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On hyperbolic 3-manifolds realizing the maximal distance between toroidal Dehn fillings

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Abstract For a hyperbolic 3-manifold M with a torus boundary component, all but finitely many Dehn fillings on the torus component yield hyperbolic 3-manifolds. In this paper, we will focus on the situation where M has two exceptional Dehn fillings, both of which yield toroidal manifolds. For such situation, Gordon gave an upper bound for the distance between two slopes of Dehn fillings. In particular, if M is large, then the distance is at most 5. We show that this upper bound can be improved by 1 for a broad class of large manifolds.

AMS Classification 57M25; 57M50

Keywords Dehn filling, toroidal filling, knot

1 Introduction

Let M be a hyperbolic 3-manifold with a torus boundary component T_0. A *slope* on T_0 is the isotopy class of an essential simple closed curve on T_0. For a slope γ on T_0, the manifold obtained by γ-*Dehn filling* is $M(\gamma) = M \cup V_\gamma$, where V_γ is a solid torus, glued to M along T_0 in such a way that γ bounds a meridian disk in V_γ. If $M(\gamma)$ is not hyperbolic, then γ is called an *exceptional slope*. By Thurston's hyperbolic Dehn surgery theorem, the number of exceptional slopes is finite. If $M(\gamma)$ fails to be hyperbolic, then it either (1) contains an essential sphere, disk, annulus or torus; or (2) contains a Heegaard sphere or torus; or (3) is a Seifert fibered manifold over the sphere with three exceptional fibers; or (4) is a counterexample to the geometrization conjecture (see [6]).

Suppose that there are two slopes α and β such that $M(\alpha)$ and $M(\beta)$ are toroidal, that is, contain essential tori. The *distance* $\Delta(\alpha,\beta)$ between them is their minimal geometric intersection number. Then Gordon [5] shows $\Delta = \Delta(\alpha,\beta) \le 8$, and there are only four manifolds $W(-1), W(5), W(5/2), W(-2)$

with $\Delta \geq 6$. Here, $W(p/q)$ is obtained by p/q-filling on one boundary torus of the Whitehead link exterior W in the usual way. In particular, these manifolds are each $\mathbb{Q}$-homology $S^1 \times D^2$, and the boundary is a single torus. Following Wu [19], let us say that M is *large* if $H_2(M, \partial M - T_0) \neq 0$. Note that M is not large if and only if M is a $\mathbb{Q}$-homology $S^1 \times D^2$ or a $\mathbb{Q}$-homology $T^2 \times I$. Hence, M is large if ∂M is not a union of at most two tori. In [6, Question 4.2], Gordon asks if there is a large hyperbolic manifold with toroidal fillings at distance 5. In this direction, [1, Theorem 3.1] shows that if ∂M is a single torus and the first betti number $\beta_1(M) \geq 3$ then the distance between two toroidal fillings is at most 4. As stated in [1, Remark 3.15], their argument also works for M whose boundary consists of at least 4 tori.

The purpose of this paper is to show that a broad class of large manifolds cannot admit two toroidal fillings at distance 5.

Theorem 1.1 *Let M be a hyperbolic 3-manifold with a torus boundary component T_0 and suppose that there are two slopes α, β on T_0 such that $M(\alpha)$ and $M(\beta)$ are toroidal. If $\Delta(\alpha, \beta) = 5$, then ∂M consists of at most two tori.*

This is sharp in the sense that there are hyperbolic 3-manifolds whose boundary is a single or two tori with two toroidal fillings at distance 5. For example, the exterior of the $(-2, 3, 7)$-pretzel knot in S^3 is hyperbolic and there are two toroidal slope 16 and 37/2. The Whitehead sister link ($(-2, 3, 8)$-pretzel link) exterior gives such an example with two torus boundary components. Also, Theorem 1.1 can be regarded as the first step to determine which hyperbolic 3-manifolds admit two toroidal slopes of distance 5. Part of the proof of Theorem 1.1 consists of carrying over the argument of [17], where we treated the case where M is the exterior of a hyperbolic knot in S^3, to the present context. Hence we assume the familiarity with [17].

Theorem 1.1 gives also a partial answer to [6, Question 5.2] which asks if there is a hyperbolic manifold whose boundary consists of three tori, having two toroidal fillings at distance 4 or 5. Combining with known facts [6], we have the following.

Corollary 1.2 *If M is a hyperbolic 3-manifold whose boundary is a union of more than two tori, then for any fixed boundary torus component T_0 of M, any two exceptional slopes of M on T_0 have distance at most 4.*

To prove Theorem 1.1, we need to consider the situation where either $M(\alpha)$ or $M(\beta)$ contains a Klein bottle. Such a phenomenon often happens in the literature [7, 8, 12].

Theorem 1.3 *Let M be a hyperbolic 3-manifold with a torus boundary component T_0 and suppose that there are two slopes α, β on T_0 such that $M(\alpha)$ contains a Klein bottle and $M(\beta)$ is toroidal. If $\Delta(\alpha, \beta) \geq 5$, then ∂M consists of at most two tori.*

In Section 2, we prepare some general lemmas about a pair of graphs coming from intersections of two essential tori. Sections 3–7 treat the case where two toroidal manifolds contain no Klein bottle. Finally, we consider the case where either contains a Klein bottle in Section 8–11. Section 10 contains the results about a reduced graph on a Klein bottle, which we need for Section 11.

2 Preliminaries

Let M be a hyperbolic 3-manifold with a torus boundary component T_0 and suppose that there are two slopes α, β on T_0 such that $M(\alpha)$ and $M(\beta)$ are toroidal. We assume that $\Delta = \Delta(\alpha, \beta) = 5$ until the end of Section 7. Then $M(\alpha)$ and $M(\beta)$ are irreducible by [13, 18].

Let $\widehat{S}$ be an essential torus in $M(\alpha)$. We may assume that $\widehat{S}$ meets the attached solid torus V_α in s meridian disks $u_1, u_2, \ldots, u_s$, numbered successively along V_α, and that s is minimal over all choices of $\widehat{S}$. Let $S = \widehat{S} \cap M$. Then S is a punctured torus properly embedded in M with s boundary components $\partial_i S = \partial u_i$, each of which has slope α. By the minimality of s, S is incompressible and boundary-incompressible in M. Similarly, we choose an essential torus $\widehat{T}$ in $M(\beta)$ which intersects the attached solid torus V_β in t meridian disks $v_1, v_2, \ldots, v_t$, numbered successively along V_β, where t is minimal as above. Then we have another incompressible and boundary-incompressible punctured torus $T = \widehat{T} \cap M$, which has t boundary components $\partial_j T = \partial v_j$. Notice that s and t are non-zero.

We may assume that S intersects T transversely. Then $S \cap T$ consists of arcs and circles. Since both surfaces are incompressible, we can assume that no circle component of $S \cap T$ bounds a disk in S or T. Furthermore, it can be assumed that $\partial_i S$ meets $\partial_j T$ in 5 points for any pair of i and j.

As seen in [10], we can choose a meridian-longitude pair m, l on T_0 so that $\alpha = m$, and $\beta = dm + 5l$ for some $d = 1, 2$. This number d is called the *jumping number* of α and β.

Lemma 2.1 *Let a_1, a_2, a_3, a_4, a_5 be the points of $\partial_i S \cap \partial_j T$, numbered so that they appear successively on $\partial_i S$. If d is the jumping number of α and*

β, then these points appear in the order of $a_d, a_{2d}, a_{3d}, a_{4d}, a_{5d}$ on $\partial_j T$ in some direction. In particular, if $d = 1$, then two points of $\partial_i S \cap \partial_j T$ are successive on $\partial_i S$ if and only if they are successive in $\partial_j T$, and if $d = 2$, then two points of $\partial_i S \cap \partial_j T$ are successive on $\partial_i S$ if and only if they are not successive in $\partial_j T$.

Proof See [10, Lemma 2.10]. □

Let G_S be the graph on $\widehat{S}$ consisting of the u_i as (fat) vertices, and the arc components of $S \cap T$ as edges. Each vertex of G_S is given a sign according to whether the core of V_α passes $\widehat{S}$ from the positive side or negative side at this vertex. Define G_T on $\widehat{T}$ similarly. Throughout the paper, two graphs on a surface are considered to be equivalent if there is a homeomorphism of the surface carrying one graph to the other. Note that G_S and G_T have no trivial loops, since S and T are boundary-incompressible.

For an edge e of G_S incident to u_i, the endpoint of e is labelled j if it is in $\partial u_i \cap \partial v_j = \partial_i S \cap \partial_j T$. Similarly, label the endpoints of each edge of G_T. Thus the labels $1, 2, \dots, t$ (resp. $1, 2, \dots, s$) appear in order around each vertex of G_S (resp. G_T) repeated 5 times. Each vertex u_i of G_S has degree $5t$, and each v_j of G_T has degree $5s$.

Let $G = G_S$ or G_T. An edge of G is a *positive* edge if it connects vertices of the same sign. Otherwise it is a *negative* edge. Possibly, a positive edge is a loop. An endpoint of a positive (resp. negative) edge around a vertex is called a *positive* (resp. *negative*) *edge endpoint*. We denote by G^+ the subgraph of G consisting of all vertices and positive edges of G.

If an edge e of G_S is incident to u_i with label j, then it is called a *j-edge at* u_i. Then e is also an i-edge at v_j in G_T. If e has labels j_1, j_2 at its endpoints, then e is called a *$\{j_1, j_2\}$-edge*. An $\{i, i\}$-edge is said to be *level*.

A cycle in G consisting of positive edges is a *Scharlemann cycle* if it bounds a disk face of G and all edges in the cycle are $\{i, i+1\}$-edges for some label i. The number of edges in a Scharlemann cycle is called the *length* of the Scharlemann cycle, and the set $\{i, i+1\}$ is called its *label pair*. A Scharlemann cycle of length two is called an *S-cycle* for short. For a label x, let G_x be the subgraph of G consisting of all vertices and all positive x-edges. Then a disk face of G_x is called an *x-face*.

Lemma 2.2 (1) (The parity rule) *An edge e is positive in a graph if and only if it is negative in the other graph.*

(2) *There is no pair of edges which are parallel in both graphs.*

(3) *If G_S (resp. G_T) has a Scharlemann cycle, then $\widehat{T}$ (resp. $\widehat{S}$) is separating.*

Proof (1) This can be found in [3]. (2) is [5, Lemma 2.1]. See [2] for (3). □

Proposition 2.3 *Either $\widehat{S}$ or $\widehat{T}$ is separating.*

Proof If G_T has more than t positive x-edges for some label x, then G_T has an x-face, which contains a Scharlemann cycle by [11]. Then $\widehat{S}$ is separating by Lemma 2.2(3).

Hence we assume that G_T has at most t positive x-edges for any label x. This means that any vertex of G_S is incident to at most t negative edges by the parity rule. Thus any vertex of G_S has at least $4t$ positive edge endpoints, and then G_S^+ has at least $2st$ edges. But this implies that G_S has more than s positive i-edges for some label i. Then G_S has an i-face, containing a Scharlemann cycle. So $\widehat{T}$ is separating by Lemma 2.2(3) again. □

Thus we can assume that $\widehat{S}$ is separating until the end of Section 7. Then s is even. Let $M(\alpha) = \mathcal{B} \cup_{\widehat{S}} \mathcal{W}$. Here $\mathcal{B}$ is called the black side of $\widehat{S}$, and $\mathcal{W}$ is the white side. A Scharlemann cycle is said to be *black* (resp. *white*) if its face lies in $\mathcal{B}$ (resp. $\mathcal{W}$).

Lemma 2.4 *G_S satisfies the following:*

(1) *If $\widehat{T}$ is non-separating, then any family of parallel positive edges in G_S contains at most $t/2$ edges. If $\widehat{T}$ is separating and $t \geq 4$, then any family of parallel positive edges in G_S contains at most $t/2+2$ edges, and moreover, if the family contains $t/2+2$ edges, then $t \equiv 0 \pmod 4$, and $M(\beta)$ contains a Klein bottle.*

(2) *Either any family of parallel negative edges in G_S contains at most t edges, or all vertices of G_T have the same sign.*

Proof (1) If $\widehat{T}$ is non-separating, then G_S cannot contain a Scharlemann cycle by Lemma 2.2(3). Thus any family of parallel positive edges in G_S contains at most $t/2$ edges by [3, Lemma 2.6.6]. Assume that $\widehat{T}$ is separating and $t \geq 4$. By [18, Lemma 1.4], any family of parallel positive edges contains at most $t/2+2$ edges. If the family contains $t/2+2$ edges, then $t \equiv 0 \pmod 4$ by [18, Corollary 1.8]. In this case, the family contains two S-cycles ρ_1 and ρ_2 with disjoint label pairs. Let $\{k_i, k_i+1\}$ be the label pair of ρ_i and let D_i be the disk face bounded by ρ_i for $i=1,2$. Let H_i be the part of V_β between

v_{k_i} and v_{k_i+1}. Then shrinking H_i into its core in $H_i \cup D_i$ gives a Möbius band B_i whose boundary is the loop on $\widehat{T}$ formed by the edges of ρ_i. In particular, ∂B_i is essential on $\widehat{T}$ [7, Lemma 3.1]. Hence ∂B_1 and ∂B_2 are disjoint, and so they bound an annulus A on $\widehat{T}$. Then the union $B_1 \cup A \cup B_2$ is a Klein bottle in $M(\beta)$.

