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# Surfaces in the complex projective plane and their mapping class groups

Susumu Hirose

**Abstract** An orientation preserving diffeomorphism over a surface embedded in a 4-manifold is called extendable, if this diffeomorphism is a restriction of an orientation preserving diffeomorphism on this 4-manifold. In this paper, we investigate conditions for extendability of diffeomorphisms over surfaces in the complex projective plane.

**AMS Classification** 57Q45; 57N05, 20F38

**Keywords** Knotted surface, plane curve, mapping class group, spin mapping class group

*Dedicated to Professor Yukio Matsumoto for his 60th birthday*

## 1 Introduction

There are deformations of embedded surfaces in 4-manifolds which induce isotopically non-trivial diffeomorphisms on surfaces. We introduce two typical examples.

For the first example, we consider a deformation of an annulus embedded in $S^3 \times [-1,1]$ so that, under this deformation, the boundary of this annulus is fixed. Let $S^1 \times [0,1]$ be an annulus embedded in $S^3 \times \{0\} \subset S^3 \times [-1,1]$, and $t\colon S^3 \times [-1,1] \to [-1,1]$ a projection to the second factor. We deform $S^1 \times [0,1]$ as in Figure 1. First, we isotope $S^1 \times [0,1]$ in $S^3$ from (1) to (3). Next, we isotope $S^1 \times [0,1]$ so that outside of the annulus $A$ of (3) $t = 0$, and inside $t > 0$. Then we isotope $S^1 \times [0,1]$ inside $A$ so that, when we push $A$ down to $S^3 \times \{0\}$, $S^1 \times [0,1]$ is as in (4). Finally, we isotope $S^1 \times [0,1]$ in $S^3$ from (4) to (6). The composition of these deformations induce a square of Dehn twist about the core circle $S^1 \times \{\frac{1}{2}\}$ of $S^1 \times [0,1]$.

For the second example, we consider a deformation of a non-singular plane curve of degree 3. A torus is defined as a quotient of the complex plane by a lattice $\mathbb{Z} + \mathbb{Z}\sqrt{-1}$. We embed this torus into the complex projective plane

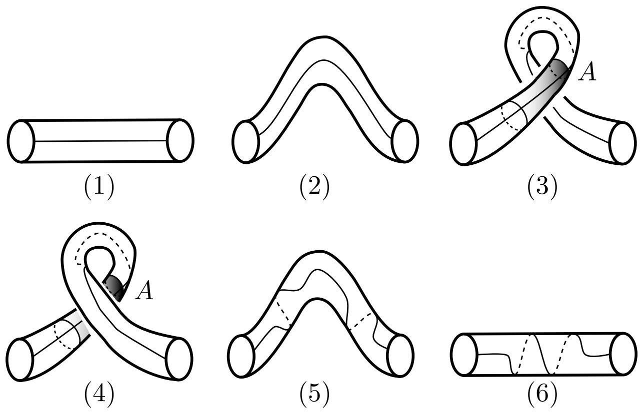



Figure 1

by using the Weierstrass $\wp$ function associated to this lattice, then the image of this embedding is a non-singular plane curve of degree 3. We deform this lattice, $\mathbb{Z}+\mathbb{Z}(\sqrt{-1}+t)$, where $0 \leq t \leq 1$ is a parameter of this deformation. Then the embedding is deformed isotopically and, finally (when $t=1$), brought back to the original position. This deformation induces a Dehn twist on the non-singular plane curve of degree 3.

In this paper, we investigate a topological meaning of the above phenomena.

We settle a general formulation. Let $M$ be a simply connected compact oriented smooth 4-manifold (possibly with boundary) and $F$ be a compact oriented smooth 2-manifold (possibly with boundary) embedded in $M$. We call the pair $(M,F)$ a *knotted surface*. In particular, if $F$ is characteristic, that is to say, $F \cdot X \equiv X \cdot X \mod 2$ for any $X \in H_2(M,\mathbb{Z})$, then we call this pair $(M,F)$ a *knotted characteristic surface*. An orientation preserving diffeomorphism $\psi$ over $F$ is *extendable* if there is an orientation preserving diffeomorphism $\Psi$ over $M$ such that $\Psi|_F = \psi$. In general, for an oriented manifold $A$ and its submanifold $B$, we denote

$$\mathrm{Diff}_+(A, \mathrm{fix}\, B) = \left\{ \psi \;\middle|\; \begin{array}{l} \text{an orientation preserving diffeomorphism over } A \\ \text{such that } \psi|_B = id_B \end{array} \right\}.$$

If $B=\phi$, we denote this group by $\mathrm{Diff}_+(M)$. The group $\pi_0(\mathrm{Diff}_+(F, \mathrm{fix}\, \partial F))$ is called the *mapping class group* of $F$ and denoted by $\mathcal{M}_F$. If $F$ is a closed oriented surface of genus $g$, this group is denoted by $\mathcal{M}_g$. We define

$$\mathcal{E}(M,F) = \{\psi \in \mathcal{M}_F \mid \psi \text{ is extendable }\}.$$

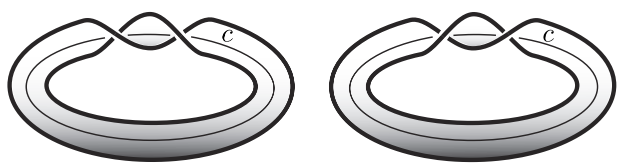
This is a subgroup of $\mathcal{M}_F$ and is a central object of this paper.

In the case where $M = S^4$, there are several works on this group. Let $(S^4, \Sigma_g)$ be the genus $g$ trivial knotted surface in $S^4$. When $g = 1$, Montesinos [19] investigated $\mathcal{E}(S^4, \Sigma_1)$, and when $g \geq 2$, the author [11] investigated $\mathcal{E}(S^4, \Sigma_g)$. Let $(S^3, k)$ be a knot in $S^3$ and $(S^4, S(k))$ (resp. $(S^4, \tilde{S}(k))$) the spun (resp. the twisted spun) of $(S^3, k)$. When $(S^3, k)$ is a torus knot, Iwase [13] investigated $\mathcal{E}(S^4, S(k))$ and $\mathcal{E}(S^4, \tilde{S}(k))$, and when $(S^3, k)$ is an arbitrary knot, the author [10] investigated these groups.

In this paper, we investigate the case where $M$ is the complex projective plane $\mathbb{CP}^2$. In §3, we treat the case where $(\mathbb{CP}^2, \Sigma_g)$ is a standard embedding of $\Sigma_g$. In §4, we treat the case where $(\mathbb{CP}^2, F)$ is a non-singular plane curve. From §5 to the end of this paper, we treat the case where $(\mathbb{CP}^2, F)$ is a connected sum of a non-singular plane curve of degree 3 and a trivial embedding.

## 2 Preliminary: A Hopf band on the boundary of the 4-ball

A link $L$ in $S^3$ is called a *fibered link* if there is a map $\phi\colon\ S^3 \setminus L \to S^1$ which is a fiber bundle projection. For each $t \in S^1$, $\phi^{-1}(t) = F$, which does not depend on $t$, is called the *fiber* of $L$. Since $\phi$ is a bundle projection, $S^3 \setminus L$ is diffeomorphic to the quotient of $F \times [0,1]$ by an equivalence $(x, 0) \sim (h(x), 1)$ where $h$ is a diffeomorphism over $F$ and called the *monodromy* of $L$.




Positive Hopf band Negative Hopf band

Figure 2

A *Hopf band* is an annulus embedded in $S^3$ as in Figure 2. In this picture, there are two types of Hopf bands. In this note, we treat both types of Hopf bands. The boundary of a Hopf band is called a *Hopf link*. The Hopf link is a fibered link whose fiber is the Hopf band and whose monodromy is a Dehn twist about the core circle of the Hopf band. Let $B$ be a Hopf band in $S^3$ which is a boundary of a 4-ball $D^4$. We push the interior of $B$ into the interior of $D^4$ and let $B'$ be the annulus obtained by this deformation and let $c$ be the core circle of $B'$.

**Proposition 2.1** *The Dehn twist $T_c$ about $c$ is extendable, i.e. there is an element $T \in \mathrm{Diff}_+(D^4, \mathrm{fix}\,\partial D^4)$ such that $T|_{B'} = T_c$.*

**Proof** Since $\partial B$ is a fibered link, whose fiber is $B$ and whose monodromy is $T_c$, there is an orientation preserving diffeomorphism $\psi$ of $S^3$ such that $\psi|_B = T_c$, and there is an isotopy $\psi_t$ $(t \in [0,1])$ with $\psi_0 = id_{S^3}$ and $\psi_1 = \psi$, which is defined by shifting fibers. Let $N(\partial D^4)$ be the regular neighborhood of $\partial D^4$ in $D^4$. We parametrize $N(\partial D^4) = S^3 \times [0,2]$ so that $S^3 \times \{0\} = \partial D^4$ and $B' = \partial B \times [0,1] \cup B \times \{1\}$. Let $T$ be a diffeomorphism defined as follows

$$T|_{N(\partial D^4)}(x,t) = \begin{cases} (\psi_t(x), t) & 0 \le t \le 1 \\ (\psi_{2-t}(x), t) & 1 \le t \le 2 \end{cases}$$
$$T|_{D^4 \setminus N(\partial D^4)} = id.$$

This is the diffeomorphism which we need. □

**Remark 2.2** Let $(S^4, \Sigma_g)$ be the genus $g$ surface standardly embedded in $S^4$. In [11], the author showed that $T_{c_4} T_{c_3} T_{c_4}^{-1} \in \mathcal{E}(S^4, \Sigma_g)$ by using Montesinos' result [19, Theorem 5.3] ($c_3$ and $c_4$ are as in Figure 7) . We show this fact by using Proposition 2.1. The 4-sphere $S^4$ is constructed from two 4-balls $D^4_+$, $D^4_-$ with attaching along the boundary $S^3 = \partial D^4_+ = \partial D^4_-$. We parametrize the regular neighborhood $N(\partial D^4_+) = S^3 \times [0,2]$ in $D^4_+$ so that $\partial D^4_+ = S^3 \times \{0\}$. The regular neighborhood $N$ of $T_{c_4}(c_3)$ in $\Sigma_g$ is a Hopf band in $S^3 \subset S^4$. We push the interior of $N$ into the interior of $D^4_+$, then we get an annulus $N'$ properly embedded in $D^4_+$. We may assume, by the above parametrization of $N(\partial D^4_+)$, $N' \cap S^3 \times \{t\} = \partial N \times \{t\}$ for $0 \le t < 2$ and $N' \cap S^3 \times \{2\} = N \times \{2\}$. We denote $D^4_+ \setminus S^3 \times [0,1)$ by $D'$. By applying Proposition 2.1 to $(D', N' \cap D')$, we show that there is an element $T \in \mathrm{Diff}_+(D', \mathrm{fix}\,\partial D')$ such that $T|_{N' \cap D'} = T_{c_4} T_{c_3} T_{c_4}^{-1}$. Therefore, we see $T_{c_4} T_{c_3} T_{c_4}^{-1} \in \mathcal{E}(S^4, \Sigma_g)$.

## 3 Surfaces standardly embedded in the complex projective plane

For the free action of $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$ on $\mathbb{C}^3 \setminus \{(0,0,0)\}$ defined by $\lambda(z_0, z_1, z_2) = (\lambda z_0, \lambda z_1, \lambda z_2)$, we take the quotient $\mathbb{CP}^2 = (\mathbb{C}^3 \setminus \{(0,0,0)\})/\mathbb{C}^*$. This space $\mathbb{CP}^2$ is a closed oriented 4-manifold and called the *complex projective plane*. This 4-manifold $\mathbb{CP}^2$ is constructed from $D^4$ by attaching a 2-handle $h^2$ along the frame 1 trivial knot $K_0$ in $\partial D^4$, and attaching a 4-handle $h^4$. A *3-dimensional handlebody* $H_g$ is an oriented 3-manifold which is constructed from a 3-ball with attaching $g$ 1-handles. Any image of embeddings of $H_g$ into $\mathbb{CP}^2$ are isotopic each other. Therefore, $(\mathbb{CP}^2, \partial H_g)$ is unique. A surface *standardly embedded* in $\mathbb{CP}^2$ is $(\mathbb{CP}^2, \partial H_g)$. We obtain:

**Theorem 3.1** *For any $g$, $\mathcal{E}(\mathbb{CP}^2, \partial H_g) = \mathcal{M}_g$.*

**Proof** Let $D^4$ be the 4-ball used to construct $\mathbb{CP}^2$ and $N(\partial D^4)$ be the regular neighborhood of $\partial D^4$ in $D^4$. We parametrize $N(\partial D^4) = S^3 \times [0, -1]$, so that $S^3 \times \{0\} = \partial D^4$ and, for $-1 \leq t < 0$, $S^3 \times \{t\}$ is in the interior of $D^4$. Since the image of embedding of $H_g$ in $\mathbb{CP}^2$ is unique up to isotopy, we assume that $H_g \subset S^3 \times \{-1\}$ and that each simple closed curve $c$ on $H_g$ which corresponds to Lickorish generator of mapping class group $\mathcal{M}_g$ is a trivial knot in $S^3 \times \{-1\}$. The regular neighborhood $N(c)$ of $c$ on $\partial H_g$ is an annulus trivially embedded in $S^3 \times \{-1\}$. At first, we deform $H_g$ in $S^3 \times \{-1\}$ so that, if we forget the second factor $[0, -1]$, $c \cup K_0$ becomes a Hopf link in $S^3$. We push $N(c)$ into $\partial(D^4 \cup h^2)$, then $N(c)$ becomes a Hopf band in $\partial h^4$. By applying Proposition 2.1, we see that $T_c$ is extendable in $h^4$, and so in $\mathbb{CP}^2$. □

## 4 Non-singular plane curves

We review here the topological description of non-singular plane curves by Akbulut and Kirby [1] (see also [6, 6.2.7]).

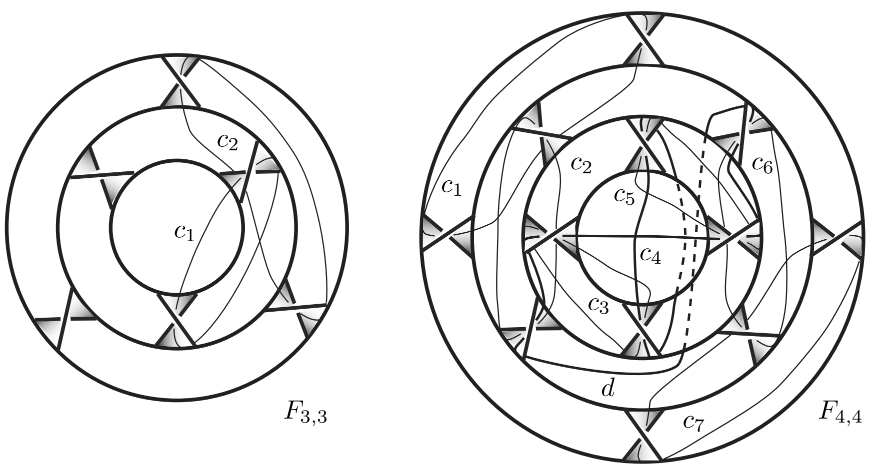



Figure 3

An $(m,n)$-*torus link* $T_{m,n}$ is an oriented link in $S^3 = \partial D^4$ consisting of $\gcd(m,n)$ oriented circles in the boundary of the tubular neighborhood $U$ of the trivial knot, representing $m\mu + n\lambda$ in $H_1(\partial U;\mathbb{Z})$, where $\mu =$ [the meridian of the trivial knot] and $\lambda =$ [the longitude of the trivial knot]. There is a canonical Seifert surface $F_{m,n}$ for $T_{m,n}$, consisting of $n$-disks connected by $m(n-1)$ twisted bands as in Figure 3. As $K_0$, we take a trivial knot given by pushing $T_{1,0}$ into the complement of $U$ ( see the left hand side of Figure 4). From here, we consider only the case where $m = n = d$. As shown in the right hand side of Figure 4, $T_{d,d}$ becomes $d$ components trivial link in $\partial(D^4 \cup h^2)$. Let $D_d$ be disjoint 2-disks in $\partial(D^4 \cup h^2)$ which bound this trivial link.

