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The Gromov width of complex Grassmannians

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Abstract We show that the Gromov width of the Grassmannian of complex k-planes in $\mathbb{C}^n$ is equal to one when the symplectic form is normalized so that it generates the integral cohomology in degree 2. We deduce the lower bound from more general results. For example, if a compact manifold N with an integral symplectic form ω admits a Hamiltonian circle action with a fixed point p such that all the isotropy weights at p are equal to one, then the Gromov width of (N,ω) is at least one. We use holomorphic techniques to prove the upper bound.

AMS Classification 53D20; 53D45

Keywords Gromov width, Moser's method, symplectic embedding, complex Grassmannian, moment map

1 Introduction

Consider the ball of capacity a

$$B(a) = \left\{ z \in \mathbb{C}^N \;\middle|\; \pi \sum_{i=1}^{N} |z_i|^2 < a \right\},$$

with the standard symplectic form $\omega_{\text{std}} = \sum dx_j \wedge dy_j$. The *Gromov width* of a $2N$-dimensional symplectic manifold (M,ω) is the supremum of the set of a's such that $B(a)$ can be symplectically embedded in (M,ω). Computations of Gromov width and, more generally, of symplectic ball packings, can be found, for example, in [4, 5, 6, 8, 16, 19].

Often in symplectic geometry, equivariant techniques give constructions whereas holomorphic techniques give obstructions. We use both.

Our main technical result is a criterion for the existence of symplectic embeddings of open subsets of $\mathbb{C}^n$ into a symplectic manifold with a Hamiltonian torus action. See Proposition 2.8.

Paul Biran has asked whether the Gromov width of a compact symplectic manifold is at least one if the symplectic form is integral. In Proposition 2.11 we answer his question positively whenever the manifold admits a Hamiltonian circle action with a fixed point p such that all the isotropy weights at p are equal to one.

As a corollary, we obtain embeddings of balls into complex Grassmannians. More precisely, we prove the following theorem.

Theorem 1 *Let* $\mathrm{Gr}(k,n)$ *be the Grassmannian of* k*-planes in* $\mathbb{C}^n$*, together with its* $\mathrm{U}(n)$*-invariant symplectic form* ω*, normalized so that* $[\omega]$ *generates the integral cohomology* $H^2(\mathrm{Gr}(k,n);\mathbb{Z})$*. There exists a symplectic embedding of* $B(a)$ *into* $\mathrm{Gr}(k,n)$ *if and only if* $a \leq 1$*.*

Next, we use holomorphic techniques to show that it is impossible to embed the ball $B(a)$ into $\mathrm{Gr}(k,n)$ if $a > 1$. The proof uses two ingredients: a slight adaptation of the proof of Gromov's non-squeezing theorem, and the calculation of certain Gromov-Witten invariants for $\mathrm{Gr}(k,n)$, which we quote from [18] (also see [11, 3]).

The Gromov width of the complex Grassmannian was independently computed by Guangcun Lu in [14]. Lu obtained the lower bound by an explicit embedding of a ball. Our results are more general in that they give lower bounds for the Gromov width of many more manifolds, such as other generalized flag manifolds, and in that they allow one to embed sets other than balls.

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2 Lower bounds for Gromov width

In this section, we construct symplectic embeddings of open subsets of $\mathbb{C}^n$ into symplectic manifolds with Hamiltonian torus actions. The key technique is

Moser's method. This section is an extension of our work in [10, §13] and is inspired by [7, §1].

Let a torus $T \cong (S^1)^{\dim T}$ with Lie algebra $\mathfrak{t}$ act effectively on a connected symplectic manifold (M, ω) by symplectic transformations. A *moment map* is a map $\Phi\colon M \longrightarrow \mathfrak{t}^*$ such that

$$\iota(\xi_M)\omega = -\,\mathrm{d}\,\langle \Phi, \xi \rangle \quad \forall\, \xi \in \mathfrak{t}, \tag{2.1}$$

where ξ_M is the corresponding vector field on M.