(2) If $t = 1$, then the second conclusion holds. If $t = 2$, then G_T has only two parallelism classes of loops [5, Lemma 5.2]. Hence if two vertices of G_T have opposite signs, then at most two negative edges can be parallel in G_S by Lemma 2.2(2). See [10, Lemma 2.3(1)] for $t > 2$. □

Lemma 2.5 *G_T satisfies the following*:

1. *If $s \geq 4$, then any family of parallel positive edges in G_T contains at most $s/2+2$ edges. Moreover, if the family contains $s/2+2$ edges, then $s \equiv 0 \pmod 4$, and $M(\alpha)$ contains a Klein bottle.*
2. *Any family of parallel negative edges in G_T contains at most s edges.*

Proof This can be proved by the same argument as in the proof of Lemma 2.4. □

For a graph G on a surface, $\overline{G}$ denotes the *reduced graph* of G obtained by amalgamating each family of parallel edges into a single edge. For an edge e of $\overline{G}$, the *weight* of e is the number of edges in the corresponding family of parallel edges in G.

3 Generic case

The proof of Theorem 1.1 occupies Sections 3–7. The case where either $M(\alpha)$ or $M(\beta)$ contains a Klein bottle will be treated from Section 8. Hence we assume that neither $M(\alpha)$ nor $M(\beta)$ contains a Klein bottle in the following 5 sections. This section treats the case where $s \geq 4$ and $t \geq 3$.

Lemma 3.1 (1) *Any family of mutually parallel positive edges in G_S (resp. G_T) contains at most $t/2+1$ (resp. $s/2+1$) edges.*

(2) *Neither G_S nor G_T contains two S-cycles with disjoint label pairs.*

Proof (1) follows from Lemmas 2.4(1) and 2.5(1).

(2) If G_S, say, contains two S-cycles with disjoint label pairs, then $M(\beta)$ contains a Klein bottle as in the proof of Lemma 2.4. □

Under the existence of Lemma 3.1, we can carry over the arguments from Lemma 4.1 to 4.13 of [17]. (In the proof of Lemma 4.12 of [17], we need to add the case where $t = 3$, but it is obvious.) Hence we have $s = 4$ or 6. To eliminate these remaining cases, we have to modify the arguments in [17], because $\widehat{T}$ is possibly non-separating, and the jumping number is one or two in the present context. (In [17], both tori were separating and the jumping number between the slopes was one.)

Proposition 3.2 *$s = 6$ is impossible.*

Proof By [17, Lemma 4.13], $\overline{G}_S^+$ consists of two components, each of which has three vertices. Also, $\overline{G}_S^+$ has a good vertex u_i of degree three, and $t \le 6$ (see the first paragraph of the proof of [17, Proposition 4.14]).

Assume $t = 6$. If u_i has more than $18 (= 3t)$ negative edge endpoints in G_S, then some label appears four times there. This implies $s = 4$ by [17, Lemma 4.7]. Hence it suffices to consider the case where u_i is incident to three families of $4 (= s/2 + 1)$ parallel positive edges. Then there are just 18 negative edge endpoints successively at u_i. Thus any label j appears three times among there. In G_T, the vertex v_j is incident to three positive i-edges. No two of them are parallel by Lemma 2.5(1), and so v_j is incident to three families of 4 parallel positive edges and three families of 6 parallel negative edges. Notice that each of the three families of positive edges contains an i-edge with label i at v_j. But it is easy to see that such labeling is impossible around v_j. The case $t = 4$ is similar to this case.

If $t = 5$, then any positive edge at u_i has weight at most two, since G_S cannot contain a Scharlemann cycle. Thus u_i is incident to at least 19 negative edges successively in G_S. Then some label appears 4 times among negative edge endpoints. This implies $s = 4$ by [17, Lemma 4.7]. The case $t = 3$ is similar to this. □

Therefore we have $s = 4$. Then $\overline{G}_S^+$ consists of two components, each of which has the form of Figure 1(1), (2) or (3) by [17, Lemmas 4.8, 4.11].

Lemma 3.3 *$\overline{G}_S^+$ does not have a component of the form as in Figure 1(1).*

Proof Let Γ be a component of $\overline{G}_S^+$ as in Figure 1(1), and let u_i be the good vertex of degree two in Γ. Since u_i is incident to at most $2(t/2 + 1) = t + 2$ positive edges in G_S, there are at least $4t - 2$ negative edge endpoints. Then

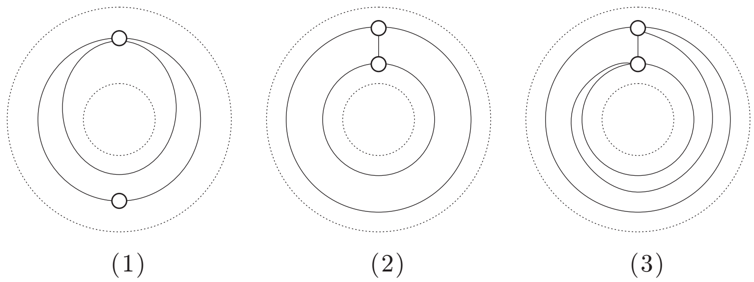


Figure 1

some label j appears 4 times there, because $4t-2 > 3t$. In G_T, v_j is incident to 4 positive i-edges. Since no two of them are parallel, v_j is incident to 4 families of 3 parallel positive edges and two families of 4 parallel negative edges. Notice that each family of positive edges contains an i-edge with label i at v_j. But this is clearly impossible, because both families of negative edges contain i-edges. (Recall that any label appears just 5 times around a vertex.) □

Lemma 3.4 *$\overline{G}_S^+$ does not have a component of the form as in Figure 1(2).*

Proof Let Γ be such a component with a good vertex u_i of degree three. Assume $t > 6$. Then u_i has at least $5t - 3(t/2+1) = 7t/2 - 3 > 3t$ negative edge endpoints. Hence some label j appears 4 times there. Then the same argument as in the proof of Lemma 3.3 works. If $t = 3$ or 5, then u_i has more than $3t$ negative edge endpoints, because G_S cannot contain a Scharlemann cycle. Then some label appears 4 times again, and so it leads to a contradiction.

Finally, the argument in the proof of [17, Lemma 4.16] works when $t = 4$ and 6. (Use Lemma 3.1(2) instead of [17, Lemma 2.7(2)].) □

Proposition 3.5 *$s = 4$ is impossible.*

Proof By Lemmas 3.3 and 3.4, $\overline{G}_S^+$ consists of two components of the form as in Figure 1(3). Notice that any vertex of $\overline{G}_S$ is incident to at most two negative edges. Let u be a vertex of G_S.

First, suppose that $\widehat{T}$ is separating. Then $t \geq 4$. Hence any family of parallel negative edges in G_S has at most t edges by Lemma 2.4. Thus u has at most $2t$ negative edge endpoints, and then it has at least $3t$ positive edge endpoints. From $4(t/2+1) \geq 3t$, we have $t = 4$. Then u is incident to three loops and two families of 3 parallel positive edges, and so there are two S-cycles with disjoint label pairs, which is impossible by Lemma 3.1.

Hence $\widehat{T}$ is non-separating. Then u has at most $4 \cdot t/2 = 2t$ positive edge endpoints by Lemma 2.4. Hence there are at least $3t$ negative edge endpoints consecutively. If there are more than $3t$, then some label appears 4 times there, which leads to a contradiction as in the proof of Lemma 3.3. Thus u has exactly $3t$ negative edge endpoints, and is incident to 4 families of $t/2$ parallel positive edges.

Let u' be the other vertex of the same component as u. Since G_S cannot contain a Scharlemann cycle, the labeling around u' is uniquely determined by the labeling around u. But then it is clear to see that there is a Scharlemann cycle of length three. □

4 The case $t = 1$

The reduced graph $\overline{G}_T$ consists of at most three edges by [5, Lemma 5.1]. We denote the weights of the edges by (w_1, w_2, w_3), and say $G_T \cong G(w_1, w_2, w_3)$ as in [5]. Notice that $G(w_1, w_2, w_3)$ is invariant under any permutations of the w's.

Lemma 4.1 $s = 2$.

Proof If $s \geq 4$, then the vertex of G_T is incident to at most $6(s/2+1) = 3s+6$ edges. From $3s + 6 \geq 5s$, we have $s \leq 3$, a contradiction. □

Thus G_T has exactly five $\{1,2\}$-edges, which are divided into at most three families of mutually parallel edges. Since any edge of G_T is positive, all edges of G_S are negative by the parity rule, and they are divided into at most 4 classes (see [8]).

Lemma 4.2 $G_T \cong G(3,1,1)$.

Proof If two parallel edges of G_T have the same edge class label, then these edges are parallel in both G_S and G_T. This is impossible by Lemma 2.2. Hence at most four edges can be parallel in G_T. Then $G_T \cong G(4,1,0)$, $G(3,2,0)$, $G(3,1,1)$ or $G(2,2,1)$. However all but $G(3,1,1)$ are impossible, because each edge must have labels 1 and 2 at its endpoints. □

Proposition 4.3 *∂M consists of a single torus.*

Proof If the jumping number is one, then G_S and G_T are determined as shown in Figure 2, where the correspondence of edges is indicated.

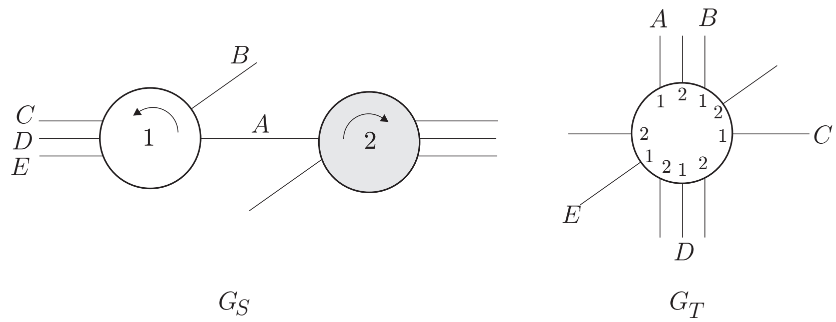


Figure 2

Hence G_T contains an S-cycle σ_1 consisting of edges A, D whose face is f_1, and a Scharlemann cycle σ_2 of length three with face f_2 consisting of edges B, C, E. They lie on the same side of $\widehat{S}$. Let us call this side the black side $\mathcal{B}$, and call the other the white side $\mathcal{W}$. Let $H = V_\alpha \cap \mathcal{B}$. Take $X = \widehat{S} \cup H \cup N(f_1 \cup f_2)$ in $\mathcal{B}$. Then ∂X consists of the torus $\widehat{S}$ and the 2-sphere. For, ∂f_1 is non-separating on the genus two surface F obtained from $\widehat{S}$ by tubing along H, and ∂f_2 is non-separating on the torus obtained from F by compressing along f_1. Since $M(\alpha)$ is irreducible, its 2-sphere bounds a ball in $\mathcal{B}$. The situation in $\mathcal{W}$ is similar. This means that $M(\alpha)$ is closed, and so ∂M is a single torus.

The case where the jumping number is two is similar. In fact, G_S and G_T are determined as shown in Figure 3, where the correspondence of edges is indicated. □

Indeed, we can calculate $\pi_1 M(\alpha)$ by Van Kampen's theorem. Then if the jumping number is one, then $\pi_1 M(\alpha) = \mathbb{Z}_5$, which contradicts that $M(\alpha)$ is toroidal.

5 The case where $s \geq 4, t = 2$

The reduced graph $\overline{G}_T$ is a subgraph of the graph as shown in Figure 4. Here, q_i denotes the weight of edge. As in [5], we say $G_T \cong G(q_1, q_2, q_3, q_4, q_5)$.

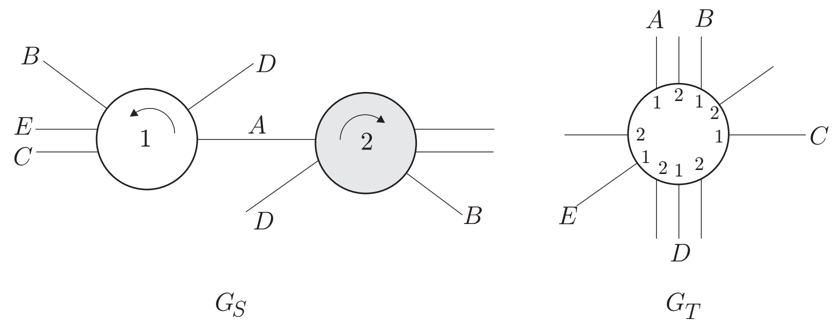


Figure 3

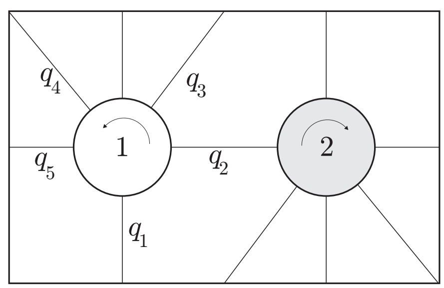


Figure 4

Lemma 5.1 *Two vertices of G_T have opposite signs.*

Proof Assume not. Each vertex of $\overline{G}_T$ has degree at most 6. Note all edges of G_T are positive. Then $5s \le 6(s/2+1) = 3s+6$, giving $s \le 3$, a contradiction. ∎

Then the arguments from Lemmas 5.1 to 5.3 of [17] work with exchanging the roles of G_S and G_T there. In particular, $q_1 = s/2$ or $s/2+1$. In the proofs of Lemmas 5.2 and 5.3 of [17], we use the fact that the jumping number is one. But the case where the jumping number is two is similar.