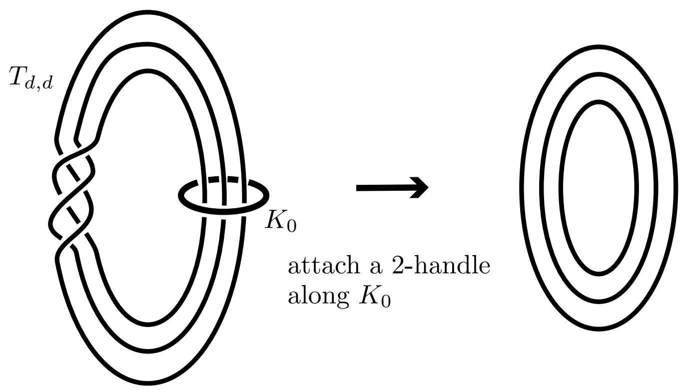



Figure 4

Let $K_d$ be a non-singular plane curve of degree $d$, then $K_d$ is a genus $\frac{(d-1)(d-2)}{2}$ closed oriented surface embedded in $\mathbb{CP}^2$. We remark that $K_d$ is unique up to isotopy, $K_d = \{[X:Y:Z] \in \mathbb{CP}^2 | X^d + Y^d + Z^d = 0\}$ and $[K_d] = d[\mathbb{CP}^1] \in H_2(\mathbb{CP}^2;\mathbb{Z})$. Akbulut and Kirby showed:

**Proposition 4.1** $K_d = F_{d,d} \cup D_d$.

Thus we obtain:

**Theorem 4.2** *When* $d = 3, 4$, $\mathcal{E}(\mathbb{CP}^2, K_d) = \mathcal{M}_{g_d}$, *where* $g_d = \frac{(d-1)(d-2)}{2}$.

**Proof** When $d = 3$, $K_3$ is homeomorphic to a 2-dimensional torus $T^2$. In $F_{3,3}$ (see Figure 3), each regular neighborhood of $c_1$ and $c_2$ is a Hopf band. Therefore, by Proposition 2.1, $T_{c_1}$ and $T_{c_2}$ are elements of $\mathcal{E}(\mathbb{CP}^2, K_3)$. On the other hand, $T_{c_1}$ and $T_{c_2}$ generate $\mathcal{M}_1$. Hence, $\mathcal{E}(\mathbb{CP}^2, K_3) = \mathcal{M}_1$.

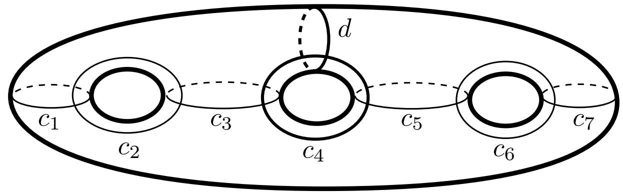



Figure 5

When $d = 4$, we do the same as the above case. We remark that the Dehn twists about the simple closed curves in Figure 5 corresponding to the simple closed curves in $F_{4,4}$ (see Figure 3) with the same symbols generate the mapping class group of genus 3 surface [16]. □

When $d \geq 5$, $\mathcal{E}(\mathbb{CP}^2, K_d)$ is unknown. It is, however, not the case that $\mathcal{E}(\mathbb{CP}^2, K_d) = \mathcal{M}_{g_d}$, because, when $d$ is odd, $K_d$ is a characteristic surface, so the Rokhlin quadratic form on $H_1(K_d; \mathbb{Z}_2)$ is well-defined (we review the definition of the Rokhlin quadratic form in the next section). By the definition of the Rokhlin quadratic form, if a diffeomorphism on $K_d$ is extendable to $\mathbb{CP}^2$, this diffeomorphism should preserve this form. Hence:

**Theorem 4.3** *When $d$ is an odd integer greater than or equal to 5, $\mathcal{E}(\mathbb{CP}^2, K_d)$ is a proper subgroup of $\mathcal{M}_{g_d}$, where $g_d = \frac{(d-1)(d-2)}{2}$.*

## 5 Connected sum of the non-singular plane curve of degree 3 and trivial knotted surface

We define knotted surfaces investigated from here to the end of this paper. The images of any embeddings of a 3-dimensional handlebody $H_g$ into $S^4$ are isotopic each other. We call this $\Sigma_g$-knot $(S^4, \partial H_g)$ *a trivial $\Sigma_g$-knot*, and this is denoted by $(S^4, \Sigma_g)$. Let $(\mathbb{CP}^2, K_3)$ be a nonsingular cubic plane curve. We define connected sum of $(\mathbb{CP}^2, K_3)$ and $(S^4, \Sigma_{g-1})$ following the construction by Boyle [3] as follows. We choose points $p$ and $q$ on $K_3$ and $\Sigma_{g-1}$ respectively, and find small 4-balls $B_1$ and $B_2$ centered at $p$ and $q$ such that the pairs $(B_1, B_1 \cap K_3)$ and $(B_2, B_2 \cap \Sigma_{g-1})$ are equivalent to the standard pair $(B^4, B^2)$. Now we glue the pairs $(S^4 \setminus \mathrm{int}(B_1), K_3 \setminus \mathrm{int}(B_1))$ and $(\mathbb{CP}^2 \setminus \mathrm{int}(B_2), \Sigma_{g-1} \setminus \mathrm{int}(B_2))$ together by an orientation-reversing diffeomorphism $f : \partial B_1 \to \partial B_2$ such that $f(\partial B_1 \cap K_3) = \partial B_2 \cap \Sigma_{g-1}$. Since the connected sum of $\mathbb{CP}^2$ and $S^4$ is diffeomorphic to $\mathbb{CP}^2$, we get a surface in $\mathbb{CP}^2$ and denote this characteristic knotted surface by $(\mathbb{CP}^2, K_3 \# \Sigma_{g-1})$. From here to the end of this paper, we investigate on the group $\mathcal{E}(\mathbb{CP}^2, K_3 \# \Sigma_{g-1})$.

For a knotted characteristic surface $(M, F)$, where $M$ is a simply connected smooth closed oriented 4-manifold, we define a quadratic form (*the Rokhlin quadratic form*) $q_F : H_1(F; \mathbb{Z}_2) \to \mathbb{Z}_2$: Let $P$ be a compact surface embedded in $M$, with its boundary contained in $F$, normal to $F$ along its boundary, and its interior is transverse to $F$. Let $P'$ be a surface transverse to $P$ obtained by sliding $P$ parallel to itself over $F$. Define $q_F([\partial P]) = \#(\mathrm{int}P \cap (P' \cup F)) \bmod 2$. This is a well-defined quadratic form with respect to the $\mathbb{Z}_2$-homology intersection form $(,)_2$ on $F$, i.e. for each pair of elements $x$, $y$ of $H_1(F; \mathbb{Z}_2)$, $q_F(x+y) = q_F(x) + q_F(y) + (x, y)_2$. By the definition of the Rokhlin quadratic from $q_F$, if $\psi \in \mathrm{Diff}_+(F)$ is extendable, then $\psi$ preserves $q_F$, that is to say, $q_F(\psi_*(x)) = q_F(x)$ for any $x \in H_1(F; \mathbb{Z}_2)$. We will show,

**Theorem 5.1** *For any $g \geq 2$,*

$$\mathcal{E}(\mathbb{CP}^2, K_3 \# \Sigma_{g-1}) = \left\{ \psi \in \mathcal{M}_g \;\middle|\; \begin{array}{c} q_{K_3 \# \Sigma_{g-1}}(\psi_*(x)) = q_{K_3 \# \Sigma_{g-1}}(x) \\ \text{for any } x \in H_1(K_3 \# \Sigma_{g-1}; \mathbb{Z}_2) \end{array} \right\}.$$

In §6, we investigate on a system of generators for the right hand side group in the equation of Theorem 5.1. In §7, we show that each element of this system of generators is extendable.

# 6 A finite set of generators for the odd spin mapping class group

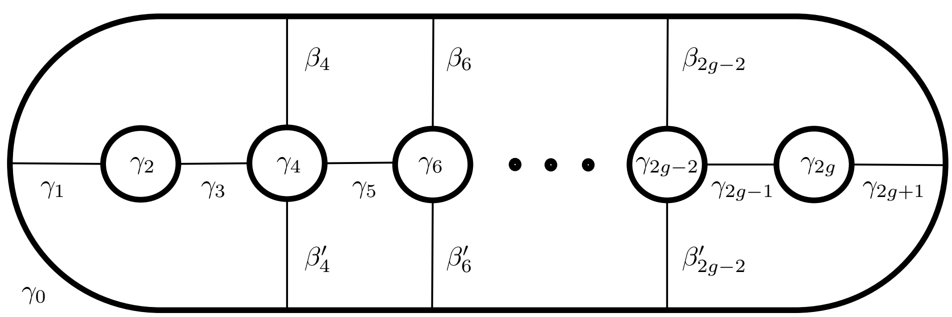



Figure 6

We settle some notations. Let $P_g$ be a planar surface constructed from a 2-disk by removing $g$ copies of disjoint 2-disks. As indicated in Figure 6, we denote the boundary components of $P_g$ by $\gamma_0, \gamma_2, \ldots, \gamma_{2g}$, and denote some properly embedded arcs of $P_g$ by $\gamma_1, \gamma_3, \ldots, \gamma_{2g+1}$, $\beta_4, \ldots, \beta_{2g-2}$ and $\beta'_4, \ldots, \beta'_{2g-2}$. On $\partial(P_g \times [-1,1]) = \Sigma_g$, we define $c_{2i-1} = \partial(\gamma_{2i-1} \times [-1,1])$ $(1 \leq i \leq g+1)$, $b_{2j} = \partial(\beta_{2j} \times [-1,1])$, $b'_{2j} = \partial(\beta'_{2j} \times [-1,1])$ $(2 \leq j \leq g-1)$, and $c_{2k} = \gamma_{2k} \times \{0\}$ $(1 \leq k \leq g)$. In Figures 7 and 8, these circles are illustrated and some of them are oriented.

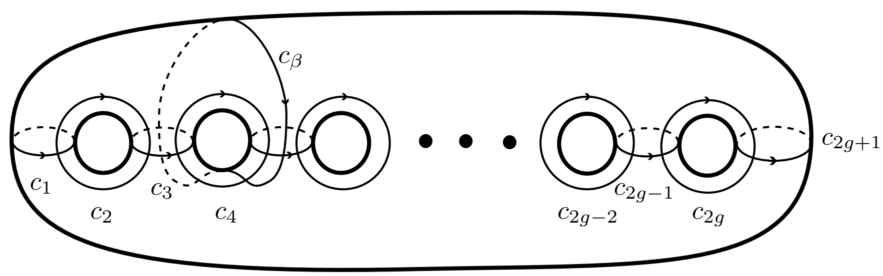



Figure 7

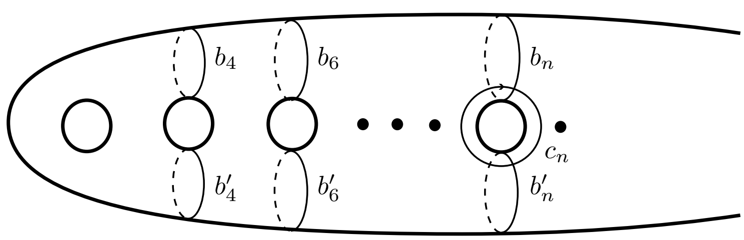



Figure 8

We set a basis of $H_1(\Sigma_g; \mathbb{Z})$ as in Figure 9, where $x_1 = [c_1$ with opposite orientation $]$, $x_i = [b_{2i}$ with proper orientation $]$ $(2 \leq i \leq g-1)$, $x_g = [c_{2g+1}]$, and $y_i = [c_{2i}$ with opposite orientation $]$.

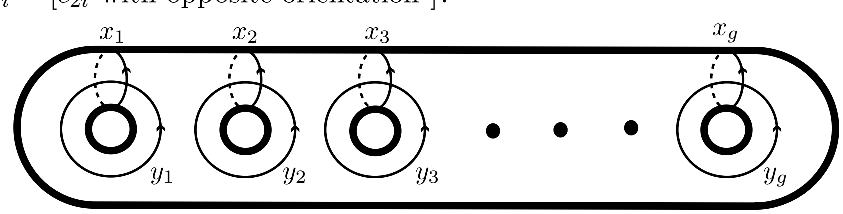



Figure 9

A map $q : H_1(\Sigma_g;\mathbb{Z}_2) \to \mathbb{Z}_2$ is called a *quadratic form* with respect to the $\mathbb{Z}_2$-homology intersection form $(,)_2$ on $\Sigma_g$ (for short, $\mathbb{Z}_2$-*quadratic form* on $\Sigma_g$) if $q(x+y) = q(x) + q(y) + (x,y)_2$, for each pair of elements $x$, $y$ of $H_1(\Sigma_g;\mathbb{Z}_2)$. For the basis $\{x_1, y_1, \dots, x_g, y_g\}$ introduced above, we define $Arf(q) = \sum_{i=1}^{g} q(x_i)q(y_i)$. We call a $\mathbb{Z}_2$-quadratic form $q$ *even* quadratic from (resp. *odd* quadratic form) if $Arf(q) = 0$ (resp. $Arf(q) = 1$). We define

$$\mathcal{SP}_g[q] = \{\psi \in \mathcal{M}_g \mid q(\psi_*(x)) = q(x) \text{ for any } x \in H_1(\Sigma_g;\mathbb{Z}_2)\}.$$

As is shown in [21], for two $\mathbb{Z}_2$-quadratic forms $q$, $q'$ on $\Sigma_g$, if $Arf(q) = Arf(q')$, then there is an element $\psi' \in \mathcal{M}_g$ so that $q(\psi'_*(x)) = q'(x)$ for any $x \in H_1(\Sigma_g;\mathbb{Z}_2)$. Therefore, if $Arf(q) = Arf(q')$, then $\mathcal{SP}_g[q]$ and $\mathcal{SP}_g[q']$ are conjugate in $\mathcal{M}_g$. By the definition of $\mathbb{Z}_2$-quadratic from, values of a quadratic form is completely determined by its value for the basis of $H_1(\Sigma_g;\mathbb{Z}_2)$. Let $q_0$ and $q_1$ be $\mathbb{Z}_2$-quadratic forms so that $q_0(x_i) = q_0(y_i) = 0$ for $1 \le i \le g$, $q_1(x_1) = q_1(y_1) = 1$ and $q_1(x_j) = q_1(y_j) = 0$ for $2 \le j \le g$. Then $q_0$ is an even quadratic form and $q_1$ an odd quadratic from. If $q$ is even, then $\mathcal{SP}_g[q]$ is conjugate to $\mathcal{SP}_g[q_0]$ in $\mathcal{M}_g$, on the other hand, if $q$ is odd, then $\mathcal{SP}_g[q]$ is conjugate to $\mathcal{SP}_g[q_1]$ in $\mathcal{M}_g$. Hence, for the sake of getting some information about groups $\mathcal{SP}_g[q]$, it suffices to consider only on $\mathcal{SP}_g[q_0]$ and $\mathcal{SP}_g[q_1]$. The group $\mathcal{SP}_g[q_0]$ is called the *even spin mapping class group*, and the group $\mathcal{SP}_g[q_1]$ is called the *odd spin mapping class group*. The spin mapping class group is defined by Harer [8], [9]. In [11], we get a system of generators for $\mathcal{SP}_g[q_0]$. In this section, we will obtain a system of generators for $\mathcal{SP}_g[q_1]$.

Let $M$ be a simply connected smooth closed oriented 4-manifold, $(M, F)$ a knotted characteristic surface and $q_F$ the Rokhlin quadratic form for $(M, F)$. Rokhlin [20] showed (see also [17] and [5]),

$$Arf(q_F) \equiv \frac{\sigma(M) - F\cdot F}{8} \mod 2,$$

where $\sigma(M)$ is the signature of $M$. By the above formula, we can see $q_{K_3\#\Sigma_{g-1}}$ is an odd quadratic form. Hence, we get a system of generators for $\mathcal{SP}_g[q_{K_3\#\Sigma_{g-1}}]$ from that for $\mathcal{SP}_g[q_1]$.

