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Smith equivalence and finite Oliver groups with Laitinen number 0 or 1

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Abstract In 1960, Paul A. Smith asked the following question. If a finite group G acts smoothly on a sphere with exactly two fixed points, is it true that the tangent G-modules at the two points are always isomorphic? We focus on the case G is an Oliver group and we present a classification of finite Oliver groups G with Laitinen number $a_G = 0$ or 1. Then we show that the Smith Isomorphism Question has a negative answer and $a_G \geq 2$ for any finite Oliver group G of odd order, and for any finite Oliver group G with a cyclic quotient of order pq for two distinct odd primes p and q. We also show that with just one unknown case, this question has a negative answer for any finite nonsolvable gap group G with $a_G \geq 2$. Moreover, we deduce that for a finite nonabelian simple group G, the answer to the Smith Isomorphism Question is affirmative if and only if $a_G = 0$ or 1.

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0.1 The Smith Isomorphism Question

Let G be a finite group. By a real *G-module* we mean a finite dimensional real vector space V with a linear action of G. Let M be a smooth G-manifold with nonempty fixed point set M^G. For any point $x \in M^G$, the tangent space $T_x(M)$ becomes a real G-module by taking the derivatives (at the point x) of the transformations $g\colon M \to M$, $z \mapsto gz$ for all $g \in G$. We refer to this G-module $T_x(M)$ as to the *tangent G-module at x*.

In 1960, Paul A. Smith [56, page 406] asked the following question.

Smith Isomorphism Question *Is it true that for any smooth action of G on a sphere with exactly two fixed points, the tangent G-modules at the two points are isomorphic?*

Following [49]–[52], two real G-modules U and V are called *Smith equivalent* if there exists a smooth action of G on a sphere S such that $S^G = \{x, y\}$ for two points x and y at which $T_x(S) = U$ and $T_y(S) = V$ as real G-modules.

In the real representation ring $RO(G)$ of G, we consider the subset $Sm(G)$ consisting of the differences $U - V$ of real G-modules U and V which are Smith equivalent. Choose a real G-module W such that $\dim W^G = 1$. Set $S = S(W)$, the G-invariant unit sphere in W. Then $S^G = \{x, y\}$ for the obvious points x and y in S, and clearly as real G-modules, $T_x(S) = T_y(S) = W - W^G$, the G-orthogonal complement of W^G in W. As a result,

$$W - W = (W - W^G) - (W - W^G) \in Sm(G).$$

Therefore, $Sm(G)$ contains the trivial subgroup 0 of $RO(G)$, and the Smith Isomorphism Question can be restated as follows. *Is it true that $Sm(G) = 0$?* As we shall see below, it may happen that $Sm(G) \neq 0$, but in general, it is an open question whether $Sm(G)$ is a subgroup of $RO(G)$.

In the following answers to the Smith Isomorphism Question, $\mathbb{Z}_n$ is the cyclic group $\mathbb{Z}/n\mathbb{Z}$ of order n, and S_3 is the symmetric group on three letters.

By [1] and [35], $Sm(\mathbb{Z}_p) = 0$ for any prime p. According to [54], $Sm(\mathbb{Z}_{p^k}) = 0$ for any odd prime p and any integer $k \geq 1$. By character theory, $Sm(S_3) = 0$ and $Sm(\mathbb{Z}_n) = 0$ for $n = 2$, 4, or 6. On the other hand, by [6]–[8], $Sm(\mathbb{Z}_n) \neq 0$ for $n = 4q$ with $q \geq 2$. So, $G = \mathbb{Z}_8$ is the smallest group with $Sm(G) \neq 0$.

We refer the reader to [1], [6]–[8], [9], [10], [17], [18], [19], [20], [28], [33], [34], [35], [46], [48], [49]–[52], [53], [54], [55], [57] for more related information.

If a finite group G acts smoothly on a homotopy sphere Σ with $\Sigma^G = \{x, y\}$, it follows from Smith theory that for every p-subgroup P of G with $p \mid |G|$, the fixed point set Σ^P is either a connected manifold of dimension ≥ 1, or $\Sigma^P = \{x, y\}$.

Henceforth, we say that a smooth action of G on a homotopy sphere Σ satisfies the 8-*condition* if for every cyclic 2-subgroup P of G with $|P| \geq 8$, the fixed point set Σ^P is connected (we recall that in [33], such an action of G on Σ is called 2-proper). In particular, the action of G on Σ satisfies the 8-condition if G has no element of order 8.

Now, two real G-modules U and V are called *Laitinen–Smith equivalent* if there exists a smooth action of G on a sphere S satisfying the 8-condition and such that $S^G = \{x, y\}$ for two points x and y at which $T_x(S) = U$ and $T_y(S) = V$ as real G-modules.

Beside $Sm(G)$, we consider the subset $LSm(G)$ of $RO(G)$ consisting of 0 and the differences $U - V$ of real G-modules U and V which are Laitinen–Smith equivalent. Again, in general, if $LSm(G) \neq 0$, it is an open question whether $LSm(G)$ is a subgroup of $RO(G)$. Clearly, $LSm(G) \subseteq Sm(G)$.

If G is a cyclic 2-group with $|G| \geq 8$, then there are no two real G-modules which are Laitinen–Smith equivalent. Therefore $LSm(G) = 0$ while $Sm(G) \neq 0$ by [6]–[8]. In particular, $LSm(G) \neq Sm(G)$. However, if G has no element of order 8, then $LSm(G) = Sm(G)$ (cf. the 8-condition Lemma in Section 0.3).

Let $IO(G)$ be the intersection of the kernels $\mathrm{Ker}\,(RO(G) \to RO(P))$ of the restriction maps $RO(G) \to RO(P)$ taken for all subgroups P of G of prime power order. Set

$$IO(G,G) = IO(G) \cap \mathrm{Ker}\,(RO(G) \to \mathbb{Z})$$

where the map $RO(G) \to \mathbb{Z}$ is defined by $U - V \mapsto \dim U^G - \dim V^G$. In [33], the abelian group $IO(G,G)$ is denoted by $IO'(G)$.

According to [33, Lemma 1.4], the difference $U - V$ of two Laitinen–Smith equivalent real G-modules U and V belongs to $IO(G,G)$. Thus, the following lemma holds.

Basic Lemma *Let G be a finite group. Then $LSm(G) \subseteq IO(G,G)$.*

Let G be a finite group. Given two elements $g,h \in G$, g is called *real conjugate* to h if g or g^{-1} is conjugate to h, written $g \sim^{\pm 1} h$. Clearly, $\sim^{\pm 1}$ is an equivalence relation in G. For any $g \in G$, the resulting equivalence class $(g)^{\pm 1}$ is called the *real conjugacy class* of g. Note that $(g)^{\pm 1} = (g) \cup (g^{-1})$, the union of the conjugacy classes (g) and (g^{-1}) of g and g^{-1}, respectively.

We denote by a_G the number of real conjugacy classes $(g)^{\pm 1}$ of elements $g \in G$ not of prime power order. In 1996, Erkki Laitinen has suggested to study the number a_G while trying to answer the Smith Isomorphism Question for specific finite groups G. Henceforth, we refer to a_G as to the *Laitinen number* of G.

The ranks of the free abelian groups $IO(G)$ and $IO(G,G)$ are computed in [33, Lemma 2.1] in terms of the Laitinen number a_G, as follows.

First Rank Lemma *Let G be a finite group. Then the following holds.*

(1) $\mathrm{rk}\, IO(G) = a_G$. *In particular, $IO(G) = 0$ if and only if $a_G = 0$.*

(2) $\mathrm{rk}\, IO(G,G) = a_G - 1$ *when* $a_G \geq 1$, *and* $\mathrm{rk}\, IO(G,G) = 0$ *when* $a_G = 0$. *In particular, $IO(G,G) = 0$ if and only if $a_G = 0$ or 1.*

In 1996, Erkki Laitinen posed the following conjecture (cf. [33, Appendix]).

Laitinen Conjecture *Let G be a finite Oliver group such that $a_G \geq 2$. Then $LSm(G) \neq 0$.*

If $a_G = 0$ or 1, $LSm(G) = 0$ by the Basic Lemma and the First Rank Lemma. So, in the Laitinen Conjecture, the condition that $a_G \geq 2$ is necessary.

One may well conjecture that $Sm(G) \cap IO(G;G) \neq 0$ for any finite Oliver group G with $a_G \geq 2$. It is very likely that $LSm(G) = Sm(G) \cap IO(G;G)$. Clearly, the inclusion $LSm(G) \subseteq Sm(G) \cap IO(G;G)$ holds by the Basic Lemma.

Before we recall the notion of Oliver group, we wish to adopt the following definition. For a given finite group G, a series of subgroups of G of the form $P \trianglelefteq H \trianglelefteq G$ is called an *isthmus series* if $|P| = p^m$ and $|G/H| = q^n$ for some primes p and q (possibly $p = q$) and some integers $m, n \geq 0$, and the quotient group H/P is cyclic (possibly $H = P$).

For a finite group G, the following three claims are equivalent.

(1) G has a smooth action on a sphere with exactly one fixed point.

(2) G has a smooth action on a disk without fixed points.

(3) G has no isthmus series of subgroups.

By the Slice Theorem, (1) implies (2). By the work of Oliver [43], (2) and (3) are equivalent, and according to Laitinen and Morimoto [32], (3) implies (1).

Following Laitinen and Morimoto [32], a finite group G is called an *Oliver group* if G has no isthmus series of subgroups. Recall that each finite nonsolvable group G is an Oliver group, and a finite abelian (more generally, nilpotent) group G is an Oliver group if and only if G has three or more noncyclic Sylow subgroups (cf. [43], [44], and [31]).

We prove that the Laitinen Conjecture holds for large classes of finite Oliver groups G such that $a_G \geq 2$, and as a consequence, we obtain that $Sm(G) \neq 0$. Moreover, we check that $Sm(G) = 0$ for specific classes of finite groups G such that $a_G \leq 1$, and therefore we can answer the Smith Isomorphism Question to the effect that $Sm(G) = 0$ if and only if $a_G \leq 1$.

We wish to recall that for a finite group G, it may happen that $Sm(G) \neq 0$ and $a_G \leq 1$ (the smallest group with these properties is $G = \mathbb{Z}_8$).

0.2 Classification and Realization Theorems

Our main algebraic theorem gives a classification of finite Oliver groups G with Laitinen number $a_G \leq 1$, and it reads as follows.

Classification Theorem *Let G be a finite Oliver group. Then the Laitinen number $a_G = 0$ or 1 if and only if one of the following conclusions holds:*

(1) $G = PSL(2,q)$ *for some* $q \in \{5,7,8,9,11,13,17\}$*; or*

(2) $G = PSL(3,3)$, $PSL(3,4)$, $Sz(8)$, $Sz(32)$, A_7, M_{11} *or* M_{22}*; or*

(3) $G = PGL(2,5)$, $PGL(2,7)$, $P\Sigma L(2,8)$, *or* M_{10}*; or*

(4) $G = PSL(3,4) \rtimes C_2 = PSL(3,4) \rtimes \langle u \rangle$*; or*

(5) $F(G) = C_2^2 \times C_3$ *and* $G = \mathrm{Stab}_{A_7}(\{1,2,3\})$ *or* $C_2^2 \rtimes D_9$*; or*

(6) $F(G)$ *is an abelian* p*-group for some odd prime* p, $G = F(G) \rtimes H$ *for* $H < G$ *with* $H = SL(2,3)$ *or* $\widehat{S}_4$, *and* $F(G)$ *is inverted by the unique involution of* H*; or*

(7) $F(G) = C_3^3$ *and* $G = F(G) \rtimes A_4$*; or*

(8) $F(G) = C_2^4$, $F^2(G) = A_4 \times A_4$, *and* $G = F^2(G) \rtimes C_4$*; or*

(9) $F(G) = C_2^8$ *and* $G = F(G) \rtimes H$ *for* $H < G$ *with* $H = PSU(3,2)$ *or* $C_3^2 \rtimes C_8$*; or*

(10) $F(G) = C_2^3$ *and* $G/F(G) = GL(3,2)$*; or*

(11) $F(G) = C_2^4$ *and* $G/F(G) = A_6$*; or*

(12) $F(G) = C_2^8$ *and* $G/F(G) = M_{10}$*; or*

(13) $F(G)$ *is a non-identity elementary abelian* 2*-group*, $G/F(G) = SL(2,4)$, $\Sigma L(2,4)$, $SL(2,8)$, $Sz(8)$ *or* $Sz(32)$, *and* $C_{F(G)}(x) = 1$ *for every* $x \in G$ *of odd order.*

Here, we consider cyclic groups C_q of order q, dihedral groups D_q of order $2q$, elementary abelian p-groups $C_p^k = C_p \times \cdots \times C_p$, alternating groups A_n, symmetric groups S_n, general linear groups $GL(n,q)$, special linear groups $SL(n,q)$, projective general linear groups $PGL(n,q)$, projective special linear groups $PSL(n,q)$, projective special unitary groups $PSU(n,q)$, the Mathieu groups M_{10}, M_{11}, and M_{22}, and the Suzuki groups $Sz(8)$ and $Sz(32)$. Recall that the group $PSL(3,4)$ admits an automorphism u of order 2, referred to as a graph-field automorphism, acting as the composition of the transpose-inverse automorphism and the squaring map (a Galois automorphism) of the field $\mathbb{F}_4$ of four elements. The fixed points of u form the group $PSU(3,2)$.

Moreover, for two finite groups N and H, $N \rtimes H$ denotes a semi-direct product of N and H (i.e., the splitting extension G associated with an exact sequence $1 \to N \to G \to H \to 1$). Also, we use the notations

$$\Sigma L(n,q) = SL(n,q) \rtimes \mathrm{Aut}(\mathbb{F}_q) \quad \text{and} \quad P\Sigma L(n,q) = PSL(n,q) \rtimes \mathrm{Aut}(\mathbb{F}_q)$$

where $\mathrm{Aut}(\mathbb{F}_q)$ is the group of all automorphisms of the field $\mathbb{F}_q$ of q elements.

For $n \geq 4$ and $n \neq 6$, there exist two groups G which are not isomorphic, do not contain a subgroup isomorphic to A_n, and occur in a short exact sequence $1 \to C_2 \to G \to S_n \to 1$. For $n = 4$, one of the groups is isomorphic to $GL(2,3)$ and the other, denoted here by $\hat{S}_4$, has exactly one element of order 2.

Finally, for a finite group G, we denote by $F(G)$ the Fitting subgroup of G (i.e., the largest nilpotent normal subgroup of G) and by $F^2(G)$ the pre-image of $F(G/F(G))$ under the quotient map $G \to G/F(G)$.