Let $p \in M$ be a fixed point. There exist $\eta_j \in \mathfrak{t}^*$, called the *isotropy weights* at p, such that the induced linear symplectic T-action on the tangent space T_pM is isomorphic to the action on $(\mathbb{C}^n, \omega_{\mathrm{std}})$ generated by the moment map

$$\Phi_{\mathbb{C}^n}(z) = \Phi(p) + \pi \sum |z_j|^2 \eta_j .$$

The isotropy weights are uniquely determined up to permutation.

By the equivariant Darboux theorem [20], a neighborhood of p in M is equivariantly symplectomorphic to a neighborhood of 0 in $\mathbb{C}^n$. The results of this section allow us to control the size of this neighborhood.

For the applications in this paper it is enough to symplectically embed the ball $B(1) \subset \mathbb{C}^n$ into manifolds with S^1-actions; see Proposition 2.11. However, we will take this opportunity to develop the relevant machinery in the more general case where we embed other subsets of $\mathbb{C}^n$, possibly unbounded, into manifolds with torus actions.

Let $\mathcal{T} \subset \mathfrak{t}^*$ be an open convex set which contains $\Phi(M)$. The quadruple $(M, \omega, \Phi, \mathcal{T})$ is a *proper Hamiltonian* $\mathbf{T}$*-manifold* if Φ is proper as a map to $\mathcal{T}$, that is, the preimage of every compact subset of $\mathcal{T}$ is compact.

For any subgroup K of T, let $M^K = \{m \in M \mid a \cdot m = m \ \forall a \in K\}$ denote its fixed point set.

Definition 2.2 A proper Hamiltonian T-manifold $(M, \omega, \Phi, \mathcal{T})$ is *centered* about a point $\alpha \in \mathcal{T}$ if α is contained in the moment map image of every component of M^K, for each $K \subseteq T$.

Example 2.3 A compact symplectic manifold with a non-trivial T-action is never centered, because it has fixed points with different moment map images.

Example 2.4 Let a torus T act linearly on $\mathbb{C}^n$ with a proper moment map $\Phi_{\mathbb{C}^n}$ such that $\Phi_{\mathbb{C}^n}(0) = 0$. Let $\mathcal{T} \subset \mathfrak{t}^*$ is be an open convex subset containing the origin. Then $\Phi_{\mathbb{C}^n}^{-1}(\mathcal{T})$ is centered about the origin.

Example 2.5 Let M be a compact symplectic toric manifold with moment map $\Phi\colon M \longrightarrow \mathfrak{t}^*$. Then $\Delta := \operatorname{image}\Phi$ is a convex polytope. The orbit type strata in M are the moment map pre-images of the relative interiors of the faces of Δ. Hence, for any $\alpha \in \Delta$,

$$\bigcup_{\substack{F \text{ face of } \Delta \\ \alpha\in F}} \Phi^{-1}(\text{rel-int } F)$$

is the largest subset of M that is centered about α.

Example 2.6 Let $(M,\omega,\Phi,\mathcal{T})$ be a proper Hamiltonian T-manifold. Then every point in $\mathfrak{t}^*$ has a neighborhood whose preimage is centered. This is a consequence of the local normal form theorem and the properness of the moment map.

Remark 2.7 In [10, Definition 1.4] we defined a proper Hamiltonian T-manifold $(M,\omega,\Phi,\mathcal{T})$ to be centered about $\alpha \in \mathcal{T}$ if α is contained in the closure of the moment map image of every orbit type stratum. This is equivalent to Definition 2.2 above because of two facts. First, the components of the fixed point sets of subgroups of T are precisely the closures of the orbit type strata. Second, because Φ is proper, the closure of the image of any subset is equal to the image of its closure.

Proposition 2.8 *Let $(M,\omega,\Phi,\mathcal{T})$ be a proper Hamiltonian T-manifold. Assume that M is centered about $\alpha \in \mathcal{T}$ and that $\Phi^{-1}(\{\alpha\})$ consists of a single fixed point p. Then M is equivariantly symplectomorphic to*

$$\left\{ z \in \mathbb{C}^n \mid \alpha + \pi \sum |z_j| \eta_j \in \mathcal{T} \right\},$$

where $\eta_1,\ldots,\eta_n$ are the isotropy weights at p.

