



All integral slopes can be Seifert fibered slopes for hyperbolic knots

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Abstract Which slopes can or cannot appear as Seifert fibered slopes for hyperbolic knots in the 3-sphere S^3 ? It is conjectured that if r -surgery on a hyperbolic knot in S^3 yields a Seifert fiber space, then r is an integer. We show that for each integer $n \in \mathbb{Z}$, there exists a tunnel number one, hyperbolic knot K_n in S^3 such that n -surgery on K_n produces a small Seifert fiber space.

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This paper is dedicated to Donald M. Davis on the occasion of his 60th birthday.

1 Introduction

Let K be a knot in the 3-sphere S^3 with a tubular neighborhood $N(K)$. Then the set of *slopes* for K (i.e., $\partial N(K)$ -isotopy classes of simple loops on $\partial N(K)$) is identified with $\mathbb{Q} \cup \{\infty\}$ using preferred meridian-longitude pair so that a meridian corresponds to ∞ . A slope γ is said to be *integral* if a representative of γ intersects a meridian exactly once, in other words, γ corresponds to an integer under the above identification. In the following, we denote by $(K; \gamma)$ the 3-manifold obtained from S^3 by Dehn surgery on a knot K with slope γ , i.e., by attaching a solid torus to $S^3 - \text{int} N(K)$ in such a way that γ bounds a meridian disk of the filled solid torus. If γ corresponds to $r \in \mathbb{Q} \cup \{\infty\}$, then we identify γ and r and write $(K; r)$ for $(K; \gamma)$.

We denote by \mathcal{L} the *set of lens slopes* $\{r \in \mathbb{Q} \mid \exists \text{ hyperbolic knot } K \subset S^3 \text{ such that } (K; r) \text{ is a lens space}\}$, where S^3 and $S^2 \times S^1$ are also considered as lens spaces. Then the cyclic surgery theorem [7] implies that $\mathcal{L} \subset \mathbb{Z}$. A

result of Gabai [10, Corollary 8.3] shows that $0 \notin \mathcal{L}$, a result of Gordon and Luecke [14] shows that $\pm 1 \notin \mathcal{L}$. In [19] Kronheimer and Mrowka prove that $\pm 2 \notin \mathcal{L}$. Furthermore, a result of Kronheimer, Mrowka, Ozsváth and Szabó [20] implies that $\pm 3, \pm 4 \notin \mathcal{L}$. Besides, Berge [4, Table of Lens Spaces] suggests that if $n \in \mathcal{L}$, then $|n| \geq 18$ and not every integer n with $|n| \geq 18$ appears in \mathcal{L} . Fintushel and Stern [9] had shown that 18-surgery on the $(-2, 3, 7)$ pretzel knot yields a lens space.

Which slope (rational number) can or cannot appear in the set of Seifert fibered slopes $\mathcal{S} = \{r \in \mathbb{Q} \mid \exists \text{ hyperbolic knot } K \subset S^3 \text{ such that } (K; r) \text{ is Seifert fibered}\}$? It is conjectured that $\mathcal{S} \subset \mathbb{Z}$ [12].

The purpose of this paper is to prove:

Theorem 1.1 *For each integer $n \in \mathbb{Z}$, there exists a tunnel number one, hyperbolic knot K_n in S^3 such that $(K_n; n)$ is a small Seifert fiber space (i.e., a Seifert fiber space over S^2 with exactly three exceptional fibers).*

Remark Since K_n has tunnel number one, it is embedded in a genus two Heegaard surface of S^3 and strongly invertible [26, Lemma 5]. See [22, Question 3.1].

Theorem 1.1, together with the previous known results, shows:

Corollary 1.2 $\mathcal{L} \subsetneq \mathbb{Z} \subset \mathcal{S}$.