We remark that the Classification Theorem stated above extends a previous result of Bannuscher and Tiedt [4] obtained for finite nonsolvable groups G such that every element of G has prime power order (i.e., such that $a_G = 0$). Our proof is largely independent of their result, but we do invoke it to establish that $F(G)$ is elementary abelian in case (13). Moreover, their result and our cases (1)–(13) allow us to list all finite Oliver groups G with $a_G = 1$, and thus with $IO(G,G) = 0$ and $IO(G) \neq 0$ (cf. the First Rank Lemma).

Let G be a finite group. By [45], there exists a smooth action of G on a disk with exactly two fixed points if and only if G is an Oliver group. For a finite Oliver group G, two real G-modules U and V are called *Oliver equivalent* if there exists a smooth action of G on a disk D such that $D^G = \{x, y\}$ for two points x and y at which $T_x(D) = U$ and $T_y(D) = V$ as real G-modules.

If $U - V \in RO(G)$ is the difference of two Oliver equivalent real G-modules U and V, then $U - V \in IO(G,G)$ by Smith theory and the Slice Theorem. On the other hand, if $U - V \in IO(G,G)$, then U and V are isomorphic as P-modules for each subgroup P of G of prime power order, and by subtracting the trivial summands, we may assume that $\dim U^G = \dim V^G = 0$. Hence, by [45, Theorem 0.4], there exists a smooth action of G on a disk D such that $D^G = \{x, y\}$ for two points x and y at which $T_x(D) = U \oplus W$ and $T_y(D) = V \oplus W$ for some real G-module W with $\dim W^G = 0$. As in $RO(G)$,

$$U - V = (U \oplus W) - (V \oplus W),$$

the element $U - V$ is the difference of two Oliver equivalent real G-modules. Consequently, $IO(G,G)$ coincides with the subset of $RO(G)$ consisting of the differences of real G-modules which are Oliver equivalent. So, the Classification Theorem and the First Rank Lemma yield the following corollary.

Classification Corollary *A finite Oliver group G has the property that two Oliver equivalent real G-modules are always isomorphic (i.e., $IO(G;G) = 0$) if and only if G is listed in cases (1)–(13) of the Classification Theorem (i.e., the Laitinen number $a_G = 0$ or 1).*

For a finite group G, we denote by $\mathcal{P}(G)$ the family of subgroups of G consisting of the trivial subgroup of G and all p-subgroups of G for all primes $p \mid |G|$.

A subgroup H of a finite group G ($H \subseteq G$) is called a *large subgroup* of G if $O^p(G) \subseteq H$ for some prime p, where $O^p(G)$ is the smallest normal subgroup of G such that $|G/O^p(G)| = p^k$ for some integer $k \geq 0$.

For a finite group G, we denote by $\mathcal{L}(G)$ the family of large subgroups of G, and a real G-module V is called *$\mathcal{L}$-free* if $\dim V^H = 0$ for each $H \in \mathcal{L}(G)$, which amounts to saying that $\dim V^{O^p(G)} = 0$ for each prime $p \mid |G|$.

Here, as in [42], a finite group G is called a *gap group* if $\mathcal{P}(G) \cap \mathcal{L}(G) = \varnothing$ and there exists a real $\mathcal{L}$-free G-module V satisfying the *gap condition* that

$$\dim V^P > 2 \dim V^H$$

for each pair (P, H) of subgroups $P < H \subseteq G$ with $P \in \mathcal{P}(G)$.

According to [42], if G is a finite group such that $\mathcal{P}(G) \cap \mathcal{L}(G) = \varnothing$, then G is a gap group under either of the following conditions:

(1) $O^p(G) \neq G$ and $O^q(G) \neq G$ for two distinct odd primes p and q.

(2) $O^2(G) = G$ (which is true when G is of odd order or G is perfect).

(3) G has a quotient which is a gap group.

Note that the condition (1) is equivalent to the condition that G has a cyclic quotient of order pq for two distinct odd primes p and q. Recall that a finite group G is nilpotent if and only if G is the product of its Sylow subgroups. Moreover, a finite nilpotent group G is an Oliver group if and only if G has three or more noncyclic Sylow subgroups. Therefore the condition (1) holds for any finite nilpotent Oliver group G.

If G is a finite Oliver group, then $\mathcal{P}(G) \cap \mathcal{L}(G) = \varnothing$ by [32], but it may happen that there is no real $\mathcal{L}$-free G-module satisfying the gap condition. In fact, by [16] or [42], the symmetric group S_n is a gap group if and only if $n \geq 6$. Hence, S_5 is an Oliver group which is not a gap group, but S_5 contains A_5 which is both an Oliver and gap group. We refer the reader to [42], [58] and [59] for more information about gap groups.

Let $LO(G)$ be the subgroup of $RO(G)$ consisting of the differences $U - V$ of real $\mathcal{L}$-free G-modules U and V which are isomorphic when restricted to any $P \in \mathcal{P}(G)$. Recall that $IO(G)$ is the intersection of the kernels of the restriction maps $RO(G) \to RO(P)$ taken for all $P \in \mathcal{P}(G)$, and $IO(G, G)$ is the intersection of $IO(G)$ and $\mathrm{Ker}\,(RO(G) \to \mathbb{Z})$ where $RO(G) \to \mathbb{Z}$ is the G-fixed point set dimension map. In particular, $LO(G) \subseteq IO(G, G)$.

Now, we are ready to state our main topological theorem.

Realization Theorem *Let G be a finite Oliver gap group. Then any element of $LO(G)$ is the difference of two Laitinen–Smith equivalent real G-modules; i.e., $LO(G) \subseteq LSm(G)$.*

The Realization Theorem and the Basic Lemma show that

$$LO(G) \subseteq LSm(G) \subseteq IO(G, G)$$

for any finite Oliver gap group G. In general, $LO(G) \neq IO(G, G)$. However, if G is perfect, $O^p(G) = G$ for any prime p, and hence $\mathcal{L}(G) = \{G\}$, and thus $LO(G) = IO(G, G)$. So, the Realization Theorem and the Basic Lemma yield the following corollary (cf. [33, Corollary 1.8] where a similar result is obtained for the realifications of complex G-modules for any finite perfect group G).

Realization Corollary *Let G be a finite perfect group. Then any element of $LO(G)$ is the difference of two Laitinen–Smith equivalent real G-modules and $LO(G) = IO(G, G)$, and thus $LO(G) = LSm(G) = IO(G, G)$.*

0.3 Answers to the Smith Isomorphism Question

By checking whether $Sm(G) = 0$, we answer the Smith Isomorphism Question for large classes of finite Oliver groups G. In order to prove that $Sm(G) = 0$ if the Laitinen number $a_G \leq 1$, we use the Classification Theorem. If the Laitinen number $a_G \geq 2$, we show that $LO(G) \neq 0$ and by the Realization Theorem, we obtain that $LSm(G) \neq 0$, and thus $Sm(G) \neq 0$.

Theorem A1 *Let G be a finite Oliver group of odd order. Then $a_G \geq 2$ and $LO(G) \neq 0$.*

Theorem A2 *Let G be a finite group with a cyclic quotient of order pq for two distinct odd primes p and q. Then $a_G \geq 2$ and $LO(G) \neq 0$.*

Theorem A3 *Let G be a finite nonsolvable group. Then*

(1) $LO(G) = 0$ *if* $a_G \leq 1$,

(2) $LO(G) \neq 0$ *if* $a_G \geq 2$, *except when* $G = \mathrm{Aut}(A_6)$ *or* $P\Sigma L(2,27)$, *and*

(3) $LO(G) = 0$ *and* $a_G = 2$ *when* $G = \mathrm{Aut}(A_6)$ *or* $P\Sigma L(2,27)$.

Theorem B1 *Let G be a finite Oliver group of odd order. Then $a_G \geq 2$ and*

$$0 \neq LO(G) \subseteq LSm(G) = Sm(G) \subseteq IO(G,G).$$

Theorem B2 *Let G be a finite Oliver group with a cyclic quotient of order pq for two distinct odd primes p and q. Then $a_G \geq 2$ and*

$$0 \neq LO(G) \subseteq LSm(G) \subseteq IO(G,G).$$

Theorem B3 *Let G be a finite nonsolvable gap group not isomorphic to $P\Sigma L(2,27)$. Then $LO(G) \neq 0$ if and only if $a_G \geq 2$,*

$$LO(G) \subseteq LSm(G) \subseteq IO(G,G);$$

and $LSm(G) \neq 0$ if and only if $a_G \geq 2$.

By [33, Theorem A], if G is a finite perfect group, $LSm(G) \neq 0$ if and only if $a_G \geq 2$. Theorem B3 extends this result in two ways. Firstly, it proves the conclusion for a large class of finite nonsolvable groups G, including all finite perfect groups. Secondly, if G is perfect, it shows that $LSm(G) = IO(G,G)$ (cf. the Realization Corollary).

If G is as in Theorems B1 or B2, the Laitinen Conjecture holds by the theorems. By Theorem B3, the Laitinen Conjecture holds for any finite nonsolvable gap group G with $a_G \geq 2$, except when $G = P\Sigma L(2,27)$. In the exceptional case, $LO(G) = 0$ and $a_G = 2$ by Theorem A3, and thus $\mathrm{rk}\, IO(G,G) = 1$ by the First Rank Lemma, so that $IO(G,G) \neq 0$. However, we do not know whether $IO(G,G) \subseteq LSm(G)$, and we are not able to confirm that $LSm(G) \neq 0$. The same is true when $G = \mathrm{Aut}(A_6)$. Recall that $P\Sigma L(2,27)$ is a gap group while $\mathrm{Aut}(A_6)$ is not a gap group (see [42, Proposition 4.1]).

Theorem C1 *Let G be a finite nonabelian simple group.*

(1) *If $a_G \leq 1$, then $Sm(G) = 0$ and G is isomorphic to one of the groups:*
$a_G = 0$: $PSL(2,q)$ *for* $q = 5,7,8,9,17$, $PSL(3,4)$, $Sz(8)$, $Sz(32)$, *or*
$a_G = 1$: $PSL(2,11)$, $PSL(2,13)$, $PSL(3,3)$, A_7, M_{11}, M_{22}.

(2) *If $a_G \geq 2$, then $LSm(G) = IO(G,G) \neq 0$, and thus $Sm(G) \neq 0$.*

Theorem C2 *Let $G = SL(n, q)$ or $Sp(n, q)$ for $n \geq 2$ where n is even in the latter case and q is any prime power in both cases.*

(1) *If $a_G \leq 1$, then $Sm(G) = 0$ and G is isomorphic to one of the groups:*
$a_G = 0$: $SL(2, 2)$, $SL(2, 4)$, $SL(2, 8)$, $SL(3, 2)$, *or*
$a_G = 1$: $SL(2, 3)$, $SL(3, 3)$.

(2) *If $a_G \geq 2$, then except for $G = Sp(4, 2)$, $LSm(G) = IO(G, G) \neq 0$, and thus $Sm(G) \neq 0$. Moreover, $Sm(G) \neq 0$ for $G = Sp(4, 2)$.*

Theorem C3 *Let $G = A_n$ or S_n for $n \geq 2$.*

(1) *If $a_G \leq 1$, then $Sm(G) = 0$ and G is one of the groups:*
$a_G = 0$: A_2, A_3, A_4, A_5, A_6, S_2, S_3, S_4, *or*
$a_G = 1$: A_7, S_5.

(2) *If $a_G \geq 2$, then $LSm(G) \supseteq LO(G) \neq 0$, and thus $Sm(G) \neq 0$. Moreover, $LSm(G) = LO(G)$ for $G = A_n$.*

We recall that A_n is a simple group if and only if $n \geq 5$. So, except for A_2, A_3 and A_4, every A_n occurs in Theorem C1. Moreover, except for $PSL(2, 2)$ and $PSL(2, 3)$, every $PSL(n, q)$ is a simple group, and the following holds: $A_5 = PSL(2, 4) = PSL(2, 5)$, $A_6 = PSL(2, 9)$, and $PSL(2, 7) = PSL(3, 2)$.

The symplectic group $Sp(n, q)$ and the projective symplectic group $PSp(n, q)$ are defined for any even integer $n \geq 2$ and any prime power q. Except for $PSp(2, 2)$, $PSp(2, 3)$, and $PSp(4, 2)$, every $PSp(n, q)$ is a nonabelian simple group, and thus occurs in Theorem C1. Moreover, in the exceptional cases, the following holds: $PSp(2, 2) = PSL(2, 2) = S_3$, $PSp(2, 3) = PSL(2, 3) = A_4$, and $PSp(4, 2) = Sp(4, 2) = S_6$. So, the cases are covered by Theorem C3.

Comment D1 The conjecture posed in [19, p. 44] asserts that if $Sm(G) = 0$ for a finite group G, then $Sm(H) = 0$ for any subgroup H of G. We are able to give counterexamples to this conjecture. In fact, according to Theorem C1 and Example E1 below, there exist (precisely four) finite simple groups G with an element of order 8, such that $Sm(G) = 0$. But G has a subgroup $H = \mathbb{Z}_8$, and we know that $Sm(H) \neq 0$ by [6]–[8].

Comment D2 Contrary to the speculation in [55, Comment (2), p. 547] that $Sm(G) \neq 0$ for any finite Oliver group G, Theorem C1 shows that there exist (precisely fourteen) finite nonabelian simple groups G such that $Sm(G) = 0$. We recall that any finite nonabelian simple group G is an Oliver group.

By using Theorems B1–B3, we can answer the Smith Isomorphism Question as follows: $Sm(G) \neq 0$ in either of the following cases.

(1) G is a finite Oliver group of odd order (and thus $a_G \geq 2$).

(2) G is a finite Oliver group with a cyclic quotient of order pq for two distinct odd primes p and q (and thus $a_G \geq 2$).

(3) G is a finite nonsolvable gap group with $a_G \geq 2$, and $G \neq P\Sigma L(2,27)$.

In turn, Theorems C1–C3 allow us to answer the Smith Isomorphism Question as follows: $Sm(G) = 0$ if and only if $a_G \leq 1$, in either of the following cases.

(1) G is a finite nonabelian simple group.

(2) $G = PSL(n,q)$ or $SL(n,q)$ for any $n \geq 2$ and any prime power q.

(3) $G = PSp(n,q)$ or $Sp(n,q)$ for any even $n \geq 2$ and any prime power q.

(4) $G = A_n$ or S_n for any $n \geq 2$.

It follows from [33, Theorem B] that for $G = A_n$, $PSL(2,p)$ or $SL(2,p)$ for any prime p, $Sm(G) = 0$ if and only if $a_G \leq 1$. However, while [33] considers the realifications of complex G-modules, we deal with real G-modules when proving that $Sm(G) \neq 0$ for $a_G \geq 2$ (cf. [33, Corollary 1.8]).