Lemma 5.2 *The case $q_1 = s/2$ is impossible.*

Proof If $q_1 = s/2$ then $G_T \cong G(s/2, s, s, s, s)$. Then the four families of parallel negative edges correspond to the same permutation σ, which is an involution [17, Lemma 5.3]. Thus any component of G_S^+ has two vertices and 8 edges. Then there are 4 mutually parallel edges, and so we have two bigons lying in the same side of $\widehat{T}$. If these bigons do not have the same pair of edge class labels, then $M(\beta)$ contains a Klein bottle by [8, Lemma 5.2]. Hence those have the same pair of edge class labels, but this is impossible by Lemma 2.2. □

Thus $q_1 = s/2 + 1$, and furthermore, Lemmas 5.8, 5.9 and 5.10 of [17] hold. (Instead of Lemma 2.7(2) of [17], we use the assumption that $M(\alpha)$ contains no Klein bottle.) Hence we have $s = 4$. But this is shown to be impossible.

Lemma 5.3 *$s = 4$ is impossible.*

Proof We use the labeling of G_T as in [17, Figure 10] (with changing t to s). In G_S, u_1 and u_4 are incident to three loops, and u_2 and u_3 are incident to two loops. In G_T, there are two S-cycles with label pair $\{2, 3\}$. The edges of them give four edges between u_2 and u_3 in G_S. Then two endpoints with label 1 of loops at u_2 cannot be successive among five occurrences of label 1. Hence the jumping number is two.

In G_S, there are two edges between u_1 and u_3, which belong to C in G_T. Hence they are not parallel in G_S by Lemma 2.2. Then there are two bigons at u_1 and u_3 which lie on the same side of $\widehat{T}$. By [8, Lemma 5.2], they must have the same pair of edge class labels. Let e be the remaining loop among three loops at u_1, not in the bigon. By Lemma 2.2, e belongs to D in G_T. Also, let c be the edge connecting u_1 and u_3 with the same label as e at u_1. Then the endpoints of c and e are consecutive at v_1 among five occurrences of label 1, which contradicts that the jumping number is two. □

6 The case where $s = 2, t > 2$

If $\widehat{T}$ is separating in $M(\beta)$, then the argument of Section 5 works with exchanging the role between G_S and G_T. Hence we suppose that $\widehat{T}$ is non-separating throughout this section. We use p_i to denote the weight in $\overline{G}_P$, instead of q_i in Figure 4. Notice that $p_1 \leq t/2$, otherwise G_S contains an S-cycle.

Lemma 6.1 $p_1 = 0$.

Proof Assume $p_1 \neq 0$. Then G_S contains a positive edge, and hence not all vertices of G_T have the same sign. By Lemma 2.4(2), $p_i \leq t$ for $i = 2, 3, 4, 5$. Since $2p_1 + p_2 + p_3 + p_4 + p_5 = 5t$, we have $p_1 = t/2$ and $p_i = t$ for any $i \neq 1$. Then Lemma 5.2 and the argument before it lead us to the conclusion. □

Thus the edges of G_S are divided into at most 4 edge classes, and then some class contains more than t edges. This implies that all vertices of G_T have the same sign by Lemma 2.4. Also any edge of G_T is a $\{1,2\}$-edge, and any disk face of G_T is a Scharlemann cycle.

Lemma 6.2 *G_T has a black Scharlemann cycle and a white Scharlemann cycle.*

Proof Since G_S has $5t$ edges, some edge class contains more than t edges. The associated permutation to the family has a single orbit by [5, Lemma 4.2]. In particular, these $t+1$ edges cut $\widehat{T}$ into a disk. Thus all faces of G_T are disks, which gives a conclusion immediately. □

We say that two (disk) faces f_1, f_2 of G_T of the same color are *isomorphic* if the cyclic sequences of edge class labels, read around their boundaries in the same direction, are equal.

Lemma 6.3 *∂M consists of at most two tori.*

Proof First, we prove:

Claim 6.4 *∂M consists of at most three tori.*

Proof of Claim 6.4 Recall that $\widehat{S}$ separates $M(\alpha)$ into $\mathcal{B}$ and $\mathcal{W}$. Let $H = V_\alpha \cap \mathcal{B}$ and let f be a black Scharlemann cycle in G_T. Then take a neighborhood $N = N(\widehat{S} \cup H \cup f)$ in $\mathcal{B}$. Thus $\partial N = \widehat{S} \cup S'$, where S' is a torus. Since $S' \cap V_\alpha = \emptyset$, and M is irreducible and atoroidal, either S' bounds a solid torus in $\mathcal{B}$ or S' is parallel to a component of ∂M. This means that $\partial\mathcal{B}$ consists of at most two torus boundary components, and similarly for $\mathcal{W}$. □

Suppose that ∂M consists of exactly three tori. This happens only when both $\mathcal{B}$ and $\mathcal{W}$ have two tori as their boundaries. Then all black disk faces of G_T are isomorphic, and so are all white disk faces of G_T by the argument of the proof of [8, Lemma 5.6]. Notice that G_T has $5t$ edges, but $\overline{G}_T$ has at most $3t$ edges, as seen by an easy Euler characteristic calculation. Hence G_T has a bigon. Thus we may assume that all black faces are bigons. By [8, Lemma 5.2], all black bigons have the same pair of edge class labels, $\{\lambda, \mu\}$, say. (For, $M(\alpha)$ contains no Klein bottle.) Since all faces of G_T are disks as in the proof of Lemma 6.2, the set of edge class labels of any white face is also $\{\lambda, \mu\}$. In G_S, this means that all edges are divided into two classes λ and μ. Thus either of them contains more than $2t$ edges. By [5, Corollary 5.5], $t = 3$. Then G_T has 15 edges. But this is impossible, because all black faces are bigons. □

7 The case where $s = t = 2$

The reduced graphs $\overline{G}_S$ and $\overline{G}_T$ are subgraphs of the graph as shown in Figure 4. Recall that we use p_i (resp. q_i) to denote the weight of edge in $\overline{G}_S$ (resp. $\overline{G}_T$).

7.1 Two vertices of G_T have the same sign

Since all edges in G_T are positive, all edges in G_S are negative. Thus the edges of G_S are divided into four edge classes. Also, any edge of G_T is a $\{1, 2\}$-edge, and any disk face of G_T is a Scharlemann cycle.

Lemma 7.1 *G_T has a black Scharlemann cycle and a white Scharlemann cycle.*

Proof If some $q_i > 2$, then G_T contains a black bigon and a white bigon. Hence we assume that $q_i \leq 2$ for any i. Since $2q_1 + q_2 + q_3 + q_4 + q_5 = 10$, $q_1 \neq 0$. If $q_1 = 1$, then $q_2 = q_3 = q_4 = q_5 = 2$, giving the conclusion. If $q_1 = 2$, then we can assume that $q_2 + q_3 = 4$ and $q_4 + q_5 = 2$ by symmetry. Then $q_2 = q_3 = 2$, giving the conclusion. □

By the same argument in the proof of Claim 6.4, ∂M consists of at most three tori.

Lemma 7.2 *∂M consists of at most two tori.*

Proof If not, then as in the proof of Lemma 6.3, all black disk faces of G_T are isomorphic, and so are all white disk faces of G_T. If $q_i \geq 3$ for some i, then all disk faces of G_T would be bigons, which is impossible. Hence $q_i \leq 2$ for any i. In particular, $q_1 > 0$. If $q_1 = 1$, then $q_2 = q_3 = q_4 = q_5 = 2$, which contradicts that all disk faces of the same color are isomorphic. If $q_1 = 2$, then we may assume that $q_2 = q_3 = 2$ and $q_4 + q_5 = 2$ by symmetry. But any case where $(q_4, q_5) = (2, 0), (1, 1)$ gives a contradiction similarly. □

7.2 Two vertices of G_T have distinct signs

We will show that there is only one possible pair for $\{G_S, G_T\}$. Lemmas 6.1 and 6.2 of [17] hold here (the jumping number two case is similar in the proof of Lemma 6.2 of [17]), and hence $p_1 = 2$ or 3.

Lemma 7.3 *If $p_1 = 2$, then the graphs are as shown in Figure 5, where the jumping number is two.*

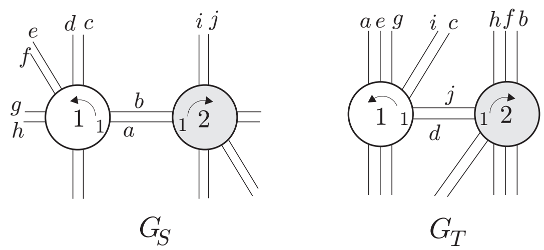


Figure 5

Proof As in the proof of [17, Lemma 6.3], there is only one possibility for G_T as shown in [17, Figure 16(4)]. In fact, if the jumping number is one, then this is also eliminated as shown there. □

Lemma 7.4 *If $p_1 = 3$, then the graphs are the same as in Figure 5 with exchanging G_S and G_T.*

Proof We may assume that $(p_2+p_3, p_4+p_5) = (4,0)$ or $(2,2)$ by symmetry. In the latter case, there are three possibilities for G_S as in the proof of [17, Lemma 6.4], and all are impossible. Thus $(p_2+p_3, p_4+p_5) = (4,0)$, giving $p_2 = p_3 = 2$. Hence $q_1 = 2$, and so we can assume that $(q_2+q_3, q_4+q_5) = (6,0)$ or $(4,2)$ by symmetry. Then $(6,0)$ contradicts Lemma 2.2. By using the parity rule, it is easy to see that G_T is as in Figure 5 (with exchanging G_S and G_T). □

Lemma 7.5 *If the graphs are as in Figure 5, then ∂M consists of at most two tori.*

Proof We may use the notation of Figure 5. Then we can assume that G_T contains 4 black bigons and two white bigons and two white 3-gons. As in the proof of Claim 6.4, the black side $\mathcal{B}$ of $\widehat{S}$ in $M(\alpha)$ has $\widehat{S}$ and at most one torus as its boundary. On the other hand, the white side $\mathcal{W}$ has a single torus $\widehat{S}$ as its boundary, because a torus obtained from $\widehat{S} \cup (V_\alpha \cap \mathcal{W})$ by attaching a bigon, will be compressed by a 3-gon. Hence ∂M consists of at most two tori. □

8 Klein bottle

In the rest of the paper, we will treat the case where either $M(\alpha)$ or $M(\beta)$ contains a Klein bottle.

Suppose that $M(\alpha)$ contains a Klein bottle $\widehat{P}$ such that $\widehat{P} \cap V_\alpha$ consists of p meridian disks $u_1, u_2, \dots, u_p$, numbered successively, of V_α, and that p is minimal among all Klein bottles in $M(\alpha)$. Let $P = \widehat{P} \cap M$. Since M is hyperbolic, $p > 0$. Remark that we do not assume that $M(\alpha)$ is toroidal.

Now, we suppose $\Delta = \Delta(\alpha, \beta) \geq 6$. Then notice that both of $M(\alpha)$ and $M(\beta)$ are irreducible by [12, 13, 14, 18]. Let $S' = \partial N(\widehat{P})$. If S' is boundary parallel in $M(\alpha)$, then $M(\alpha) = N(\widehat{P})$, and hence ∂M consists of two tori. If $M(\alpha)$ is also toroidal, then ∂M is at most two tori by [5]. Hence we assume S' is compressible in $M(\alpha)$. But this implies that S' bounds a solid torus by the irreducibility of $M(\alpha)$, and so ∂M is a single torus. Therefore we assume that $\Delta = 5$ in the rest of the paper. Then both of $M(\alpha)$ and $M(\beta)$ are irreducible.

Lemma 8.1 *P is incompressible and boundary-incompressible in M.*

Proof Suppose that P is compressible in M. Let D be a disk in M such that $D \cap P = \partial D$ and ∂D does not bound a disk on P. Note that ∂D is orientation-preserving on P.

If ∂D is non-separating on $\widehat{P}$, then we get a non-separating 2-sphere in $M(\alpha)$ by compressing $\widehat{P}$ along D. This contradicts the irreducibility of $M(\alpha)$. If ∂D bounds a disk on $\widehat{P}$, then we replace the disk with D, and get a new Klein bottle in $M(\alpha)$ with fewer intersections with V_α than $\widehat{P}$. This contradicts the choice of $\widehat{P}$. Thus ∂D is essential and separating on $\widehat{P}$. Compressing $\widehat{P}$ along D gives two disjoint projective planes in $M(\alpha)$. Since $M(\alpha)$ is irreducible, this is also impossible. Thus we have shown that P is incompressible.

Next, let E be a disk in M such that $E \cap P = \partial E \cap P$, $\partial E = a \cup b$, where $a \subset P$ is an essential (i.e., not boundary-parallel) arc in P and $b \subset \partial M$. If a joins distinct components of ∂P, then a compressing disk for P is obtained from two parallel copies of E and the disk obtained by removing a neighborhood of b from the annulus in ∂M cobounded by those components of ∂P meeting a. Hence ∂a is contained in the same component $\partial_1 P$, say, of ∂P. If $p > 1$, then b bounds a disk D' in ∂M together with a subarc of $\partial_1 P$. Then $E \cup D'$ gives a compressing disk for P in M. Therefore $p = 1$. Then we can move the core of V_α onto an orientation-reversing loop in $\widehat{P}$ by using E. This implies that M contains a properly embedded Möbius band, which contradicts the fact that M is hyperbolic. □

Thus we can define two graphs G_P on $\widehat{P}$ and G_T on $\widehat{T}$ from the arcs in $P \cap T$ as in Section 2. We can label each endpoint of edges of these graphs as before. Note that neither G_P nor G_T has a trivial loop. Lemma 2.1 holds without any change.