Remarks

(1) For the *set of reducing slopes* $\mathcal{R} = \{r \in \mathbb{Q} \mid \exists \text{ hyperbolic knot } K \subset S^3 \text{ such that } (K; r) \text{ is reducible}\}$, Gordon and Luecke [13] have shown that $\mathcal{R} \subset \mathbb{Z}$. In fact, the cabling conjecture [11] asserts that $\mathcal{R} = \emptyset$.

(2) For the *set of toroidal slopes* $\mathcal{T} = \{r \in \mathbb{Q} \mid \exists \text{ hyperbolic knot } K \subset S^3 \text{ such that } (K; r) \text{ is toroidal}\}$, Gordon and Luecke [15] have shown that $\mathcal{T} \subset \mathbb{Z}/2$ (integers or half integers). In [28], Teragaito shows that $\mathbb{Z} \subset \mathcal{T}$ and conjectures that $\mathcal{T} \subsetneq \mathbb{Z}/2$.

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2 Hyperbolic knots with Seifert fibered surgeries

Our construction is based on an example of a longitudinal Seifert fibered surgery given in [17].

Let $k \cup c$ be a 2-bridge link given in Figure 1, and let K_n be a knot obtained from k by $\frac{1}{-n+4}$ -surgery along c .

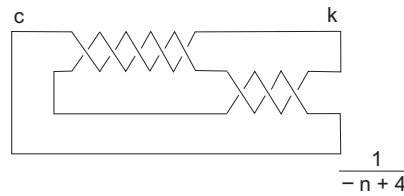


Figure 1: K_n

We shall say that a Seifert fiber space is of *type* $S^2(n_1, n_2, n_3)$ if it has a Seifert fibration over S^2 with three exceptional fibers of indices n_1, n_2 and n_3 ($n_i \geq 2$). Since K_4 is unknotted, $(K_4; 4)$ is a lens space $L(4, 1)$. For the other n 's, we have:

Lemma 2.1 $(K_n; n)$ is a small Seifert fiber space of type $S^2(3, 5, |4n - 15|)$ for any integer $n \neq 4$.

Proof Since the linking number of k and c is one (with suitable orientations), $(K_n; n)$ has surgery descriptions as in Figure 2.

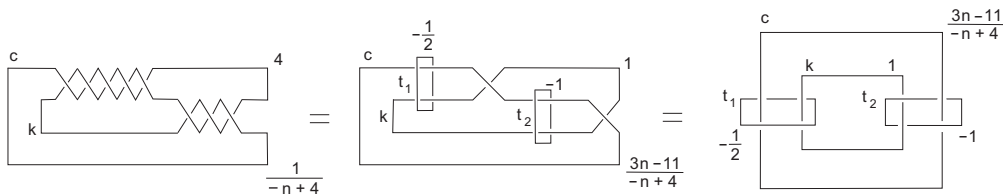


Figure 2: Surgery descriptions of $(K_n; n)$

Let us take the quotient by the strong inversion of S^3 with an axis L as shown in Figure 3.

Then we obtain a branch knot b' which is the image of the axis L . The Montesinos trick ([25], [6]) shows that $-\frac{1}{2}, -1, \frac{3n-11}{-n+4}$ and 1-surgery on t_1, t_2, c and