We introduce some notations used for describing a system of generators for $\mathcal{SP}_g[q_1]$. For a simple closed curve $a$ on $\Sigma_g$, $T_a$ denotes the Dehn twist about $a$. The order of composition of maps is the functional one: $T_bT_a$ means we apply $T_a$ first, then $T_b$. For elements $a$, $b$ and $c$ of a group, we write $\overline{c} = c^{-1}$, and $a * b = ab\overline{a}$ . We define some elements of $\mathcal{M}_g$ as follows:

$$\begin{aligned}
&C_i = T_{c_i},\ B_i = T_{b_i},\ B'_i = T_{b'_i},\\
&X_i = C_{i+1}C_i\overline{C_{i+1}}\ ,\ X_i^* = \overline{C_{i+1}}\ C_iC_{i+1}\ \ (4 \le i \le 2g),\\
&Y_{2j} = C_{2j}B_{2j}\overline{C_{2j}}\ ,\ Y_{2j}^* = \overline{C_{2j}}\ B_{2j}C_{2j}\ \ (2 \le j \le g-1),\\
&D_i = C_i^2\ \ (1 \le i \le 2g+1),\\
&DB_{2j} = B_{2j}^2\ \ (2 \le j \le g-1),\\
&T_1 = B_4C_5C_7\cdots C_{2g+1}.
\end{aligned}$$

When $g \ge 3$, $G_g$ denotes the subgroup of $\mathcal{M}_g$ generated by $C_1$, $C_2$, $C_3$, $X_i$ $(4 \le i \le 2g)$, $Y_{2j}$ $(2 \le j \le g-1)$, $D_i$ $(1 \le i \le 2g+1)$, $DB_{2j}$ $(2 \le j \le g-1)$, and $T_1$. It is clear that $X_i^*$ and $Y_{2j}^*$ are elements of $G_g$. When $g = 2$, the subgroup of $\mathcal{M}_2$ generated by $C_1$, $C_2$, $C_3$, $X_4$, and $D_j$ $(1 \le j \le 5)$ is denoted by $G_2$. For two simple closed curves $l$ and $m$ on $\Sigma_g$, $l$ and $m$ are called $G_g$*-equivalent* (denoted by $l \underset{G_g}{\sim} m$) if there is an element $\phi$ of $G_g$ such that $\phi(l) = m$.

We show that $G_g = \mathcal{SP}_g[q_1]$. That is to say, we show,

**Theorem 6.1** *If $g = 2$, $\mathcal{SP}_2[q_1]$ is generated by $C_1$, $C_2$, $C_3$, $X_4$, and $D_j$ $(1 \le j \le 5)$. If $g \ge 3$, $\mathcal{SP}_g[q_1]$ is generated by $C_1$, $C_2$, $C_3$, $X_i$ $(4 \le i \le 2g)$, $Y_{2j}$ $(2 \le j \le g-1)$, $D_k$ $(1 \le k \le 2g+1)$, $DB_{2l}$ $(2 \le l \le g-1)$, and $T_1$.*

We prove Theorem 6.1 by using the same method as in the proof of Theorem 3.1 in [11]. By an easy calculation, we can check that each generator of $G_g$ is an element of $\mathcal{SP}_g[q_1]$, therefore, $G_g \subset \mathcal{SP}_g[q_1]$. Hence, we should show $\mathcal{SP}_g[q_1] \subset G_g$. In the case where $g = 2$, we use the Reidemeister-Schreier method to show $\mathcal{SP}_g[q_1] \subset G_2$ (§6.4). In the case where $g \ge 3$, we use other method to show $\mathcal{SP}_g[q_1] \subset G_g$. Here, we present this method in outline.

The integral symplectic group is denoted by $\mathrm{Sp}(2g,\mathbb{Z})$ and the $\mathbb{Z}_2$ symplectic group by $\mathrm{Sp}(2g,\mathbb{Z}_2)$. The generators of these groups are known (on $\mathrm{Sp}(2g,\mathbb{Z})$ see for example [12], on $\mathrm{Sp}(2g,\mathbb{Z}_2)$ see for example [7, Chap.3]), and these generators are induced by the action of $\mathcal{M}_g$ on $H_1(\Sigma_g,\mathbb{Z})$ or $H_1(\Sigma_g,\mathbb{Z}_2)$. Therefore, the homomorphism $\Phi\colon \mathcal{M}_g \to \mathrm{Sp}(2g,\mathbb{Z})$, defined by the action of $\mathcal{M}_g$ on $H_1(\Sigma_g,\mathbb{Z})$, is a surjection, and $\Psi\colon \mathrm{Sp}(2g,\mathbb{Z}) \to \mathrm{Sp}(2g,\mathbb{Z}_2)$, defined by changing the coefficient from $\mathbb{Z}$ to $\mathbb{Z}_2$, is a surjection. In §6.1, we show $\ker\Phi \subset G_g$. In §6.2, we introduce a finite system of generators for $\ker\Psi$, and, for each generator, we show that one of its inverse by $\Phi$ is an element of $G_g$. Hence, we conclude $\ker\Psi\circ\Phi \subset G_g$. In §6.3, we introduce a finite system of generators for $\Psi \circ \Phi(\mathcal{SP}_g[q_1])$, and, for each generator, we show that one of its inverse by $\Psi \circ \Phi$ is an element of $G_g$. As a consequence, we show $\mathcal{SP}_g[q_1] \subset G_g$.

## 6.1 Step 1 for the case where $g \geq 3$

There is a natural surjection $\Phi\colon \mathcal{M}_g \to \mathrm{Sp}(2g, \mathbb{Z})$ defined by the action of $\mathcal{M}_g$ on $H_1(\Sigma_g; \mathbb{Z})$. The kernel of $\Phi$ is denoted by $\mathcal{I}_g$ and called *the Torelli group*. In this subsection, we prove the following lemma:

**Lemma 6.2** *The Torelli group $\mathcal{I}_g$ is a subgroup of $G_g$.*

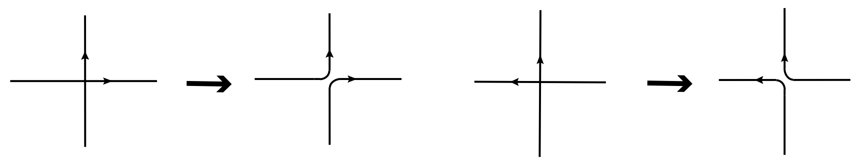


Figure 10

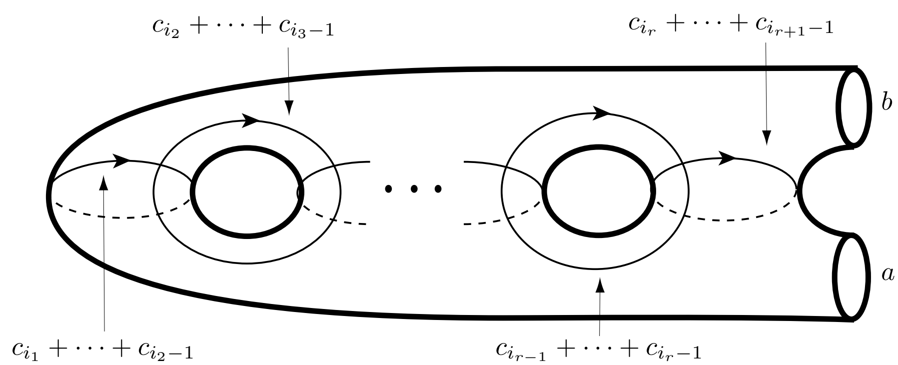



Figure 11

Johnson [14] showed that, when $g$ is larger than or equal to 3, $\mathcal{I}_g$ is finitely generated. We review his result. For oriented simple closed curves shown in Figure 7, we refer to $(c_1, c_2, \ldots, c_{2g+1})$ and $(c_\beta, c_5, \ldots, c_{2g+1})$ as *chains*. For oriented simple closed curves $d$ and $e$ which intersect transversely in one point, we construct an oriented simple closed curve $d+e$ from $d \cup e$ as follows: choose a disk neighborhood of the intersection point and in it make a replacement as indicated in Figure 10. For a consecutive subset $\{c_i, c_{i+1}, \ldots, c_j\}$ of a chain, let $c_i + \cdots + c_j$ be the oriented simple closed curve constructed by repeated applications of the above operations. Let $(i_1, \ldots, i_{r+1})$ be a subsequence of

$(1, 2, \dots, 2g+2)$ (resp. $(\beta, 5, \dots, 2g+2)$). We construct the union of circles $\mathcal{C} = (c_{i_1} + \cdots + c_{i_2 - 1}) \cup (c_{i_2} + \cdots + c_{i_3 - 1}) \cup \cdots \cup (c_{i_r} + \cdots + c_{i_{r+1} - 1})$. If $r$ is odd, a regular neighborhood of $\mathcal{C}$ is homeomorphic to the compact surface indicated in Figure 11 whose boundaries are $a$ and $b$. Let $\phi = T_b T_a^{-1}$, then $\phi$ is an element of $\mathcal{I}_g$. We denote $\phi$ by $[i_1, \dots, i_{r+1}]$, and call this *the odd subchain map* of $(c_1, c_2, \dots, c_{2g+1})$ (resp. $(c_\beta, c_5, \dots, c_{2g+1})$) with *length* $r+1$. Johnson [14] showed the following theorem:

**Theorem 6.3** [14, Main Theorem] *For $g \geq 3$, the odd subchain maps of the two chains $(c_1, c_2, \dots, c_{2g+1})$ and $(c_\beta, c_5, \dots, c_{2g+1})$ generate $\mathcal{I}_g$.*

We use the following results by Johnson [14].

**Lemma 6.4** [14] *(a) $C_j$ commutes with $[i_1, i_2, \cdots]$ if and only if $j$ and $j+1$ are either both contained in or are disjoint from the $i$'s.*
*(b) If $i \neq j+1$, then $\overline{C_j} * [\cdots, j, i, \cdots] = [\cdots, j+1, i, \cdots]$.*
*(c) If $k \neq j$, then $C_j * [\cdots, k, j+1, \cdots] = [\cdots, k, j, \cdots]$.*
*(d) $[1,2,3,4][1,2,5,6,\dots,2n]B_4 * [3,4,5,\dots,2n] = [5,6,\dots,2n][1,2,3,4,\dots,2n]$, where $3 \leq n \leq g$.*

**Remark 6.5** Johnson showed (d) only in the case where $n = g$. But we can apply the proof of Lemma 10 of [14] for the case where $3 \leq n < g$, since we can regard each surfaces in Figure 18 of [14] as a surface of genus $n$ which is a submanifold of $\Sigma_g$.

We prove that any odd subchain map of $(c_1, c_2, c_3, \dots, c_{2g+1})$ or $(c_\beta, c_5, c_6, \dots, c_{2g})$ is a product of elements of $G_g$. The following lemma shows that any odd subchain map of $(c_\beta, c_5, c_6, \dots, c_{2g})$ is a product of an odd subchain map of $(c_1, c_2, c_3, \dots, c_{2g+1})$ and elements of $G_g$.

**Lemma 6.6** *For any odd subchain map $h$ of $(c_\beta, c_5, c_6, \dots, c_{2g+1})$, there is an element $g$ of $G_g$ such that $g * h$ is an odd subchain map of $(c_1, c_2, c_3, \dots, c_{2g+1})$.*

**Proof** If there is not $\beta$ in the sequence which define $h$, then $h$ is an odd subchain map of $(c_1, c_2, c_3, \dots, c_{2g+1})$. Hence, it suffices to treat the case where the sequence defining $h$ includes $\beta$. If $g = C_{2g+1}^{\epsilon_{2g+1}} \cdots C_7^{\epsilon_7} C_5^{\epsilon_5} B^{-1}$ $(\epsilon_i = \pm 1)$, then, under any choice of signs of $\epsilon_i$, $g \in G_g$. We can choose signs of $\epsilon_i$ so that $g * h$ is an odd subchain map of $(c_1, c_2, c_3, \dots, c_{2g+1})$. ☐

From here to the end of this subsection, odd subchain maps mean only those of $(c_1, c_2, c_3, \dots, c_{2g+1})$. The following lemma shows that any odd subchain map, whose length is at least 5 and which begins from $1, 2, 3, 4, 5$, is a product of shorter odd subchain maps and elements of $G_g$.

**Lemma 6.7** *For any* $6 \leq n_6 < n_7 < \cdots < n_{2k} \leq 2g+2$,

$$(C_4^2) * [1,2,3,5][1,2,4,n_6,n_7,\dots,n_{2k}](C_4 B_4 \overline{C_4}\ ) * [3,4,5,n_6,n_7,\dots,n_{2k}] = \\ = [4,n_6,n_7,\dots,n_{2k}][1,2,3,4,5,n_6,n_7,\dots,n_{2k}]$$

**Proof** By (a) of Lemma 6.4, $\overline{C_4}\ * [3,4,5,\dots,2k] = [3,4,5,\dots,2k]$, and by (d) of Lemma 6.4,

$$[1,2,3,4][1,2,5,6,\dots,2k] \cdot (B_4 \overline{C_4}\ ) * [3,4,5,\dots,2k] = \\ = [5,6,\dots,2k][1,2,3,4,\dots,2k].$$

By applying $C_4$ to the above equation and remarking that $C_4 * [1,2,3,4] = (C_4^2) * (\overline{C_4}\ * [1,2,3,4]) = (C_4^2) * [1,2,3,5]$, we get,

$$(C_4^2) * [1,2,3,5] \cdot [1,2,4,6,\cdots,2k] \cdot (C_4 B_4 \overline{C_4}\ ) * [3,4,5,6,\cdots,2k] = \\ = [4,6,7,\cdots,2k][1,2,3,4,5,6,\cdots,2k].$$

After proper applications of $\overline{C_6}$ , $\overline{C_7}$ , ..., $\overline{C_{2g+1}}$ , we get the equation we need. □

**Lemma 6.8** *(1) When* $i-k \geq 3$, $(\overline{C_{i-1}}\ C_{i-2} C_{i-1}) * [\dots,k,i,j,\dots] = [\dots,k,i-2,j,\dots]$.
*(2) When* $i-k \geq 2$, $(C_i C_{i-1} \overline{C_i}\ ) * [\dots,k,i,i+1,\dots] = [\dots,k,i-1,i,\dots]$.

**Proof** Lemma 6.4 shows (1) and (2). □

For any odd subchain map $[i_1, i_2, \dots, i_r]$, we introduce a notation $[[\tau_1, \tau_2, \dots, \tau_{2g+2}]]$ : $\tau_k = 1$ if $k$ is a member of $\{i_1, i_2, \dots, i_r\}$, and $\tau_k = 0$ if $k$ is not a member of $\{i_1, i_2, \dots, i_r\}$. For $[[\tau_1, \tau_2, \dots, \tau_{2g+2}]]$, $\tau_i$ $(1 \leq i \leq 2g+2)$ is called the $i$-th *tack* of $[[\tau_1, \tau_2, \dots, \tau_{2g+2}]]$, and if $\tau_i = 0$ (resp. 1) then $\tau_i$ is called a 0-tack (resp. a 1-tack). The number of 1-tacks in $[[\tau_1, \tau_2, \dots, \tau_{2g+2}]]$ is called the *length* of $[[\tau_1, \tau_2, \dots, \tau_{2g+2}]]$. Lemma 6.8 (1) means that, when $k \geq 3$, if there is a sequence of 0-tacks which begins from the $k+1$-st tack and whose length is at least 2, then the 1-tack subsequent to this 0-tack sequence is moved to left by 2-steps under the action of $G_g$. Lemma 6.8 (2) means that, when $k \geq 3$, if there is a sequence of 0-tacks which begins from the $k+1$-st tack and whose length is at least 1, then the adjacent two 1-tacks subsequent to this 0-tack sequence is moved to left by 1-step under the action of $G_g$. Therefore, for any $[[\tau_1, \tau_2, \ldots, \tau_{2g+2}]]$, we see,

$$[[\tau_1, \tau_2, \ldots, \tau_{2g+2}]] \underset{G_g}{\sim} [[\tau_1, \tau_2, \tau_3, 1, \ldots, 1, 0, 1, \ldots, 0, 1, 0, \ldots, 0]],$$

where $1, \ldots, 1$ is a sequence of 1-tacks ($b$ denotes the length of this sequence), $0, 1, \ldots, 0, 1$ is a sequence arranged 0-tacks and 1-tacks alternatively ($t$ denotes the number of 1-tacks in this sequence ), $0, \ldots, 0$ is a sequence of 0-tacks. Since $C_1, C_2, C_3 \in G_g$, if there is one 1-tack among $\tau_1, \tau_2, \tau_3$, then $[[\tau_1, \tau_2, \tau_3, \ldots]] \underset{G_g}{\sim} [[1, 0, 0, \ldots]]$, if there are two 1-tacks among $\tau_1, \tau_2, \tau_3$, then $[[\tau_1, \tau_2, \tau_3, \ldots]] \underset{G_g}{\sim} [[1, 1, 0, \ldots]]$. The number of 1-tacks in $\tau_1, \tau_2, \tau_3$ is denoted by $h$.