Since $\widehat{P}$ is non-orientable, we cannot give a sign to a vertex of G_P as in G_T. Hence assign an orientation to each vertex of G_P as a meridian disk of V_α. That is, all vertices of G_P determine the same homology class in $H_2(V_\alpha, \partial V_\alpha)$. By using this, we give a sign to each edge of G_P as follows.

Let e be an edge of G_P. Assume that e is a loop based at u. Then e is *positive* if a regular neighborhood $N(u \cup e)$ on $\widehat{P}$ is an annulus, *negative* otherwise. Assume that e connects distinct vertices u_i and u_j. Then $N(u_i \cup e \cup u_j)$ is a disk. Then e is *positive* if we can give an orientation to the disk $N(u_i \cup e \cup u_j)$ so that the induced orientations on u_i and u_j are compatible with those of u_i and u_j simultaneously. Otherwise, e is *negative*. Then the parity rule (Lemma 2.2(1)) still holds without change. In fact, the above definition works for G_T, and so this is a natural generalization of the usual parity rule. Also, Lemma 2.2(2) is true.

Lemma 8.2 *G_P satisfies the following*:

(1) *If $t \geq 3$, then any family of parallel positive edges contains at most $t/2+2$ edges. Moreover, if it contains $t/2+2$ edges, then $t \equiv 0 \pmod 4$, and, up to relabelling of vertices of G_T, it contains $\{1,2\}$ S-cycle and $\{t/2, t/2+1\}$ S-cycle.*

(2) *Either all the vertices of G_T have the same vertex, or any family of negative edges contains at most t edges. In particular, if G_P contains a positive edge, any family of negative edges contains at most t edges.*

Proof (1) is [18, Lemma 1.4 and Corollary 1.8]. (2) is the same as Lemma 2.4(2). □

If $p \geq 3$, a *generalized S-cycle* in G_T is the triplet of mutually parallel positive edges e_{-1}, e_0, e_1, where e_{-1} and e_1 have the same label pair $\{i-1, i+1\}$, and e_0 is a level i-edge for some i.

Lemma 8.3 *G_T has neither a Scharlemann cycle nor a generalized S-cycle.*

Proof For a Scharlemann cycle, see [16, Lemma 3.2]. (It treats the case of S-cycles, but the argument works for general case.) □

Lemma 8.4 *Assume $p \geq 2$. Then G_T satisfies the following*:

(1) *Any family of parallel positive edges contains at most $p/2+1$ edges. Moreover, if it contains $p/2+1$ edges, then the first and last edge are level.*

(2) *Any family of parallel negative edges contains at most p edges.*

Proof (1) Assume that G_T contains a family A of mutually parallel positive edges which connect v_i and v_j (possibly, $i=j$), and that A contains more than $p/2+1$ edges.

When $p=2$, no edge of A is level. Otherwise, there would be a pair of edges which are parallel in both graphs. But this is impossible by Lemma 2.2(2). Hence A contains an S-cycle, a contradiction by Lemma 8.3.

Suppose $p>2$. Note that some label appears at both v_i and v_j. If A contains no level edge, then A contains an S-cycle. This is impossible by Lemma 8.3. Hence A must contain a level edge. Moreover, a level edge is the first or last edge of A. Otherwise, A contains a generalized S-cycle, which is also impossible by Lemma 8.3. We may assume that the first edge of A is level. Then A contains an S-cycle if p is odd. If p is even, the second to last edge is level, and hence there is a generalized S-cycle. The second assertion is easy to see.

(2) Assume that G_T contains $p+1$ parallel negative edges, connecting v_i and v_j. Consider the associated permutation σ to these edges as follows. Let $a_1, a_2, \dots, a_p, b_1$ be the edges labelled successively. We may assume that a_k has label k at v_i, label $\sigma(k)$ at v_j. Let θ be the orbit of σ containing 1, and let C_θ be the cycle in G_P corresponding to θ. Then C_θ does not bound a disk in $\widehat{P}$ by [5, Lemma 2.3]. Note that there are two possibilities for C_θ, that is, separating or non-separating in $\widehat{P}$, since C_θ is orientation-preserving in $\widehat{P}$. Consider the edge b_1. Since b_1 is positive in G_P, either b_1 is parallel to a_1 in G_P, or, the cycle consisting of the edges $a_2, \dots, a_p, b_1$ bounds a disk in $\widehat{P}$. But the former contradicts Lemma 2.2(2), and the latter is impossible by [5, Lemma 2.3] again. □

In this section, we treat the case that G_P or G_T has a single vertex.

Proposition 8.5 *If $t=1$, then ∂M is a single torus.*

Proof Suppose $t=1$. If $p=1$, then G_P has a single vertex with degree 5, which is impossible. Recall that the edges of G_T are divided into at most three edge classes as in Section 4. Also, at most $p/2+1$ edges can be parallel in G_T. If $p \geq 3$, then $6(p/2+1) \geq 5p$ gives $p=3$. But then at most two edges can be parallel in G_T, giving $6 \cdot 2 \geq 15$, a contradiction. Thus $p=2$, and hence $G_T \cong G(2,2,1)$. (Recall the notation in Section 4.) Then G_S is uniquely determined, and the correspondence between the edges of G_P and G_T is shown in Figure 6. Here, the jumping number must be one, and two end circles of the cylinder are identified through a suitable involution to form the Klein bottle $\widehat{P}$. Note that the edge connecting two vertices with labels 1 and 2 is negative in G_P.

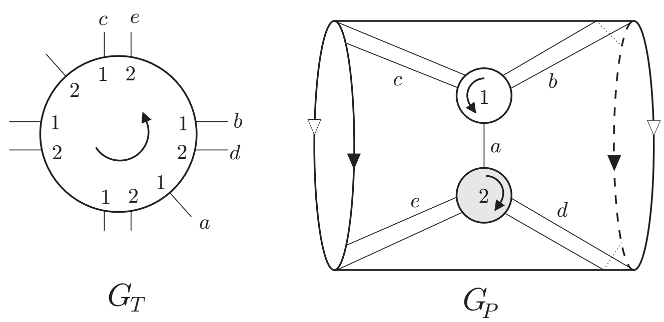


Figure 6

Let $N(\widehat{P})$ be a regular neighborhood of $\widehat{P}$ in $M(\alpha)$. Then $N(\widehat{P})$ is the twisted I-bundle over $\widehat{P}$, and $\partial N(\widehat{P})$ is a torus. Let us write $M(\alpha) = N(\widehat{P}) \cup W$. Then $T \cap W$ consists of two bigons and two 3-gons. Also, $V_\alpha \cap W$ consists of two 1-handles H_1, H_2. Let F be the genus three closed surface obtained from ∂W by performing surgery along H_1 and H_2. Then attaching a bigon and two 3-gons to F yield the 2-sphere. Since $M(\alpha)$ is irreducible, $M(\alpha)$ must be closed. The result immediately follows from this. □

Lemma 8.6 *If $p=1$, then $\overline{G}_P$ is a subgraph of either graph shown in Figure 7.*

Proof An orientation-preserving loop on a Klein bottle is non-separating or separating. Also, there are only two classes of orientation-reversing loops. The result follows immediately. (See [15, Lemma 2.1].) □

Thus we say $G_P \cong H(p_1,p_2,p_3)$ for (i), or $H'(p_1,p_2,p_3)$ for (ii), where p_1 denotes the weight of the positive loop, and the others denote the weight of negative loops in each class. Clearly, $H(p_1,p_2,p_3) \cong H(p_1,p_3,p_2)$, $H'(p_1,p_2,p_3) \cong$

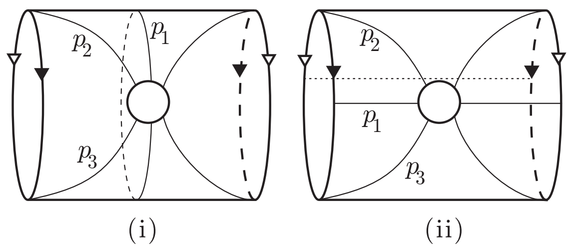


Figure 7

$H'(p_1, p_3, p_2)$ and $H(0, p_2, p_3) \cong H'(0, p_2, p_3)$. Also, $2(p_1+p_2+p_3) = 5t$ implies that t is even.

Proposition 8.7 *If $p = 1$ and $t = 2$, then ∂M consists of at most two tori.*

Proof First, we claim $p_1 \neq 0$. If $p_1 = 0$, then $G_P \cong H(0, 5, 0)$, $H(0, 4, 1)$ or $H(0, 3, 2)$. For $H(0, 5, 0)$, G_T contains 5 edges connecting v_1 and v_2. Since there are at most 4 edge classes, this contradicts Lemma 2.2(2). For $H(0, 4, 1)$, G_T has two loops at each vertex, which must be parallel. So, this contradicts Lemma 2.2(2) again. For $H(0, 3, 2)$, $G_T \cong G(1, 1, 1, 1, 0)$ by using Lemma 2.2(2). Then a jumping number argument eliminates this as follows. By examining the endpoints of a loop at v_1, we see that the jumping number is two. Let a and b be the edges connecting v_1 and v_2 such that their end points at v_1 are consecutive. Then they are parallel in G_P and adjacent. (In fact, they belong to the family of 3 mutually parallel negative edges of G_P.) By Lemma 2.1, the endpoints of a and b with label 1 are not consecutive at u_1 among five occurrences of label 1. Then their endpoints with label 2 are consecutive among five occurrences of label 2 at u_1. But a and b are consecutive at v_2 also, which contradicts Lemma 2.1.

Notice that $1 \leq p_1 \leq 5$. If $p_1 = 5$, then we have a pair of edges which are parallel in both graphs, a contradiction. In the following, we consider all possibilities for G_P.

Seven cases $H(4, 1, 0)$, $H(3, 2, 0)$, $H(2, 3, 0)$, $H(2, 2, 1)$, $H(1, 4, 0)$, $H(1, 2, 2)$, $H'(1, 3, 1)$ are impossible by the parity rule. For the four cases $H'(4, 1, 0)$, $H'(3, 1, 1)$, $H'(2, 3, 0)$, $H'(2, 2, 1)$, G_P contains an S-cycle. Hence $\widehat{T}$ is separating, and so the faces of G_P can be colored by two colors in such a way that two sides of an edge have distinct colors. This fact eliminates these four cases. For $H(1, 3, 1)$ and $H'(1, 4, 0)$, G_T contains two loops which are parallel in both graphs.

For $H'(1,2,2)$, $G_T \cong G(2,1,0,0,0)$. At v_1, there is no correct arrangement of edges to satisfy Lemma 2.1. For $H(3,1,1)$, $G_T \cong G(1,1,1,1,0)$. As in the proof of Proposition 8.5, let $M(\alpha) = N(\widehat{P}) \cup W$. Then $T \cap W$ contains two 3-gons. Attaching these 3-gons to $N(\partial W \cup (V_\alpha \cap W))$ yields a 2-sphere. Since $M(\alpha)$ is irreducible, $M(\alpha)$ is closed. Thus ∂M is a single torus. Finally, for $H'(3,2,0)$, $G_T \cong G(1,1,1,1,0)$ again. Take one 3-gon in $T \cap W$. Attaching it to $N(\partial W \cup (V_\alpha \cap W))$ yields a torus S', missing V_α. Thus S' is boundary parallel or compressible. In the former, ∂M consists of two tori. In the latter, either S' bounds a solid torus in W, which implies that ∂M is a single torus, or S' is contained in a 3-ball in $M(\alpha)$, which implies that S' bounds a knot exterior X. Since a Klein bottle cannot lie in a knot exterior, X lies in W. In any case, ∂M is a single torus. □

Proposition 8.8 *If $p = 1$ and $t > 2$, then ∂M consists of at most two tori.*

Proof By Lemma 8.2, $p_1 \le t/2+2$. Hence $p_2+p_3 = 5t/2-p_1 \ge 2t-2$. Then an Euler characteristic calculation shows that G_T^+ has a disk face D. Let us write $M(\alpha) = N(\widehat{P}) \cup W$. Then $\partial N(\partial W \cup (V_\alpha \cap W) \cup D)$ consists of two tori, since ∂D runs on the 1-handle $V_\alpha \cap W$ in the same direction. This implies that ∂M consists of at most two tori as in the last paragraph of the proof of Proposition 8.7. □

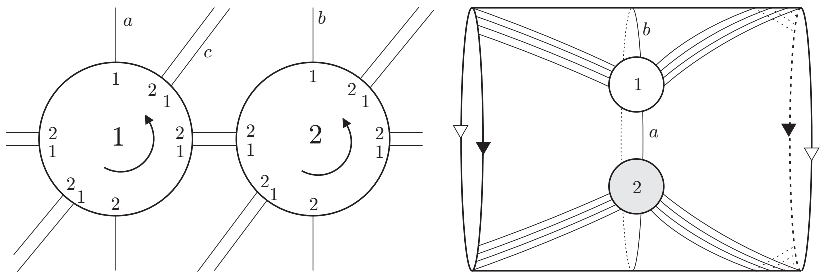
9 Klein bottle; the case $t = 2$

By Section 8, we may assume $p \ge 2$.