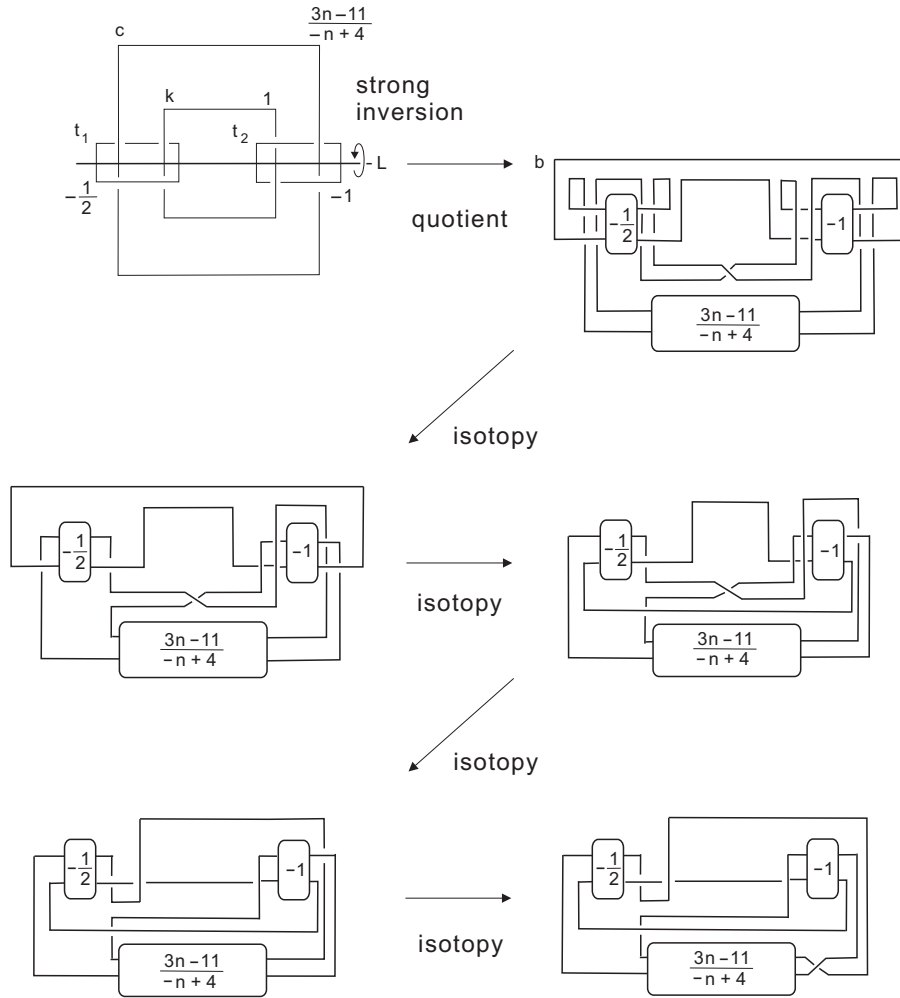


Figure 3

k in the upstairs correspond to $-\frac{1}{2}, -1, \frac{3n-11}{-n+4}$ and 1-untangle surgery on b' in the downstairs, where an r -untangle surgery is a replacement of $\frac{1}{0}$ -untangle by r -untangle. (We adopt Bleiler's convention [5] on the parametrization of rational tangles.) These untangle surgeries convert b' into a link b (Figure 3).

Following the sequence of isotopies in Figures 3 and 4, we obtain a Montesinos link $M(\frac{2}{5}, -\frac{2}{3}, \frac{n-4}{4n-15})$.

Since $(K_n; n)$ is the double branched cover of S^3 branched over the Montesinos