**Lemma 6.9** *Any odd subchain map is a product of elements of $G_g$ and the odd subchain maps whose $h$ and $b$ are (1) $h = 3, b = 1$, (2) $h = 3, b = 0$, (3) $h = 2, b = 0$, (4) $h = 1, b = 0$, (5) $h = 0, b = 0$.*

**Proof** We treat the case where $h = 3$. If $b \geq 2$, by Lemma 6.7, this odd subchain map is a product of elements of $G_g$ and shorter odd subchain maps.

We treat the case where $h = 2$. If $b \geq 3$,

$$[[1, 1, 0, 1, 1, 1, \ldots]] \underset{C_3}{\longrightarrow} [[1, 1, 1, 0, 1, 1, \ldots]] \underset{\text{Lemma 6.8(2)}}{\longrightarrow} [[1, 1, 1, 1, 1, 0, \ldots]],$$

by Lemma 6.7, the last odd subchain map is a product of elements of $G_g$ and shorter odd subchain maps. If $b = 2$,

$$[[1, 1, 0, 1, 1, 0, \ldots]] \underset{C_3}{\longrightarrow} [[1, 1, 1, 0, 1, 0, \ldots]],$$

the last odd subchain map is in the case where $h = 3, b = 0$. If $b = 1$, $t$ should be at least 1, and

$$\begin{aligned}[[1, 1, 0, 1, 0, 1, 0, \ldots]] &\underset{C_3}{\longrightarrow} [[1, 1, 1, 0, 0, 1, 0, \ldots]] \\ &\underset{\text{Lemma 6.8(1)}}{\longrightarrow} [[1, 1, 1, 1, 0, 0, 0, \ldots]],\end{aligned}$$

the last odd subchain map is in the case where $h = 3, b = 1$.

We treat the case where $h = 1$. If $b \geq 5$,

$$\begin{aligned}&[[1, 0, 0, 1, 1, 1, 1, 1, \ldots]] \underset{C_2 C_3}{\longrightarrow} [[1, 1, 0, 0, 1, 1, 1, 1, \ldots]] \\ &\underset{\text{Lemma 6.8(2)}}{\longrightarrow} [[1, 1, 0, 1, 1, 0, 1, 1, \ldots]] \underset{\text{Lemma 6.8(2)}}{\longrightarrow} [[1, 1, 0, 1, 1, 1, 1, 0, \ldots]] \\ &\underset{C_3}{\longrightarrow} [[1, 1, 1, 0, 1, 1, 1, 0, \ldots]] \underset{\text{Lemma 6.8(2)}}{\longrightarrow} [[1, 1, 1, 1, 1, 0, 1, 0, \ldots]],\end{aligned}$$

by Lemma 6.7, the last odd subchain map is a product of elements of $G_g$ and the shorter odd subchain maps. If $b=4$,

$$[[1,0,0,1,1,1,1,0,\ldots]] \underset{C_2C_3}{\longrightarrow} [[1,1,0,0,1,1,1,0,\ldots]] \underset{\text{Lemma 6.8(2)}}{\longrightarrow} [[1,1,0,1,1,0,1,0,\ldots]] \underset{C_3}{\longrightarrow} [[1,1,1,0,1,0,1,0,\ldots]],$$

the last odd subchain map is in the case where $h=3, b=0$. If $b=3$ and $t=0$,

$$[[1,0,0,1,1,1,0,0,\ldots]] \underset{C_2C_3}{\longrightarrow} [[1,1,0,0,1,1,0,0,\ldots]] \underset{\text{Lemma 6.8(2)}}{\longrightarrow} [[1,1,0,1,1,0,0,0,\ldots]] \underset{C_3}{\longrightarrow} [[1,1,1,0,1,0,0,0,\ldots]],$$

the last odd subchain map is in the case where $h=3, b=0$. If $b=3$ and $t\geq 2$,

$$[[1,0,0,1,1,1,0,1,\ldots]] \underset{C_2C_3}{\longrightarrow} [[1,1,0,0,1,1,0,1,\ldots]] \underset{\text{Lemma 6.8(2)}}{\longrightarrow} [[1,1,0,1,1,0,0,1,\ldots]] \underset{\text{Lemma 6.8(1)}}{\longrightarrow} [[1,1,0,1,1,1,0,0,\ldots]] \underset{C_3}{\longrightarrow} [[1,1,1,0,1,1,0,0,\ldots]] \underset{\text{Lemma 6.8(2)}}{\longrightarrow} [[1,1,1,1,1,0,0,0,\ldots]],$$

by Lemma 6.7, the last odd subchain map is a product of elements of $G_g$ and shorter odd subchain maps. If $b=2$,

$$[[1,0,0,1,1,0,\ldots]] \underset{C_2C_3}{\longrightarrow} [[1,1,0,0,1,0,\ldots]],$$

the last odd subchain map is in the case where $h=2, b=0$. If $b=1$, $t$ should be at least $2$,

$$[[1,0,0,1,0,1,0,1,\ldots]] \underset{C_2C_3}{\longrightarrow} [[1,1,0,0,0,1,0,1,\ldots]] \underset{\text{Lemma 6.8(1)}}{\longrightarrow} [[1,1,0,1,0,1,0,0,\ldots]],$$

the last odd subchain map is in the case where $h=2, b=1$, which we treat before.

We treat the case where $h=0$. If $b\geq 7$,

$$[[0,0,0,1,1,1,1,1,1,1,\ldots]] \underset{C_1C_2C_3}{\longrightarrow} [[1,0,0,0,1,1,1,1,1,1,\ldots]] \underset{\text{Lemma 6.8(2)}}{\longrightarrow} [[1,0,0,1,1,1,1,1,1,0,\ldots]] \underset{C_2C_3}{\longrightarrow} [[1,1,0,0,1,1,1,1,1,0,\ldots]] \underset{\text{Lemma 6.8(2)}}{\longrightarrow} [[1,1,0,1,1,1,1,0,1,0,\ldots]] \underset{C_3}{\longrightarrow} [[1,1,1,0,1,1,1,0,1,0,\ldots]] \underset{\text{Lemma 6.8(2)}}{\longrightarrow} [[1,1,1,1,1,0,1,0,1,0,\ldots]],$$

by Lemma 6.7, the last odd subchain map is a product of $G_g$ and shorter odd subchain maps. If $b=6$,

$$[[0,0,0,1,1,1,1,1,1,0,\dots]] \underset{C_1C_2C_3}{\longrightarrow} [[1,0,0,0,1,1,1,1,1,0,\dots]]$$
$$\underset{\text{Lemma 6.8(2)}}{\longrightarrow} [[1,0,0,1,1,1,1,0,1,0,\dots]] \underset{C_2C_3}{\longrightarrow} [[1,1,0,0,1,1,1,0,1,0,\dots]]$$
$$\underset{\text{Lemma 6.8(2)}}{\longrightarrow} [[1,1,0,1,1,0,1,0,1,0,\dots]] \underset{C_3}{\longrightarrow} [[1,1,1,0,1,0,1,0,1,0,\dots]],$$

the last odd subchain map is in the case where $h=3, b=0$. If $b=5$, $t$ should be at least 1 and,

$$[[0,0,0,1,1,1,1,1,0,1,\dots]] \underset{C_1C_2C_3}{\longrightarrow} [[1,0,0,0,1,1,1,1,0,1,\dots]]$$
$$\underset{\text{Lemma 6.8(2)}}{\longrightarrow} [[1,0,0,1,1,1,1,0,0,1,\dots]] \underset{\text{Lemma 6.8(1)}}{\longrightarrow} [[1,0,0,1,1,1,1,1,0,0,\dots]]$$
$$\underset{C_2C_3}{\longrightarrow} [[1,1,0,0,1,1,1,1,0,0,\dots]] \underset{\text{Lemma 6.8(2)}}{\longrightarrow} [[1,1,0,1,1,1,1,0,0,0,\dots]]$$
$$\underset{C_3}{\longrightarrow} [[1,1,1,0,1,1,1,0,0,0,\dots]] \underset{\text{Lemma 6.8(2)}}{\longrightarrow} [[1,1,1,1,1,0,1,0,0,0,\dots]],$$

by Lemma 6.7, the last odd subchain map is a product of elements of $G_g$ and shorter odd subchain maps. If $b=4$,

$$[[0,0,0,1,1,1,1,0,\dots]] \underset{C_1C_2C_3}{\longrightarrow} [[1,0,0,0,1,1,1,0,\dots]]$$
$$\underset{\text{Lemma 6.8(2)}}{\longrightarrow} [[1,0,0,1,1,0,1,0,\dots]] \underset{C_2C_3}{\longrightarrow} [[1,1,0,0,1,0,1,0,\dots]],$$

the last odd subchain map is in the case where $h=2, b=0$. If $b=3$ and $t=1$,

$$[[0,0,0,1,1,1,0,1,0,\dots]] \underset{C_1C_2C_3}{\longrightarrow} [[1,0,0,0,1,1,0,1,0,\dots]]$$
$$\underset{\text{Lemma 6.8(2)}}{\longrightarrow} [[1,0,0,1,1,0,0,1,0,\dots]] \underset{\text{Lemma 6.8(1)}}{\longrightarrow} [[1,0,0,1,1,1,0,0,0,\dots]]$$
$$\underset{C_2C_3}{\longrightarrow} [[1,1,0,0,1,1,0,0,0,\dots]] \underset{\text{Lemma 6.8(2)}}{\longrightarrow} [[1,1,0,1,1,0,0,0,0,\dots]]$$
$$\underset{C_3}{\longrightarrow} [[1,1,1,0,1,0,0,0,0,\dots]],$$

the last odd subchain map is in the case where $h=3, b=0$. If $b=3$ and $t\neq 1$, then $t$ should be at least 3 and,

$$[[0,0,0,1,1,1,0,1,0,1,0,1,\dots]]$$
$$\underset{C_1C_2C_3}{\longrightarrow} [[1,0,0,0,1,1,0,1,0,1,0,1,\dots]]$$
$$\underset{\text{Lemma 6.8(2)}}{\longrightarrow} [[1,0,0,1,1,0,0,1,0,1,0,1,\dots]]$$

$$\underset{\text{Lemma 6.8(1)}}{\longrightarrow} [[1,0,0,1,1,1,0,1,0,1,0,0,\ldots]]$$
$$\underset{C_2C_3}{\longrightarrow} [[1,1,0,0,1,1,0,1,0,1,0,0,\ldots]]$$
$$\underset{\text{Lemma 6.8(2)}}{\longrightarrow} [[1,1,0,1,1,0,0,1,0,1,0,0,\ldots]]$$
$$\underset{\text{Lemma 6.8(1)}}{\longrightarrow} [[1,1,0,1,1,1,0,1,0,0,0,0,\ldots]]$$
$$\underset{C_3}{\longrightarrow} [[1,1,1,0,1,1,0,1,0,0,0,0,\ldots]]$$
$$\underset{\text{Lemma 6.8(2)}}{\longrightarrow} [[1,1,1,1,1,0,0,1,0,0,0,0,\ldots]],$$

by Lemma 6.7, the last odd subchain map is a product of elements of $G_g$ and shorter odd subchain maps. If $b=2$,

$$[[0,0,0,1,1,0,\ldots]] \underset{C_1C_2C_3}{\longrightarrow} [[1,0,0,0,1,0,\ldots]],$$

the last odd subchain map is in the case where $h=1, b=0$. If $b=1$, then $t$ should be at least 3 and,

$$[[0,0,0,1,0,1,0,1,0,1,\ldots]] \underset{C_1C_2C_3}{\longrightarrow} [[1,0,0,0,0,1,0,1,0,1,\ldots]]$$
$$\underset{\text{Lemma 6.8(1)}}{\longrightarrow} [[1,0,0,1,0,1,0,1,0,0,\ldots]] \underset{C_2C_3}{\longrightarrow} [[1,1,0,0,0,1,0,1,0,0,\ldots]]$$
$$\underset{\text{Lemma 6.8(1)}}{\longrightarrow} [[1,1,0,1,0,1,0,0,0,0,\ldots]] \underset{C_3}{\longrightarrow} [[1,1,1,0,0,1,0,0,0,0,\ldots]]$$
$$\underset{\text{Lemma 6.8(1)}}{\longrightarrow} [[1,1,1,1,0,0,0,0,0,0,\ldots]],$$

the last odd subchain map is in the case where $h=3, b=1$.

This Lemma follows from the above case by case arguments and the induction on the length $(=h+b+t)$ of odd subchain maps. □

**Lemma 6.10** *Any odd subchain maps of the* 6 *cases listed in Lemma 6.9 are products of elements of* $G_g$ *and odd subchain maps* $[[1,1,1,1,0,\ldots,0]]$, $[[1,1,1,0,1,0,\ldots,0]]$, $[[1,1,1,0,1,0,1,0,1,0,\ldots,0]]$, *and* $[[1,1,0,0,1,0,1,0,\ldots,0]]$, *where* $0,\ldots,0$ *are sequences of* 0*-tacks.*

**Proof** By checking figures of chain maps, for examples $[[1,1,1,1,0,1,0,1,\ldots]]$ indicated in Figure 12 and $[[0,0,0,0,1,0,1,0,\ldots]]$ indicated in Figure 13, we see that if a odd subchain map begins from $[[0,0,0,0,\ldots,$ or $[[1,1,1,1,\ldots,$ then this map commutes with $B_4$, hence $B_4*$ does not effect on this map.

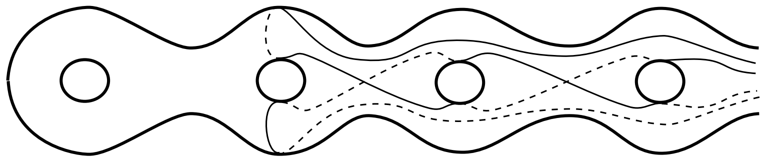


Figure 12

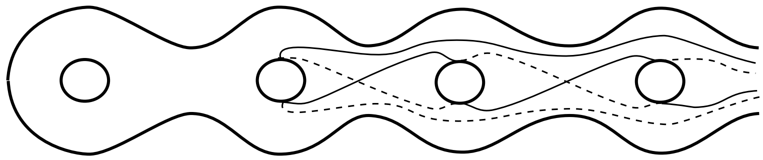


Figure 13

We treat the case where $h = 3$, $b = 1$. If $t = 0$, then this odd subchain map is $[[1, 1, 1, 1, 0, \ldots, 0]]$. If $t \neq 0$, then $t$ should be at least 2 and,

$$[[1, 1, 1, 1, 0, 1, 0, 1, \ldots]] \xrightarrow[T_1]{} [[1, 1, 1, 1, 1, 0, 1, 0, \ldots]],$$

by Lemma 6.7, the last odd subchain map is a product of elements of $G_g$ and shorter odd subchain maps.