By using the Realization Theorem, the Basic Lemma, the First Rank Lemma, and Theorems A1–A3, we are able to prove Theorems B1–B3.

Proofs of Theorems B1–B3 Let G be as in Theorems B1–B3. Then, by the Realization Theorem and the Basic Lemma,

$$LO(G) \subseteq LSm(G) \subseteq IO(G,G).$$

If G is as in Theorem B1 (resp., B2), $a_G \geq 2$ and $LO(G) \neq 0$ by Theorem A1 (resp., A2). Suppose that G is as in Theorem B3. According to our assumption, $G \neq P\Sigma L(2,27)$ and $G \neq \mathrm{Aut}(A_6)$ as G is a gap group while $\mathrm{Aut}(A_6)$ is not (cf. [42, Proposition 4.1]). If $a_G \leq 1$, $IO(G,G) = 0$ by the First Rank Lemma, and thus $LO(G) = LSm(G) = 0$. If $a_G \geq 2$, $LO(G) \neq 0$ by Theorem A3, and thus $LSm(G) \neq 0$. □

Now, we adopt the following definition for any finite group G. We say that G satisfies the *8-condition* if for every cyclic 2-subgroup P of G with $|P| \geq 8$, $\dim V^P > 0$ for any irreducible G-module V. In particular, if G is without elements of order 8, G satisfies the 8-condition. Recall that in [33], G satisfying the 8-condition is called 2-proper (cf. [33, Example 2.5]).

If a finite group G satisfies the 8-condition and G acts smoothly on a homotopy sphere Σ with $\Sigma^G \neq \varnothing$, then the action of G on Σ satisfies the 8-condition (cf. Section 0.1), and thus the following lemma holds (cf. [33, Lemma 2.6]).

8-condition Lemma *For each finite group G satisfying the 8-condition, any two Smith equivalent real G-modules are also Laitinen–Smith equivalent; i.e., $Sm(G) \subseteq LSm(G)$, and thus $Sm(G) = LSm(G)$.*

Example E1 In the following list (C1), each group G satisfies the 8-condition and $a_G = 0$ or 1, where G is one of the groups:

$a_G = 0$: $PSL(2,q)$ for $q = 2,3,5,7,8,9,17$, $PSL(3,4)$, $Sz(8)$ or $Sz(32)$,

$a_G = 1$: $PSL(2,11)$, $PSL(2,13)$, $PSL(3,3)$, A_7, M_{11} or M_{22}.

If $G = PSL(2,2) = S_3$ or $G = PSL(2,3) = A_4$, then $a_G = 0$ and G has no element of order 8 (cf. [33, Proposition 2.4]). In list (C1), except for $PSL(2,2)$ and $PSL(2,3)$, every G is a nonabelian simple group, and some inspection in [11] or [23] confirms that $a_G = 0$ for $G = PSL(2,q)$ with $q = 5,7,8,9,17$, and $a_G = 0$ for $G = PSL(3,4)$, $Sz(8)$ or $Sz(32)$. Also, $a_G = 1$ corresponding to an element of order 6 when $G = PSL(2,11)$, $PSL(2,13)$, $PSL(3,3)$, A_7, M_{11} or M_{22}. Further inspection in [11] or [23] shows that in list (C1), G has an element of order 8 if and only if $G = PSL(2,17)$, $PSL(3,3)$, M_{11} or M_{22}, and the groups all satisfy the 8-condition. All finite groups G without elements of order 8 also satisfy the 8-condition. Therefore, each group G in list (C1) satisfies the 8-condition.

Example E2 In the following list (C2), each group G satisfies the 8-condition and $a_G = 0$ or 1, where G is one of the groups:

$a_G = 0$: $SL(2,2)$, $SL(2,4)$, $SL(2,8)$, $SL(3,2)$, $Sp(2,2)$, $Sp(2,4)$ or $Sp(2,8)$,

$a_G = 1$: $SL(2,3)$, $SL(3,3)$ or $Sp(2,3)$.

As $Sp(2,q) = SL(2,q)$ for any prime power q, it suffices to check the result for the special linear groups. First, recall that $SL(2,q) = PSL(2,q)$ when q is a power of 2. Clearly, $a_G = 0$ when $G = SL(2,2) = PSL(2,2) = S_3$, and by Example E1, $a_G = 0$ when $G = SL(2,4) = PSL(2,4) = PSL(2,5) = A_5$, or $G = SL(2,8) = PSL(2,8)$, or $G = SL(3,2) = PSL(3,2) = PSL(2,7)$. Moreover, for $G = SL(3,3) = PSL(3,3)$, $a_G = 1$ corresponding to an element of order 6. The same holds for $G = SL(2,3)$ because G has elements of orders 1, 2, 3, 4, and 6, and the elements of order 6 are all real conjugate in G (cf. [33, Proposition 2.3]). By the discussion above and Example E1, we see that in list (C2), G has an element of order 8 if and only if $G = SL(3,3)$, and $SL(3,3) = PSL(3,3)$ satisfies the 8-condition. So, each group G in list (C2) satisfies the 8-condition.

Example E3 In the following list (C3), each group G is without elements of order 8 and $a_G = 0$ or 1, or $a_G \geq 2$, where G is one of the groups:

$a_G = 0$: A_2, A_3, A_4, A_5, A_6, S_2, S_3 or S_4,

$a_G = 1$: A_7 or S_5,

$a_G \geq 2$: A_8, A_9, S_6 or S_7.

First, we consider the case $G = A_n$. For $n \leq 6$, $a_G = 0$ because each element of G has prime power order. For $n = 7$, $a_G = 1$ corresponding to the element (12)(34)(567) of order 6. For $n \geq 8$, $a_G \geq 2$ because the elements (12)(34)(567) and (123456)(78) have order 6 and are not real conjugate in G.

Now, we consider the case $G = S_n$. For $n \leq 4$, $a_G = 0$ because each element of G has prime power order. For $n = 5$, $a_G = 1$ corresponding to the element (12)(345) of order 6. For $n \geq 6$, $a_G \geq 2$ because the elements (12)(345) and (123456) have order 6 and are not real conjugate in G.

As a result, if $G = A_n$ (resp., S_n), $a_G \leq 1$ if and only if $n \leq 7$ (resp., $n \leq 5$). Moreover, if $G = A_n$ or S_n for $n \leq 7$, G has no element of order 8 because any permutation of order 8 must involve an 8-cycle in its cycle decomposition. Also, if $G = A_8$ or A_9, G has no element of order 8 because an 8-cycle is not an even permutation. Therefore, each group G in list (C3) is without elements of order 8, and thus G satisfies the 8-condition.

By using the Classification Theorem, Examples E1–E3, the 8-condition Lemma, the Basic Lemma, the First Rank Lemma, and Theorems B1–B3, we are able to prove Theorems C1–C3.

Proofs of Theorems C1–C3 Let G be as in Theorems C1–C3. Then, by the Classification Theorem and Examples E1–E3, $a_G = 0$ or 1 if and only if G is as in claims (1) of Theorems C1–C3.

If $a_G \leq 1$, G satisfies the 8-condition by Examples E1–E3, and thus

$$Sm(G) = LSm(G) = IO(G;G) = 0$$

by the 8-condition Lemma, the Basic Lemma and the First Rank Lemma.

If $a_G \geq 2$, G is as in Theorems B1–B3, and therefore

$$0 \neq LO(G) \subseteq LSm(G) \subseteq IO(G;G).$$

Moreover, except for $G = S_n$ or $Sp(4;2) = S_6$, G is a perfect group, and thus $LO(G) = LSm(G) = IO(G;G)$ (cf. the Realization Corollary obtained from the Realization Theorem and the Basic Lemma). □

0.4 Second Rank Lemma

Let G be a finite group. In Sections 0.1 and 0.2, we defined the following series of free abelian subgroups of $RO(G)$: $LO(G) \subseteq IO(G,G) \subseteq IO(G)$. Recall that $IO(G)$ consists of the differences $U - V$ of real G-modules U and V which are isomorphic when restricted to any $P \in \mathcal{P}(G)$, $IO(G,G)$ is obtained from $IO(G)$ by imposing the additional condition that $\dim U^G = \dim V^G$, and $LO(G)$ consists of the differences $U - V \in IO(G)$ such that U and V are both $\mathcal{L}$-free. Now, for any normal subgroup H of G, we put

$$IO(G,H) = IO(G) \cap \operatorname{Ker}\left(RO(G) \xrightarrow{\operatorname{Fix}^H} RO(G/H)\right)$$

where $\operatorname{Fix}^H(U - V) = U^H - V^H$ and the H-fixed point sets U^H and V^H are considered as the canonical G/H-modules. As $RO(G/G) = \mathbb{Z}$ and

$$\operatorname{Ker}\left(RO(G) \xrightarrow{\operatorname{Fix}^G} RO(G/G)\right) = \operatorname{Ker}\left(RO(G) \xrightarrow{\operatorname{Dim}^G} \mathbb{Z}\right),$$

the two definitions of $IO(G,G)$ coincide. In general, $IO(G,H) \subseteq IO(G,G)$. In fact, if $U - V \in IO(G,H)$, then $U - V \in IO(G)$ and in addition $U^H \cong V^H$ as G/H-modules, so that $\dim U^G = \dim(U^H)^{G/H} = \dim(V^H)^{G/H} = \dim V^G$, proving that $U - V \in IO(G,G)$. Therefore $IO(G,H) \subseteq IO(G,G)$.

Henceforth, we denote by $b_{G/H}$ the number of real conjugacy classes $(gH)^{\pm 1}$ in G/H of cosets gH containing elements of G not of prime power order.

In general, $a_G \geq b_{G/H} \geq a_{G/H}$. Clearly, $a_G = b_{G/G} = 0$ when each element of G has prime power order, and $a_G = b_{G/G} = 1$ when G has elements not of prime power order and any two such elements are real conjugate in G. Otherwise, $a_G > b_{G/G} = 1$. Therefore, $a_G = b_{G/G}$ if and only if $a_G = 0$ or 1.

We compute the rank $\operatorname{rk} IO(G,H)$. For $H = G$, the computation goes back to [33, Lemma 2.1] (cf. the First Rank Lemma in Section 0.1 of this paper).

Second Rank Lemma *Let G be a finite group and let $H \trianglelefteq G$. Then*

$$\operatorname{rk} IO(G,H) = a_G - b_{G/H} \text{ and thus } \operatorname{rk} IO(G,G) = a_G - b_{G/G}.$$

In particular, $IO(G,H) = 0$ if $a_G \leq 1$, and $IO(G,G) = 0$ if and only if $a_G \leq 1$.

Proof In [33, Lemma 2.1], the rank of $IO(G)$ is computed as follows. The rank of the free abelian group $IO(G)$ is equal to the dimension of the real vector space $\mathbb{R} \otimes_{\mathbb{Z}} IO(G)$ which consists of the real valued functions on G that are constant on the real conjugacy classes $(g)^{\pm 1}$ and that vanish when g is of prime power order. Therefore $\operatorname{rk} IO(G) = a_G$.

Now, for a normal subgroup H of G, we compute the rank of the kernel

$$IO(G;H) = \operatorname{Ker}\left(IO(G) \xrightarrow{\mathrm{Fix}^H} RO(G/H)\right).$$

First, for any representation $\rho : G \to \mathrm{GL}(V)$, consider the representation $\mathrm{Fix}^H \rho : G/H \to \mathrm{GL}(V^H)$ given by $(\mathrm{Fix}^H \rho)(gH) = \rho(g)|_{V^H}$ for each $g \in G$. Let $\pi : V \to V$ be the projection of V onto V^H, that is,

$$\pi = \frac{1}{|H|} \sum_{h \in H} \rho(h) : V \to V.$$

Then the trace of $(\mathrm{Fix}^H \rho)(gH) : V^H \to V^H$ is the same as the trace of the endomorphism

$$\rho(g) \circ \pi = \frac{1}{|H|} \sum_{h \in H} \rho(gh) : V \to V.$$

So, if χ is the character of ρ, then the character $\mathrm{Fix}^H \chi$ of $\mathrm{Fix}^H \rho$ is given by

$$(\mathrm{Fix}^H \chi)(gH) = \frac{1}{|H|} \sum_{h \in H} \chi(gh).$$

This formula extends (by linearity) to $\mathbb{R} \otimes_{\mathbb{Z}} RO(G)$. Now, consider the basis of $\mathbb{R} \otimes_{\mathbb{Z}} IO(G)$ consisting of the functions $f_{(g)^{\pm 1}}$ which have the value 1 on $(g)^{\pm 1}$ and 0 otherwise, defined for all classes $(g)^{\pm 1}$ represented by elements $g \in G$ not of prime power order. Then, by the formula above applied to $\chi = f_{(g)^{\pm 1}}$,

$$(\mathrm{Fix}^H f_{(g)^{\pm 1}})(gH) = \frac{|(g)^{\pm 1} \cap gH|}{|H|}$$

and $\mathrm{Fix}^H f_{(g)^{\pm 1}}$ vanishes outside of $(gH)^{\pm 1}$. Therefore, the map

$$\mathrm{Fix}^H : IO(G) \to RO(G/H)$$

has image of rank $b_{G/H}$, and its kernel $IO(G;H)$ is of rank $a_G - b_{G/H}$. □

We wish to note that if G is a finite group and $H \lhd G$ (i.e. $H \trianglelefteq G$ and $H \neq G$), then one of the following conclusions holds:

(1) $a_G = b_{G/H} = 0$ if each $g \in G$ has prime power order, and otherwise

(2) $a_G = b_{G/H} = 1$ (holds, e.g., for $G = S_5$ and $H = G^{\mathrm{sol}} = A_5$), or

(3) $a_G = b_{G/H} > 1$ (holds, e.g., for $G = \mathrm{Aut}(A_6)$ and $H = G^{\mathrm{sol}}$), or

(4) $a_G > b_{G/H} = 1$ (holds, e.g., for $G = S_6$ and $H = G^{\mathrm{sol}} = A_6$), or

(5) $a_G > b_{G/H} > 1$ (holds, e.g., for $G = A_5 \times \mathbb{Z}_3$ and $H = G^{\mathrm{sol}} = A_5$).

Let G be a finite group with two subgroups $H \trianglelefteq G$ and $K \trianglelefteq G$. We claim that if H is a subgroup of K, $H \subseteq K$, then $IO(G;H)$ is a subgroup of $IO(G;K)$. In fact, take an element

$$U - V \in IO(G;H) = \mathrm{Ker}\,(IO(G) \xrightarrow{\mathrm{Fix}^H} RO(G/H))$$

and consider the G-orthogonal complements $U - U^H$ and $V - V^H$ of the real G-modules U and V. Then $U - V = (U - U^H) - (V - V^H)$ because $U^H = V^H$ as G/H-modules, and $(U - U^H)^K = (V - V^H)^K = \{0\}$ because $H \subseteq K$. Therefore, it follows that

$$U - V = (U - U^H) - (V - V^H) \in IO(G;K) = \mathrm{Ker}\,(IO(G) \xrightarrow{\mathrm{Fix}^K} RO(G/K));$$

proving the claim that $IO(G;H) \subseteq IO(G;K)$.