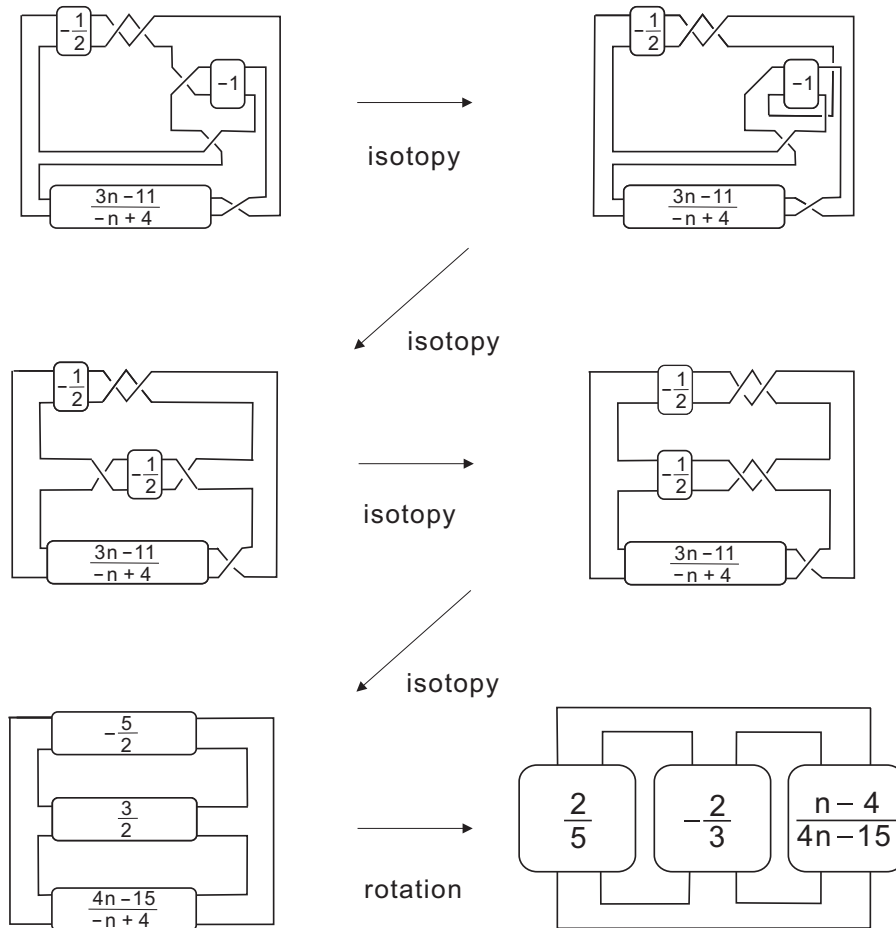


Figure 4: Continued from Figure 3

link $M(\frac{2}{5}, -\frac{2}{3}, \frac{n-4}{4n-15})$, $(K_n; n)$ is a Seifert fiber space of type $S^2(3, 5, |4n - 15|)$ as desired. \square

Lemma 2.2 *The knot K_n is hyperbolic if $n \neq 3, 4, 5$.*

Proof Note that the 2-bridge link given in Figure 1 is not a $(2, p)$ -torus link, and hence by [23] it is a hyperbolic link. If $n \neq 3, 4, 5$, then $|-n + 4| > 1$ and it follows from [1, Theorem 1] (also [3, Theorem 1.2]) that K_n is a hyperbolic knot. See also [16, Corollary A.2], [24, Theorem 1.2] and [2, Theorem 1.1]. \square

Remark It follows from [21], [18] that K_n is a nontrivial knot except when $n = 4$. An experiment using Weeks' computer program "SnapPea" [31] suggests that K_3 and K_5 are hyperbolic, but we will not use this experimental results.

Lemma 2.3 *The knot K_n has tunnel number one for any integer $n \neq 4$.*

Proof Since the link $k \cup c$ is a two-bridge link, the tunnel number of $k \cup c$ is one with unknotting tunnel τ ; A regular neighborhood $N(k \cup c \cup \tau)$ is a genus two handlebody and $S^3 - \text{int}N(k \cup c \cup \tau)$ is also a genus two handlebody, see Figure 5.

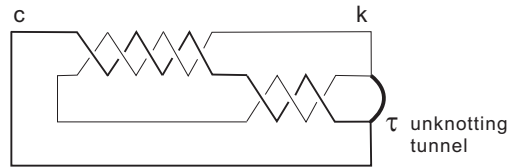


Figure 5

Then the general fact below (in which $k \cup c$ is not necessarily a two-bridge link) shows that the tunnel number of K_n is less than or equal to one. Since our knot K_n ($n \neq 4$) is knotted in S^3 , the tunnel number of K_n is one. \square

Claim 2.4 *Let $k \cup c$ be a two component link in S^3 which has tunnel number one. Assume that c is unknotted in S^3 . Then every knot obtained from k by twisting along c has tunnel number at most one.*