We treat the case where $h = 3$, $b = 0$. In this case, $t$ should be an odd integer at least 1. If $t = 1$, then this map is $[[1, 1, 1, 0, 1, 0, \ldots, 0]]$. If $t = 3$, then this map is $[[1, 1, 1, 0, 1, 0, 1, 0, 1, 0, \ldots, 0]]$. If $t \geq 5$,

$$\begin{aligned}
&[[1, 1, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, \ldots]] \\
&\xrightarrow[\overline{C_3}]{} [[1, 1, 0, 1, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, \ldots]] \\
&\xrightarrow[\text{Lemma 6.8(2)}]{} [[1, 1, 0, 0, 1, 1, 1, 0, 1, 0, 1, 0, 1, 0, \ldots]] \\
&\xrightarrow[\overline{C_3}\,\overline{C_2}]{} [[1, 0, 0, 1, 1, 1, 1, 0, 1, 0, 1, 0, 1, 0, \ldots]] \\
&\xrightarrow[\text{Lemma 6.8(2)}]{} [[1, 0, 0, 0, 1, 1, 1, 1, 1, 0, 1, 0, 1, 0, \ldots]] \\
&\xrightarrow[\overline{C_3}\,\overline{C_2}\,\overline{C_1}]{} [[0, 0, 0, 1, 1, 1, 1, 1, 1, 0, 1, 0, 1, 0, \ldots]] \\
&\xrightarrow[\text{Lemma 6.8(2)}]{} [[0, 0, 0, 0, 1, 1, 1, 1, 1, 1, 1, 0, 1, 0, \ldots]] \\
&\xrightarrow[\overline{T_1}]{} [[0, 0, 0, 0, 1, 1, 1, 1, 1, 1, 0, 1, 0, 1, \ldots]] \\
&\xrightarrow[\text{Lemma 6.8(2)}]{} [[0, 0, 0, 1, 1, 1, 1, 1, 1, 0, 0, 1, 0, 1, \ldots]]
\end{aligned}$$

$$
\begin{aligned}
&\xrightarrow[C_1C_2C_3]{} [[1,0,0,0,1,1,1,1,1,0,0,1,0,1,\ldots]] \\
&\xrightarrow[\text{Lemma 6.8(1)}]{} [[1,0,0,0,1,1,1,1,1,1,0,0,0,1,\ldots]] \\
&\xrightarrow[\text{Lemma 6.8(2)}]{} [[1,0,0,1,1,1,1,1,1,0,0,0,0,1,\ldots]] \\
&\xrightarrow[C_2C_3]{} [[1,1,0,0,1,1,1,1,1,0,0,0,0,1,\ldots]] \\
&\xrightarrow[\text{Lemma 6.8(2)}]{} [[1,1,0,1,1,1,1,0,1,0,0,0,0,1,\ldots]] \\
&\xrightarrow[C_3]{} [[1,1,1,0,1,1,1,0,1,0,0,0,0,1,\ldots]] \\
&\xrightarrow[\text{Lemma 6.8(2)}]{} [[1,1,1,1,1,0,1,0,1,0,0,0,0,1,\ldots]],
\end{aligned}
$$

by Lemma 6.7, the last odd subchain map is a product of elements of $G_g$ and shorter odd subchain maps.

We treat the case where $h = 2$, $b = 0$. In this case, $t$ should be even integer at least $2$. If $t = 2$, this map is $[[1,1,0,0,1,0,1,0,\ldots,0]]$. If $t \geq 4$,

$$
\begin{aligned}
&[[1,1,0,0,1,0,1,0,1,0,1,0,\ldots]] \\
&\xrightarrow[\overline{C_3}\,\overline{C_2}]{} [[1,0,0,1,1,0,1,0,1,0,1,0,\ldots]] \\
&\xrightarrow[\text{Lemma 6.8(2)}]{} [[1,0,0,0,1,1,1,0,1,0,1,0,\ldots]] \\
&\xrightarrow[\overline{C_3}\,\overline{C_2}\,\overline{C_1}]{} [[0,0,0,1,1,1,1,0,1,0,1,0,\ldots]] \\
&\xrightarrow[\text{Lemma 6.8(2)}]{} [[0,0,0,0,1,1,1,1,1,0,1,0,\ldots]] \\
&\xrightarrow[\overline{T_1}]{} [[0,0,0,0,1,1,1,1,0,1,0,1,\ldots]] \\
&\xrightarrow[\text{Lemma 6.8(2)}]{} [[0,0,0,1,1,1,1,0,0,1,0,1,\ldots]] \\
&\xrightarrow[C_1C_2C_3]{} [[1,0,0,0,1,1,1,0,0,1,0,1,\ldots]] \\
&\xrightarrow[\text{Lemma 6.8(1)}]{} [[1,0,0,0,1,1,1,1,0,1,0,0,\ldots]] \\
&\xrightarrow[\text{Lemma 6.8(2)}]{} [[1,0,0,1,1,1,1,0,0,1,0,0,\ldots]] \\
&\xrightarrow[\text{Lemma 6.8(1)}]{} [[1,0,0,1,1,1,1,1,0,0,0,0,\ldots]] \\
&\xrightarrow[C_2C_3]{} [[1,1,0,0,1,1,1,1,0,0,0,0,\ldots]] \\
&\xrightarrow[\text{Lemma 6.8(2)}]{} [[1,1,0,1,1,1,1,0,0,0,0,0,\ldots]]
\end{aligned}
$$

$$
\begin{aligned}
&\xrightarrow[C_3]{} [[1,1,1,0,1,1,1,0,0,0,0,0,\ldots]] \\
&\xrightarrow[\text{Lemma 6.8(2)}]{} [[1,1,1,1,1,0,1,0,0,0,0,0,\ldots]],
\end{aligned}
$$

by Lemma 6.7, the last odd subchain map is a product of elements of $G_g$ and shorter odd subchain maps.

We treat the case where $h = 1$, $b = 0$. In this case, $t$ should be an odd integer at least $3$. If $t = 3$,

$$
\begin{aligned}
&[[1,0,0,0,1,0,1,0,1,0,\ldots]] \xrightarrow[\overline{C_3}\,\overline{C_2}\,\overline{C_1}]{} [[0,0,0,1,1,0,1,0,1,0,\ldots]] \\
&\xrightarrow[\text{Lemma 6.8(2)}]{} [[0,0,0,0,1,1,1,0,1,0,\ldots]] \xrightarrow[\overline{T_1}]{} [[0,0,0,0,1,1,0,1,0,1,\ldots]] \\
&\xrightarrow[\text{Lemma 6.8(2)}]{} [[0,0,0,1,1,0,0,1,0,1,\ldots]] \xrightarrow[\text{Lemma 6.8(1)}]{} [[0,0,0,1,1,1,0,1,0,0,\ldots]] \\
&\xrightarrow[C_1C_2C_3]{} [[1,0,0,0,1,1,0,1,0,0,\ldots]] \xrightarrow[\text{Lemma 6.8(2)}]{} [[1,0,0,1,1,0,0,1,0,0,\ldots]] \\
&\xrightarrow[\text{Lemma 6.8(1)}]{} [[1,0,0,1,1,1,0,0,0,0,\ldots]] \xrightarrow[C_2C_3]{} [[1,1,0,0,1,1,0,0,0,0,\ldots]] \\
&\xrightarrow[\text{Lemma 6.8(2)}]{} [[1,1,0,1,1,0,0,0,0,0,\ldots]] \xrightarrow[C_3]{} [[1,1,1,0,1,0,0,0,0,0,\ldots]].
\end{aligned}
$$

If $t \geq 5$,

$$
\begin{aligned}
&[[1,0,0,0,1,0,1,0,1,0,1,0,\ldots]] \\
&\xrightarrow[\overline{C_3}\,\overline{C_2}\,\overline{C_1}]{} [[0,0,0,1,1,0,1,0,1,0,1,0,\ldots]] \\
&\xrightarrow[\text{Lemma 6.8(2)}]{} [[0,0,0,0,1,1,1,0,1,0,1,0,\ldots]] \\
&\xrightarrow[\overline{T_1}]{} [[0,0,0,0,1,1,0,1,0,1,0,1,\ldots]] \\
&\xrightarrow[\text{Lemma 6.8(2)}]{} [[0,0,0,1,1,0,0,1,0,1,0,1,\ldots]] \\
&\xrightarrow[\text{Lemma 6.8(1)}]{} [[0,0,0,1,1,1,0,1,0,1,0,0,\ldots]] \\
&\xrightarrow[C_1C_2C_3]{} [[1,0,0,0,1,1,0,1,0,1,0,0,\ldots]] \\
&\xrightarrow[\text{Lemma 6.8(2)}]{} [[1,0,0,1,1,0,0,1,0,1,0,0,\ldots]] \\
&\xrightarrow[\text{Lemma 6.8(1)}]{} [[1,0,0,1,1,1,0,1,0,0,0,0,\ldots]] \\
&\xrightarrow[C_2C_3]{} [[1,1,0,0,1,1,0,1,0,0,0,0,\ldots]] \\
&\xrightarrow[\text{Lemma 6.8(2)}]{} [[1,1,0,1,1,0,0,1,0,0,0,0,\ldots]]
\end{aligned}
$$

$$\begin{aligned}
&\xrightarrow[\text{Lemma 6.8(1)}]{} [[1,1,0,1,1,1,0,0,0,0,0,0,\ldots]]\\
&\xrightarrow[C_3]{} [[1,1,1,0,1,1,0,0,0,0,0,0,\ldots]]\\
&\xrightarrow[\text{Lemma 6.8(2)}]{} [[1,1,1,1,1,0,0,0,0,0,0,0,\ldots]],
\end{aligned}$$

by Lemma 6.7, the last odd subchain map is a product of elements of $G_g$ and shorter odd subchain maps.

We treat the case where $h=0,\ b=0$. In this case, $t$ should be an even integer at least $4$. If $t=4$,

$$\begin{aligned}
&[[0,0,0,0,1,0,1,0,1,0,1,0,\ldots]]\\
&\xrightarrow[\overline{T_1}]{} [[0,0,0,0,0,1,0,1,0,1,0,1,\ldots]]\\
&\xrightarrow[\text{Lemma 6.8(1)}]{} [[0,0,0,1,0,1,0,1,0,1,0,0,\ldots]]\\
&\xrightarrow[C_1C_2C_3]{} [[1,0,0,0,0,1,0,1,0,1,0,0,\ldots]]\\
&\xrightarrow[\text{Lemma 6.8(1)}]{} [[1,0,0,1,0,1,0,1,0,0,0,0,\ldots]]\\
&\xrightarrow[C_2C_3]{} [[1,1,0,0,0,1,0,1,0,0,0,0,\ldots]]\\
&\xrightarrow[\text{Lemma 6.8(1)}]{} [[1,1,0,1,0,1,0,0,0,0,0,0,\ldots]]\\
&\xrightarrow[C_3]{} [[1,1,1,0,0,1,0,0,0,0,0,0,\ldots]]\\
&\xrightarrow[\text{Lemma 6.8(1)}]{} [[1,1,1,1,0,0,0,0,0,0,0,0,\ldots]].
\end{aligned}$$

If $t\geq 6$,

$$\begin{aligned}
&[[0,0,0,0,1,0,1,0,1,0,1,0,1,0,1,0,\ldots]]\\
&\xrightarrow[\overline{T_1}]{} [[0,0,0,0,0,1,0,1,0,1,0,1,0,1,0,1,\ldots]]\\
&\longrightarrow \text{(as in the previous case)} \longrightarrow [[1,1,1,1,0,0,0,0,0,0,0,0,1,0,1,\ldots]]\\
&\xrightarrow[\text{Lemma 6.8(1)}]{} [[1,1,1,1,0,1,0,1,\ldots]],
\end{aligned}$$

the last odd subchain map is in the case where $h=3, b=1$, which we treat before. □

**Lemma 6.11** *The odd subchain maps* $[[1,1,1,1,0,\ldots,0]]$, $[[1,1,1,0,1,0,\ldots,0]]$ *and* $[[1,1,0,0,1,0,1,0,\ldots,0]]$ *are elements of* $G_g$.

**Proof** In a proof of this Lemma, we use "braid relation", which is explained as follows. Let $a$ and $b$ are simple closed curves on $\Sigma_g$ intersecting transversely in one point, then $T_aT_bT_a^{-1} = T_b^{-1}T_aT_b$, in other word, $T_a * T_b = \overline{T_b} * T_a$.

Let $b_4'$ be the simple closed curve on $\Sigma_g$ indicated in Figure 8 and let $B_4' = T_{b_4'}$. The odd subchain map $[[1,1,1,1,0,\dots,0]]$ is equal to $B_4\overline{B_4'}$. Since $b_4' = C_4C_3C_2C_1C_1C_2C_3C_4(b_4)$,

$$\begin{aligned}B_4\overline{B_4'} =& B_4C_4C_3C_2C_1C_1C_2C_3C_4\overline{B_4}\,\overline{C_4}\,\overline{C_3}\,\overline{C_2}\,\overline{C_1}\,\overline{C_1}\,\overline{C_2}\,\overline{C_3}\,\overline{C_4}\\ =& (B_4C_4C_3C_2) * (C_1C_1) \cdot (B_4C_4C_3) * (C_2C_2) \cdot (B_4C_4) * (C_3C_3) \cdot\\ & \cdot B_4 * (C_4C_4) \cdot (\overline{C_4}\,\overline{C_3}\,\overline{C_2}) * (\overline{C_1}\,\overline{C_1}) \cdot (\overline{C_4}\,\overline{C_3}) * (\overline{C_2}\,\overline{C_2}) \cdot\\ & \cdot \overline{C_4} * (\overline{C_3}\,\overline{C_3}) \cdot \overline{C_4}\,\overline{C_4}.\end{aligned}$$

This equation means that $B_4\overline{B_4'}$ is a product of squares Dehn twists. By using braid relations of $\mathcal{M}_g$, we can see that these squares of Dehn twists are elements of $G_g$ as follows,

$$\begin{aligned}(B_4C_4C_3C_2) * (C_1C_1) &= (\overline{C_1} \cdot \overline{C_2} \cdot \overline{C_3} \cdot B_4) * (C_4C_4)\\ &= (\overline{C_1} \cdot \overline{C_2} \cdot \overline{C_3}) * (B_4C_4\overline{B_4} \cdot B_4C_4\overline{B_4}),\\ (B_4C_4C_3) * (C_2C_2) &= (\overline{C_2} \cdot \overline{C_3} \cdot B_4) * (C_4C_4)\\ &= (\overline{C_2} \cdot \overline{C_3}) * (B_4C_4\overline{B_4} \cdot B_4C_4\overline{B_4}),\\ (B_4C_4) * (C_3C_3) &= (\overline{C_3} \cdot B_4) * (C_4C_4) = \overline{C_3} * (B_4C_4\overline{B_4} \cdot B_4C_4\overline{B_4}),\\ B_4 * (C_4C_4) &= B_4C_4\overline{B_4} \cdot B_4C_4\overline{B_4},\\ (\overline{C_4}\,\overline{C_3}\,\overline{C_2}) * (C_1C_1) &= (C_1 \cdot C_2 \cdot C_3) * (C_4C_4),\\ (\overline{C_4}\,\overline{C_3}) * (C_2C_2) &= (C_2 \cdot C_3) * (C_4C_4),\\ \overline{C_4} * (C_3C_3) &= C_3 * (C_4C_4).\end{aligned}$$

Since $\overline{C_4} * [[1,1,1,1,0,\dots,0]] = [[1,1,1,0,1,0,\dots,0]]$,

$$\begin{aligned}&[[1,1,1,0,1,0,\dots,0]] = \overline{C_4} * (B_4\overline{B_4'})\\ &= (\overline{C_4}\,B_4C_4C_3C_2) * (C_1C_1) \cdot (\overline{C_4}\,B_4C_4C_3) * (C_2C_2) \cdot (\overline{C_4}\,B_4C_4) * (C_3C_3) \cdot\\ &\cdot (\overline{C_4}\,B_4) * (C_4C_4) \cdot (\overline{C_4}\,\overline{C_4}\,\overline{C_3}\,\overline{C_2}) * (\overline{C_1}\,\overline{C_1}) \cdot (\overline{C_4}\,\overline{C_4}\,\overline{C_3}) * (\overline{C_2}\,\overline{C_2}) \cdot\\ &\cdot (\overline{C_4}\,\overline{C_4}) * (\overline{C_3}\,\overline{C_3}) \cdot \overline{C_4} * (\overline{C_4}\,\overline{C_4})\\ &= (\overline{C_4}\,B_4C_4 \cdot C_3 \cdot C_2) * (C_1C_1) \cdot (\overline{C_4}\,B_4C_4 \cdot C_3) * (C_2C_2) \cdot\\ &\cdot (\overline{C_4}\,B_4C_4) * (C_3C_3) \cdot (\overline{C_4}\,B_4C_4) * (C_4C_4) \cdot (\overline{C_4}\,\overline{C_4} \cdot \overline{C_3} \cdot \overline{C_2}) * (\overline{C_1}\,\overline{C_1}) \cdot\\ &\cdot (\overline{C_4}\,\overline{C_4} \cdot \overline{C_3}) * (\overline{C_2}\,\overline{C_2}) \cdot (\overline{C_4}\,\overline{C_4}) * (\overline{C_3}\,\overline{C_3}) \cdot (\overline{C_4}\,\overline{C_4})\end{aligned}$$

This equation shows that $[[1,1,1,0,1,0,\dots,0]] \in G_g$.

