

| $n_3^h(g)$ | $h=0$ | $h=1$ | $h=2$ | $h=3$ | $h=4$ | $h=5$ | $h=6$ | $h=7$ |
|---|---|---|---|---|---|---|---|---|
| $g=2$ | $-3$ | 2 | 73 | 50 | ? | ? | ? | ? |
| $g=3$ | $-27$ | 36 | $-18$ | 4 | 2641 | 9604 | ? | ? |
| $g=4$ | $-243$ | 486 | $-405$ | 180 | $-45$ | 6 | 692352 | 836310 |
| $g=5$ | $-2187$ | 5832 | $-6804$ | 4536 | $-1890$ | 504 | $-84$ | 8 |
| $g=6$ | $-19683$ | 65610 | $-98415$ | 87480 | $-51030$ | 20412 | $-5670$ | 1080 |
| $g=7$ | $-177147$ | 708588 | $-1299078$ | 1443420 | $-1082565$ | 577368 | $-224532$ | 64152 |

| $n_4^h(g)$ | $h=0$ | $h=1$ | $h=2$ | $h=3$ | $h=4$ | $h=5$ | $h=6$ | $h=7$ |
|---|---|---|---|---|---|---|---|---|
| $g=2$ | 0 | $-60$ | 30 | ? | ? | ? | ? | ? |
| $g=3$ | 0 | 0 | $-4032$ | 3520 | ? | ? | ? | ? |
| $g=4$ | 0 | 0 | 0 | $-261120$ | 367104 | ? | ? | ? |
| $g=5$ | 0 | 0 | 0 | 0 | $-16760832$ | 32735232 | ? | ? |

| $n_5^h(g)$ | $h=0$ | $h=1$ | $h=2$ | $h=3$ | $h=4$ | $h=5$ | $h=6$ | $h=7$ |
|---|---|---|---|---|---|---|---|---|
| $g=2$ | $-5$ | 10 | $-7$ | 2 | $-1935 + a_{5;2}$ | | ? | ? |
| $g=3$ | $-125$ | 500 | $-850$ | 800 | $-455$ | 160 | $-34$ | 4 |
| $g=4$ | $-3125$ | 18750 | $-50625$ | 81250 | $-86250$ | 63750 | $-33625$ | 12750 |
| $g=5$ | $-78125$ | 625000 | $-2312500$ | 5250000 | $-8181250$ | 9275000 | $-7910000$ | 5175000 |

Table 2: The local BPS invariants $n_d^h(g)$ for small $d$, $g$, and $h$. Note: the value of in the above table is $\frac{1}{8}(D_{5;2} - D_{5;2} - \frac{1}{27}D_{5;2})$.

| $C_{d;g}$ | $g=1$ | $g=2$ | $g=3$ | $g=4$ | $g=5$ | $g=6$ |
|---|---|---|---|---|---|---|
| $d=2$ | 3=2 | 15=2 | 63=2 | 255=2 | 1023=2 | 4095=2 |
| $d=3$ | 4=3 | 220=3 | 7924=3 | 281740=3 | 10095844=3 | 362968060=3 |
| $d=4$ | 7=4 | 5275=4 | 2757307=4 | $a_{4;4}$ − 408421=4 | $a_{4;5}$ − 13985413=4 | $a_{4;6}$ − 492346021=4 |

| $D_{d;g}$ | $g=1$ | $g=2$ | $g=3$ | $g=4$ | $g=5$ | $g=6$ | $g=7$ |
|---|---|---|---|---|---|---|---|
| $d=2$ | 2 | 8 | 32 | 128 | 512 | 2048 | 8192 |
| $d=3$ | 16 | 640 | 23296 | 839680 | 30232576 | 1088389120 | 39182073856 |

| $D_{d;g}$ | $g=1$ | $g=2$ | $g=3$ | $g=4$ | $g=5$ | $g=6$ | $g=7$ |
|---|---|---|---|---|---|---|---|
| $d=2$ | 2 | 8 | 32 | 128 | 512 | 2048 | 8192 |
| $d=3$ | 7 | 235 | 7987 | 281995 | 10096867 | 362972155 | 13062280147 |

| $D_{d;g}$ | $g=1$ | $g=2$ | $g=3$ | $g=4$ | $g=5$ | $g=6$ | $g=7$ |
|---|---|---|---|---|---|---|---|
| $d=2$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $d=3$ | 3 | 135 | 5103 | 185895 | 6711903 | 241805655 | 8706597903 |

Table 3: The Hurwitz numbers $C_{d;g}$, $D_{d;g}$, $D_{d;g}$, and $D_{d;g}$ for small $d$ and $g$.