Proof For simplicity, assume that $\alpha = 0$. Let

$$\Phi_{\mathbb{C}^n}(z) = \pi \sum |z_j|^2 \eta_j.$$

There exist a convex neighborhood V of 0 in $\mathcal{T}$ and a T-equivariant symplectomorphism from $\Phi^{-1}(V) \subseteq M$ to $\Phi_{\mathbb{C}^n}^{-1}(V) \subseteq \mathbb{C}^n$:

$$F\colon \Phi^{-1}(V) \longrightarrow (\Phi_{\mathbb{C}^n})^{-1}(V).$$

These exist for the following reasons. The equivariant Darboux theorem gives an equivariant symplectomorphism from a neighborhood of p in M to a neighborhood of 0 in $\mathbb{C}^n$. Because $\Phi^{-1}(\{0\})$ consists of a single point, the moment map $\Phi_{\mathbb{C}^n}\colon \mathbb{C}^n \longrightarrow \mathfrak{t}^*$ is proper. (See, for example, Lemma 5.4 in [10].) Hence, the neighborhoods of $p \in M$ and $0 \in \mathbb{C}^n$ contain the preimages of a neighborhood of $0 \in \mathfrak{t}^*$.

The *Euler vector field* on a vector space is the generator of the flow $x \mapsto e^t x$. Let X be the Euler vector field on $\mathfrak{t}^*$; then $-X$ generates the flow $x \mapsto e^{-t}x$.

Since $(M, \omega, \Phi, \mathcal{T})$ is a proper Hamiltonian T-manifold that is centered about the origin $0 \in \mathfrak{t}^*$, by Lemma 13.2 of [10][1] there exists a smooth invariant vector field $\widetilde{X}$ on M such that $\Phi_*(\widetilde{X}) = X$. The vector field $-\widetilde{X}$ generates a flow $\widetilde{\psi}_t\colon M \longrightarrow M$, which satisfies

$$\Phi \circ \widetilde{\psi}_t = e^{-t}\,\Phi$$

wherever it is defined. Hence, since Φ is proper and $\mathcal{T}$ is convex and contains the origin, $\widetilde{\psi}_t$ is defined for all $t \geq 0$. Because $\widetilde{X}$ is T-invariant, $\widetilde{\psi}_t$ is T-equivariant. Define a family of T-invariant symplectic forms on M by

$$\omega_t = e^t\,(\widetilde{\psi}_t)^*\omega.$$

Let $\widehat{X}$ be half the Euler vector field on $\mathbb{C}^n$. Then $-\widehat{X}$ generates the flow $\widehat{\psi}_t\colon \mathbb{C}^n \longrightarrow \mathbb{C}^n$ given by

$$\widehat{\psi}_t(z) = e^{-t/2}\,z.$$

Note that $\Phi_{\mathbb{C}^n} \circ \widehat{\psi}_t = e^{-t}\,\Phi_{\mathbb{C}^n}$ and $(\widehat{\psi}_t)^*\omega_{\mathrm{std}} = e^{-t}\,\omega_{\mathrm{std}}$. So

$$\widetilde{F}_t := (\widehat{\psi}_t)^{-1} \circ F \circ \widetilde{\psi}_t\colon (\Phi^{-1}(e^tV), \omega_t) \longrightarrow (\Phi_{\mathbb{C}^n}^{-1}(e^tV \cap \mathcal{T}), \omega_{\mathrm{std}})$$

is an equivariant symplectomorphism.

We will now apply Moser's method. Let λ be a T-invariant one form such that $d\lambda = \omega$. Let

$$\lambda_t = e^t(\widetilde{\psi}_t)^*\lambda.$$

Then $d\lambda_t = \omega_t$. Also, $\iota(\xi_M)\lambda_t = \Phi^\xi$, because both sides take the value 0 at p, and they have the same differential. Hence, $\beta_t := \frac{d}{dt}\lambda_t$ is T-invariant and $\iota(\xi_M)\beta_t = 0$ for all $\xi \in \mathfrak{t}$. Let Y_t be the time-dependent vector field on M which is determined by $i_{Y_t}\omega_t = -\beta_t$. Then $Y_t \in \ker d\Phi$ because

$$\langle d\Phi(Y_t), \xi\rangle = -\omega_t(\xi_M, Y_t) = -i_{\xi_M}\beta_t = 0$$