Proof Let τ be an unknotting tunnel and V a regular neighborhood of $k \cup c \cup \tau$ in S^3 ; V is a genus two handlebody. Since τ is an unknotting tunnel for $k \cup c$, by definition, $W = S^3 - \text{int}V$ is also a genus two handlebody. Take a small tubular neighborhood $N(c) \subset \text{int}V$ and perform $-\frac{1}{n}$ -surgery on c using $N(c)$. Then we obtain a knot k_n as the image of k and obtain a genus two handlebody $V(c; -\frac{1}{n})$. Note that $V(c; -\frac{1}{n})$ and W define a genus two Heegaard splitting of S^3 , see Figure 6, where c_n^* denotes the core of the filled solid torus.

Then it is easy to see that an arc τ_n given by Figure 6 is an unknotting tunnel for k_n as desired. \square

Now we are ready to prove Theorem 1.1. Lemmas 2.1, 2.2 and 2.3 show that our knots K_n enjoy the required properties, except for $n = 3, 4, 5$. To prove

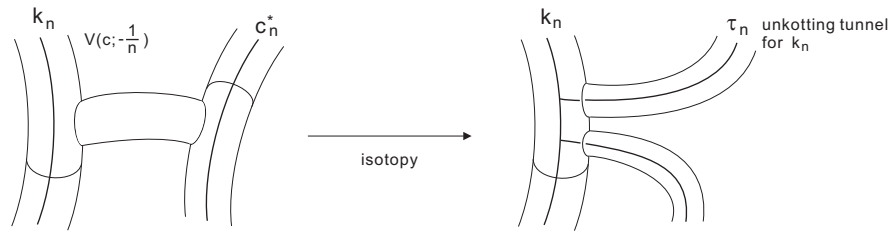


Figure 6

Theorem 1.1, we find hyperbolic knots K'_n so that $(K'_n; n)$ is Seifert fibered for $n = 3, 4, 5$ (instead of showing that K_3, K_5 are hyperbolic). As the simplest way, let K'_3, K'_4 and K'_5 be the mirror image of K_{-3}, K_{-4} and K_{-5} , respectively. Since K_{-3}, K_{-4} and K_{-5} are tunnel number one, hyperbolic knots by Lemmas 2.2 and 2.3, their mirror images K'_3, K'_4 and K'_5 are also tunnel number one, hyperbolic knots. It is easy to observe that $(K'_3; 3)$ (resp. $(K'_4; 4), (K'_5; 5)$) is the mirror image of $(K_{-3}; -3)$ (resp. $(K_{-4}; -4), (K_{-5}; -5)$). By Lemma 2.1, $(K_{-3}; -3), (K_{-4}; -4)$ and $(K_{-5}; -5)$ are Seifert fibered, and hence $(K'_3; 3), (K'_4; 4)$ and $(K'_5; 5)$ are also Seifert fibered. Putting K_n as K'_n for $n = 3, 4, 5$, we finish a proof of Theorem 1.1. \square

3 Identifying exceptional fibers

In [24], Miyazaki and Motegi conjectured that if K admits a Seifert fibered surgery, then there is a trivial knot $c \subset S^3$ disjoint from K which becomes a Seifert fiber in the resulting Seifert fiber space, and verified the conjecture for several Seifert fibered surgeries [24, Section 6], see also [8]. Furthermore, computer experiments via “SnapPea” [31] suggest that such a knot c is realized by a short closed geodesic in the hyperbolic manifold $S^3 - K$, for details see [24, Section 9], [27].

In this section, we verify the conjecture for Seifert fibered surgeries given in Theorem 1.1.

Recall that K_n is obtained from k by $\frac{1}{-n+4}$ -surgery on the trivial knot c (i.e., $(n - 4)$ -twist along c), see Figure 1. Denote by c_n the core of the filled solid torus. Then $K_n \cup c_n$ is a link in S^3 such that c_n is a trivial knot.