For any finite group G, we consider the group $IO(G;H)$, where H is:

G^{sol} : the smallest normal subgroup of G such that G/H is solvable,

G^{nil} : the smallest normal subgroup of G such that G/H is nilpotent,

$O^p(G)$: the smallest normal subgroup of G such that G/H is a p-group.

Clearly, G is perfect if and only if $G^{\mathrm{sol}} = G$, and G is solvable if and only if G^{sol} is trivial. And similarly, G is nilpotent if and only if G^{nil} is trivial. Moreover, $G^{\mathrm{sol}} \subseteq G^{\mathrm{nil}} = \bigcap_p O^p(G)$ taken for all primes $p \mid |G|$.

Subgroup Lemma *Let G be a finite group and let p be a prime. Then*

$$IO(G;G^{\mathrm{sol}}) \subseteq IO(G;G^{\mathrm{nil}}) \subseteq LO(G) \subseteq IO(G;O^p(G)) \subseteq IO(G;G).$$

Proof By the claim above, $IO(G;G^{\mathrm{sol}}) \subseteq IO(G;G^{\mathrm{nil}})$ because $G^{\mathrm{sol}} \subseteq G^{\mathrm{nil}}$. Now, set $H = G^{\mathrm{nil}}$. For a real G-module V, consider V^H as a real G-module with the canonical action of G. Then the G-orthogonal complement $V - V^H$ of V^H in V is L-free because $H \subseteq O^p(G)$ for each prime p. Take an element $U - V \in IO(G;H)$. Then $U^H = V^H$ as G-modules, so that

$$U - V = (U - U^H) - (V - V^H) \in LO(G);$$

proving that $IO(G;G^{\mathrm{nil}}) \subseteq LO(G)$. Any element of $LO(G)$ is the difference of two real L-free G-modules U and V such that $U - V \in IO(G)$. As U and V are L-free, $\dim U^{O^p(G)} = \dim V^{O^p(G)} = 0$, and thus $U - V \in IO(G;O^p(G))$, proving that $LO(G) \subseteq IO(G;O^p(G))$. Clearly, $IO(G;O^p(G)) \subseteq IO(G;G)$ by the claim above. □

By the Subgroup Lemma and the Second Rank Lemma,

$$a_G - b_{G/G^{\mathrm{nil}}} \le \mathrm{rk}\, LO(G) \le \min\{a_G - b_{G/O^p(G)} : p \mid |G|\}$$

for any finite group G. In particular, $a_G - b_{G/G^{\mathrm{sol}}} \le \mathrm{rk}\, LO(G) \le a_G - b_{G/G}$.

Example E4 Let $G = A_n$ for $n \ge 2$. By the First Rank Lemma, we know that $\mathrm{rk}\, IO(G,G) = 0$ when $a_G \le 1$, and $\mathrm{rk}\, IO(G,G) = a_G - 1$ when $a_G \ge 1$. Moreover, by Theorem C3,

$$Sm(G) \subseteq LSm(G) = LO(G) = IO(G,G).$$

Now, assume that $G = A_8$ or A_9. Then G has no element of order 8, and thus $Sm(G) = LSm(G)$. By straightforward computation, we check that $a_G = 3$ (resp., 6) for $G = A_8$ (resp., A_9). As a result, we obtain that

(1) $Sm(A_8) = LSm(A_8) = LO(A_8) = IO(A_8, A_8) = \mathbb{Z}^2$ and

(2) $Sm(A_9) = LSm(A_9) = LO(A_9) = IO(A_9, A_9) = \mathbb{Z}^5$.

Generalizing the case where $G = A_8$ or A_9, note that the 8-condition Lemma and the Realization Corollary yield the following corollary.

8-condition Corollary *Let G be a finite group satisfying the 8-condition. If G is perfect, then $Sm(G) = LSm(G) = LO(G) = IO(G,G)$.*

Example E5 Let $G = S_n$ and $H = A_n$ for $n \ge 2$. Then $G^{\mathrm{sol}} = H = O^2(G)$ and $O^p(G) = G$ for each odd prime p. Therefore, $LO(G) = IO(G,H)$ by the Subgroup Lemma, and $\mathrm{rk}\, LO(G) = a_G - b_{G/H}$ by the Second Rank Lemma. It follows from Example E3 that $b_{G/H} = 0$ for $n = 2$, 3 or 4, $b_{G/H} = 1$ for $n = 5$ or 6, and $b_{G/H} = 2$ for $n \ge 7$. Also, $a_G = 0$ for $n = 2$, 3 or 4, and $a_G = 1$ for $n = 5$. Thus, $\mathrm{rk}\, LO(G) = a_G - b_{G/H} = 0$ for $n = 2$, 3, 4 or 5. For $n \ge 6$, $a_G \ge 2$ and by Theorem C3 and the Basic Lemma, we see that $0 \ne LO(G) \subseteq LSm(G) \subseteq IO(G,G)$.

Now, let $G = S_6$ (resp., S_7) and let $H \trianglelefteq G$ be as above. By straightforward computation, we check that $a_G = 2$ (resp., 5). As we noted above, $b_{G/H} = 1$ (resp., 2), and thus $\mathrm{rk}\, LO(G) = a_G - b_{G/H} = 1$ (resp., 3). Moreover, by the First Rank Lemma, $\mathrm{rk}\, IO(G,G) = a_G - 1 = 1$ (resp., 4). As G has no element of order 8, $Sm(G) = LSm(G)$. As a result, we obtain that

(1) $Sm(S_6) = LSm(S_6) = LO(S_6) = IO(S_6, S_6) = \mathbb{Z}$ and

(2) $Sm(S_7) = LSm(S_7) \supseteq LO(S_7) = \mathbb{Z}^3$ and $IO(S_7, S_7) = \mathbb{Z}^4$.

0 Outline of material

Let G be a finite group. For the convenience of the reader, we give a glossary of subsets and subgroups (defined above) of the real representation ring $RO(G)$. First, recall that the following two subsets of $RO(G)$ consist of the differences $U - V$ of real G-modules U and V such that:

$$Sm(G): \ U \text{ and } V \text{ are Smith equivalent};$$
$$LSm(G): \ U \text{ and } V \text{ are Laitinen–Smith equivalent}.$$

The following four subgroups of $RO(G)$ consist of the differences $U - V$ of real G-modules U and V such that $U \cong V$ as P-modules for each $P \in \mathcal{P}(G)$, and:

$$IO(G): \ \text{there is no additional restriction on } U \text{ and } V;$$
$$LO(G): \ \text{the } G\text{-modules } U \text{ and } V \text{ are both } \mathcal{L}\text{-free};$$
$$IO(G,G): \ \dim U^G = \dim V^G;$$
$$IO(G,H): \ U^H \cong V^H \text{ as } G/H\text{-modules};$$

where $IO(G,H)$ is defined for any normal subgroup H of G.

In Section 0.1, for a finite group G, we recalled the question of Paul A. Smith about the tangent G-modules for smooth actions of G on spheres with exactly two fixed points. Then we stated the Basic Lemma and the First Rank Lemma. Moreover, we restated the Laitinen Conjecture from [33].

In Section 0.2, we stated the Classification and Realization Theorems (our main algebraic and topological theorems) and by using the theorems, we obtained the Classification and Realization Corollaries.

In Section 0.3, we stated Theorems A1–A3, B1–B3, and C1–C3. We answered the Smith Isomorphism Question and confirmed that the Laitinen Conjecture holds for many groups G. Then we have proved that Theorems B1–B3 follow from the Realization Theorem, the Basic Lemma, the First Rank Lemma, and Theorems A1–A3. Moreover, we stated the 8-condition Lemma and we gave Examples E1–E3. Finally, we have proved that Theorems C1–C3 follow from the Classification Theorem, Examples E1–E3, the 8-condition Lemma, the Basic Lemma, the First Rank Lemma, and Theorems B1–B3.

In Section 0.4, we stated and proved the Second Rank Lemma and the Subgroup Lemma. We also gave Examples E4 and E5 with $G = A_n$ and S_n, respectively. Moreover, we obtained the 8-condition Corollary for any finite perfect group G satisfying the 8-condition.

As we pointed out above, the Basic Lemma, the First Rand Lemma, and the 8-condition Lemma all three go back to [33]. Therefore, it remains to prove the Classification Theorem, the Realization Theorem, and Theorems A1–A3.

In Section 1, we prove Theorems A1 and A2. To prove Theorem A1, we obtain our first major result about the Laitinen number a_G. The result asserts that if G is a finite Oliver group of odd order and without cyclic quotient of order pq for two distinct odd primes p and q, then $a_G > b_{G/G^{\mathrm{nil}}}$ (Proposition 1.6), and thus $LO(G) \neq 0$ by the Second Rank Lemma and the Subgroup Lemma. If G is a finite group with a cyclic quotient of order pq for two distinct odd primes p and q, then $a_G \geq 4$ and $LO(G) \neq 0$ by an explicit construction of two real L-free G-modules U and V, which we give at the end of Section 1. This completes the proof of Theorem A1, and proves Theorem A2.

In Section 2, we prove the Classification Theorem by using the fundamental results of [21]–[23], including those restated in Theorems 2.2–2.4 of this paper, as well as by using Burnside's p^aq^b Theorem, the Feit–Thompson Theorem, the Brauer–Suzuki Theorem, and the Classification of the finite simple groups.

In Section 3, we prove Theorem A3. To present the proof, we analyze first the cases where $a_G = b_{G/G^{\mathrm{sol}}}$. As a result, we obtain our next major result about the Laitinen number a_G. The result asserts that if G is a finite nonsolvable group with $a_G = b_{G/G^{\mathrm{sol}}}$, then either $a_G \leq 1$ or $a_G = 2$ and $G = \mathrm{Aut}(A_6)$ or $P\Sigma L(2,27)$ (Proposition 3.1). By using the Second Rank Lemma, this allows us to find the cases where $IO(G, G^{\mathrm{sol}}) \neq 0$ (Corollary 3.13), and then by using the Subgroup Lemma, we are able to complete the proof of Theorem A3.

In Section 4, we prove the Realization Theorem. To present the proof, we recall first in Theorems 4.1 and 4.2 some equivariant thickening and surgery results which follow from [40] and [41], respectively. Then, in Theorems 4.3 and 4.4, we construct smooth actions of G on spheres with prescribed real G-modules at the fixed points. The required proof follows easily from Theorem 4.4.

We use information from [5], [15], [30] on transformation group theory and from [12], [13], [21]–[25], [27], [29] on group theory and representation theory.

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1 Proofs of Theorems A1 and A2

Let G be a finite group. We denote by $\mathrm{NPP}(G)$ the set of elements g of G which are not of prime power order, and we refer to the elements of $\mathrm{NPP}(G)$ as NPP elements of G. Also, we denote by $\overline{\mathrm{NPP}}(G)$ the set of real conjugacy classes which are subsets of $\mathrm{NPP}(G)$. Therefore, the Laitinen number a_G is the number of elements in $\overline{\mathrm{NPP}}(G)$.

Let $H \trianglelefteq G$. Then, by the Second Rank Lemma, $IO(G;H) \neq 0$ if and only if $a_G > b_{G/H}$. Clearly, $a_G > b_{G/H}$ if and only if $\mathrm{NPP}(G)$ contains two elements x and y not real conjugate in G, but such that the cosets xH and yH are real conjugate in G/H.

Lemma 1.1 *Let $H \trianglelefteq G$. Then the following three conclusions hold.*

(1) *Some coset gH meets two members of $\overline{\mathrm{NPP}}(G)$ if and only if $a_G > b_{G/H}$.*

(2) *If H contains two distinct members of $\overline{\mathrm{NPP}}(G)$, then $a_G > b_{G/H}$.*

(3) *If $a_G = b_{G/H}$, then $a_{G/K} = b_{(G/K)/(H/K)}$ for any $K \trianglelefteq G$ with $K \leq H$.*

Proof The first conclusion is immediate from the remarks above, while the second one is a special case of the first. To prove the third conclusion, suppose that $a_{G/K} > b_{(G/K)/(H/K)}$. Then some coset $\overline{g}(H/K)$ meets two members of $\overline{\mathrm{NPP}}(G/K)$. Assume that x and y are two elements of G such that $\overline{x}$ and $\overline{y}$ are not of prime power order in G/K and are not in the same real conjugacy class of G/K. Then neither x nor y is of prime power order and $xH = yH$. If $zxz^{-1} \in \{y, y^{-1}\}$ for an element $z \in G$, then $\overline{z}\,\overline{x}\,\overline{z}^{-1} \in \{\overline{y}, \overline{y}^{-1}\}$ contrary to assumption. Therefore, x and y are not in the same real conjugacy class of G, and thus $a_G > b_{G/H}$ by the first conclusion, proving the third one. □

Lemma 1.2 *Let G be a finite group and assume that $K \trianglelefteq L \leq H \leq G$ is a sequence of subgroups of G such that L/K contains NPP elements of two different orders. Then H contains NPP elements of two different orders, and $a_G > b_{G/H} \geq a_{G/H}$ when $H \trianglelefteq G$.*

Proof Suppose xK and yK are NPP elements of L/K of different orders. If the elements x and y have different orders, we are done. If not, we may assume that the order of x is larger than the order of xK, in which case the cyclic group generated by x contains two NPP elements of different orders. So in any case, H contains two NPP elements of different orders. Moreover, if $H \trianglelefteq G$, then $a_G > b_{G/H}$ by Lemma 1.1. Clearly $b_{G/H} \geq a_{G/H}$. □

Lemma 1.3 *Let G be a finite group containing a nonsolvable subgroup B and a cyclic subgroup $C \neq 1$ such that BC is a subgroup of G isomorphic to $B \times C$. Then G has NPP elements of different orders, and thus $a_G \geq 2$. Moreover, if $B \subseteq G^{\mathrm{sol}}$, then $a_G > b_{G/G^{\mathrm{sol}}}$.*

Proof For a prime divisor p of the order of C, choose an element $g \in C$ of order p. Since B is nonsolvable, it follows from Burnside's $p^a q^b$ Theorem that the order of B has (at least) three distinct prime divisors q, r, and s, say with $p \neq r$ and $p \neq s$. Choose two elements x and y in B of orders r and s, respectively. By the assumption, BC is a subgroup of G (which amounts to saying that $BC = CB$) isomorphic to $B \times C$. Thus, the elements gx and gy have orders pr and ps, respectively, proving that $a_G \geq 2$. If $B \subseteq G^{\mathrm{sol}}$, then the coset gG^{sol} contains the elements gx and gy which are not real conjugate in G, and thus $a_G > b_{G/G^{\mathrm{sol}}}$ by Lemma 1.1. □