Lemma 9.1 *Two vertices of G_T have opposite signs.*

Proof Assume not. Then $q_i \le p/2+1$ for any i. Thus $5p \le 6(p/2+1) = 3p+6$ gives $p \le 3$. If $p = 3$, then $q_i \le 2$, and so $15 = 5p \le 12$, a contradiction. Assume $p = 2$. Since $q_i \le 2$ for any i, $q_1 = 1$ or 2. Hence $G_T \cong G(1,2,2,2,2)$ or $G(2,q_2,q_3,q_4,q_5)$ with $q_2+q_3 = q_4+q_5 = 3$.

For $G(1,2,2,2,2)$, the labels of G_T are determined, up to exchange of 1 and 2, and then G_P is uniquely determined. See Figure 8. Consider the edges a, b and c as there. The endpoints of a and c are consecutive at v_1, but those of b and c are not consecutive at v_2, among the five occurrences of label 1. Any location of c contradicts Lemma 2.1 at u_1.



G_T G_P

Figure 8

Suppose $G_T \cong G(2, q_2, q_3, q_4, q_5)$ with $q_2 + q_3 = q_4 + q_5 = 3$. Then $G_T \cong G(2, 2, 1, 2, 1)$ or $G(2, 2, 1, 1, 2)$. In any case, each vertex of G_P is incident to 4 negative loops, where are parallel. But two of them are level, and the others are not level, a contradiction. □

Lemma 9.2 *If p is even, then $q_1 = p/2$ or $p/2 + 1$. If p is odd, then $q_1 = (p + 1)/2$.*

Proof By Lemma 8.4, $q_1 \le p/2 + 1$ and $q_i \le p$ for $i \ne 1$. Since $2q_1 + q_2 + q_3 + q_4 + q_5 = 5p$, we have $q_1 \ge p/2$, giving the conclusion. □

We consider three cases.

9.1 $q_1 = p/2$

Then $G_T \cong G(p/2, p, p, p, p)$. Let A_i be the family of parallel negative edges of weight q_i for $i = 2, 3, 4, 5$. Then they associate to the same permutation σ.

Lemma 9.3 *σ is not the identity.*

Proof If σ is the identity, then each family A_i contains a $\{1, 1\}$-edge and $\{p, p\}$-edge. Let $G(1, p)$ be the subgraph of G_P spanned by u_1 and u_p. Then $G(1, p)$ has the form as in Figure 9. But a jumping number argument gives a contradiction. □

Lemma 9.4 *If $p = 2$, then ∂M consists of at most two tori.*

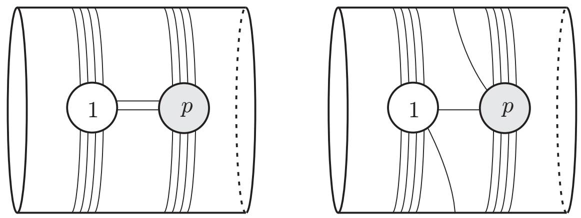


Figure 9

Proof Let us write $M(\alpha) = N(\widehat{P}) \cup W$ again. Then $T \cap W$ consists of four bigons and four 3-gons. Let us choose a bigon D_1 and a 3-gon D_2. It is easy to see that if $X = N(\partial W \cup (V_\alpha \cap W) \cup D_1 \cup D_2)$ then $\partial X = \partial W \cup S'$, where S' is a torus missing V_α. The result follows from this as in the proof of Proposition 8.7. □

Hence we assume $p \geq 3$ hereafter.

Lemma 9.5 *If σ is not the identity, then σ^2 is the identity. In particular, each orbit of σ has length two.*

Proof The proof of [17, Lemma 5.3] works here. □

Lemma 9.6 $q_1 = p/2$ *is impossible.*

Proof We may assume that the edges of A_1 have labels $1, 2, \ldots, p$ at u_1. We follow the argument of [17, Lemma 5.4]. Then the component H of G_P containing $G(1, p/2+1)$ and $G(p/2, p/2+1)$ has the form as in Figure 11 of [17]. (Here, we do not need the assumption $p > 4$.) But a jumping number argument eliminates this configuration (even for the case that the jumping number is two). □

9.2 $q_1 = p/2 + 1$

Since G_T cannot contain a Scharlemann cycle, $G_T \cong G(p/2+1, p, p-1, p, p-1)$ or $G(p/2+1, p, p-1, p-1, p)$. Notice that the first and last edges of the positive loops at each vertex of G_T are level. We may assume that the edges of A_1 have labels $1, 2, \ldots, p$ at u_1. Let σ be the associated permutation to A_1. Then $\sigma(i) \equiv i - 1$ or $i + p/2 - 1 \pmod p$, since $p/2$ and p are the only labels of positive level edges in G_T.

Lemma 9.7 *If $p > 2$, then $\sigma(i) \equiv i + p/2 - 1 \pmod{p}$.*

Proof Assume $\sigma(i) \equiv i - 1 \pmod{p}$. Then the edges of A_1 form an essential cycle C through all vertices, which is separating or non-separating on $\widehat{P}$. Notice that G_T has a $\{1,1\}$-edge in A_2. After putting negative loops at $u_{p/2}$ and u_p, we cannot locate a positive loop at u_1. □

Lemma 9.8 $p = 2$.

Proof Assume not. Suppose $p/2$ is odd. Then σ has at least two orbits. Thus the edges of A_1 form at least two essential cycles on $\widehat{P}$, where $u_{p/2}$ and u_p lie on distinct orbits. Notice that A_2 contains a $\{p/2, p\}$-edge. Since $u_{p/2}$ and u_p are incident to negative loops, the edges of A_1 form just two cycles, which are separating on $\widehat{P}$. Furthermore, u_1 and u_{p-1} lie one the same cycle. Although there is a $\{1, p-1\}$-loop among positive loops at v_1, we cannot locate it in G_P.

Suppose $p/2$ is even. Then σ has a single orbit. Thus the edges of A_1 form an essential cycle C on $\widehat{P}$. Notice that G_T contains a $\{1, p/2+1\}$-edge e in A_2 and a $\{1, p-1\}$-loop f at v_1. After putting the negative loops at $u_{p/2}$ and u_p, we cannot locate e (resp. f) in G_P when C is non-separating (resp. separating) on $\widehat{P}$. □

Finally, we eliminate the case $p = 2$. We denote by σ the associated permutation to A_1.

Lemma 9.9 *If $G_T \cong G(2,2,1,2,1)$, then ∂M is a single torus.*

Proof If σ is the identity, then each vertex of G_P is incident to two positive loops and two negative loops. Hence these positive loops are separating on $\widehat{P}$. Also, G_T has two negative $\{1,2\}$-edges. There are two possibilities for the arrangement of these two edges in G_P. But both contradict Lemma 2.1 by looking the endpoint of the edge of A_2 at u_1.

Thus $\sigma = (12)$. Each vertex of G_P is incident to one positive loop and two negative loops, and there are 4 positive edges between u_1 and u_2. Then G_P^+ is contained in an annulus, whose core is separating on $\widehat{P}$. By Lemma 2.2, the 4 positive edges between u_1 and u_2 are divided into two edge classes. Then the jumping number is two, and G_P is uniquely determined. Let $M(\alpha) = N(\widehat{P}) \cup W$. Then $T \cap W$ consists of four bigons and four 3-gons. Let D_1 be the bigon contained in the parallelism between two loops at v_1, and D_2 a bigon between v_1 and v_2. Also, let D_3 be any 3-gon. Then $X = N(\partial W \cup (V_\alpha \cap W) \cup D_1 \cup D_2 \cup D_3)$ has ∂W and a 2-sphere as its boundary. Since $M(\alpha)$ is irreducible, this implies $M(\alpha)$ is closed. Hence ∂M is a single torus. □

Lemma 9.10 *$G_T \cong G(2,2,1,1,2)$ is impossible.*

Proof By the same argument as in the proof of Lemma 9.9, $\sigma = (12)$. Again, the 4 edges between u_1 and u_2 are divided into two edge classes. In fact, they form two S-cycles, whose faces lie on the same side Y of $\widehat{T}$. By examining the edge correspondence, we see that the jumping number is two. But we cannot draw two loops of the faces of those S-cycles on a genus two surface obtained from $\widehat{T}$ by tubing along $V_\beta \cap Y$, simultaneously. □

9.3 $q_1 = (p+1)/2$

Lemma 9.11 *The case that $q_1 = (p+1)/2$ is impossible.*

Proof Since $q_2+q_3+q_4+q_5 = 4p-1$, $G_T \cong G((p+1)/2, p, p, p, p-1)$. Then two families A_2 and A_3 associate to the same permutation σ. By Lemma 9.5, σ^2 must be the identity, but this is impossible, because p is odd. Thus σ is the identity. Also, if τ is the associated permutation to A_4, then $\tau(i) \equiv i-1 \pmod p$. Hence the edges in A_4 form an essential orientation-preserving cycle on $\widehat{P}$. Since any vertex is incident to a positive loop, corresponding to the edges of A_2, G_P would contain a trivial loop. □

10 Reduced graphs

In this section, we prepare some results about the reduced graph of a graph G (or its subgraph) on a Klein bottle $\widehat{P}$, which will be needed in the last section. We need only the assumption that G has no trivial loops and that the edges of G are divided into positive edges and negative edges.

Let Λ be a component of $\overline{G}^+$. If there is a disk D in $\widehat{P}$ such that $\operatorname{Int} D$ contains Λ, then we say that Λ has a *disk support*. Also, if there is an annulus A in $\widehat{P}$ such that $\operatorname{Int} A$ contains Λ and Λ does not have a disk support, then we say that Λ has an *annulus support*.

Now, suppose that Λ has a support E, where E is a disk or an annulus. A vertex x of Λ is called an *outer vertex* if there is an arc ξ connecting x to ∂E whose interior is disjoint from Λ. Define an *outer edge* similarly. Then $\partial\Lambda$ denotes the subgraph of Λ consisting of all outer vertices and all outer edges of Λ. A vertex x of Λ is called a *cut vertex* if $\Lambda - x$ has more components than Λ.

Suppose that Λ has an annulus support A. A vertex x of Λ is a *pinched vertex* if there is a spanning arc of A which meets Λ in only x. An edge e of Λ is a *pinched edge* if there is a spanning arc of A which meets Λ in only one point on e. Clearly, both endpoints of a pinched edge are pinched vertices.

We say that Λ is an *extremal component* of $\overline{G}^+$ if Λ has a support which is disjoint from the other components of $\overline{G}^+$.

Lemma 10.1 *$\overline{G}^+$ has an extremal component with a disk support or an annulus support.*

Proof Let Λ be a component of $\overline{G}^+$. Choose a spanning tree H of Λ, and contract H into one point. Then we get a bouquet Λ' in $\widehat{P}$. Note that any loop in Λ' is orientation-preserving. If all loops in Λ' are inessential in $\widehat{P}$, then Λ' has a disk support. There are two isotopy classes of orientation-preserving essential loops in $\widehat{P}$. But these two classes cannot exist simultaneously. Therefore, if some loop in Λ' is essential, then Λ' has an annulus support, and so does Λ.

If $\overline{G}^+$ has a component with a disk support, then there exists an extremal component with a disk support. Otherwise, any component of $\overline{G}^+$ has an annulus support, and hence any component is extremal. □

Let x be a vertex of G. Then x is called an *interior vertex* if there is no negative edge incident to x in G. Since $\overline{G}$ and $\overline{G}^+$ have the same vertex set as G, we may call a vertex of $\overline{G}$ or $\overline{G}^+$ an interior vertex when it is an interior vertex of G. In particular, if x is in an extremal component of $\overline{G}^+$ with a disk or an annulus support, and it is not an outer vertex, then x is an interior vertex.

A vertex x is said to be *good* if all positive edge endpoints around x are successive in G. Thus an interior vertex is good. When x is a vertex of an extremal component Λ of $\overline{G}^+$ with a disk or an annulus support, x is good if

(i) x is not a cut vertex of Λ if Λ has a disk support; or

(ii) x is neither a cut vertex nor a pinched vertex of Λ if Λ has an annulus support.

Proposition 10.2 *If each interior vertex of $\overline{G}$ has degree at least 6, then $\overline{G}^+$ has either a good vertex of degree at most 4, or a vertex of degree at most 2.*

Proof See [17, Proposition 3.4] (and its proof). If an extremal component of $\overline{G}^+$ is a single vertex or a cycle, then we have the second conclusion. □

If G has no interior vertex, then we have a stronger conclusion.

Lemma 10.3 *Suppose that G has no interior vertex. Let Λ be an extremal component of $\overline{G}^+$. If Λ has an annulus support and Λ is not a cycle, then either*

(1) *Λ has two non-pinched good vertices of degree at most 4 on the same side of Λ;*

(2) *Λ has a non-pinched vertex of degree at most two*; *or*

(3) *Λ is as shown in Figure* 10(1), (2), (3) *or* (4) *with possibly no pinched edge.*

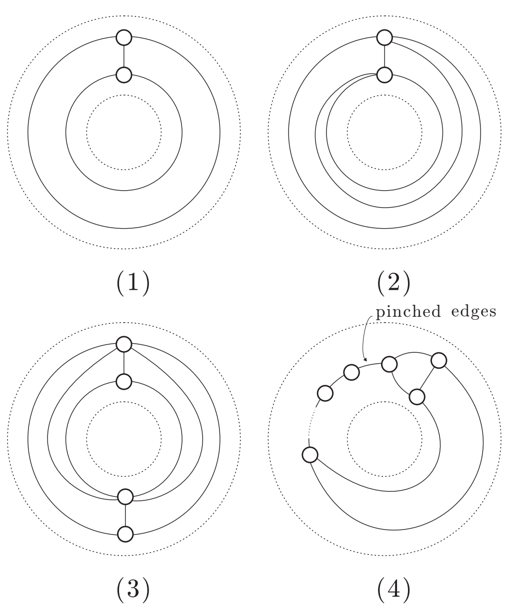


Figure 10

Proof Let V be the number of vertices of Λ.