[1] For our application in Proposition 2.11, we only need to consider the case that $\dim T = 1$, and $\Phi\colon M \longrightarrow \mathbb{R}$ is a submersion away from p. Since Φ is homogeneous near p with respect to Darboux coordinates, we can lift the vector field X to M without referring to Lemma 13.2 of [10].

for all $\xi \in \mathfrak{t}$. Since Φ is proper, this implies that Y_t integrates to an isotopy, $G_t \colon M \longrightarrow M$. Since ω_t and β_t are T-invariant, the vector field Y_t is also T-invariant; consequently, G_t is T-equivariant. Finally,

$$\frac{d}{dt}(G_t^*\omega_t) = G_t^*(L_{Y_t}\omega_t) + G_t^*(\frac{d}{dt}\omega_t) = G_t^*(-d\beta_t) + G_t^*(d\beta_t) = 0.$$

Hence, $G_t^*\omega_t$ is independent of t. Because $G_0^*\omega_0 = \omega$,

$$G_t \colon (M,\omega) \longrightarrow (M,\omega_t)$$

is a symplectomorphism for all $t \geq 0$.

Thus, the composition

$$\widetilde{F}_t \circ G_t \colon (\Phi^{-1}(e^t\, V), \omega) \longrightarrow (\Phi_{\mathbb{C}^n}^{-1}(e^t\, V \cap \mathcal{T}), \omega_{\mathrm{std}})$$

is an equivariant symplectomorphism. If $\mathcal{T}$ is bounded, then we can choose t sufficiently large so that $\mathcal{T} \subset e^t\, V$; hence we are done.

Now suppose that $\mathcal{T}$ is not bounded. We will modify our constructions of the vector field $\widetilde{X}$ and the one-form λ so that $\widetilde{F}_t = \widetilde{F}_s$ and $G_s = G_t$ on the intersection $\Phi^{-1}(e^tV) \cap \Phi^{-1}(e^sV)$.

First, we modify our construction of $\widetilde{X}$ so that, in addition to satisfying $\Phi_*(\widetilde{X}) = X$, after possibly shrinking V

$$F_*(\widetilde{X}) = \widehat{X} \quad \text{on} \quad \Phi^{-1}(V). \tag{2.9}$$

To do this, we construct $\widetilde{X}$ on $M \smallsetminus \{p\}$ as before, and then patch with the vector field $F^*(\widehat{X})$ on $\Phi^{-1}(V)$, using an invariant partition of unity subordinate to the sets $M \smallsetminus \{p\}$ and $\Phi^{-1}(V)$.

The property (2.9) implies that $F \circ \widetilde{\psi}_t = \widehat{\psi}_t \circ F$ on $\Phi^{-1}(V)$, so

$$\widetilde{F}_t = \widetilde{F}_s \quad \text{on} \quad \Phi^{-1}(e^tV) \cap \Phi^{-1}(e^sV).$$

Therefore, since the union of the sets e^tV over all $t \geq 0$ is all of $\mathfrak{t}^*$, we can define an equivariant diffeomorphism

$$\widetilde{F} \colon M \longrightarrow \Phi_{\mathbb{C}^n}^{-1}(\mathcal{T})$$

by

$$\widetilde{F} = F_t \quad \text{on} \quad \Phi^{-1}(e^tV).$$

Clearly,

$$\widetilde{F}^*(\omega_{\mathrm{std}}) = \omega_t \quad \text{on} \quad \Phi^{-1}(e^tV).$$

Next, we modify our construction of the invariant one-form λ so that, in addition to satisfying $d\lambda = \omega$, after possibly shrinking V

$$F^*(\lambda_{\mathrm{std}}) = \lambda \quad \text{on} \quad \Phi^{-1}(V). \tag{2.10}$$