Lemma 1.4 *Let G be a finite group of odd order and let $H \trianglelefteq G$. Suppose that p is a prime and P is an abelian p-subgroup of H with $P \trianglelefteq G$. Suppose also that q is a prime, $q \neq p$, and $x \in H$ of order q with $V = C_P(x) \neq 1$. Then $a_G > b_{G/H}$.*

Proof Suppose $a_G = b_{G/H}$. By Lemma 1.1, H contains at most one member of $\overline{\mathrm{NPP}}(G)$. On the other hand, every element of $Vx \setminus \{x\}$ has order pq and so all of these elements lie in the same member of $\overline{\mathrm{NPP}}(G)$. Take $y, y' \in V \setminus \{1\}$ and set $h = yx$. Now, take $g \in G$ with $h^g = ghg^{-1} \in \{y'x, (y'x)^{-1}\}$. Then $y^g x^g \in \{y'x, (y'x)^{-1}x^{-1}\}$ and so $x^g \in \{x, x^{-1}\}$. As $|G|$ is odd, $x^{-1} \notin x^G$. Hence $x^g = x$, and thus $g \in C_G(x)$. As $C_G(x)$ normalizes $V = C_P(x)$, $C_G(x)$ transitively permutes the set $Vx \setminus \{x\}$. But $|Vx \setminus \{x\}| = |V| - 1$ is even, whereas $|C_G(x)|$ is odd, contradicting Lagrange's Theorem. Thus $a_G > b_{G/H}$. □

Lemma 1.5 *Let G be a finite group of odd order, and let $H \trianglelefteq G$. Suppose that $a_G = b_{G/H}$. Then $F(H)$ is a p-group for some prime p, and the Sylow q-subgroups of H are cyclic for all primes $q \neq p$.*

Proof The result is trivial if $|H| = 1$. Otherwise let p be a prime divisor of $|F(H)|$ and let P be a nontrivial abelian normal subgroup of H. By Lemma 1.4, $C_P(x) = 1$ for all elements x of H of prime order q with $q \neq p$. Therefore $F(H)$ is a p-group. Moreover, if H contains a noncyclic abelian q-subgroup A for some prime $q \neq p$, then it follows from Theorem 2.3 below that $C_P(x) \neq 1$ for some element $x \in A$ of order q, a contradiction. Hence, as $|G|$ is odd, all Sylow q-subgroups of H are cyclic for $q \neq p$, as claimed. □

Now, we obtain our first major result about the Laitinen number a_G. First, we wish to recall that a finite group G is an Oliver group if and only if G does not have subgroups $P \trianglelefteq H \trianglelefteq G$ such that H/P is cyclic, P is a p-group and G/H is a q-group for some primes p and q, not necessarily distinct.

Proposition 1.6 *Let G be a finite Oliver group of odd order. Suppose that each cyclic quotient of G has prime power order. Then $a_G > b_{G/G^{\mathrm{nil}}} \geq 1$.*

Proof Set $H = G^{\mathrm{nil}}$. As $a_G = 0$ if and only if $b_{G/H} = 0$, we are done once we prove that $a_G > b_{G/H}$. So, assume on the contrary that $a_G = b_{G/H}$. Then Lemma 1.5 asserts that $F(H) = P$ is a p-group for some odd prime p. By assumption, for any prime $q \neq p$, G has no cyclic quotient of order pq. Hence G/H is an r-group for some prime r, and thus $H \neq F(H)$ because G is an Oliver group. By Lemma 1.1 (3) applied for $K = P$, $a_{G/P} = b_{(G/P)/(H/P)}$. Thus, by Lemma 1.5 applied to G/P, $F(H/P)$ is a q-group for some prime q. As $P = F(H)$, we have that $q \neq p$. Let F_2 be the pre-image in H of $F(H/P)$ and write $F_2 = PQ$ with Q a q-group. Again, by Lemma 1.5, Q is cyclic. Moreover, by Lemma 1.4, $C_P(Q) = 1$ and so $N = N_G(Q)$ is a complement to P in G by the Frattini argument. As Q is cyclic, $\mathrm{Aut}(Q)$ is abelian and so $N/C_G(Q)$ is abelian. Hence $PC_G(Q)$ is a normal subgroup of G with abelian quotient, whence $H \subseteq PC_G(Q)$, since H is the smallest normal subgroup of G with nilpotent quotient. But QP/P is the Fitting subgroup of H/P, whence $C_H(Q) \subseteq C_H(QP/P) = QP$. Thus $H = QP$. But then G is not an Oliver group, contrary to assumption. □

Proofs of Theorems A1 and A2 For G as in Theorems A1 and A2, we shall prove that $a_G \geq 2$ and $LO(G) \neq 0$.

First, assume that G is a finite Oliver group of odd order. If each cyclic quotient of G has prime power order, then $a_G > b_{G/G^{\mathrm{nil}}} \geq 1$ by Proposition 1.6, and thus $IO(G, G^{\mathrm{nil}}) \neq 0$ by the Second Rank Lemma in Section 0.4. In particular, $a_G \geq 2$ and $LO(G) \neq 0$ by the Subgroup Lemma in Section 0.4.

Now, assume that G is a finite (not necessarily Oliver) group with a cyclic quotient of order pq for two distinct odd primes p and q. We will prove that $a_G \geq 4$ and $LO(G) \neq 0$. As a result, we will complete the proof of Theorem A1 and show that Theorem A2 holds.

Take $H \trianglelefteq G$ with $G/H = \mathbb{Z}_{pq}$ and note that $\mathbb{Z}_{pq}$ contains $\varphi(pq) = (p-1)(q-1)$ elements of order pq, and hence $\frac{1}{2}(p-1)(q-1)$ real conjugacy classes of elements of order pq. We may assume that $p \geq 3$ and $q \geq 5$, and as $a_G \geq b_{G/H} \geq a_{G/H}$, we see that $a_G \geq 4$. We will prove that $LO(G) \neq 0$ by constructing a nonzero element of $LO(G)$. Set $n = pq$. Let ζ_n be the primitive n-th root of unity. Assume that $H = 1$ so that $G = \mathbb{Z}_n = \langle g \mid g^n = 1 \rangle$. Take $U = U_1 \oplus U_2$ and $V = V_1 \oplus V_2$, where U_i and V_i ($i = 1, 2$) are the irreducible 1-dimensional complex G-modules with characters

$$\chi_U(g) = \chi_{U_1}(g) + \chi_{U_2}(g) = \zeta_n + \zeta_n^2$$
$$\chi_V(g) = \chi_{V_1}(g) + \chi_{V_2}(g) = \zeta_n^a + \zeta_n^b,$$

and the integers a and b are chosen so that the following holds:

$$a \equiv 1 \pmod p; \quad a \equiv 2 \pmod q$$
$$b \equiv 2 \pmod p; \quad b \equiv 1 \pmod q$$

(for example, if $p = 3$ and $q = 5$, $a = 7$ and $b = 11$). Then U and V are complex L-free G-modules isomorphic when restricted to $P = \mathbb{Z}_p$ or $\mathbb{Z}_q$. The realifications $r(U)$ and $r(V)$ are not isomorphic as real G-modules (remember p and q are odd) but $r(U)$ and $r(V)$ are isomorphic when restricted to $P = \mathbb{Z}_p$ or $\mathbb{Z}_q$. So, as a result, we obtain that $0 \neq r(U) - r(V) \in LO(G)$. If $H \neq 1$, the epimorphism $G \to G/H \cong \mathbb{Z}_n$ (mapping large subgroups of G onto large subgroups of G/H) allows us to consider the complex $\mathbb{Z}_n$-modules U and V constructed above as complex L-free G-modules. As before, we obtain that $0 \neq r(U) - r(V) \in LO(G)$, completing the proofs. □

2 Proof of the Classification Theorem

In this section, we wish to classify finite Oliver groups G with Laitinen number $a_G = 0$ or 1, in such a way that we obatin a proof of the Classification Theorem stated in Section 0.2.

Theorem 2.1 *Let G be a finite Oliver group. Then $a_G = 0$ or 1 if and only if one of the conclusions (1)–(13) in the Classification Theorem holds.*

In the proof, our analysis will make repeated use of a few basic concepts and theorems. Recall that for a finite group H, the Fitting subgroup $F(H)$ of H is the largest normal nilpotent subgroup of H, $E(H)$ denotes the largest normal semisimple subgroup of H, and $F^*(H) = E(H)F(H)$ is the generalized Fitting subgroup of H, as defined by Helmut Bender. In the proof of Theorem 2.1, we shall use the fundamental results of [22, Theorems 3.5, 3.6] describing the structure and embedding of $F^*(H)$ for a finite group H.

Theorem 2.2 (Fitting–Bender Theorem) *For a finite group H, the following holds: $[E(H); F(H)] = 1$ and $C_H(F^*(H)) = Z(F(H))$. If H is solvable, then $F^*(H) = F(H)$.*

Theorem 2.3 *If $E = C_p \times C_p$ acts on an abelian q-group V, where p and q are distinct primes, then $V = \langle C_V(e) : e \in E^{\#} \rangle$.*

Theorem 2.4 *For two finite groups H and K, let $F = K \rtimes H$ be a Frobenius group with kernel K and complement H. If F acts faithfully on a vector space V over a field of characteristic p, where $(p; |K|) = 1$, then $C_V(H) \neq 0$.*

We shall also use Burnside's $p^a q^b$ Theorem which asserts that a finite nonsolvable group has order which is always divisible by at least three distinct primes, the Feit–Thompson Theorem which asserts that finite groups of odd order are solvable; the Brauer–Suzuki Theorem which asserts that if G is a finite group with no nontrivial normal subgroup of odd order and with a Sylow 2-subgroup of 2-rank 1, then G has a unique involution z lying in $Z(G)$. Finally, we shall use the Classification of the finite simple groups (cf. [21]–[25]).

The proof of Theorem 2.1 will be accomplished in a sequence of lemmas, the first two of which will address the following general situation: finite groups H without NPP elements; that is, each element of H is of prime power order. Such finite groups H are called *CP groups*, and CP groups have been studied by several authors including Higman [26], Suzuki [60], Bannuscher–Tiedt [4], and Delgado–Wu [14].

We remark that finite simple CP groups were first classified by Michio Suzuki in a deep paper [60], whose main theorem is one of the fundamental results in the proof of the classification of finite simple groups.

The following lemma goes back to Higman [26].

Lemma 2.5 *Let H be a finite solvable CP group. Then one of the following conclusions holds:*

(1) *H is a p-group for some prime p; or*

(2) *$H = K \rtimes C$ is a Frobenius group with kernel K and complement C, where K is a p-group and C is a q-group of q-rank 1 for two distinct primes p and q; or*

(3) *$H = K \rtimes C \rtimes A$ is a 3-step group, in the sense that $K \rtimes C$ is a Frobenius group as in the conclusion (2) with C cyclic, and $C \rtimes A$ is a Frobenius group with kernel C and complement A, a cyclic p-group.*

Corollary 2.6 *If G is a finite Oliver CP group, then $F^*(G)$ is nonsolvable.*

Proof As none of the groups in the conclusions of Lemma 2.5 is an Oliver group, the result follows by Lemma 2.5. □

Now, we analyze the situation where a finite nonabelian simple group L is without NPP elements or all NPP elements of L have the same order.

Lemma 2.7 *Let L be a finite nonabelian simple group. Assume that L is without NPP elements or all NPP elements of L have the same order. Then L is isomorphic to one of the following groups:*

(1) $PSL(2,q)$ *with* $q \equiv \pm 3 \pmod 8$*; or*

(2) $PSL(2,q)$ *with* $q = 9$ *or* q *a Fermat or Mersenne prime; or*

(3) $PSL(2,2^n)$ *or* $Sz(2^n)$, $n \geq 3$*; or*

(4) $PSL(3,3)$, $PSL(3,4)$, A_7, M_{11} *or* M_{22}.

Proof We survey the finite simple groups freely making use of the information in [23] and [11]. If $L = A_n$ for some $n \geq 8$, then L contains elements of orders 6 and 15, contrary to assumption. By inspection of [23, Tables 5.3], we see that if L is a sporadic simple group, then $L = M_{11}$ or M_{22}.

Hence, we may assume that L is a finite simple group of Lie type defined over a field of characteristic p. Assume that L is not isomorphic to $PSL(2,q)$ or $Sz(2^n)$. Then by consideration of subsystem subgroups ([23, Section 2.6]), we see that one of the following statements is true about L:

(1) L has a subgroup K with $K/Z(K) \cong PSL(4,p)$, $PSU(4,p)$, $PSp(6,p)$, $G_2(p)$ or ${}^2F_4(2)'$; or

(2) $L = PSL(3,q)$, $PSU(3,q)$ or $PSp(4,q)$.

Suppose first that p is odd. Then $PSp(4,p)$, $PSL(4,p)$, $PSU(4,p)$ and $G_2(p)$ all contain subgroups isomorphic to a commuting product of $SL(2,p)$ and a cyclic group of order 4 (see [23, Table 4.5.1]). Hence, each contains elements of order 6 and 12. Thus, we are reduced to the cases $L = PSL(3,q)$ or $L = PSU(3,q)$. In both cases, L contains a subgroup isomorphic to $SL(2,q)$. If $q > 3$, then $SL(2,q)$ contains an element of odd prime order $r > 3$ and hence elements of orders 6 and $2r$, contrary to assumption. Thus, we may assume that $L = PSL(3,3)$ or $PSU(3,3)$. We readily check that $PSU(3,3)$ contains elements of order 12 (cf. [11]), completing the case when p is odd.

Now suppose that $p = 2$. Now $SL(3, 2^n)$ contains a subgroup isomorphic to $GL(2, 2^n)$, whence $PSL(3, 2^n)$ contains $H = J \times C$, where $J = SL(2, 2^n)$ and C is cyclic of order $2^n - 1$ or $\frac{2^n-1}{3}$. If $n > 2$, this contradicts Lemma 1.3. Similarly, $SU(3, 2^n)$ contains a subgroup isomorphic to $GU(2, 2^n)$, whence $PSU(3, 2^n)$ contains $H_1 = J_1 \times C_1$ with $J_1 = SL(2, 2^n)$ and C_1 cyclic of order $2^n + 1$ or $\frac{2^n+1}{3}$. If $n > 1$, this again contradicts Lemma 1.3. Finally, $PSp(4, 2^n) = Sp(4, 2^n)$ contains a subgroup isomorphic to $GL(2, 2^n)$, again giving a contradiction when $n > 1$.