(1) First, consider the case where Λ has no cut vertex.

If Λ has no pinched vertex, then $\partial\Lambda$ consists of two cycles. Note that any vertex lies on $\partial\Lambda$, because G has no interior vertex. If some edge, not in $\partial\Lambda$, connects two vertices on the same cycle of $\partial\Lambda$, then the cycle contains a vertex of degree two, giving the conclusion (2). Hence we can assume that any edge of Λ, not in $\partial\Lambda$, connects two vertices on distinct sides.

If $V=2$, then Λ is as shown in Figure 10(1) or (2). If $V=3$, then it is easy to see that (1) or (2) holds. Let $V=4$. If one cycle of $\partial\Lambda$ contains one vertex, then we have (1) or (2). Hence we may assume that each cycle of $\partial\Lambda$ contains two vertices. Clearly, any vertex has degree at most 5. If all vertices have degree at most 4, then we have (1). So, assume that some vertex has degree 5. Then we have (2), or Λ is Figure 10(3). Hereafter we assume $V\geq 5$.

If there are more than two vertices of degree at most 4, then we have (1). Hence we assume that all vertices but at most two vertices x,y have degree at least 5. Take a double of Λ along $\partial\Lambda$, and let E and F be the number of edges, faces, respectively, as a graph on a torus. Then $V-E+F=0$ and $3F\leq 2E$, giving $E\leq 3V$. For a vertex of Λ with degree at least 5, it has degree at least 8 in the double. Hence $\deg(x)+\deg(y)+8(V-2)\leq 2E\leq 6V$, where $\deg(-)$ denotes degree in the double. Then $\deg(x)+\deg(y)\leq 16-2V\leq 6$. If either x or y has degree two in Λ, then (2) holds. But, if not, $\deg(x)\geq 4$ and $\deg(y)\geq 4$, a contradiction.

Next, suppose that Λ has a pinched vertex. If necessary, contract all pinched edges, and denote the resulting graph by Λ'. If Λ' contains more than one pinched vertices, then consider a part H between two consecutive pinched vertices. If Λ' contains only one pinched vertex x, then take a spanning arc ξ of the annulus support of Λ' with $\xi\cap\Lambda'=x$, and split along ξ to obtain H. In any case, H has more than two vertices, and has a disk support. Let x_1 and x_2 denote the two vertices coming from pinched vertices of Λ'.

If $H=\partial H$, then any vertex, except x_1 and x_2, gives the conclusion (2). Otherwise, H has an edge not on ∂H. If there is an edge incident to x_i not on ∂H, then H contains a good vertex of degree two by considering an outermost edge. Thus we may assume that $\deg(x_1)=\deg(x_2)=2$. Let V', E', F' be the numbers of vertices, edges and faces of H as a graph in a disk. Then $V'\geq 4$, and $V'-E'+F'=1$, $3F'+V'\leq 2E'$, giving $E'\leq 2V'-3$.

Claim 10.4 *If $V'>4$, then H has either a good vertex of degree two, or at least 3 good vertices of degree at most 4, except x_1 and x_2.*

Proof of Claim 10.4 Assume that any vertex, except x_1, x_2, y, z has degree at least 5. Then $\deg(x_1)+\deg(x_2)+\deg(y)+\deg(z)+5(V'-4)\le 2E'$. So, $\deg(y)+\deg(z)\le 10-V'\le 5$. Thus y or z has degree two. □

Thus if $V'>4$, then we have the conclusion (1) or (2). When $V'=4$, the other two vertices of H than x_1 and x_2 are connected with a single edge, and so have degree 3. If Λ' contains more than one pinched vertices, then there are at least two parts such as H. Then we have the conclusion (1). Otherwise, Λ must be the form as in Figure 10(4).

(2) Consider the case where Λ has a cut vertex.

If some block has a disk support, then we can see that (2) holds by [18, Lemma 3.2]. Thus we can assume that any block has an annulus support. Then either $\partial\Lambda$ consists of two cycles, or Λ has a single pinched vertex. Hence the first or the second part of the previous case gives the result, respectively. □

Proposition 10.5 *Suppose that G lies on a Klein bottle and has $p\ge 3$ vertices. If G has no interior vertex, then either*

(1) *$\overline{G}^+$ has a good vertex of degree 4, which is not incident to a negative loop in G;*

(2) *$\overline{G}^+$ has a vertex of degree at most 3, which is not incident to a negative loop in G;*

(3) *$\overline{G}^+$ has a vertex of degree at most 2, which is incident to a single negative loop in $\overline{G}$; or*

(4) *$\overline{G}$ is either of the graphs shown in Figure 11, where the end circles of the cylinder are identified suitably to form a Klein bottle, and the thicker edges are positive.*

Proof Let Λ be an extremal component of $\overline{G}^+$. Assume that Λ has a disk support. If Λ is not a single vertex, then it has two good vertices of degree at most two [18, Lemma 3.2]. If either of them is not incident to a negative loop, then (2) holds. Otherwise, either we have (3), or both are incident to more than one negative loops in $\overline{G}$. Then there is another extremal component Λ' with a disk support. Notice that at most two vertices of G can be incident to a negative loop. Hence any vertex of Λ' of degree at most two is not incident to a negative loop, which gives (2) again. If Λ is a single vertex, then either we have (2) or (3), or Λ has more than one negative loops in $\overline{G}$. But the latter implies

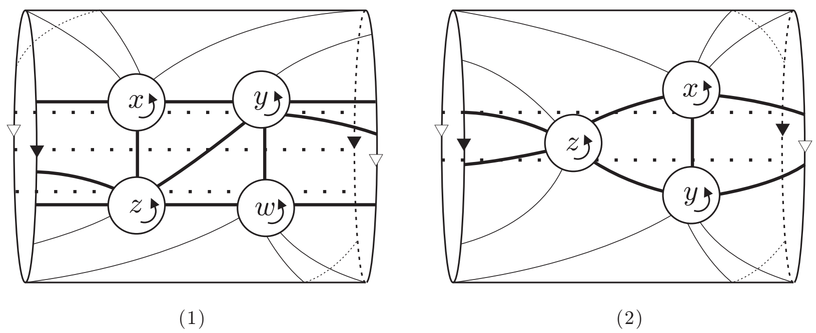


Figure 11

the existence of another extremal component with a disk support, which gives (2) as above.

Thus we assume that $\overline{G}^+$ has no component with a disk support. Hence any component has an annulus support by Lemma 10.1, and anyone is extremal. First, if all components of $\overline{G}^+$ are cycles, we can choose a vertex of degree two, which is not incident to a negative loop, since $p \geq 3$. So we have (2). Thus we may choose so that Λ is not cycle. Then one of (1), (2), (3) in Lemma 10.3 holds for Λ.

If (1) of Lemma 10.3 happens, then either of such two vertices is not incident to a negative loop. Therefore we have the conclusion (1) or (2). If (2) of Lemma 10.3 happens, then a non-pinched vertex of degree at most two in Λ satisfies (2) or (3). Finally, assume that Λ satisfies (3) of Lemma 10.3. If Λ is as Figure 10(1) or (2), then it has a good vertex of degree at most 4, which is not incident to a negative loop, because $\overline{G}^+$ has other component. This gives the conclusion (1) or (2).

If Λ is as Figure 10(3) and $\overline{G}^+$ has other component, then Λ has a good vertex of degree 3, which is not incident to a negative loop. This is (2). Hence we assume $\overline{G}^+ = \Lambda$ has the form Figure 10(3). Let A be its annulus support. If the core of A is non-separating on the Klein bottle, we have (2) again. Hence the core of A is assumed to be separating. Let x and w be the vertices of degree 3 in Λ. If either of them is not incident to a negative loop, then (2) holds. Hence we assume that both are incident to a negative loop. Since there is no interior vertex, $\overline{G}$ must be the graph of Figure 11(1).

If Λ is as Figure 10(4) and $\overline{G}^+$ has other component, then Λ has a good vertex of degree 3, which is not incident to a negative loop. This is (2) again. So, suppose that $\overline{G}^+ = \Lambda$ has the form Figure 10(4). Let x and y be the non-pinched vertices of Λ of degree 3. If either of them is not incident to a negative loop, then (2) holds. Assume that both are incident to a negative loop. If $p > 3$, then any pinched vertex satisfies (2). If $p = 3$, then the unique pinched vertex satisfies (1) or $\overline{G}$ is the graph of Figure 11(2). □

Lemma 10.6 *If G has only two vertices, then $\overline{G}^+$ is one of the following*:

(1) *a single edge*;

(2) *a cycle of length two*;

(3) *the graph as shown in Figure* 10(1), (2) *or Figure* 12(1) *or* (2);

(4) *two isolated vertices*;

(5) *two loops*; *or*

(6) *an isolated vertex and a loop.*

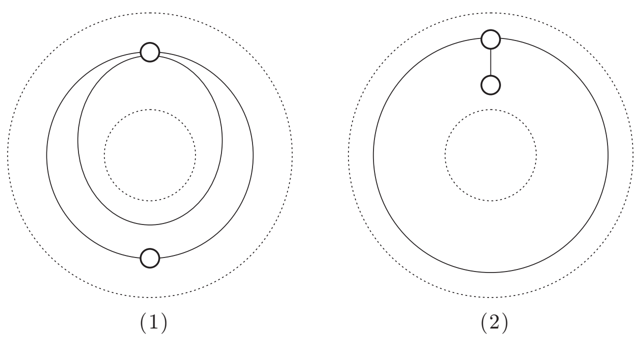


Figure 12

Proof Let u_1 and u_2 be the vertices of G. Notice that the number of loops in G at u_1 is equal to that of loops at u_2.

Assume that $\overline{G}^+$ is connected. If $\overline{G}^+$ has a disk support, then it is a single edge. For, if there is a loop at u_1 say, then there would be a trivial loop at u_2. If $\overline{G}^+$ has an annulus support, then it is easy to see that (2) or (3) holds.

Next assume that $\overline{G}^+$ is not connected. Let H_1, H_2 be the components of $\overline{G}^+$ containing u_1 and u_2, respectively. If H_1 has a disk support, then $H_1 = u_1$. Also if H_1 has an annulus support, then H_1 is a positive loop. Thus either (4), (5) or (6) holds. □

11 Klein bottle; generic case

Finally, we consider the case that $p \geq 2$ and $t \geq 3$.

Lemma 11.1 *Assume $t = 4$ and that G_T contains m x-edges connecting v_1 and v_2 and n x-edges connecting v_3 and v_4 for some label x. Then*

(1) *if $m, n \geq 4$, then $m = n = 4$;*

(2) *if all the x-edges are level and $m, n \geq 2$, then $m = n = 2$.*

Proof Let $G(1,2)$ be the subgraph of G_T consisting of v_1, v_2 and m x-edges between them. Define $G(3,4)$ similarly.

(1) If $G(1,2)$, say, has a disk support disjoint from $G(3,4)$ on $\widehat{T}$, then $G(1,2)$ contains three mutually parallel x-edges. But this means that G_T has $p+1$ parallel edges, which is impossible by Lemma 8.4. Hence neither $G(1,2)$ nor $G(3,4)$ has a disk support on $\widehat{T}$. Then both have annulus supports. If $m > 4$, then $G(1,2)$ would have three mutually parallel edges again, and so $m = 4$. Similarly, we have $n = 4$.