Consider the one-form $\lambda_{\rm std} = \frac{1}{2}\sum_{i=1}^n (x_i dy_i - y_i dx_i)$ on $\mathbb{C}^n$. Let λ_V be any T-invariant one-form on M such that $\lambda_V = F^* \lambda_{\rm std}$ on $\Phi^{-1}(V)$. Then $\omega' := \omega - d\lambda_V$ is a closed two-form on M which vanishes on $\Phi^{-1}(V)$. Since M is diffeomorphic to $\Phi_{\mathbb{C}^n}^{-1}(\mathcal{T})$, $\mathcal{T}$ is convex, and $\Phi_{\mathbb{C}^n}$ is homogeneous, M is contractible; similarly, $\Phi^{-1}(V)$ is contractible. Consequently, the relative cohomology $H^2(M, \Phi^{-1}(V))$ is zero. This implies that ω' has a primitive one-form λ' which vanishes on $\Phi^{-1}(V)$; we may choose it to be T-invariant. Let $\lambda = \lambda' + \lambda_V$.

Since $(\widehat{\psi}_t)^* \lambda_{\rm std} = e^{-t} \lambda_{\rm std}$, the property (2.10) implies that

$$\lambda_t = \widetilde{F}_t^* (\lambda_{\rm std}) \quad \text{on} \quad \Phi^{-1}(e^t V).$$

Hence, $\lambda_t = \lambda_s$ on $\Phi^{-1}(e^t V) \cap \Phi^{-1}(e^s V)$. Therefore $\beta_t = 0$, and hence $Y_t = 0$, on $\Phi^{-1}(e^t V)$. Consequently,

$$G_t = G_s \quad \text{on} \quad \Phi^{-1}(e^t V) \cap \Phi^{-1}(e^s V).$$

Therefore, since the union of the sets $e^t V$ over all $t \geq 0$ is all of $\mathfrak{t}^*$, we can define an equivariant diffeomorphism

$$G \colon M \longrightarrow M$$

by

$$G = G_t \quad \text{on} \quad \Phi^{-1}(e^t V).$$

Clearly,

$$G^*(\omega_t) = \omega \quad \text{on} \quad \Phi^{-1}(e^t V).$$

Hence, $\widetilde{F} \circ G : (M, \omega) \longrightarrow (\Phi_{\mathbb{C}^n}^{-1}(\mathcal{T}), \omega_{\rm std})$ is an equivariant symplectomorphism, as required. □

Proposition 2.8 answer Biran's question affirmatively in a special case:

Proposition 2.11 *Let N be a compact manifold with an integral symplectic form ω. Suppose that it admits a Hamiltonian circle action with a fixed point p such that all the isotropy weights at p are equal to one. Then there exists a symplectic embedding of the ball $(B(1), \omega_{\rm std})$ in (N, ω).*

Proof We assume, without loss of generality, that N is connected. Let ξ_N denote the vector field that generates the circle action. Our convention is that the circle group is $S^1 = \mathbb{R}/\mathbb{Z}$, so that ξ_N generates a flow of period one. Let

$$\Phi \colon N \longrightarrow \mathbb{R}$$

be the moment map, so that $\iota(\xi_N)\omega = -d\Phi$. For simplicity, assume that $\Phi(p) = 0$

Because the isotropy weights are positive, p is an isolated local minimum for the moment map. Since the moment map fibers are connected [2, 9], $\Phi^{-1}(\{0\}) = \{p\}$.

By Stokes's theorem, for any fixed point q, the difference $\Phi(q) - \Phi(p)$ is equal to the integral of ω over the cycle obtained from a curve connecting p to q by "sweeping" the curve by the circle action. Because $[\omega]$ is integral, this implies that $\Phi(q) - \Phi(p)$ is an integer. So p is the only fixed point that is mapped to $[0, 1)$.

Let $M = \Phi^{-1}([0,1))$. Consider a subgroup $K \subset S^1$ and let $Y \subset M^K$ be a connected component of its fixed point set. Since Y is closed in M and $\Phi\colon M \longrightarrow [0,1)$ is proper, the image of Y is a closed subset of $[0,1)$, so it has a minimum. Any point in Y which is mapped to this minimum must be a fixed point. Hence $p \in Y$, and so $0 \in \Phi(Y)$. This shows that M is centered about 0.