Since $\overline{C_4}\,\overline{C_3}\,\overline{C_6}\,\overline{C_5}\,\overline{C_4} * [[1,1,1,1,0,\dots,0]] = [[1,1,0,0,1,0,1,0,\dots,0]]$,

$$\begin{aligned}
&[[1,1,0,0,1,0,1,0,\dots,0]] = \overline{C_4}\,\overline{C_3}\,\overline{C_6}\,\overline{C_5}\,\overline{C_4} * (B_4\overline{B_4'})\\
&= (\overline{C_4}\,\overline{C_3}\,\overline{C_6}\,\overline{C_5}\,\overline{C_4}\,B_4C_4C_3C_2) * (C_1C_1)\cdot(\overline{C_4}\,\overline{C_3}\,\overline{C_6}\,\overline{C_5}\,\overline{C_4}\,B_4C_4C_3) * (C_2C_2)\cdot\\
&\cdot(\overline{C_4}\,\overline{C_3}\,\overline{C_6}\,\overline{C_5}\,\overline{C_4}\,B_4C_4) * (C_3C_3)\cdot(\overline{C_4}\,\overline{C_3}\,\overline{C_6}\,\overline{C_5}\,\overline{C_4}\,B_4) * (C_4C_4)\cdot\\
&\cdot(\overline{C_4}\,\overline{C_3}\,\overline{C_6}\,\overline{C_5}\,\overline{C_4}\,\overline{C_4}\,\overline{C_3}\,\overline{C_2}) * (\overline{C_1}\,\overline{C_1})\cdot(\overline{C_4}\,\overline{C_3}\,\overline{C_6}\,\overline{C_5}\,\overline{C_4}\,\overline{C_4}\,\overline{C_3}) * (\overline{C_2}\,\overline{C_2})\cdot\\
&\cdot(\overline{C_4}\,\overline{C_3}\,\overline{C_6}\,\overline{C_5}\,\overline{C_4}\,\overline{C_4}) * (\overline{C_3}\,\overline{C_3})\cdot(\overline{C_4}\,\overline{C_3}\,\overline{C_6}\,\overline{C_5}\,\overline{C_4}) * (\overline{C_4}\,\overline{C_4}).
\end{aligned}$$

This equation describes $[[1,1,0,0,1,0,1,0,\dots,0]]$ as a product of squares of Dehn twists. By using braid relations of $\mathcal{M}_g$, we show that these squares of Dehn twists are elements of $G_g$ as follows,

$$\begin{aligned}
&(\overline{C_4}\,\overline{C_3}\,\overline{C_6}\,\overline{C_5}\,\overline{C_4}\,B_4C_4C_3C_2) * (C_1C_1)\\
&\qquad = (\overline{C_1}\cdot\overline{C_4}\,B_4C_4\cdot\overline{C_2}\cdot C_3\cdot\overline{C_6}\,C_5C_6) * (C_4C_4),\\
&(\overline{C_4}\,\overline{C_3}\,\overline{C_6}\,\overline{C_5}\,\overline{C_4}\,B_4C_4C_3) * (C_2C_2)\\
&\qquad = (\overline{C_4}\,B_4C_4\cdot\overline{C_2}\cdot C_3\cdot\overline{C_6}\,C_5C_6) * (C_4C_4),\\
&(\overline{C_4}\,\overline{C_3}\,\overline{C_6}\,\overline{C_5}\,\overline{C_4}\,B_4C_4) * (C_3C_3) = (\overline{C_4}\,B_4C_4\cdot C_3\cdot\overline{C_4}\,\overline{C_4}\cdot\overline{C_6}) * (C_5C_5)\\
&\qquad = (\overline{C_4}\,B_4C_4\cdot C_3\cdot\overline{C_4}\,\overline{C_4}) * (\overline{C_6}\,C_5C_6\cdot\overline{C_6}\,C_5C_6),\\
&(\overline{C_4}\,\overline{C_3}\,\overline{C_6}\,\overline{C_5}\,\overline{C_4}\,B_4) * (C_4C_4)\\
&\qquad = (C_3\cdot\overline{C_4}\,B_4C_4\cdot\overline{C_6}\,C_5C_6\cdot\overline{C_4}\,\overline{C_4}) * (C_3C_3),\\
&(\overline{C_4}\,\overline{C_3}\,\overline{C_6}\,\overline{C_5}\,\overline{C_4}\,\overline{C_4}\,\overline{C_3}\,\overline{C_2}) * (\overline{C_1}\,\overline{C_1})\\
&\qquad = (C_1\cdot C_3\cdot C_2\cdot\overline{C_4}\,\overline{C_4}\cdot\overline{C_6}\,\overline{C_5}\,C_6\cdot\overline{C_4}\,\overline{C_4}) * (C_3C_3),\\
&(\overline{C_4}\,\overline{C_3}\,\overline{C_6}\,\overline{C_5}\,\overline{C_4}\,\overline{C_4}\,\overline{C_3}) * (\overline{C_2}\,\overline{C_2})\\
&\qquad = (C_3\cdot C_2\cdot\overline{C_4}\,\overline{C_4}\cdot\overline{C_6}\,\overline{C_5}\,C_6\cdot\overline{C_4}\,\overline{C_4}) * (C_3C_3),\\
&(\overline{C_4}\,\overline{C_3}\,\overline{C_6}\,\overline{C_5}\,\overline{C_4}\,\overline{C_4}) * (\overline{C_3}\,\overline{C_3}) = (C_3\cdot\overline{C_6}\,C_5C_6\cdot\overline{C_6}\,C_5C_6) * (C_4C_4),\\
&(\overline{C_4}\,\overline{C_3}\,\overline{C_6}\,\overline{C_5}\,\overline{C_4}) * (\overline{C_4}\,\overline{C_4}) = (C_3\cdot\overline{C_4}\,C_5C_4) * (C_6C_6).
\end{aligned}$$

□

**Lemma 6.12** *The odd subchain map* $[[1,1,1,0,1,0,1,0,1,0,\dots,0]]$ *is an element of* $G_g$.

**Proof** We can show that this odd subchain map is $G_g$-equivalent to $[[0,0,0,0,1,1,1,1,1,1,0,\dots,0]]$ as follows,

$$[[1,1,1,0,1,0,1,0,1,0,\dots,0]] \underset{\overline{C_3}}{\longrightarrow} [[1,1,0,1,1,0,1,0,1,0,\dots,0]]$$

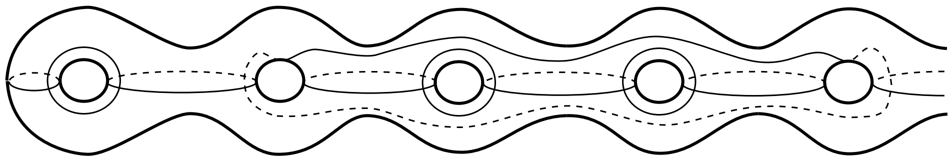


Figure 14

$$\xrightarrow[\text{Lemma 6.8(2)}]{} [[1,1,0,0,1,1,1,0,1,0,\dots,0]]$$
$$\xrightarrow[\overline{C_3}\,\overline{C_2}]{} [[1,0,0,1,1,1,1,0,1,0,\dots,0]]$$
$$\xrightarrow[\text{Lemma 6.8(2)}]{} [[1,0,0,0,1,1,1,1,1,0,\dots,0]]$$
$$\xrightarrow[\overline{C_3}\,\overline{C_2}\,\overline{C_1}]{} [[0,0,0,1,1,1,1,1,1,0,\dots,0]]$$
$$\xrightarrow[\text{Lemma 6.8(2)}]{} [[0,0,0,0,1,1,1,1,1,1,0,\dots,0]].$$

If $g=4$, $[[0,0,0,0,1,1,1,1,1,1]] = B_4\overline{B_4'} = [[1,1,1,1,0,0,0,0,0,0]]$, which we have already treated in Lemma 6.11. If $g \geq 5$, as we see in Figure 14,

$$[[0,0,0,0,1,1,1,1,1,1,0,\dots,0]] = [[1,1,1,1,0,0,0,0,0,0,1,\dots,1]],$$

in the notation of the last odd subchain map, $\dots$ is a sequence of 1-tacks. By Lemma 6.8 (2),

$$[[1,1,1,1,0,0,0,0,0,0,1,\dots,1]] \underset{G_g}{\sim} [[1,1,1,1,1,\dots,1,0,0,0,0,0,0]],$$

which is a product of elements of $G_g$ and shorter odd subchain maps. □

Therefore, Lemma 6.2 is proved.

## 6.2 Step 2 for the case where $g \geq 3$

Let $\Phi_2$ be the natural homomorphism from $\mathcal{M}_g$ to $\mathrm{Sp}(2g,\mathbb{Z}_2)$ defined by the action of $\mathcal{M}_g$ on the $\mathbb{Z}_2$-coefficient first homology group $H_1(\Sigma_g;\mathbb{Z}_2)$. In this section, we will show the following lemma.

**Lemma 6.13** *$\ker\Phi_2$ is a subgroup of $G_g$.*

We denote the kernel of the natural homomorphism from $\mathrm{Sp}(2g,\mathbb{Z})$ to $\mathrm{Sp}(2g,\mathbb{Z}_2)$ by $\mathrm{Sp}^{(2)}(2g)$. We set a basis of $H_1(\Sigma_g;\mathbb{Z})$ as in Figure 9, and define the intersection form $(,)$ on $H_1(\Sigma_g;\mathbb{Z})$ to satisfy $(x_i, y_j) = \delta_{i,j}$, $(x_i, x_j) = (y_i, y_j) = 0$ $(1 \leq i, j, \leq g)$. An element $a$ of $H_1(\Sigma_g;\mathbb{Z})$ is called *primitive* if there is no element $n(\neq 0, \pm 1)$ of $\mathbb{Z}$, and no element $b$ of $H_1(\Sigma_g;\mathbb{Z})$ such that $a = nb$. For a primitive element $a$ of $H_1(\Sigma_g;\mathbb{Z})$, we define an isomorphism $T_a : H_1(\Sigma_g;\mathbb{Z}) \to H_1(\Sigma_g;\mathbb{Z})$ by $T_a(v) = v + (a, v)a$. This isomorphism is the action of Dehn twist about a simple closed curve representing $a$ on $H_1(\Sigma_g;\mathbb{Z})$. We call $T_a^2$ *the square transvection about* $a$. Johnson [15] showed the following result.

**Lemma 6.14** $\mathrm{Sp}^{(2)}(2g)$ *is generated by square transvections.*

In [11], we showed,

**Lemma 6.15** $\mathrm{Sp}^{(2)}(2g)$ *is generated by the square transvections about the primitive elements* $\sum_{i=1}^{g}(\epsilon_i x_i + \delta_i y_i)$, *where* $\epsilon_i = 0, 1$ *and* $\delta_i = 0, 1$.

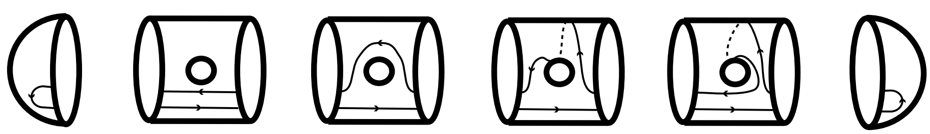



Figure 15

For each element $[(\epsilon_1, \delta_1), \cdots, (\epsilon_g, \delta_g)] = \sum_{i=1}^{g}(\epsilon_i x_i + \delta_i y_i)$ (where $\epsilon_i = 0, 1$, $\delta_i = 0, 1$) of $H_1(\Sigma_g;\mathbb{Z})$, we construct an oriented simple closed curve on $\Sigma_g$ which represent this homology class. For each $i$-th block, if $(\epsilon_i, \delta_i) = (0, 0)$, we prepare (0) of Figure 15, if $(\epsilon_i, \delta_i) = (0, 1)$, we prepare (1) of Figure 15, if $(\epsilon_i, \delta_i) = (1, 1)$, we prepare (2) of Figure 15, if $(\epsilon_i, \delta_i) = (1, 0)$, we prepare (3) of Figure 15. After that, we glue them along the boundaries and cap the left boundary component by (-) of Figure 15 and the right boundary component by (+) of Figure 15. We denote this oriented simple closed curve on $\Sigma_g$ by $\{(\epsilon_1, \delta_1), \cdots, (\epsilon_g, \delta_g)\}$. Here, we remark that the action of $T_{\{(\epsilon_1,\delta_1),\cdots,(\epsilon_g,\delta_g)\}}$ on $H_1(\Sigma_g;\mathbb{Z})$ equals $T_{[(\epsilon_1,\delta_1),\cdots,(\epsilon_g,\delta_g)]}$, and, for any $\phi$ of $\mathcal{M}_g$, $\phi \circ T_{\{(\epsilon_1,\delta_1),\cdots,(\epsilon_g,\delta_g)\}} \circ \phi^{-1} = T_{\phi(\{(\epsilon_1,\delta_1),\cdots,(\epsilon_g,\delta_g)\})}$.

**Lemma 6.16** *For any* $\{(\epsilon_1, \delta_1), \cdots, (\epsilon_g, \delta_g)\}$, *there is an element* $\phi$ *of* $G_g$ *such* *that*

$$
\begin{aligned}
\phi(\{(\epsilon_1,\delta_1),\cdots,(\epsilon_g,\delta_g)\}) &= \{(0,0),(0,1),(0,0),(0,0),\cdots,(0,0)\} \\
or\ &= \{(0,0),(1,1),(0,0),(0,0),\cdots,(0,0)\} \\
or\ &= \{(0,0),(0,0),(1,1),(0,0),\cdots,(0,0)\} \\
or\ &= \{(0,1),(0,0),(0,0),\cdots,(0,0)\} \\
or\ &= \{(1,1),(0,0),(0,0),\cdots,(0,0)\} \\
or\ &= \{(0,0),(0,0),(0,0),\cdots,(0,0)\}.
\end{aligned}
$$

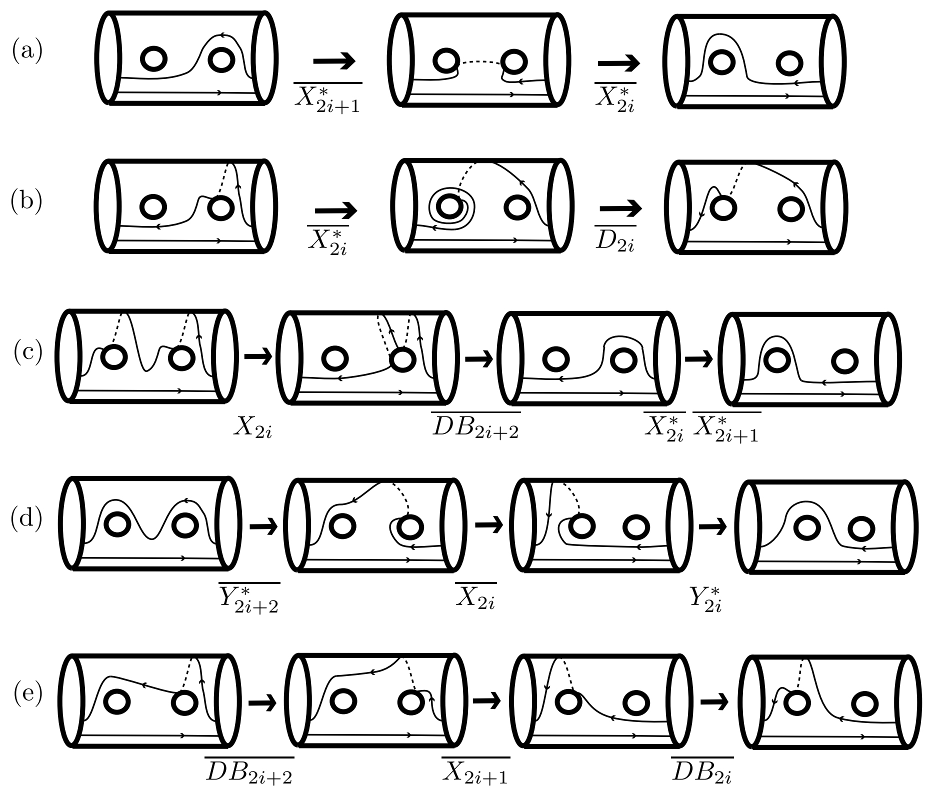



Figure 16

**Proof** If the $i$-th block is (3), by the action of $\overline{Y_{2i}}$ if $2 \leq i \leq g-1$, $C_2\overline{C_1}\,\overline{C_2}$ if $i=1$, and $C_{2g}\overline{C_{2g+1}}\,\overline{C_{2g}}$ if $i=g$, this block is changed to (1). Therefore, it suffices to show this lemma in the case where each block is not (3). First we investigate actions of elements of $G_g$ on adjacent blocks, say the $i$-th block and the $i+1$-st block, where $i \geq 2$. Each picture of Figure 16 shows the action of $G_g$ on this adjacent blocks.