(2) Since two level x-edges cannot be parallel, the result follows from a similar argument to (1). □

Lemma 11.2 *If $t = 4$, then there are no three successive (distinct) positive edges of weight 4 incident to a vertex of $\overline{G}_P$.*

Proof Let e_1, e_2, e_3 be successive positive edges incident to a vertex x of $\overline{G}_P$. If each e_i has weight 4, then we may assume that each family of mutually parallel positive edges corresponding to e_i contains a $\{1,2\}$ S-cycle and a $\{3,4\}$ S-cycle by Lemma 8.2. Then this contradicts Lemma 11.1(1). □

Lemma 11.3 *Let u_i be a vertex of G_P. Suppose that u_i is incident to k non-loop negative edges and n negative loops in G_P and that u_i has degree m in $\overline{G}_P^+$. Then,*

(1) *$k \leq t$;*

(2) *$k + 2n \geq (10 - m)t/2 - 2m$ when $t \equiv 0 \pmod 4$, and $k + 2n \geq (10 - m)t/2 - m$, otherwise; and*

(3) *If $\widehat{T}$ is non-separating, then $k + 2n \geq (10 - m)t/2$.*

Proof In G_T, there are n positive level i-edges and k positive non-level i-edges. By Lemma 8.4, no two of positive i-edges are parallel. Also, a negative level i-edge cannot be parallel to a negative i-edge, and a negative non-level i-edge can be parallel to at most one negative i-edge. Thus $\overline{G}_T$ has at least $n+k+(5t-(2n+k))/2 = (5t+k)/2$ edges. Since $\overline{G}_T$ has at most $3t$ edges, we have $(5t+k)/2 \leq 3t$, giving (1). Also, $(k+2n)+m(t/2+2) \geq 5t$ and $(k+2n)+m(t/2+1) \geq 5t$ give (2). If $\widehat{T}$ is non-separating, then $(k+2n)+m\cdot t/2 \geq 5t$, giving (3). □

Proposition 11.4 *Suppose that G_P contains a positive edge. Let u_i be a vertex of G_P, which has degree m in $\overline{G}_P^+$. Assume that u_i is incident to at most one negative loop in $\overline{G}_P$, and let $n\,(\geq 0)$ denote the weight of the negative loop. If $n=0$, then $m \geq 4$, and otherwise, $m \geq 2$. When the equality holds, $\widehat{T}$ is separating and $t=4$.*

Proof By Lemma 8.2, $n \leq t$. Then the conclusion immediately follows from Lemma 11.3. □

11.1 Case 1: $p \geq 3$

Proposition 11.5 *Each vertex of $\overline{G}_T$ has degree 6.*

Proof Let v_i be a vertex of $\overline{G}_T$. If $\deg(v_i) \geq 6$, then an easy Euler characteristic calculation shows $\deg(v_i)=6$. Since each edge has weight at most p by Lemma 8.4, $\deg(v_i) \geq 5$. Hence suppose $\deg(v_i)=5$ for contradiction. Then v_i is incident to 5 negative edges with weight p, because $p/2+1 < p$. By the parity rule, all i-edges of G_P are positive. In particular, any vertex of G_P is incident to five i-edges, none of which are parallel by Lemma 8.2. Thus any vertex of $\overline{G}_P^+$ has degree at least 5. By Proposition 10.2, $\overline{G}_P$ has an interior vertex u of degree at most 5, hence just 5. Then $5(t/2+2) \geq 5t$ gives $t \leq 4$. But, if $t=3$, then G_P cannot contain a pair of parallel positive edges. So $t=4$. Thus u is incident to 5 positive edges of weight 4. In G_P, each edge corresponds to a family of 4 parallel positive edges. By Lemma 8.2(1), we may assume that each family contains a $\{1,2\}$ S-cycle and a $\{3,4\}$ S-cycle. Then we can see that u is not incident to a loop. But this contradicts Lemma 11.2. □

Lemma 11.6 *G_P contains a positive edge.*

Proof If not, we can choose a vertex which is not incident to a negative loop, because $p \geq 3$ and at most two vertices can be incident to a negative loop. But this contradicts Lemma 11.3(1). □

We now consider two cases, according to the existence of an interior vertex in G_P.

11.1.1 Case: G_P has no interior vertex

Lemma 11.7 *$\overline{G}_P^+$ cannot have a good vertex of degree 4, which is not incident to a negative loop in G_P.*

Proof Let u_i be such a vertex, and let k be the number of negative edge endpoints at u_i. By Proposition 11.4 and Lemma 11.3, $t = k = 4$. Hence u_i is incident to 4 families of 4 mutually parallel positive edges, successively, because u_i is good. By examining the labels, we see that there is no positive loop at u_i. But this contradicts Lemma 11.2. □

Lemma 11.8 *$\overline{G}_P^+$ cannot have a vertex of degree at most 3, which is not incident to a negative loop in G_P.*

Proof This is an immediate consequence of Proposition 11.4. □

Lemma 11.9 *$\overline{G}_P^+$ cannot have a vertex of degree at most two, which is incident to a single negative loop in $\overline{G}_P$.*

Proof Let u_i be such a vertex. Let n be the number of negative loops and k be the number of negative non-loop edges at u_i in G_P. By Proposition 11.4, $\widehat{T}$ is separating and $t = 4$. Then u_i has at most 8 positive edge endpoints, and so at least 12 negative edge endpoints. Notice that G_T has n positive level i-edges and at least $12-2n$ positive non-level i-edges, and that no two of them are parallel. Hence $\overline{G}_T$ has at least $n+(12-2n) = 12-n \geq 8$ positive i-edges, and then at most 4 negative edges.

Let Λ be an extremal component of $\overline{G}_T^+$. If Λ has a disk support, then we see that it contains a vertex of degree at most one. Then such a vertex is incident to at least 5 negative edges in $\overline{G}_T$. But this is impossible, because $\overline{G}_T$ has at most 4 negative edges. Thus $\overline{G}_T^+$ has no component with a disk support, and then each component has an annulus support.

Let $G(1,3)$ be the (possibly, disconnected) subgraph of $\overline{G}_T^+$ spanned by v_1 and v_3, and define $G(2,4)$ similarly. If $G(1,3)$ is disconnected, then it consists of two loops. Then $G(2,4)$ contains at least 6 edges. But this is impossible by an Euler characteristic calculation. This implies that both of $G(1,3)$ and $G(2,4)$ are connected. Then we see that both of them contain 4 edges and have the form as in Figure 10(2). Each vertex of $\overline{G}_T$ has at most two negative edges. Thus $4(p/2+1)+2p \geq 5p$ gives $p \leq 4$.

If $p=3$, then each positive edge at v_1 has weight at most two, and so the total weight cannot be $5p=15$. If $p=4$, then the 4 positive edges at v_1 have weight 3 and two negative edges have weight 4. But there is an S-cycle among loops at v_1, which is impossible by Lemma 8.3. □

Lemma 11.10 *$\overline{G}_P$ is not the graph as shown in Figure* 11(1).

Proof Let x and y be the vertices as shown there. If k is the number of negative edges at y in G_P, then $k \leq t$ by Lemma 11.3. For y, Lemma 11.3(2) gives that $\widehat{T}$ is separating and $t=4$. Let n be the number of negative loops at x. By Lemma 11.2, x has at most 11 positive edge endpoints, and then $k+2n \geq 9$. Hence $n=3$ or 4.

Assume $n=3$. Notice that k must be odd by the parity rule. Hence $k=3$. Thus the three positive edges at x have weights $\{4,4,3\}$. By examining the labels at x, we see that all vertices of G_T have degree at least 6 in $\overline{G}_T$ and some vertex has degree more than 6 there. (For example, see Figure 13. In this case, the degrees of v_i in $\overline{G}_T$ are 7, 7, 6 and 6 for $i=1,2,3,4$, respectively.) Clearly, this is impossible.

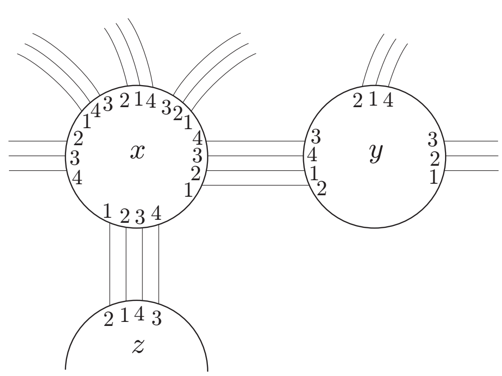


Figure 13

Assume $n = 4$. By the parity rule, k must be even. Then $k = 2$ or 4.

First assume $k = 2$. Then the three positive edges at x have weights $\{4, 4, 2\}$ or $\{4, 3, 3\}$. If one family between x and y has weight 4, then look at the two vertices, which do not appear at non-loop negative edge endpoints at x. In $\overline{G}_T$, they cannot have degree 6, which contradicts Proposition 11.5. Thus the only possibility is that the two edges between x and y have weight 3, and the edge between x and w has weight 4. Then the two non-loop edges at x are level by Lemma 11.1(1). But two vertices of $\overline{G}_T$ cannot have degree 6 as above again.

Thus we have $k = 4$. Then the associated permutation to the family of 4 negative loops at x is the identity. Hence any vertex of G_T is incident to a loop. If the associated permutation σ to the family of 4 non-loop negative edges at x is also the identity, then a level x-loop and a $\{x, y\}$-loop are incident to each vertex of G_T. These two loops are parallel, which contradicts Lemma 8.4. Thus σ is the permutation $(13)(24)$. We may assume that the labels in G_P are as in Figure 14, where a, b and c denotes the number of edges in the families. Hence $\overline{G}_T^+$ has two components, each of which has the form as in Figure 10(2). Notice that G_P contains a $\{1, 2\}$-edge and a $\{3, 4\}$-edge. This implies that any negative edge of G_T connects either v_1 and v_2, or v_3 and v_4. By the parity rule, any positive edge of G_P is a $\{1, 2\}$-edge or a $\{3, 4\}$-edge.

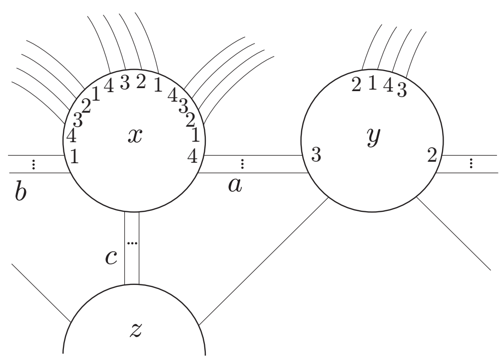


Figure 14

If $a = 2$ or 4, then there would be a $\{1, 4\}$- or $\{2, 3\}$-edge between y and z, a contradiction. Similarly, $b \neq 2,\ 4$. Since $a + b + c = 8$ and none of them is zero, $(a, b, c) = (1, 3, 4)$, $(3, 1, 4)$ or $(3, 3, 2)$. If $(a, b, c) = (1, 3, 4)$ or $(3, 1, 4)$, then the family between x and z contains an extended S-cycle, a contradiction. Finally, if $(a, b, c) = (3, 3, 2)$, then G_P contains S-cycles with label set $\{1, 2\}$ and $\{3, 4\}$ and a Scharlemann cycle of length 3 with label set

$\{1,2\}$. But, under the existence of a Scharlemann cycle with label set $\{3,4\}$, all Scharlemann cycles with label set $\{1,2\}$ must have the same length [9, Theorem 5.7]. □

Lemma 11.11 *$\overline{G}_P$ is not the graph as shown in Figure* 11(2).

Proof Let x and y be the vertices which are incident to a negative loop, and z the other one. By applying Lemma 11.3 and Proposition 11.4 to z, $t=k=4$, where k is the number of negative edge at z in G_P. Then each positive edge at z has weight 4. Since z is not good, the weights of two negative edges at z are $\{1,3\}$ or $\{2,2\}$. By examining the labels, the former contradicts Lemmas 8.2(1) and 11.1(1), and the latter contradicts Lemma 8.2(1). □

Proposition 11.12 *G_P must have an interior vertex.*

Proof If not, (1), (2), (3) or (4) of Proposition 10.5 holds. But all of these are impossible by Lemmas 11.7, 11.8, 11.9, 11.10 and 11.11. □

11.1.2 Case: G_P has an interior vertex

Let u_i be an interior vertex of G_P. Thus all edges incident to u_i are positive, and hence all i-edges in G_T are negative.

Lemma 11.13 $t=4$.

Proof The argument of the proof of [17, Lemma 4.5] works without any change. Therefore G_P has an S-cycle with label j for any label j. In particular, $\widehat{T}$ is separating and t is even. If $t \geq 6$, then G_P would contain three S-cycles with disjoint label pairs as in the proof of [17, Proposition 4.6]. □

Lemma 11.14 *Any vertex of G_P has at most 4 negative edge endpoints.*

Proof Let l be the number of (positive) loops at u_i in G_P. Then G_T has n negative level i-edges and $20-2l$ negative non-level i-edges. Among these i-edges, none is parallel to a level one, and at most two non-level ones can be parallel. Thus $\overline{G}_T$ has at least $l+(20-2l)/2=10$ negative edges, and hence at most two positive edges.

If a vertex u_j has more than 4 negative edge endpoints, then u_j is incident to at least 3 negative edges (possibly, loops). This means that G_T has at least 3 positive j-edges. Thus G_T contains a pair of parallel positive j-edges, and so an S-cycle or a generalized S-cycle. This is impossible by Lemma 8.3. □

Proposition 11.15 *G_P cannot contain an interior vertex.*

Proof Assume not. By Lemma 11.2, any interior vertex of G_P has degree at least 6 in $\overline{G}_P$. Hence $\overline{G}_P^+$ has either a good vertex of degree at most 4 or a vertex of degree at most two by Proposition 10.2. In fact, there is a good vertex x of degree 4 by Lemma 11.14. Then x is incident to 4 successive positive edges of weight 4. (Examining the labels shows that there is no positive loop at x as before.) But this contradicts Lemma 11.2. □

By Propositions 11.12 and 11.15, we have shown that $p \geq 3$ is impossible.

11.2 Case 2: $p = 2$

There are 6 possibilities (1)-(6) for $\overline{G}_P^+$ as stated in Lemma 10.6. In particular, any vertex of $\overline{G}_P^+$ has degree at most 4.

Lemma 11.16 *G_P has no interior vertex.*

Proof For an interior vertex, $4(t/2 + 2) \geq 5t$, giving $t \leq 2$. □

Lemma 11.17 *Each vertex of G_P is incident to a single negative loop in $\overline{G}_P$.*

Proof Assume that there is a vertex which is not incident to a negative loop. If G_P has no positive edges, then we have a contradiction by Lemma 11.3. Thus G_P has a positive edge, then Proposition 11.4 implies $m = t = 4$. Hence G_P has a good vertex of degree 4. But Lemma 11.7 works here. Clearly, all negative loops at a vertex are parallel. □

Thus we can eliminate (1), (6) and Figure 12(2) of (3) in Lemma 10.6 as the possibility of $\overline{G}_P^+$ by Proposition 11.4. We now proceed to rule out the remaining possibilities of $\overline{G}_P^+$.