We know that $PSL(4, 2) = A_8$, $PSU(4, 2) = PSp(4, 3)$ and $G_2(2)' = U_3(3)$. By inspection in [11], $Sp(6, 2)$ and ${}^2F_4(2)'$ have elements of orders 6 and 10. We conclude that the only examples with $p = 2$ are $PSL(3, 2) = PSL(2, 7)$, $PSL(3, 4)$ and $Sp(4, 2)' = A_6$.

Finally, suppose that $L = PSL(2, q)$ and $q \equiv \varepsilon \pmod 8$, $\varepsilon = \pm 1$. Then L has a cyclic subgroup of order $\frac{q-\varepsilon}{2}$. If r is an odd prime divisor of $q - \varepsilon$, then L has elements of order $2r$ and $4r$, contrary to assumption. Hence, q is a Fermat or Mersenne prime, or $q = 9$, completing the proof. □

Lemma 2.8 *Suppose that $F^*(G) = L$ is a finite nonabelian simple group and $a_G = b_{G/L}$. Then G is isomorphic to one of the following groups:*

(1) *$PSL(2, q)$, $q \in \{5, 7, 8, 9, 11, 13, 17\}$; or*

(2) *$Sz(8)$, $Sz(32)$, A_7, $PSL(3, 3)$, $PSL(3, 4)$, M_{11} or M_{22}; or*

(3) *$PGL(2, 5)$, $PGL(2, 7)$, $P\Gamma L(2, 8)$, M_{10}, $\mathrm{Aut}(A_6)$, $P\Gamma L(2, 27)$ or the extension $PSL(3, 4) = PSL(3, 4) \rtimes \langle u \rangle$ of $PSL(3, 4)$ by an involutory graph-field automorphism u of order 2.*

If G is a CP group, then G is isomorphic to one of the following groups: $PSL(2, q)$, $q \in \{5, 7, 8, 9, 17\}$; or $Sz(8)$, $Sz(32)$, $PSL(3, 4)$ or M_{10}. Moreover, if $G = \mathrm{Aut}(A_6)$ or $P\Gamma L(2, 27)$, then $a_G = 2$. In all other cases, $a_G \leq 1$.

Proof Certainly L is one of the groups listed in Lemma 2.7. First, suppose that $G \neq L$. Note that the hypotheses imply that for any $x \in G$, all NPP elements of the coset Lx have the same order. By easy inspection (or making use of [11]), we see that if

$$L \in \{PSL(3, 3), PSL(3, 4), A_7, M_{11}, M_{22}\},$$

then $L = PSL(3, 4)$ and G is as described.

Suppose next that $L = Sz(2^n)$ and let $x \in G \setminus L$ be of prime order p. Then x induces a field automorphism on L and p divides n, whence p is odd. But

$C_L(x)$ has a subgroup $H = Sz(2)$, which has a cyclic subgroup of order 4. Hence G has elements of orders $2p$ and $4p$, contrary to assumption.

Suppose that $L = PSL(2, p^n)$. If $x \in G \setminus L$ has prime order r and induces a field automorphism on L, then $C_L(x)$ contains a subgroup $H = PSL(2, p)$. If $p > 3$, then Lx contains elements of orders $2r$, $3r$ and pr, at least two of which are not prime powers, a contradiction. Hence $p \in \{2, 3\}$ and Lx contains elements of orders $2r$ and $3r$, whence $r \in \{2, 3\}$ and $C_L(x)$ is a $\{2, 3\}$-group. Hence $C_L(x) = PSL(2, 2)$, $PSL(2, 3)$ or $PGL(2, 3)$. Thus $p^n \in \{4, 8, 9, 27\}$. Now by inspection, we get the cases listed in Lemma 2.8.

Suppose now that $L = PSL(2, q)$ for $q > 9$, and G has no element inducing a non-trivial field automorphism on L. As $L \neq G$, it follows that q is odd. Then by Lemma 2.7, q is an odd power of a prime and so $G = PGL(2, q)$. Let $q \equiv \varepsilon \pmod 4$, $\varepsilon = \pm 1$. Then $G \setminus L$ has an element x of order $q + \varepsilon$, and two elements y, y' in $\langle x \rangle$ are G-conjugate if and only if $y' = y^{-1}$. However Lx contains $\varphi(q + \varepsilon)$ elements of order $q + \varepsilon$, whence $\varphi(q + \varepsilon) = 2$ and $q + \varepsilon = 6$, contrary to the assumption that $q > 9$.

As $PSL(2, 4) \cong PSL(2, 5)$, we conclude the following: if $L = PSL(2, q)$, then $q \in \{5, 7, 8, 9, 27\}$, as claimed. The precise possibilities for G as stated in the proposition may then be inferred easily from [11].

Now suppose that $G = L = PSL(2, q)$. First we make a numerical remark. Suppose $2^n + 1 = 3^m$ for some natural numbers m and n. If m is odd, then $3^m - 1 \equiv 2 \pmod 8$ and so $m = 1$. If $m = 2r$, then $2^n = (3^r - 1)(3^r + 1)$ and so $m = 2$.

Now suppose that $G = PSL(2, q)$ with $q = 2^n > 8$. Then G has cyclic subgroups D_1 and D_2 of orders $2^n - 1$ and $2^n + 1$ respectively. If n is odd, then 3 divides $2^n + 1$, but $2^n + 1$ is not a power of 3 by the first paragraph. Hence $\mathrm{NPP}(G) \cap D_2$ contains $\varphi(2^n + 1)$ elements of order $2^n + 1$, lying in $\frac{\varphi(2^n+1)}{2}$ real classes. Thus, $\varphi(2^n + 1) \leq 2$ because $a_G \leq 1$, whence $2^n + 1 = 3$, a contradiction. Thus $n = 2s$ is even and 3 divides $2^n - 1 = (2^r - 1)(2^r + 1)$. As $n > 2$, D_1 is not a 3-group and, as above, $\frac{\varphi(2^n+1)}{2} \leq 1$. Then $2^n - 1 = 3$, again a contradiction.

Finally suppose that $G = PSL(2, q)$ with q odd and $q > 17$. Then G has cyclic subgroups T_1 and T_{-1} of orders $\frac{q-1}{2}$ and $\frac{q+1}{2}$ respectively. If $q \equiv 3 \pmod 8$ and $q \equiv \varepsilon \pmod 4$, then $T_\varepsilon \cap \mathrm{NPP}(G)$ has $\varphi(\frac{q-\varepsilon}{2})$ elements of order $\frac{q-\varepsilon}{2}$ lying in $\frac{\varphi(\frac{q-\varepsilon}{2})}{2}$ real classes. Hence $\varphi(\frac{q-\varepsilon}{2}) = 2$, whence $\frac{q-\varepsilon}{4} = 3$, a contradiction.

Hence by Lemma 2.7, q is a Fermat or Mersenne prime. Again assume that $q \equiv \varepsilon \pmod 4$. As $q \neq 3$, 3 divides $q + \varepsilon$. Suppose that $q + \varepsilon = 2 \cdot 3^m$ for some $m \geq 2$. As $q - \varepsilon = 2^k$, we have $2^k + 2\varepsilon = 2 \cdot 3^m$. Hence $2^{k-1} = 3^m - \varepsilon$. If $\varepsilon = -1$, then $3^m + 1 \equiv 2$ or 4 (mod 8), whence $q \leq 9$, contrary to assumption. If $\varepsilon = 1$, then by the first paragraph, $m \leq 2$ and $q \leq 17$, again a contradiction. It follows that $\frac{q+\varepsilon}{2}$ is not a prime power. But then $T_{-\varepsilon} \cap \mathrm{NPP}(G)$ has $\varphi(\frac{q+\varepsilon}{2})$ elements of order $\frac{q+\varepsilon}{2}$ lying in $\frac{\varphi(\frac{q+\varepsilon}{2})}{2}$ real classes. As usual this implies that $\varphi(\frac{q+\varepsilon}{2}) = 2$, again a contradiction.

Finally suppose that $G = L = Sz(2^n)$ with $n \geq 7$ and set $q = 2^n$. Then G has cyclic subgroups T_ε with $|T_\varepsilon| = q + \varepsilon\sqrt{2q} + 1$, for $\varepsilon = \pm 1$. As $q = 2^n$, n odd, we have that 5 divides $q^2 + 1 = |T_1||T_{-1}|$. Thus 5 divides $|T_\varepsilon|$ for some ε. We shall argue that $|T_\varepsilon|$ is not a power of 5 when $n \geq 7$. For suppose that it is. Let $n = 2m + 1$. Then

$$q + \varepsilon\sqrt{2q} + 1 = 2^{2m+1} + \varepsilon 2^{m+1} + 1 = 5^k$$

for some $k \geq 1$. Consideration of the 2-part of $5^k - 1$ shows that the smallest positive k for which $5^k - 1$ is divisible by 2^{m+1}, $m \geq 1$, is $k = 2^{m-1}$. But $2^{2m+1} + 2^{m+1} + 1 < 2^{2m+2}$, while $5^{2^{m-1}} > 4^{2^{m-1}} = 2^{2^m}$. Thus for equality to hold, we must have $2^m < 2m + 2$, which holds only for $m \leq 2$, i.e. for $n \leq 5$.

Thus for $G = Sz(2^n)$, $n \geq 7$, the cyclic subgroup T_ε is generated by elements in $\mathrm{NPP}(G)$. Let $h = |T_\varepsilon|$. As $N_G(T_\varepsilon)/T_\varepsilon$ has order 4 and $a_G = 1$, we must have $\varphi(h) = 4$, whence $h = 5$, a final contradiction. □

Henceforth, we assume that G is a finite Oliver group. Moreover, we assume that G satisfies the following two properties:

(1) all elements of $\mathrm{NPP}(G)$ have the same order; and

(2) if $K \trianglelefteq G$ and $K \cap \mathrm{NPP}(G) \neq \emptyset$, then $\mathrm{NPP}(G) \subseteq K$.

We shall call G an *EP group* if the properties (1) and (2) above hold. Of course, both of these properties hold when $a_G \leq 1$. By Lemma 1.2, the class of EP groups is closed under taking subgroups and homomorphic images.

Lemma 2.9 *Suppose that G is an EP group and $F(G)$ is not a p-group. Then G is solvable and the following conclusions hold:*

(1) *$F(G) = P \times Q$ with P an elementary abelian p-group of order p^a and Q an elementary abelian q-group of order q^b; and either*

(2) *$G/F(G)$ is an r-group of r-rank 1, r prime, with $r \notin \{p, q\}$; or*

(3) *$G/F(G)$ is a nonabelian metacyclic Frobenius group of order $p^c q^d$.*

Proof Clearly $\mathrm{NPP}(G) \subseteq Z(F(G))$, whence $F(G) = P \times Q$ with P and Q elementary abelian, as in (1). If $G = F(G)$, then G is not an Oliver group, contrary to assumption. Therefore $G \neq F(G)$. Set $\overline{G} = G/F(G)$. If $\overline{G}$ has r-rank greater than 1 for some prime r, then from Theorem 2.3 it follows that $G \setminus F(G)$ contains elements of order rs for $s \in \{p,q\} \setminus \{r\}$, a contradiction to $\mathrm{NPP}(G) \subseteq F(G)$. So $\overline{G}$ has r-rank 1 for every prime divisor r of $|\overline{G}|$.

Now, suppose that $F(\overline{G}) = 1$. Then by the Feit–Thompson Theorem, $\overline{G}$ has no nontrivial normal subgroup of odd order. Moreover $\overline{G}$ has 2-rank 1, whence by the Brauer–Suzuki Theorem, $1 \neq Z(\overline{G}) \leq F(\overline{G})$, a contradiction. Thus $F(\overline{G}) \neq 1$ and, as $\mathrm{NPP}(\overline{G}) = \varnothing$, $F(\overline{G}) = \overline{R}$ is an r-group of r-rank 1 for some prime r. Moreover, note that $C_{\overline{G}}(Z(\overline{R}))$ is a normal r-subgroup of $\overline{G}$, whence $C_{\overline{G}}(Z(\overline{R})) = \overline{R}$. Moreover, note that $\overline{G}/\overline{R}$ is isomorphic to a cyclic r'-subgroup of $\mathrm{Aut}(Z(\overline{R}))$. If $\overline{G} = \overline{R}$, then $r \notin \{p,q\}$ because G is an Oliver group, and (2) holds. Otherwise $\overline{G} = \overline{R} \rtimes \overline{C}$ with both $\overline{R}$ and $\overline{C}$ cyclic. As in Lemma 2.5, C is an s-group for some prime $s \neq r$. Choose $s \in \{p,q\} \setminus \{r\}$. As $\overline{R} \rtimes \overline{C}$ is a Frobenius group, $C_P(\overline{C}) \neq 1$ and so $s = p$. If also $r \neq q$, then the same argument would yield $s = q$, a contradiction. Hence $r = q$ and $s = p$ with $p < q$, whence (3) holds. □

Lemma 2.10 *Suppose that G is a finite Oliver group with $a_G \leq 1$ and with $F(G)$ not a p-group. Then $F(G) = C_2^2 \times C_3$ and one of the following holds:*

(1) $G = \mathrm{Stab}_{A_7}(\{1,2,3\})$; *or*

(2) $G = C_2^2 \rtimes D_9$.

Proof We continue the notation of Lemma 2.9, and we note that the following holds: $|\mathrm{NPP}(G)| = (p^a - 1)(q^b - 1)$. Moreover, $\mathrm{NPP}(G)$ is a union of one or two G-classes of equal cardinality.

If $|\overline{G}| = r^c$ for some prime r, and $R \in \mathrm{Syl}_r(G)$, then $C_R(P) = 1 = C_R(Q)$, whence $r^c \geq \min\{p^a - 1, q^b - 1\}$. On the other hand, $(p^a - 1)(q^b - 1) \leq 2r^c$, which is a contradiction.

Hence $|\overline{G}| = p^c q^d$ with $p < q$. Let $F(\overline{G}) = \overline{R}$ with $|\overline{R}| = q^d$, and let R be the full preimage of $\overline{R}$ in G. Then, as $F(G)$ is abelian, $1 \neq C_Q(R) \trianglelefteq G$. As all elements of $F(G)$ of order pq lie in the same real G-conjugacy class, this forces $C_Q(R) = Q$. Moreover $\overline{R}$ acts semi-regularly on $P^\#$, whence q^d divides $p^a - 1$. Let $\overline{S} \in \mathrm{Syl}_p(\overline{G})$. Then $\overline{S}$ acts semi-regularly on $Q^\#$, whence p^c divides $q^b - 1$. Finally $(p^a - 1)(q^b - 1) = p^c q^d$ or $2p^c q^d$. If both p and q are odd, then 4 divides the left-hand side of the equation but not the right. Hence $p = 2$. Then both $q^b - 1$ and $q^d + 1$ are powers of 2, whence $q = q^d = 3$ and $p^a = 4$. As $\overline{G}$ acts faithfully on P, it follows that $\overline{G} = S_3$. In particular $p^c = 2$, whence $q^b = 3$. Thus $F(G) = C_2^2 \times C_3$. Moreover, if $R_0 \in \mathrm{Syl}_3(G)$, then $G = P \rtimes N_G(R_0)$ and R_0 is inverted by an involution in $N_G(R_0)$. Thus either (1) or (2) holds. □

We keep our assumption that G is a finite Oliver group G. Recall that G is an EP group if all elements of $\mathrm{NPP}(G)$ have the same order, and the following holds: if $K \trianglelefteq G$ and $K \cap \mathrm{NPP}(G) \neq \varnothing$, then $\mathrm{NPP}(G) \subseteq K$.