$$(a) \text{ shows } \{\bullet\bullet\bullet, (0,0), (0,1), \bullet\bullet\bullet\} \underset{G_g}{\sim} \{\bullet\bullet\bullet, (0,1), (0,0), \bullet\bullet\bullet\},$$

$$(b) \text{ shows } \{\bullet\bullet\bullet, (0,0), (1,1), \bullet\bullet\bullet\} \underset{G_g}{\sim} \{\bullet\bullet\bullet, (1,1), (0,1), \bullet\bullet\bullet\},$$

$$(c) \text{ shows } \{\bullet\bullet\bullet, (1,1), (1,1), \bullet\bullet\bullet\} \underset{G_g}{\sim} \{\bullet\bullet\bullet, (0,1), (0,0), \bullet\bullet\bullet\},$$

$$(d) \text{ shows } \{\bullet\bullet\bullet, (0,1), (0,1), \bullet\bullet\bullet\} \underset{G_g}{\sim} \{\bullet\bullet\bullet, (0,1), (0,0), \bullet\bullet\bullet\},$$

$$(e) \text{ shows } \{\bullet\bullet\bullet, (0,1), (1,1), \bullet\bullet\bullet\} \underset{G_g}{\sim} \{\bullet\bullet\bullet, (1,1), (0,0), \bullet\bullet\bullet\},$$

where $\bullet\bullet\bullet$ indicates the part which is not changed by the action of $G_g$. Let $x = \{(\epsilon_1, \delta_1), \cdots, (\epsilon_g, \delta_g)\}$, each of whose block is $(0,0)$ or $(0,1)$ or $(1,1)$. If there are the $j$-th blocks $(1,1)$ $(j \geq 2)$, by (b) and (e), they are gathered to a sequence of $(1,1)$ blocks which begins from the second block. If there are the $j$-th blocks $(0,1)$ $(j \geq 2)$, by (a), they are gathered to a sequence of $(0,1)$ blocks subsequent to the previous sequence of $(1,1)$ blocks. Hence, we showed,

$$x \underset{G_g}{\sim} \{(\epsilon_1, \delta_1), (1,1), \cdots, (1,1), (0,1), \cdots, (0,1), (0,0), \cdots, (0,0)\}.$$

By (a) and (d), the sequence of $(0,1)$ blocks is altered to $(0,1), (0,0), \cdots, (0,0)$ or $(0,0), \cdots, (0,0)$. By (c), the sequence of $(1,1)$ blocks is altered to $(1,1)$, $(0,1), (0,0), \cdots, (0,1), (0,0)$ (when the length of the sequence is odd) or to $(0,1), (0,0), \cdots, (0,1), (0,0)$ (when the length of the sequence is even). By (a) and (d), $(1,1), (0,1), (0,0), \cdots, (0,1), (0,0)$ is altered to $(1,1), (0,1), (0,0), \cdots,$ $(0,0), (0,0)$ or $(1,1), (0,0), (0,0), \cdots, (0,0), (0,0)$, and $(0,1), (0,0), \cdots, (0,1),$ $(0,0)$ to $(0,1), (0,0), \cdots, (0,0), (0,0)$ or $(0,0), (0,0), \cdots, (0,0), (0,0)$. Therefore, we showed,

$$\begin{aligned}
x &\underset{G_g}{\sim} \{(\epsilon_1, \delta_1), (1,1), (0,0), (0,0), \cdots, (0,0), (0,0), (0,0), \cdots, (0,0)\}, \\
\text{or } &\underset{G_g}{\sim} \{(\epsilon_1, \delta_1), (1,1), (0,0), (0,0), \cdots, (0,0), (0,1), (0,0), \cdots, (0,0)\}, \\
\text{or } &\underset{G_g}{\sim} \{(\epsilon_1, \delta_1), (1,1), (0,1), (0,0), \cdots, (0,0), (0,0), (0,0), \cdots, (0,0)\}, \\
\text{or } &\underset{G_g}{\sim} \{(\epsilon_1, \delta_1), (1,1), (0,1), (0,0), \cdots, (0,0), (0,1), (0,0), \cdots, (0,0)\}, \\
\text{or } &\underset{G_g}{\sim} \{(\epsilon_1, \delta_1), (0,1), (0,0), (0,0), \cdots, (0,0), (0,0), (0,0), \cdots, (0,0)\}, \\
\text{or } &\underset{G_g}{\sim} \{(\epsilon_1, \delta_1), (0,1), (0,0), (0,0), \cdots, (0,0), (0,1), (0,0), \cdots, (0,0)\}, \\
\text{or } &\underset{G_g}{\sim} \{(\epsilon_1, \delta_1), (0,0), (0,0), (0,0), \cdots, (0,0), (0,0), (0,0), \cdots, (0,0)\},
\end{aligned}$$

$$\text{or} \underset{G_g}{\sim} \{(\epsilon_1,\delta_1),(0,0),(0,0),(0,0),\cdots,(0,0),(0,1),(0,0),\cdots,(0,0)\}.$$

In the second case,

$$\begin{aligned}&\{(\epsilon_1,\delta_1),(1,1),(0,0),\cdots,(0,0),(0,1),(0,0),\cdots,(0,0)\}\\ &\underset{G_g}{\sim} \{(\epsilon_1,\delta_1),(1,1),(0,1),(0,0),\cdots,(0,0)\}(\text{ by } (a)).\end{aligned}$$

In the 4-th case,

$$\begin{aligned}&\{(\epsilon_1,\delta_1),(1,1),(0,1),(0,0),\cdots,(0,0),(0,1),(0,0),\cdots,(0,0)\}\\ &\underset{G_g}{\sim} \{(\epsilon_1,\delta_1),(1,1),(0,1),(0,1),(0,0),\cdots,(0,0)\}(\text{ by } (a))\\ &\underset{G_g}{\sim} \{(\epsilon_1,\delta_1),(1,1),(0,1),(0,0),(0,0),\cdots,(0,0)\}(\text{ by } (d)).\end{aligned}$$

In the 6-th case,

$$\begin{aligned}&\{(\epsilon_1,\delta_1),(0,1),(0,0),\cdots,(0,0),(0,1),(0,0),\cdots,(0,0)\}\\ &\underset{G_g}{\sim} \{(\epsilon_1,\delta_1),(0,1),(0,1),(0,0),\cdots,(0,0)\}(\text{ by } (a))\\ &\underset{G_g}{\sim} \{(\epsilon_1,\delta_1),(0,1),(0,0),(0,0),\cdots,(0,0)\}(\text{ by } (d)).\end{aligned}$$

In the 8-th case,

$$\begin{aligned}&\{(\epsilon_1,\delta_1),(0,0),\cdots,(0,0),(0,1),(0,0),\cdots,(0,0)\}\\ &\underset{G_g}{\sim} \{(\epsilon_1,\delta_1),(0,1),(0,0),\cdots,(0,0)\}(\text{ by } (a)).\end{aligned}$$

Therefore,

$$\begin{aligned}x &\underset{G_g}{\sim} \{(\epsilon_1,\delta_1),(1,1),(0,0),(0,0),\cdots,(0,0)\},\\ &\text{or} \underset{G_g}{\sim} \{(\epsilon_1,\delta_1),(1,1),(0,1),(0,0),\cdots,(0,0)\},\\ &\text{or} \underset{G_g}{\sim} \{(\epsilon_1,\delta_1),(0,1),(0,0),(0,0),\cdots,(0,0)\},\\ &\text{or} \underset{G_g}{\sim} \{(\epsilon_1,\delta_1),(0,0),(0,0),(0,0),\cdots,(0,0)\}.\end{aligned}$$

There are 7 cases remained to consider,

$$\begin{aligned}&\{(0,0),(1,1),(0,1),(0,0),\cdots,(0,0)\}, \quad \{(0,1),(1,1),(0,0),(0,0),\cdots,(0,0)\},\\ &\{(0,1),(1,1),(0,1),(0,0),\cdots,(0,0)\}, \quad \{(0,1),(0,1),(0,0),(0,0),\cdots,(0,0)\},\\ &\{(1,1),(1,1),(0,0),(0,0),\cdots,(0,0)\}, \quad \{(1,1),(1,1),(0,1),(0,0),\cdots,(0,0)\},\\ &\{(1,1),(0,1),(0,0),(0,0),\cdots,(0,0)\}.\end{aligned}$$

By (b), the first one is $G_g$-equivalent to $\{(0,0),(0,0),(1,1),(0,0),\cdots,(0,0)\}$. Here, we observe actions of $G_g$ on the first and the second blocks,

$$\{(0,1),(1,1),\cdots\} \underset{C_1}{\longrightarrow} \{(1,1),(1,1),\cdots\} \underset{\overline{DB_4}\cdot C_3\cdot C_2}{\longrightarrow} \{(0,0),(0,1),\cdots\},$$
$$\{(0,1),(0,1),\cdots\} \underset{C_1}{\longrightarrow} \{(1,1),(0,1),\cdots\} \underset{C_3C_2}{\longrightarrow} \{(0,0),(1,1),\cdots\}.$$

By the above observation, we see,

$$\begin{aligned}
\{(0,1),(1,1),(0,0),\cdots,(0,0)\} &\underset{G_g}{\sim} \{(1,1),(1,1),(0,0),\cdots,(0,0)\}\\
&\underset{G_g}{\sim} \{(0,0),(0,1),(0,0),\cdots,(0,0)\},\\
\{(0,1),(1,1),(0,1),\cdots,(0,0)\} &\underset{G_g}{\sim} \{(1,1),(1,1),(0,1),\cdots,(0,0)\}\\
&\underset{G_g}{\sim} \{(0,0),(0,1),(0,1),\cdots,(0,0)\}\\
&\underset{G_g}{\sim} \{(0,0),(0,1),(0,0),\cdots,(0,0)\},\\
\{(0,1),(0,1),(0,0),\cdots,(0,0)\} &\underset{G_g}{\sim} \{(1,1),(0,1),(0,0),\cdots,(0,0)\}\\
&\underset{G_g}{\sim} \{(0,0),(1,1),(0,0),\cdots,(0,0)\}.
\end{aligned}$$

Hence, we showed that any $x$ is $G_g$-equivalent to the elements listed in the statement of this Lemma. □

Since

$$\begin{aligned}
&T^2_{\{(0,1),(0,0),\cdots,(0,0)\}} = D_2, \quad T^2_{\{(1,1),(0,0),\cdots,(0,0)\}} = (C_1C_2C_1^{-1})^2,\\
&T^2_{\{(0,0),(1,1),(0,0),\cdots,(0,0)\}} = (Y_2^*)^2, \quad T^2_{\{(0,0),(0,1),(0,0),\cdots,(0,0)\}} = D_4,\\
&T^2_{\{(0,0),(0,0),(1,1),(0,0),\cdots,(0,0)\}} = (Y_4^*)^2, \quad T^2_{\{(0,0),\cdots,(0,0)\}} = id,
\end{aligned}$$

these are elements of $G_g$. By this fact and Lemma 6.2, Lemma 6.13 is proved.

## 6.3 Step 3 for the case where $g \geq 3$

As in the previous subsection, let $\Phi_2\colon \mathcal{M}_g \to \mathrm{Sp}(2g,\mathbb{Z}_2)$ be the natural homomorphism. Let $q_1\colon H_1(\Sigma_g;\mathbb{Z}_2) \to \mathbb{Z}_2$ be the quadratic form associated with the intersection form $(,)_2$ of $H_1(\Sigma_g;\mathbb{Z}_2)$ which satisfies, for the basis $x_i, y_i$ of $H_1(\Sigma_g;\mathbb{Z}_2)$ indicated on Figure 9, $q_1(x_1) = q_1(y_1) = 1$, and $q_1(x_i) = q_1(y_i) = 0$ when $i \neq 1$. We define $\mathrm{O}_{q_1}(2g,\mathbb{Z}_2) = \{\phi \in \mathrm{Aut}(H_1(\Sigma_g;\mathbb{Z}_2)) | q_1(\phi(x)) = q_1(x)$ for any $x \in H_1(\Sigma_g;\mathbb{Z}_2)\}$, then $\mathcal{SP}_g[q_1] = \Phi_2^{-1}(\mathrm{O}_{q_1}(2g,\mathbb{Z}_2))$. Because of Lemma 6.13, if we show $\Phi_2(G_g) = \mathrm{O}_{q_1}(2g,\mathbb{Z}_2)$, then $G_g = \mathcal{SP}_g[q_1]$ follows.

For any $z \in H_1(\Sigma_g;\mathbb{Z}_2)$ such that $q_1(z)=1$, we define $\mathbb{T}_z(x) = x + (z,x)_2\, z$. Then $\mathbb{T}_z$ is an element of $\mathrm{O}_{q_1}(2g,\mathbb{Z}_2)$, and we call this a $\mathbb{Z}_2$*-transvection about* $z$. Dieudonné [4] showed the following Theorem (see also [7, Chap.14]).

**Theorem 6.17** [4, Proposition 14 on p.42] *When $g \geq 3$, $\mathrm{O}_{q_1}(2g,\mathbb{Z}_2)$ is generated by $\mathbb{Z}_2$-transvections.*

Let $\Lambda_g$ be the set of $z$ of $H_1(\Sigma_g;\mathbb{Z}_2)$ such that $q(z)=1$. For any elements $z_1$ and $z_2$ of $\Lambda_g$, we define $z_1 \Box z_2 = z_1 + (z_2,z_1)_2\, z_2$. Here, we remark that $\mathbb{T}_{z_1}^2 = \mathrm{id}$, $\mathbb{T}_{z_2}\mathbb{T}_{z_1}\mathbb{T}_{z_2}^{-1} = \mathbb{T}_{z_1 \Box z_2}$ and $(z_1 \Box z_2)\Box z_2 = z_1$. An element $\epsilon_1 x_1 + \delta_1 y_1 + \cdots + \epsilon_g x_g + \delta_g y_g$ of $H_1(\Sigma_g;\mathbb{Z}_2)$ is denoted by $[(\epsilon_1,\delta_1),\cdots,(\epsilon_g,\delta_g)]$, and each $(\epsilon_i,\delta_i)$ is called the $i$-th block. We remark that $q([(\epsilon_1,\delta_1),\cdots,(\epsilon_g,\delta_g)]) = (\epsilon_1+\delta_1+\epsilon_1\delta_1) + \epsilon_2\delta_2 + \cdots \epsilon_g\delta_g$.