Lemma 11.18 *(2) of Lemma 10.6 is impossible.*

Proof By Proposition 11.4, $\widehat{T}$ is separating and $t = 4$. As in the proof of Lemma 11.9, $\overline{G}_T$ has at least 8 positive edges and at most 4 negative edges. By Lemma 8.4, $\overline{G}_T^+$ cannot have an isolated vertex or two vertices of degree one. Thus $\overline{G}_T^+$ has two components, each of which is as shown in Figure 10(2).

Hence $\overline{G}_T$ has exactly 8 positive edges, and so each vertex of G_P is incident to 4 negative loops (see the proof of Lemma 11.9). Let k be the number of non-loop negative edges at u_1 in G_P. Then G_T contains k positive $\{1,2\}$-edges. Since a positive $\{1,2\}$-edge cannot be parallel to a positive level edge in G_T, and since G_T cannot contain an S-cycle, we see that $k=4$. Thus each positive edge of $\overline{G}_P$ has weight 4.

The core of the annulus support of $\overline{G}_P^+$ is separating or non-separating on $\widehat{P}$. First, consider the case where it is separating. Then the 4 negative edges between u_1 and u_2 are divided into at most two families. Since the associated permutation σ to the negative loops at u_1 is the identity or $(13)(24)$, the numbers of edges of those families are $\{4,0\}$ or $\{2,2\}$.

For $\{4,0\}$, we may assume that G_P has the labels as in Figure 15. In this case, σ is the identity. Hence any vertex of G_T is incident to a loop. But there are only 4 edges between v_1 and v_2. This implies that any loop at v_1 is not level, a contradiction.

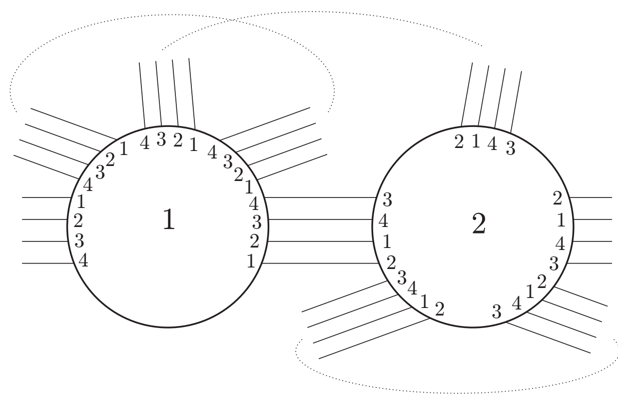


Figure 15

For $\{2,2\}$, we may assume that G_P has the labels as in Figure 16. Then $\sigma=(13)(24)$. Thus the negative loops at u_1 form two essential cycles on $\widehat{T}$. Also, each vertex of G_T is incident to a loop, and there are two parallel pairs between v_i and v_{i+1} for $i=1,3$. Hence G_T is uniquely determined. But the arrangement of edges with label 1 around v_1 contradicts Lemma 2.1. For, when we look at the two 1-edges connecting v_1 with v_3, the jumping number must be two, but when we look at the two 1-edges connecting v_1 with v_2, it must be one.

Thus the core of the annulus support of $\overline{G}_P^+$ is non-separating on $\widehat{P}$. We may

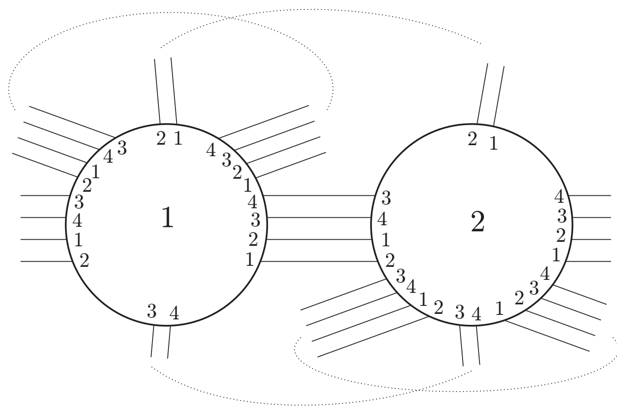


Figure 16

assume that there are two S-cycles with label pair $\{1, 2\}$ and $\{3, 4\}$. These 4 edges form two essential cycles on $\widehat{T}$. Again, denote by σ the associated permutation to the negative loops at u_1. Then σ is the identity or (13)(24). If σ is the identity, then G_T contains a graph as in Figure 17. In G_P, there are 4 negative edges between u_1 and u_2. This means that G_T has 4 positive $\{1, 2\}$-edges. Since each vertex of G_T has a level positive loop, these 4 positive $\{1, 2\}$-edges connect v_1 and v_3 or v_2 and v_4, and furthermore three of them connect the same pair of vertices. But then, there is a pair of parallel positive $\{1, 2\}$-edges, which forms an S-cycle.

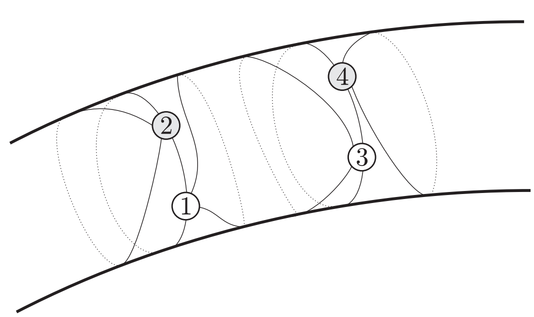


Figure 17

Thus $\sigma = (13)(24)$, and then G_P and hence G_T are uniquely determined as in Figure 18. At u_1, two occurrences of label 1 at $\{1, 2\}$ S-cycles are not consecutive among 5 occurrences of label 1. But these points are consecutive at v_1. Hence the jumping number is two. Then the edge e is located as in

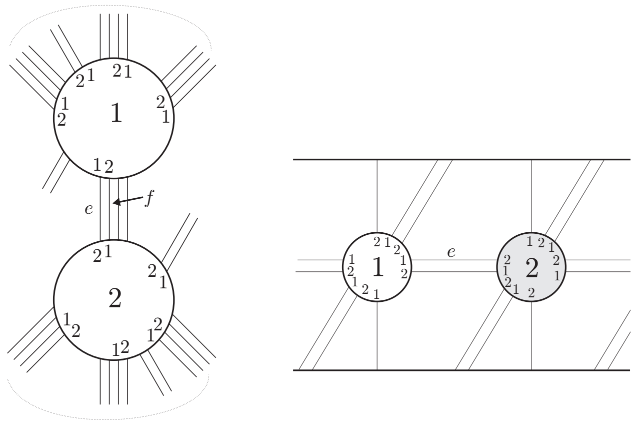


Figure 18

Figure 18. Consider the location of the edge f around v_2. Then f would be parallel to e, a contradiction. □

Lemma 11.19 (3) *of Lemma* 10.6 *is impossible.*

Proof Three cases are remaining. For all cases, the core of the annulus support of $\overline{G}_P^+$ is separating on $\widehat{P}$ by Lemma 11.17. Also, when $\overline{G}_P^+$ is Figure 10(1) or (2), there is no non-loop negative edge in G_P.

Assume that $\overline{G}_P^+$ is as shown in Figure 10(1). Hence $t = 4$ by Lemma 11.3(2), and then each vertex is incident to 4 negative loops, and each positive edge of $\overline{G}_P$ has weight 4. But there would be an extended S-cycle among positive loops, a contradiction.

Assume that $\overline{G}_P^+$ is as shown in Figure 10(2). By Lemma 11.3, $\widehat{T}$ is separating, and $t = 4$ or 8. But, if $t = 8$, then each positive edge of $\overline{G}_P$ has weight 6 and each negative loop has weight 8. Then there would be an extended S-cycle among positive loops. Thus $t = 4$.

Let n be the number of negative loops at u_1. Then $n = 3$ or 4 by Lemmas 8.2(2) and 11.2. If $n = 3$, then a negative loop at u_1 contradicts the parity rule. Thus $n = 4$. By the same reason, u_2 is also incident to 4 negative loops. Then each vertex of G_T is incident to two loops, which are level.

Let l be the number of positive loops at u_1. Then $l = 2$, 3 or 4.

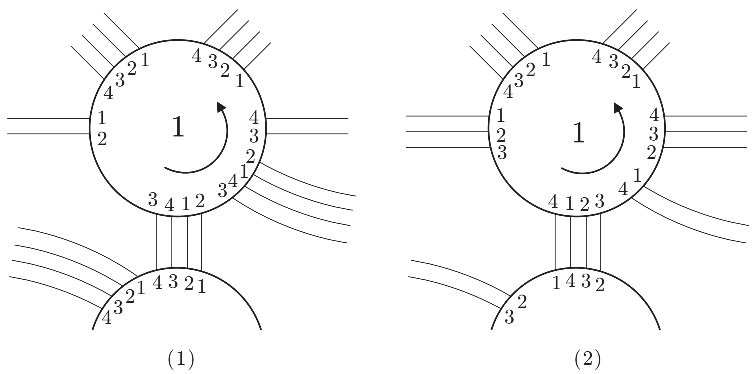


Figure 19

If $l = 2$, we may assume that G_P has the labels as in Figure 19(1). Then G_T has 4 edges between v_1 and v_2. But a loop at v_1 is level, a contradiction.

If $l = 3$, then two positive edges between u_1 and u_2 have weight $\{4, 2\}$ or $\{3, 3\}$. In the former, we may assume that G_P has the labels as in Figure 19(2). There are two S-cycles with label pair $\{2, 3\}$. In G_T, these 4 edges are divided into two edge classes. But such edge class cannot contain both a level edge and a non-level edge. In the latter, we may assume that three positive loops at u_1 contain an S-cycle with label pair $\{2, 3\}$ and a $\{1, 4\}$-edge. Although there are two possibilities for the labels at u_2, there is always a $\{2, 3\}$-edge between u_1 and u_2. Thus a similar argument to the former yields a contradiction.

If $l = 4$, then there is an extended S-cycle among positive loops at u_1, a contradiction.

Finally, assume that $\overline{G}_P^+$ is as shown in Figure 12(1). We may assume that u_2 is incident to a positive loop. By Proposition 11.4 (applying to u_1), $\widehat{T}$ is separating and $t = 4$. Furthermore, u_1 is incident to 4 negative loops, 4 non-loop negative edges and two families of 4 parallel positive edges. Thus we can assume that G_P has the labels as in Figure 20. Then the associated permutation to the negative loops is the identity, and that to the family A, say, of non-loop negative edges is $(13)(24)$. Let e be the positive $\{2, 3\}$-loop at u_2. The edges of A form two essential cycles on $\widehat{T}$. Put e between v_2 and v_3. Also, each vertex is incident to a loop. Then we cannot locate the edges of an S-cycle with label pair $\{1, 2\}$ so as to form an essential cycle. □

Lemma 11.20 *If $\overline{G}_P^+$ is (4) of Lemma 10.6, then ∂M consists of at most two tori.*

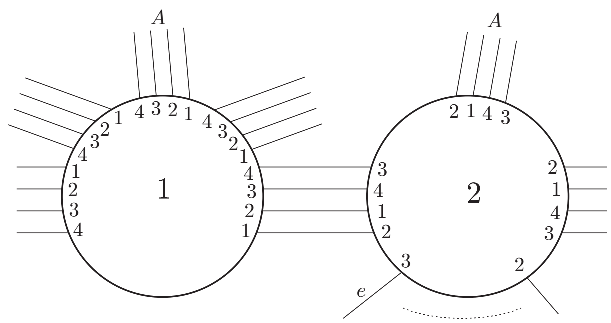


Figure 20

Proof Let n and k be the number of negative loops, non-loop negative edges, respectively, at u_1 in G_P. Then $k + 2n = 5t$, and $k \leq t$ by Lemma 11.3. Thus $n \geq 2t$.

Let A be the family of negative loops at u_1, and let σ be the associated permutation. By Lemma 8.2, all vertices of G_T have the same sign. In fact, σ has a single orbit [5, Lemma 4.2]. Let $a_1, a_2, \dots, a_t, b_1$ be the successive $t+1$ edges in A, and let $D_1, D_2, \dots, D_t$ be the disks between them. Then the edges $a_1, a_2, \dots, a_t$ form an essential cycle on $\widehat{T}$, and $a_2, a_3, \dots, a_t, b_1$ form a distinct essential cycle. Let

$$X = N(\widehat{T} \cup V_\beta \cup \bigcup_{i=1}^{t} D_i).$$

Then ∂X is a torus T', disjoint from V_β. Thus either T' is boundary parallel in $M(\beta)$, which implies that ∂M is a union of two tori, or T' is compressible. In the latter, T' bounds a solid torus or is contained in a ball. But if T' lies in a ball, then T would be compressible. Hence ∂M is a single torus. □

Remark that this case can be eliminated by a lengthy argument using a jumping number.

Lemma 11.21 (5) *of Lemma* 10.6 *is impossible.*

Proof Since each vertex of G_P is incident to a negative loop by Lemma 11.17, the two loops of $\overline{G}_P^+$ are separating on $\widehat{P}$. As in the proof of Lemma 11.18, $t = 4$ and each vertex of G_P is incident to 4 negative loops and 4 positive loops. Then the family of 4 loops contains an extended S-cycle, a contradiction. □

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