Lemma 2.11 *If G is an EP group, then one of the conclusions holds:*

(1) *$F^*(G)$ is a p-group for some prime p; or*

(2) *$F^*(G)$ is one of the nonabelian simple groups listed in Lemma 2.7; or*

(3) *G is solvable and satisfies the conclusions of Lemma 2.9.*

Moreover if $a_G \leq 1$, then either $F^*(G)$ is a p-group or one of the conclusions of Lemma 2.8 or 2.10 holds.

Proof Suppose that L is a normal quasisimple subgroup of $F^*(G)$. Then, by Burnside's $p^a q^b$ Theorem, there exist distinct primes p, q and r dividing $|L|$. Thus if $C_{F^*(G)}(L) \neq 1$, then $\mathrm{NPP}(G)$ contains elements of two distinct orders, a contradiction. Hence $C_{F^*(G)}(L) = 1$, whence $L = F^*(G)$ is a nonabelian simple group and one of the conclusions of Lemma 2.7 (resp. 2.8) holds. On the other hand, if $F^*(G) = F(G)$, then either $F^*(G)$ is a p-group or one of the conclusions of Lemma 2.9 (resp. 2.10) holds, as claimed. □

Henceforth, we shall assume that $F^*(G) = P$ is a p-group. Clearly $G \neq P$ and we set $\overline{G} = G/P$. Also we let L be the full pre-image in G of $F^*(\overline{G})$.

Lemma 2.12 *Suppose that G is an EP group. Then either $\overline{L}$ is a q-group for some prime $q \neq p$ or $\overline{L}$ is a nonabelian simple group.*

Proof Suppose that the conclusion of Lemma 2.12 does not hold. Then, by arguing as in Lemmas 1.1 and 1.2, we see that $\overline{G}$ is an EP group, provided $\overline{G}$ is an Oliver group. If L is nonsolvable, then $\overline{G}$ is an Oliver group, whence Lemma 2.11 applied to $\overline{G}$ yields that $\overline{L}$ is a nonabelian simple group.

Suppose that L is solvable but $\overline{L}$ is not a q-group for any prime q. Since all elements of $\mathrm{NPP}(G)$ have the same order, $\overline{L} = \overline{Q} \times \overline{R}$ where Q is an elementary abelian q-group and R is an elementary abelian r-group for some primes q, with r different from p. Moreover, G contains no elements of order pq or pr, whence $|Q| = q$ and $|R| = r$. Since $\overline{L} = F^*(\overline{G})$ and $\overline{L}$ is cyclic of order qr, we conclude that G/L is abelian. As $\mathrm{NPP}(G) \subseteq L$, in fact G/L is abelian of prime power order. But then as P is a p-group and L/P is cyclic, G is not an Oliver group, contrary to assumption. □

Henceforth, we assume that G is a finite Oliver group with Laitinen number $a_G \leq 1$. In the next three lemmas, we treat the case where $\overline{L}$ is a q-group.

Lemma 2.13 *If $\overline{L}$ is a q-group of q-rank 1, then $q = 2$ and $G = P \rtimes K$, where $K = SL(2,3)$ or $\hat{S}_4$ and P is an abelian p-group of odd order inverted by the unique involution of K.*

Proof Suppose that $\overline{L}$ is a cyclic q-group. As $\overline{L} = F^*(\overline{G})$, $\overline{G}/\overline{L}$ is a cyclic q'-group and $\overline{G}$ is a metacyclic Frobenius group with kernel $\overline{L}$. If $\overline{x} \in \overline{G} \setminus \overline{L}$, $C_P(\overline{x}) \neq 1$ whence G has elements of order pr, where r is the order of x and $(r,p) = 1$. In this case $\overline{G}/\overline{L}$ has prime order r, and otherwise $\overline{G}/\overline{L}$ is a p-group. In either case, as $\overline{L}$ is cyclic, G is not an Oliver group, a contradiction.

Hence $\overline{L} = Q_8$ and $[\overline{G},\overline{G}] = SL(2,3)$. As $\mathrm{NPP}(\overline{G}) \neq \emptyset$, P is inverted by z for any involution z of L. As $P = C_G(P)$, it follows that $G = P \rtimes K$ and K contains a unique involution, whence the lemma holds. □

Lemma 2.14 *Suppose that $\overline{L}$ is a q-group of q-rank greater than 1. Then $\overline{G}$ is a solvable group without NPP elements. Moreover, P is a finite elementary abelian p-group and $H = P \rtimes Q \rtimes C$, where $L = P \rtimes Q$, $Q \in \mathrm{Syl}_q(G)$ and $N_G(Q) = Q \rtimes C$ is a Frobenius group with kernel Q and complement C such that C is a p-group.*

Proof Let $Z = \{z \in Z(P) : z^p = 1\}$. Then Z is a nontrivial elementary abelian normal p-subgroup of G. Let E be an elementary q-subgroup of L of q-rank greater than 1. Then, according to Theorem 2.3, $\mathrm{NPP}(L)$ contains an element x of order pq with $x^q \in Z$. Hence $\mathrm{NPP}(G/Z) = \emptyset$ and so, applying Theorem 2.3 again in G/Z, we conclude that $P = Z$. If $L = G$, then G is not an Oliver group, contrary to assumption. Thus $L \neq G$.

Let $L = P \rtimes Q$, $Q \in \mathrm{Syl}_q(L)$. Suppose that $C_P(Q) = A \neq 1$. Then every element $x \in \mathrm{NPP}(G)$ satisfies $x^q \in A$. But then $C_P(e) = A$ for all $e \in E^\#$, whence $P = A$ by Theorem 2.3. But then $Q \leq C_G(P) = P$, a contradiction. Therefore $C_P(Q) = 1$ and by making use of a Frattini argument, we see that $N_G(Q)$ is a complement to P in G.

Let $N = N_G(Q)$. Then N is without NPP elements. Suppose that r is a prime divisor of $|N|$ with $r \notin \{p,q\}$ and let R be a nontrivial r-subgroup of N. As $\mathrm{NPP}(N) = \varnothing$, it follows that $Q \rtimes R$ is a Frobenius group with kernel Q acting faithfully on P. Hence $C_P(R) \neq 1$, a contradiction. Hence N is a $\{p,q\}$-group. In particular, N is solvable by Burnside's theorem and so Lemma 2.5 applies to N, yielding that either $N = Q \rtimes C$ is a Frobenius group with kernel Q and complement C a p-group, as claimed, or $N = Q \rtimes C \rtimes A$ with C a cyclic p-group and A a cyclic q-group disjoint from Q (as $CA = N_N(C)$ is a complement to Q in N). Suppose the latter and let $y \in A$ of order q and $z \in Q \cap Z(QA)$ of order q. Then $U = \langle y, z\rangle = C_q \times C_q$ and so $P = \langle C_P(u) : u \in U^{\#}\rangle$ by Theorem 2.3. However $\mathrm{NPP}(G) \subseteq L$ and $U \cap L = \langle z\rangle$, whence $C_P(u) = 1$ for all $u \in U \setminus \langle z\rangle$. Thus $P = C_P(z)$, a contradiction. □

Finally, we complete the analysis of the case when $\overline{L}$ is a q-group.

Lemma 2.15 *Suppose that $G = P \rtimes Q \rtimes C$ as in Lemma 2.14. Then one of the following conclusions hold:*

(1) $P = C_3^3$ *and* $QC = A_4$*; or*

(2) $P = C_2^4$, $PQ = A_4 \times A_4$ *and* $C = C_4$*; or*

(3) $P = C_2^8$ *and* $QC = (C_3 \times C_3) \rtimes C_8$*; or*

(4) $P = C_2^8$ *and* $QC = (C_3 \times C_3) \rtimes Q_8$.

Proof Let $x \in Q$ of order q with $C_P(x) = V$ of maximum order. The elements of $V^{\#}x$ are in $\mathrm{NPP}(G)$. As $(vx)^{-1} \in Vx^{-1}$, either $C_G(x)$ is transitive on $V^{\#}x$ or $q = 2$ and $C_G(x)$ has two equal-sized orbits on $V^{\#}x$. In any case, as QC is a Frobenius group, $C_G(x) = VC_Q(x)$ and $V\langle x\rangle$ acts trivially on $V^{\#}x$. Hence $|V^{\#}| = |V^{\#}x| = q^b$ for some $b \geq 0$. Let $|V| = p^a$. Then either $p = 2$ and $b = 1$ or $q = 2$ and $p^a \in \{p, 9\}$. In all cases we set $Z = \{z \in Z(Q) : z^q = 1\}$. Thus Z is a normal elementary abelian q-subgroup of QC and $PZ \trianglelefteq G$.

Suppose first that $q = 2$. As ZC is a Frobenius group, Z contains a Klein 4-subgroup U and so $\mathrm{NPP}(G) \subseteq PZ \trianglelefteq G$. Thus in particular $x \in Z$. Suppose further that $|V| = p$. Then $|P| \leq p^3$. Moreover, as P is a faithful QC-module, it follows that $|C| \leq \dim V \leq 3$, whence $p = 3 = \dim V$ and $P = C_3^3$. Now, QC is isomorphic to a Frobenius $\{2,3\}$-subgroup of $SL(3,3)$, and thus $QC = A_4$. Therefore (1) holds.

Next suppose that $q = 2$ and $|V| = 9$. Note that $C_Q(v) \leq Z$ for all $v \in P^{\#}$. In particular $C_Q(V) = A$ is an elementary abelian q-group with $C_P(a) = V$ for all $a \in A^\#$, by maximal choice of V. If $A \neq \langle x\rangle$, then by Theorem 2.3, $P = V$, a contradiction. Hence $C_Q(V) = \langle x\rangle$ and $Q/\langle x\rangle$ is isomorphic to a subgroup of $GL(V) = GL(2,3)$. In particular $|Q| \le 32$. As C is fixed point free on Q, $|Q| = 2^m$, m even. As $|Q/Z| \le 4$, $[Q,Q]$ is cyclic, whence $[Q,Q] = 1$ and $Q = C_2^2$, C_2^3 or $C_4 \times C_4$. On the other hand, $Q/\langle x\rangle$ is isomorphic to an abelian subgroup of $GL(2,3)$ of order 8, whence $Q/\langle x\rangle = C_8$, a contradiction.

Finally suppose that $p = 2$ and $|V| = 2^a = q + 1$. As QC is a Frobenius group with C a 2-group, Q is abelian. Note that C permutes the set

$$\mathcal{Z} = \{z \in Z^\# : C_P(z) \neq 1\}$$

in one or two equal-sized orbits. As $z \in \mathcal{Z}$ if and only if $\langle z\rangle^\# \subseteq \mathcal{Z}$, it follows that $|\mathcal{Z}| = k(q-1)$, $k \ge 1$. But also, as $|C|$ is a power of 2, $|\mathcal{Z}| = 2^c$ for some $c \ge 1$. Hence $q = 2^d + 1 = 2^a - 1$, whence $q = 3$ and $|V| = 4$. Now P is a completely reducible Z-module and as before $C_P(Z) = 1$, whence $P = P_1 \times \cdots \times P_r$ with P_i an irreducible Z-module and $|P_i| = 4$ for all i. Then $C_Z(P_1 \times P_2) = 1$, whence Q acts faithfully on $P_1 \times P_2 = C_2^4$. Hence $Q = Z = C_3 \times C_3$. As QC is a Frobenius group, either $C = Q_8$ or C is cyclic with $|C| \le 8$. In any case the involution of C inverts Q. By Theorem 2.3, at least two cyclic subgroups of Q have non-trivial fixed points on P and as $a_G = 1$, C permutes these subgroups transitively. Hence $|C| \ge 4$. If $|C| = 4$, then only two cyclic subgroups of Q have non-trivial fixed points on P and so $|P| = 16$ and $L = A_4 \times A_4$. Therefore (2) holds.

If $|C| = 8$, then $\dim V \ge 8$. On the other hand, $\dim V \le 8$ as Q has only four cyclic subgroups of order 3. Therefore, equality holds and G is as described either in (3) or (4), completing the proof. □

We have now completed the analysis of the case where $\overline{L}$ is a q-group. Thus, for the remainder of the analysis in the proof of Theorem 2.1, we may assume that $\overline{L}$ is a nonabelian simple group.

Lemma 2.16 *If the p-group P is of odd order, then P is elementary abelian and one of the following conclusions holds:*

(1) $p = 3$ *and* $\overline{G} = PSL(2,q)$, $q \in \{5,7,9,17\}$, *or* $\overline{G} = M_{10}$*; or*

(2) $p = 7$ *and* $\overline{G} = SL(2,8)$ *or* $Sz(8)$*; or*

(3) $p = 31$ *and* $\overline{G} = Sz(32)$.

Proof Let $Z = \{z \in Z(P) : z^p = 1\}$. Then by Theorem 2.3, G contains an element x of order $2p$ with $x^2 \in Z$ and so every element of NPP(G) has this property. In particular $P = Z$ is elementary abelian, as usual. Moreover $\mathrm{NPP}(\overline{G}) = \varnothing$ and so $\overline{G} = PSL(2,q)$, $q \in \{5,7,8,9,17\}$, M_{10}, $Sz(8)$, $Sz(32)$ or $PSL(3,4)$. If $\overline{G}$ contains a subgroup isomorphic to $C_3 \times C_3$ or to A_4, then G contains elements of order $3p$ and so $p = 3$.

Thus if $p > 3$, then $G = SL(2,8)$, $Sz(8)$ or $Sz(32)$. Moreover, consideration of the Borel subgroups of $\overline{G}$ in these cases shows that $p = 7$, 7 or 31, respectively, as claimed.