**Lemma 6.18** *Under the operation $\Box$, $\Lambda_g$ is generated by $x_1$, $y_1$, $x_1+x_2$, $x_i+y_i$ $(2 \leq i \leq g)$, $x_i+y_i+x_{i+1}$ $(2 \leq i \leq g-1)$, and $x_i+x_{i+1}+y_{i+1}$ $(2 \leq i \leq g-1)$.*

**Proof** For an element $[(\epsilon_1,\delta_1),\cdots,(\epsilon_g,\delta_g)]$ of $H_1(\Sigma_g;\mathbb{Z}_2)$, let the $j$-th block be the right most block which is $(1,1)$. When $j \geq 3$, there exist 4 cases of the combination of the $(j-1)$-st block and the $j$-th block: $[\cdots,(1,1),(1,1),\cdots]$, $[\cdots,(0,0),(1,1),\cdots]$, $[\cdots,(0,1),(1,1),\cdots]$, $[\cdots,(1,0),(1,1),\cdots]$. In each case, we can reduce $j$ at least 1. In fact,

$$\begin{aligned}
&[\cdots,(1,1),(1,1),\cdots]\Box(x_{j-1}+x_j+y_j) = [\cdots,(0,1),(0,0),\cdots],\\
&[\cdots,(0,0),(1,1),\cdots]\Box(x_{j-1}+y_{j-1}+x_j) = [\cdots,(1,1),(0,1),\cdots],\\
&[\cdots,(0,1),(1,1),\cdots]\Box(x_{j-1}+x_j+y_j) = [\cdots,(1,1),(0,0),\cdots],\\
&([\cdots,(1,0),(1,1),\cdots]\Box(x_{j-1}+y_{j-1}))\Box(x_{j-1}+x_j+y_j)\\
&\qquad\qquad = [\cdots,(1,1),(0,0),\cdots].
\end{aligned}$$

When $j=2$, since $q([(\epsilon_1,\delta_1),\cdots,(\epsilon_g,\delta_g)])=1$, $[(\epsilon_1,\delta_1),\cdots,(\epsilon_g,\delta_g)]$ must be $[(0,0),(1,1),\cdots]$. Because of an equation

$$([(0,0),(1,1),\cdots]\Box(x_1+x_2))\Box y_1 = [(1,1),(0,1),\cdots],$$

we can reduce $j$ to 1. When $j=1$, if every $i$-th $(i \geq 2)$ block is $(0,0)$, then it is $x_1+y_1$, which is equal to $x_1 \Box y_1$. If there exist at least one of the $i$-th

$(i \geq 2)$ blocks which are $(1,0)$ or $(0,1)$, then,

$$[\cdots, (0,0), \overset{i}{(1,0)}, \cdots]\square(x_{i-1}+x_i+y_i) = [\cdots, (1,0), (0,1), \cdots],$$
$$[\cdots, (1,0), \overset{i}{(0,0)}, \cdots]\square(x_{i-1}+y_{i-1}+x_i) = [\cdots, (0,1), (1,0), \cdots],$$
$$[\cdots, (0,0), \overset{i}{(0,1)}, \cdots]\square(x_{i-1}+x_i+y_i) = [\cdots, (1,0), (1,0), \cdots],$$
$$[\cdots, (0,1), \overset{i}{(0,0)}, \cdots]\square(x_{i-1}+y_{i-1}+x_i) = [\cdots, (1,0), (1,0), \cdots].$$

Therefore, we can alter this to an element, each $i$-th $(i \geq 2)$ block of which is $(1,0)$ or $(0,1)$. If the $i$-th block of this is $(0,1)$, then

$$[\cdots, (0,1), \cdots]\square(x_i+y_i) = [\cdots, (1,0), \cdots].$$

Therefore, it suffices to consider the case where the first block is $(1,1)$ and other blocks are $(1,0)$. In this case,

$$([\cdots, (1,0), (1,0)]\square(x_{g-1}+y_{g-1}+x_g))\square(x_{g-1}+y_{g-1}) = [\cdots, (1,0), (0,0)].$$

By applying the same operation repeatedly, we get $[(1,1), (1,0), (0,0), \cdots, (0,0)]$, which is equal to $y_1\square(x_1+x_2)$. □

This lemma and Theorem 6.17 shows that

**Corollary 6.19** $\mathrm{O}_{q_1}(2g, \mathbb{Z}_2)$ *is generated by* $\mathbb{T}_{x_1}$, $\mathbb{T}_{y_1}$, $\mathbb{T}_{x_1+x_2}$, $\mathbb{T}_{x_i+y_i}$ $(2 \leq i \leq g)$, $\mathbb{T}_{x_i+y_i+x_{i+1}}$ $(2 \leq i \leq g-1)$, *and* $\mathbb{T}_{x_i+x_{i+1}+y_{i+1}}$ $(2 \leq i \leq g-1)$. □

Since $G_g$ is a subgroup of $\mathcal{SP}_g[q_1]$, $\Phi_2(G_g) \subset \mathrm{O}_{q_1}(2g, \mathbb{Z}_2)$. On the other hand, the fact that $\Phi_2(C_1) = \mathbb{T}_{x_1}$, $\Phi_2(C_2) = \mathbb{T}_{y_1}$, $\Phi_2(C_3) = \mathbb{T}_{x_1+x_2}$, $\Phi_2(X_{2i}) = \mathbb{T}_{x_i+y_i+x_{i+1}}$ $(2 \leq i \leq g-1)$, $\Phi_2(X_{2i+1}) = \mathbb{T}_{x_i+x_{i+1}+y_{i+1}}$ $(2 \leq i \leq g-1)$, $\Phi_2(Y_{2j}) = \mathbb{T}_{x_j+y_j}$ $(2 \leq j \leq g-1)$, $\Phi_2(X_{2g}) = \mathbb{T}_{x_g+y_g}$, and Corollary 6.19, show $\Phi_2(G_g) \supset \mathrm{O}_{q_1}(2g, \mathbb{Z}_2)$. Therefore we proved that $\mathcal{SP}_g[q_1] = G_g$ when $g \geq 3$.

## 6.4 Genus 2 case: Reidemeister-Schreier method

Birman and Hilden showed the following Theorem.

**Theorem 6.20** [2] *$\mathcal{M}_2$ is generated by $C_1, C_2, C_3, C_4, C_5$ and its defining relations are:*

*(1)* $C_iC_j = C_jC_i$, *if* $|i-j| \geq 2$, $i, j = 1, 2, 3, 4, 5$,

*(2)* $C_iC_{i+1}C_i = C_{i+1}C_iC_{i+1}$, $i = 1, 2, 3, 4$,

*(3)* $(C_1C_2C_3C_4C_5)^6 = 1$,

*(4)* $(C_1C_2C_3C_4C_5C_5C_4C_3C_2C_1)^2 = 1$,

*(5)* $C_1C_2C_3C_4C_5C_5C_4C_3C_2C_1 \rightleftarrows C_i$, $i = 1, 2, 3, 4, 5$,

*where* $\rightleftarrows$ *means "commute with".*

We call (1) (2) of the above relations *braid relations*. We will use the well-known method, called *the Reidemeister–Schreier method* [18, §2.3], to show $\mathcal{SP}_2[q_1] \subset G_2$. We review (a part of) this method.

Let $G$ be a group generated by finite elements $g_1, \ldots, g_m$ and $H$ be a finite index subgroup of $G$. For two elements $a$, $b$ of $G$, we write $a \equiv b \mod H$ if there is an element $h$ of $H$ such that $a = hb$. A finite subset $S$ of $G$ is called *a coset representative system* for $G$ mod $H$, if, for each elements $g$ of $G$, there is only one element $\overline{\overline{g}} \in S$ such that $g \equiv \overline{\overline{g}} \mod H$. The set $\{s g_i \overline{\overline{s g_i}}^{-1} \mid i = 1, \ldots, m,\ s \in S\}$ generates $H$.

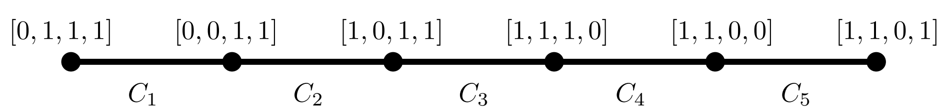



Figure 17

For the sake of giving a coset representative system for $\mathcal{M}_2$ modulo $\mathcal{SP}_2[q_1]$, we will draw a graph $\Gamma$ which represents the action of $\mathcal{M}_2$ on the quadratic forms of $H_1(\Sigma_2; \mathbb{Z}_2)$ with Arf invariants 1. Let $[\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4]$ denote the quadratic form $q'$ of $H_1(\Sigma_2; \mathbb{Z}_2)$ such that $q'(x_1) = \epsilon_1$, $q'(y_1) = \epsilon_2$, $q'(x_2) = \epsilon_3$, $q'(y_2) = \epsilon_4$. Each vertex of $\Gamma$ corresponds to a quadratic form. For each generator $C_i$ of $\mathcal{M}_2$, we denote its action on $H_1(\Sigma_2; \mathbb{Z}_2)$ by $(C_i)_*$. For the quadratic form $q'$ indicated by the symbol $[\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4]$, let $\delta_1 = q'((C_i)_* x_1)$, $\delta_2 = q'((C_i)_* y_1)$, $\delta_3 = q'((C_i)_* x_2)$, and $\delta_4 = q'((C_i)_* y_2)$. Then, we connect two vertices, corresponding to $[\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4]$, $[\delta_1, \delta_2, \delta_3, \delta_4]$ respectively, by the edge with the letter $C_i$. We remark that this action is a right action. For simplicity, we omit the edge whose ends are the same vertex. As a result, we get a graph $\Gamma$ as in Figure 17. The words $S = \{1,\ C_5,\ C_4,\ C_4C_3,\ C_4C_3C_2,\ C_4C_3C_2C_1\}$, which correspond to the edge paths beginning from $[1, 1, 0, 0]$ on $\Gamma$, define a coset representative system for $\mathcal{M}_2$ modulo $\mathcal{SP}_2[q_1]$. For each element $g$ of $\mathcal{M}_2$, we can give a $\overline{\overline{g}} \in S$ with using this graph. For example, say $g = C_2C_4C_5C_3$, we follow an edge path assigned to this word which begins from $[1, 1, 0, 0]$, (note that we read words from left to right) then we arrive at the vertex

Table 1: Generators of $\mathcal{SP}_2[q_1]$

| | $C_1$ | $C_2$ | $C_3$ |
|---|---|---|---|
| $1$ | $1$ | $1$ | $1$ |
| $C_5$ | $C_1$ | $C_2$ | $C_3$ |
| $C_4$ | $C_1$ | $C_2$ | $1$ |
| $C_4C_3$ | $C_1$ | $1$ | $C_3^{-1}D_4C_3$ |
| $C_4C_3C_2$ | $1$ | $C_2^{-1}C_3^{-1}D_4C_3C_2$ | $C_2$ |
| $C_4C_3C_2C_1$ | $C_1^{-1}C_2^{-1}C_3^{-1}D_4C_3C_2C_1$ | $C_1$ | $C_2$ |

| | $C_4$ | $C_5$ |
|---|---|---|
| $1$ | $1$ | $1$ |
| $C_5$ | $1$ | $D_5$ |
| $C_4$ | $D_4$ | $D_5^{-1}X_4D_5$ |
| $C_4C_3$ | $C_3$ | $D_5^{-1}X_4D_5$ |
| $C_4C_3C_2$ | $C_3$ | $D_5^{-1}X_4D_5$ |
| $C_4C_3C_2C_1$ | $C_3$ | $D_5^{-1}X_4D_5$ |

$[1,0,1,1]$. The element in $T$ which begins from $[1,1,0,0]$ and ends at $[1,0,1,1]$ is $C_4C_3$. Hence, $\overline{\overline{C_2C_4C_5C_3}} = C_4C_3$. We list in Table 1 the set of generators $\{sC_i\overline{\overline{sC_i}}^{-1} \mid i = 1,\ldots,5,\ s \in S\}$ of $\mathcal{SP}_g[q_1]$. In Table 1, vertical direction is a coset representative system $S$, horizontal direction is a set of generators $\{C_1,\ C_2,\ C_3,\ C_4,\ C_5\}$. We can check this table by Figure 17 and braid relations. For example,

$$\begin{aligned}
&C_4C_3C_2C_1 \cdot C_2\overline{\overline{C_4C_3C_2C_1 \cdot C_2}}^{-1} = C_4C_3C_2C_1C_2(C_4C_3C_2C_1)^{-1} \\
&= C_4C_3C_2C_1C_2C_1^{-1}C_2^{-1}C_3^{-1}C_4^{-1} = C_4C_3C_2C_2^{-1}C_1C_2C_2^{-1}C_3^{-1}C_4^{-1} \\
&= C_4C_3C_1C_3^{-1}C_4^{-1} = C_1.
\end{aligned}$$

This table shows that $\mathcal{SP}_2[q_1] \subset G_2$ .

# 7 Proof of Theorem 5.1

We embed $H_{g-1}$ standardly in $S^3 = \partial D_4$ such that there is a 2-sphere separating $F_{3,3}$ and $H_{g-1}$, and make a connected sum $F_{3,3}\#\partial H_{g-1}$ as indicated in Figure 18. Then, we can see $(\mathbb{CP}^2, K_3\#\Sigma_{g-1}) = (\mathbb{CP}^2, (F_{3,3}\#\partial H_{g-1})\cup D_3)$, where $K_3$ is the non-singular plane curve of degree 3 and $D_3$ is parallel three disks which is used to construct $K_3$ in §4. We identify $K_3\#\Sigma_{g-1}$ with $\Sigma_g$ so that

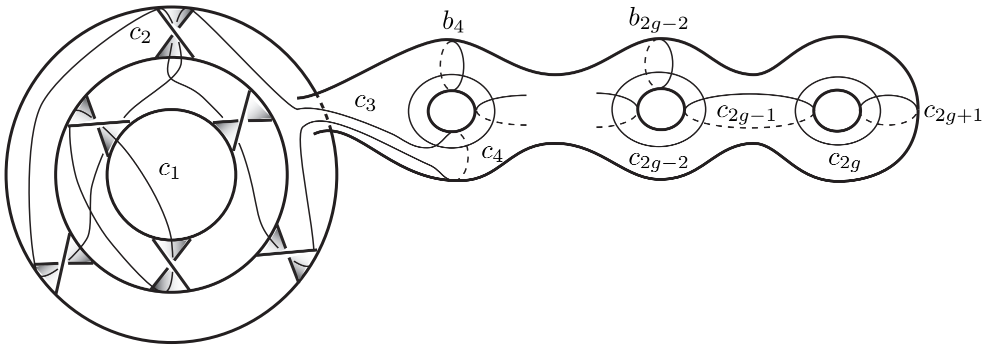



Figure 18

simple closed curves with the same symbol are identified. Then $q_{K_3\#\Sigma_{g-1}} = q_1$. We will show that each elements of $\mathcal{SP}_g[q_{K_3\#\Sigma_{g-1}}] = \mathcal{SP}_g[q_1]$ is extendable.

Each regular neighborhood of $c_1$, $c_2$, $c_3$, $C_{i+1}(c_i)$ $(4 \leq i \leq 2g)$, and $C_{2j}(b_{2j})$ $(2 \leq j \leq g-1)$ is Hopf band. Therefore, by Proposition 2.1, $C_1$, $C_2$, $C_3$, $C_{i+1}C_i\overline{C_{i+1}}$ $(4 \leq i \leq 2g)$, and $C_{2j}B_{2j}\overline{C_{2j}}$ $(2 \leq j \leq g-1)$ are elements of $\mathcal{E}(\mathbb{CP}^2, K_3\#\Sigma_{g-1})$. Each regular neighborhood of $c_i$ $(4 \leq i \leq 2g+1)$, $b_{2j}$ $(2 \leq i \leq g-1)$ is an annulus standardly embedded in $S^3 = \partial D^4$. We can deform this annulus as indicated in Figure 1. Therefore, $C_i^2$ $(4 \leq i \leq 2g+1)$, $B_{2j}^2$ $(2 \leq j \leq g-1)$ are elements of $\mathcal{E}(\mathbb{CP}^2, K_3\#\Sigma_{g-1})$. Finally, the extendability of $B_4C_5C_7\ldots C_{2g+1}$ follows from the proof of Lemma 2.2 in [11]. Therefore, we showed $\mathcal{SP}_g[q_{K_3\#\Sigma_{g-1}}] \subset \mathcal{E}(\mathbb{CP}^2, K_3\#\Sigma_{g-1})$. On the other hand, by the definition of the Rokhlin quadratic form $q_{K_3\#\Sigma_{g-1}}$, we see $\mathcal{E}(\mathbb{CP}^2, K_3\#\Sigma_{g-1}) \subset \mathcal{SP}_g[q_{K_3\#\Sigma_{g-1}}]$. Theorem 5.1 follows.

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*Department of Mathematics, Faculty of Science and Engineering*
*Saga University, Saga, 840-8502 Japan.*

Email: `hirose@ms.saga-u.ac.jp`

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