Proposition 2.11 then follows from Proposition 2.8. □

3 Lower bounds for Grassmannians

In this section we construct an embedding of the ball $B(1)$ into the complex Grassmannian $\mathrm{Gr}(k,n)$, thus showing that the Gromov width of the Grassmannian is at least one.

Proposition 3.1 *Let $\mathrm{Gr}(k,n)$ be the Grassmannian of k-planes in $\mathbb{C}^n$, together with its $\mathrm{U}(n)$-invariant symplectic form ω, normalized so that $[\omega]$ generates the integral cohomology $H^2(\mathrm{Gr}(k,n);\mathbb{Z})$. There exists a symplectic embedding of $B(1)$ into $\mathrm{Gr}(k,n)$.*

Proof Let the circle group S^1 act on $\mathbb{C}^n$ by

$$a \cdot (z_1, \dots, z_n) = (z_1, \dots, z_k, az_{k+1}, \dots, az_n).$$

Take the induced action on $\mathrm{Gr}(k,n)$. Then $p = \mathbb{C}^k \times \{0\}$ is a fixed point for this action. Since $T_p \mathrm{Gr}(k,n) \cong \mathrm{Hom}(\mathbb{C}^k, \mathbb{C}^{n-k})$, the isotropy action is complex multiplication by $a \in S^1$. Proposition 3.1 now follows from Proposition 2.11. □

4 Upper bounds for Gromov width

In this section, we give a short review of Gromov-Witten invariants and how they can be used to give upper bounds to Gromov widths. This material appears in detail in many places; our treatment is adapted from [17].

Let (M,ω) be a compact symplectic manifold. A homology class $B \in H_2(M)$ is *spherical* if it is in the image of the Hurewicz homomorphism $\pi_2(M) \longrightarrow H_2(M)$. A homology class $B \in H_2(M)$ is *indecomposable* if it does not decompose as a sum $B = B_1 + \cdots + B_k$ of spherical classes such that $\omega(B_i) > 0$.

The form ω *tames* an almost complex structure J on M if

$$\omega(v, Jv) > 0$$

for all non-zero $v \in TM$. Given ω, there are many almost complex structures J on M that are tamed by ω; however, the Chern classes $c_i(TM)$ of the complex vector bundles (TM, J) are independent of this choice. We say that (M,ω) is *monotone* if there exists a positive constant $\lambda > 0$ so that $\omega(B) = \lambda c_1(TM)(B)$ for all spherical classes $B \in H_2(M;\mathbb{Z})$.

Assume that (M,ω) is monotone of dimension $2n$. Fix an indecomposable spherical class $A \in H_2(M;\mathbb{Z})$. Let N_i, for $i = 1,\dots,s$, be submanifolds such that $\sum_{i=1}^{s} \operatorname{codim} N_i = 2n + 2c_1(TM)(A) + 2s - 6$. Let B_i be the homology class represented by N_i. The Gromov invariant[2]

$$\Phi_A(B_1,\dots,B_s) \in \mathbb{Z},$$

which is defined, for example, in [17], has the following property. Let $\mathcal{J}$ denote the space of almost complex structures that are tamed by ω. Let $\mathcal{J}_{\mathrm{reg}}(A) \subseteq \mathcal{J}$ denote the set of regular almost complex structures for the class A (see [17]). Given $J \in \mathcal{J}_{\mathrm{reg}}(A)$, for generic deformations N_i' of N_i and generic points $t_i \in \mathbb{CP}^1$, the number of J-holomorphic maps $\mathbb{CP}^1 \longrightarrow M$ in the class A which send each t_i into N_i', counted with appropriate signs, is equal to $\Phi_A(B_1,\dots,B_s)$. In particular, if $[p]$ is the homology class of a point and X, Y are submanifolds such that $\Phi_A([p],[X],[Y]) \neq 0$, then, for $J \in \mathcal{J}_{\mathrm{reg}}(A)$, for every point $p \in M$ and every neighborhood U of p there exists a J-holomorphic sphere in the class A that passes through U.