Lemma 3.1 *After n -surgery on K_n , c_n becomes an exceptional fiber of index $|4n - 15|$ in the resulting Seifert fiber space $(K_n; n)$.*

Proof Following the sequences given by Figures 3 and 4, we have a Montesinos link with three arcs γ , τ_1 and τ_2 as in Figure 7, where $n = 1$ in the final Montesinos link, and γ , τ_1 , τ_2 and κ are the images of c , t_1 , t_2 and k , respectively.

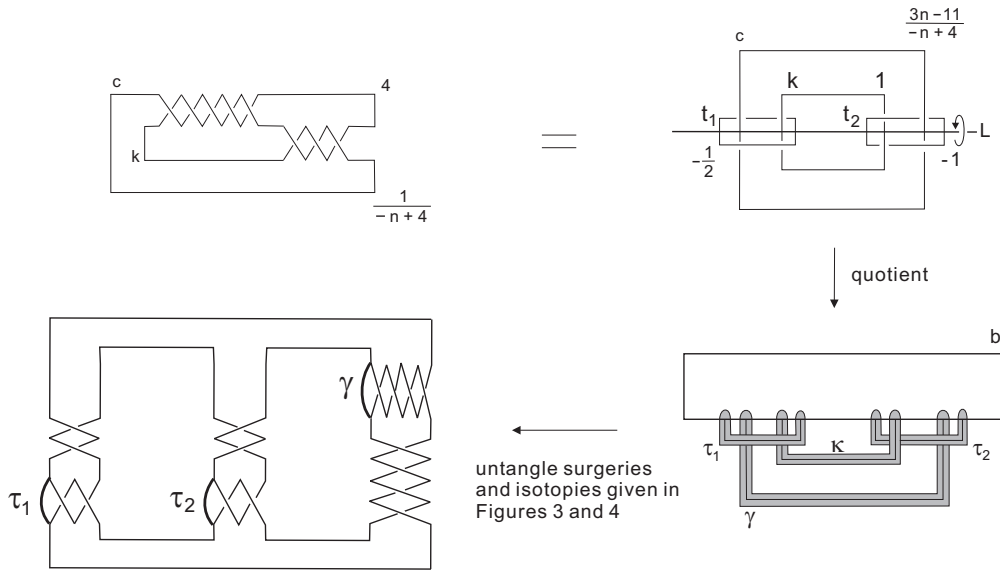


Figure 7: Positions of exceptional fibers

From Figure 7 we recognize that t_1, t_2 and c become exceptional fibers of indices 5, 3 and $|4n - 15|$, respectively in $(K_n; n)$. \square

For $n \neq 3, 4, 5$, c_n becomes an exceptional fiber of index $|4n - 15|$, which is the unique maximal index, in $(K_n; n)$. Experiments via “SnapPea” [31] suggest that c_n is a shortest closed geodesic in $S^3 - K_n$ ($n \neq 3, 4, 5$). For sufficiently large $|n|$, hyperbolic Dehn surgery theorem [29], [30] shows that c_n is the unique shortest closed geodesic in $S^3 - K_n$.

Let us assume that $n = 3, 4, 5$. Then we have put K_n as the mirror image of K_{-n} in the proof of Theorem 1.1. Let $k' \cup c'$ be the mirror image of the link $k \cup c$. Then K_n is obtained also from k' by $\frac{1}{-n-4}$ -surgery on c' (i.e., $(n + 4)$ -twist along c'); we denote the core of the filled solid torus by c'_n . Note that there is an orientation reversing diffeomorphism from $(K_{-n}; -n)$ to $(K_n; n)$ sending c_{-n} (regarded as a fiber in $(K_{-n}; -n)$) to c'_n (regarded as a fiber in $(K_n; n)$). Thus the above observation implies that c'_n becomes an exceptional fiber of index $|4n + 15|$, which is the unique maximal index, in $(K_n; n)$ ($n = 3, 4, 5$).

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