Proposition 4.1 *Let (M,ω) be a monotone symplectic manifold. Let $A \in H_2(M;\mathbb{Z})$ be an indecomposable spherical class. Let*

$$\lambda = \int_A \omega.$$

[2]This invariant can be defined under more general assumptions.

Let $[p]$ denote the homology class of a point. Suppose that there exist submanifolds X and Y of M so that $\dim X + \dim Y = 4n - 2c_1(TM)(A)$ and so that

$$\Phi_A([p],[X],[Y]) \neq 0.$$

If $a > \lambda$, there does not exist a symplectic embedding $B(a) \longrightarrow M$.

Proof Suppose that there exists a symplectic embedding

$$B(a) \overset{\rho}{\hookrightarrow} M$$

where $a > \lambda$. The standard complex structure J_{std} on $B(a) \subset \mathbb{C}^n$ transports through ρ to a complex structure on $\rho(B(a))$. By a well known technique, after passing to a smaller a such that $a > \lambda$ there exists a tamed almost complex structure J on M such that ρ intertwines J_{std} with J. (Since Riemannian metrics can be patched together by partitions of unity, this follows from the existence of an $\mathrm{Sp}(\mathbb{R}^{2n})$-equivariant projection from the space of inner products on $\mathbb{R}^{2n}$ to the subspace of those inner products that are compatible with the symplectic form.)

Because $\mathcal{J}_{\mathrm{reg}}(A)$ is of second category in $\mathcal{J}$, and by the characterization of Gromov-Witten invariants discussed above, there exists a sequence J_i of almost complex structures converging to J, and points p_i which converge to $p := \rho(0)$, and, for each i, a J_i-holomorphic curves $\mathbb{CP}^1 \longrightarrow M$ in the class A which passes through p_i.

Since the class A is indecomposable, by Gromov's compactness theorem [8, 17] there exists a subsequence of the J_i's which converges weakly to a J-holomorphic curve C in the class A. In particular, $\int_C \omega = \lambda$. Since p_i converge to p, this limit curve contains p.

The pre-image of C under ρ is a holomorphic curve in the ball $B(a)$ which passes through the center of the ball and which is closed in $B(a)$. But the smallest area of a such a curve is that of a disk through the center, which is a. (See [1, page 99].) This contradicts the assumption that $a > \lambda$. □

5 Upper bounds for Grassmannians

It remains to prove the following proposition.

Proposition 5.1 *Let* $\mathrm{Gr}(k,n)$ *be the Grassmannian of* k*-planes in* $\mathbb{C}^n$*, together with its natural* $\mathrm{U}(n)$ *invariant symplectic form, normalized so that* $[\omega]$ *generates the integral cohomology* $H^2(\mathrm{Gr}(k,n);\mathbb{Z})$*. If* $a > 1$*, there does not exist a symplectic embedding of* $B(a)$ *into* $\mathrm{Gr}(k,n)$*.*

Proof The real dimension of $M = \mathrm{Gr}(k,n)$ is $2k(n-k)$. Let A be the generator of $H_2(\mathrm{Gr}(k,n),\mathbb{Z}) \cong \mathbb{Z}$. Clearly, A is irreducible.

The standard complex structure is tamed by ω, and $c_1(TM)(A) = n$. It is easy to check that $\mathrm{Gr}(k,n)$ is monotone.

Fix a hyperplane $W \subset \mathbb{C}^n$ and let $X \subset \mathrm{Gr}(k,n)$ be the set of k-planes that are contained in W. Fix a vector $y \in \mathbb{C}^n$ and let $Y \subset \mathrm{Gr}(k,n)$ be the set of k-planes that contain y. Let $p \in \mathrm{Gr}(k,n)$ be any point.

These submanifolds represent homology classes $[p]$, $[X]$, and $[Y]$ in $H_0(M)$, $H_{2k(n-1)}(M)$, and $H_{2n(k-1)}(M)$, respectively.

In [18], it was shown that $\Phi_A([p],[X],[Y]) = 1$. The Proposition then follows from Proposition 4.1. □

Theorem 1 follows immediately from Propositions 3.1 and 5.1.

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