Suppose finally that $p = 3$. If $\overline{G} = SL(2,8)$, $Sz(8)$, $Sz(32)$ or $PSL(3,4)$, then $\overline{L}$ contains a Frobenius group with kernel a 2-group and complement of order 7, 7, 31 or 5, respectively. But then G would contain elements of order 21, 21, 93 or 15, respectively, a contradiction. □

Lemma 2.17 *$F^*(G)$ is a 2-group.*

Proof Suppose first that $p > 3$. Let $U \in \mathrm{Syl}_2(G)$. Then by consideration of the Frobenius group $\overline{B} = N_{\overline{G}}(\overline{U})$, we have $\dim(P) \geq p$. Hence if $V = C_P(z)$ for $z \in Z(U)^\#$, then $\dim(V) \geq \frac{1}{3}p > 2$. On the other hand $C_G(z)/V\langle z\rangle$ permutes $V^\#$ in at most two equal orbits. Hence $p^3 - 1 \leq |U|$, which is false in all cases.

Thus we may assume that $p = 3$ and $G = PSL(2,5)$, $PSL(2,7)$, $PSL(2,9)$, $PSL(2,17)$ or M_{10} and with $\dim(P) \geq 4$, 6, 4, 16, 8, respectively. Again let $U \in \mathrm{Syl}_2(G)$, $z \in Z(U)^\#$ and $V = C_P(z)$. Then $C_G(z) = VU$. Thus, as above, if $\dim(V) = d$, then $3^d - 1$ is a power of 2, whence $d \leq 2$ and so $\dim(P) \leq 3d \leq 6$. Hence $\dim(V) = 2$, $\dim(P) = 6$ and $|U| \leq 8$. Thus $\overline{G} = PSL(2,7)$ or $PSL(2,9)$. However, in both cases, $U/\langle z\rangle = C_2 \times C_2$, which cannot act semiregularly on $V^\#$ by Theorem 2.3, a contradiction. □

Lemma 2.18 *One of the following conclusions holds:*

(1) $\overline{G} = PSL(2,q)$, $q \in \{5,7,8,9,17\}$; *or*

(2) $\overline{G} = Sz(8)$, $Sz(32)$, $PSL(3,4)$, $PGL(2,5)$ *or* M_{10}.

Proof If $\mathrm{NPP}(\overline{G}) = \varnothing$, $\overline{G}$ is listed above. So, assume that $\mathrm{NPP}(\overline{G}) \neq \varnothing$. Then G has no element x of odd order, such that $C_P(x) \neq 1$. In particular, it follows from Theorem 2.3 that G has a cyclic Sylow 3-subgroup, whence $\overline{G} = PGL(2,5)$, $PGL(2,7)$, $PSL(2,11)$ or $PSL(2,13)$. Note that in the last three cases, $\overline{G}$ contains a Frobenius subgroup of order 21, 55 or 39, respectively, whence G contains elements x of order 3, 5, 3, respectively, with $C_P(x) \neq 1$, which is a contradiction. Hence $\overline{G} = PGL(2,5)$. □

Lemma 2.19 *If $\overline{G}$ is one of the groups $PSL(2,7)$, $PSL(2,9)$, $PSL(2,17)$, $PSL(3,4)$, or M_{10}, then one of the following conclusions holds:*

(1) $P = C_2^3$ *and* $\overline{G} = GL(3,2)$*; or*

(2) $P = C_2^4$ *and* $\overline{G} = A_6 = Sp(4,2)'$*; or*

(3) $P = C_2^8$ *and* $\overline{G} = M_{10}$.

Proof Suppose that $\overline{G} = PSL(2,9)$, M_{10} or $PSL(3,4)$. Let $T \in \mathrm{Syl}_3(G)$. Then $T = C_3 \times C_3$ and $N_{\overline{G}}(\overline{T}) = \overline{T} \rtimes \overline{Q}$ with $\overline{Q} = C_4$, Q_8, Q_8, respectively, and with $\overline{T} \rtimes \overline{Q}$ a Frobenius group. Hence $\dim(P) \geq 4, 8, 8$, respectively. On the other hand, if $x \in T^\#$ and $V = C_P(x)$, then $C_G(x) = VT$ acts transitively on $V^\#$, whence $\dim(V) \leq 2$. As $\overline{Q}$ transitively permutes the set $\mathcal{T}$ of non-identity cyclic subgroups $\langle y \rangle$ of T with $C_P(y) \neq 1$, we have that $|\mathcal{T}| = 2, 4, 4$, respectively. Hence $\dim(P) \leq 4, 8, 8$, respectively, whence equality holds in all cases. But then if $\overline{G} = PSL(3,4)$ and $g \in G$ of order 7, then $C_G(y) \neq 1$, whence G contains elements of order 6 and 14, a contradiction.

Next suppose that $\overline{G} = PSL(2,7)$. Let $x \in G$ be an element of order 3. Then $\langle x \rangle$ is a Frobenius complement in a subgroup F of order 21, whence $C_P(x) \neq 1$. Thus G contains elements of order 6 and therefore G contains no elements of order 14. So P is a sum of faithful F-module, hence a sum of free $\langle x \rangle$-modules. On the other hand, as $C_G(x) = C_P(x)\langle x \rangle$, we must have $|C_P(x)| = 2$. Hence P is a single free $\langle x \rangle$-module, i.e. $|P| = 8$. Finally suppose that $\overline{G} = PSL(2,17)$. By inspection of the 2-modular character table for $\overline{G}$, we see that if $x \in G$ of order 3, then $\dim(C_P(x)) \geq 3$. But $C_G(x) = C_P(x)X$ with $|X| = 9$, whence $C_G(x)$ is not transitive on $C_P(x)^\#$, a contradiction. □

Completion 2.20 Now, we complete the proof of Theorem 2.1 as follows. The possibilities for $\overline{G}$ listed in Lemma 2.18 and not discussed in Lemma 2.19 are precisely those groups which are listed in the final conclusion (13) of the Classification Theorem. For each of these cases, if $x \in G$ is of odd order, then $\hat{C} = C_G(x)/\langle C_V(x), x \rangle$ must transitively permute $C_P(x)^\#$. However $|\hat{C}| \leq 2$, except in the cases when $\overline{G} = SL(2,8)$ or $Sz(32)$ and both x and $\hat{C}$ have order $p = 3$ or 5, respectively. Consideration of the 2-modular representations of these two groups shows that if W is an nontrivial irreducible 2-modular representation of $\overline{G}$ with $C_W(x) \neq 0$, then $|C_W(x)| > p + 1$, and so $\hat{C}$ cannot act transitively on $C_P(x)^\#$. Thus in all of these cases, we must have $C_V(x) = 0$ for all $x \in G$ of odd order. Let $H = G^2$. Thus $H = G$ in all cases, except when $\overline{G} = \Sigma L(2,4) = S_5$. Then by the above remarks, H is a CP group and so by the theorem of Bannuscher–Tiedt [4], the structure of H is as specified in the Classification Theorem, completing the proof of Theorem 2.1. □

3 Proof of Theorem A3

The main goal of this section is to prove the following proposition which contains our next major result about the Laitinen number a_G (cf. Proposition 1.6).

Proposition 3.1 *Let G be a finite nonsolvable group. If $a_G = b_{G/G^{\mathrm{sol}}}$, then either $a_G \leq 1$ or $a_G = 2$ and $G = \mathrm{Aut}(A_6)$ or $P\Gamma L(2,27)$.*

By inspection in [11], we see that for $G = \mathrm{Aut}(A_6)$, $a_G = 2$ corresponding to elements of order 6 and 10, and for $G = P\Gamma L(2,27)$, $a_G = 2$ corresponding to elements of order 6 and 14.

Below, we assume that G is a finite nonsolvable group and we set $H = G^{\mathrm{sol}}$. As we know, we always have $a_G \leq b_{G/H}$. We shall analyze the situation where $a_G = b_{G/H}$. Clearly, in this situation each coset gH meets at most one real conjugacy class $(x)^{\pm 1}$ with $x \in \mathrm{NPP}(G)$.

The proof of Proposition 3.1 will proceed via a sequence of lemmas. As the arguments are very similar to those in Section 2, we shall be a bit sketchy. First, we remark that if $H = G$ (which amounts to saying that G is perfect), then $b_{G/H} \leq 1$ and there is nothing to prove because $a_G = 0$ (resp., 1) if and only if $b_{G/H} = 0$ (resp., 1). So, we may assume that $H < G$. Let S denote the solvable radical of G (i.e., S is the largest normal solvable subgroup of G).

Lemma 3.2 *$S \leq H$ and $G/S \cong PGL(2,5)$, $PGL(2,7)$, $P\Gamma L(2,8)$, M_{10}, $\mathrm{Aut}(A_6)$, $P\Gamma L(2,27)$ or $PSL(3,4)$.*

Proof Let S_0 be the solvable radical of H. Set $\overline{G} = G/S_0$ and note that as G is nonsolvable, $\overline{G}$ has a subnormal nonabelian simple subgroup $\overline{L} \trianglelefteq\trianglelefteq \overline{H}$. By Lemmas 2.5 (3) and 2.7, we see that $C_{\overline{G}}(\overline{L}) = 1$. Hence $\overline{L} = F^*(\overline{G}) \neq \overline{G}$. Then the possibilities for $\overline{G}$ follow from Lemma 2.8. As $\overline{S} = 1$, we see that $S_0 = S$ and the proof is complete. □

Lemma 3.3 *Either $S = 1$ or S is a p-group for some prime p.*

Proof Suppose that $S \neq 1$. Now, by Lemma 2.9, $F(G)$ is a p-group for some prime p. Let $\overline{G} = G/F(G)$ and $\overline{L} = F^*(\overline{G})$. Suppose that $\overline{L}$ is a nonabelian simple group. As G has only one nonabelian composition factor by Lemma 3.2, $H \leq L$, where L is the pre-image of $\overline{L}$ in G. Then $S = F(G)$ and therefore S is a p-group, as claimed.

Now, by Lemma 2.12, we may assume that $\overline{L}$ is a q-group for some prime $q \neq p$. As $C_{\overline{G}}(\overline{L}) \leq \overline{L}$, it follows that $\mathrm{Aut}(\overline{L})$ is nonsolvable, whence $\overline{L}$ has q-rank at least 2. Thus L contains elements of order pq and so every NPP element of H lies in L. Since either p or q is odd, it follows that H/L has 2-rank 1. But then by the Brauer–Suzuki Theorem, H/L contains NPP elements, whence so does $H \setminus L$, a contradiction. □

Now, we wish to prove that $S = 1$. In order to prove it, we assume the contrary and argue to a contradiction. Henceforth, we set $\overline{G} = G/S$ and $\overline{H} = H/S$ (remember $S \leq H$ by Lemma 3.2).

Lemma 3.4 *S is either a 2-group or an elementary abelian p-group for some odd prime p, and in the latter case, every NPP element of H has order $2p$.*

Proof As $\overline{H}$ contains a Klein 4-group, we may apply the usual argument to obtain the result. □

Lemma 3.5 *$\overline{H}$ is not isomorphic to $PSL(2,27)$.*

Proof Suppose that the contrary claim holds: $\overline{H}$ is isomorphic to $PSL(2,27)$. Then H contains an element x of order 14 with $x^{14} \notin S$. Hence H has no NPP element y with $y^r \in S$ for $r \in \{2,3\}$. However, as $\overline{H}$ has 2-rank 2 and 3-rank 3, this contradicts Theorem 2.3 for $r \neq p$. □

Lemma 3.6 *S is a 2-group and $\overline{G}$ is not isomorphic to M_{10}.*

Proof Suppose first that S is a 2-group and $\overline{G} = M_{10}$. Then every element of $G \setminus H$ is a 2-element, whence $b_{G/H} \leq 1$, contrary to hypothesis.

Thus it remains to prove that S is a 2-group. Suppose S is not a 2-group. Then S is an elementary abelian p-group for some odd prime p. Suppose that there is an involution $x \in G \setminus H$. Then by inspection x centralizes a coset Hy of odd order, whence Hx contains NPP elements outside Sx. But then Sx contains no NPP elements, i.e. x inverts S and $H = [H,x]$ centralizes S, a contradiction. Thus there is no involution in $G \setminus H$, and therefore we have the following two possibilities: $\overline{G} = P\Gamma L(2,8)$ or M_{10}.

Suppose now that $\overline{G} = P\Gamma L(2,8)$. As $\overline{H}$ contains a Frobenius subgroup of order 56, S must be a 7-group and H must contain elements of order 14. Thus a 3-element of H acts without fixed points on S. But then by Theorem 2.3, some element $x \in G \setminus H$ of order 3 must have fixed points on S and so Hx contains elements of orders 6 and 21, a contradiction.

Finally suppose that $\overline{G} = M_{10}$. Then $\overline{H}$ contains an A_4-subgroup and threfore S is a 3-group and the NPP elements of H have order 6. Let t be an involution of H. As H contains only one real G-class of NPP elements, $C_G(t)$ permutes transitively the nonidentity elements of $C_S(t)$. Now $|C_G(t)/C_S(t)| = 16$ and $C_S(t)$ acts trivially on itself by conjugation. Hence $|C_S(t)| \leq 9$. Moreover, as H contains elements of order 6, H contains no elements of order 15. Hence, an element of H of order 5 acts fixed point freely on S, whence $\dim(S)$ is a multiple of 4. By Theorem 2.3, on the other hand, $\dim(S) \leq 3\dim(C_S(t))$, whence $|C_S(t)| = 9$. Let T be a Sylow 2-subgroup of G with $t \in Z(T)$. As t acts trivially on $C_S(t)$, $T/\langle t\rangle$ must act regularly on the eight elements of $C_S(t) \setminus \{1\}$. But $T/\langle t\rangle$ is a dihedral group of order 8 and hence has no such regular action, a final contradiction. □

In Lemmas 3.7–3.11 below, x will be an element of H of order 3 with $U = C_S(x)$ and with $|U| = 2^a > 1$, if possible. Set $\widehat{C} = C_G(x)/\langle U, x\rangle$.

Lemma 3.7 *U is elementary abelian and $\widehat{C}$ transitively permutes the set $Ux \setminus \{x\}$ of cardinality $2^a - 1$. Moreover no chief H-factor in S is a trivial $\overline{H}$-module.*

Proof Note that $|H/S|$ is divisible by at least two odd primes and H has NPP elements of order $2p$ for at most one odd prime p. Therefore, no chief H-factor in S is a trivial $\overline{H}$-module.

If $U = 1$, the lemma holds trivially. Suppose $U \neq 1$. As all NPP elements of H have order 6, all elements of U have order 2 and so U is elementary abelian. Moreover all elements of $Ux \setminus \{x\}$ are G-conjugate, hence $C_G(x)$-conjugate and since $\langle U, x\rangle$ is contained in the kernel of the conjugation action on Ux, the result follows. □

Lemma 3.8 *$\overline{G}$ is not isomorphic to $PGL(2,5)$.*

Proof Suppose $\overline{G} = PGL(2,5)$. As $b_{G/H} = 2$, H must contain NPP elements. By using Lemma 3.7, we obtain that every chief H-factor of S is isomorphic to a 4-dimensional irreducible H/S-module. Thus if $y \in H$ of order 5, we have $C_S(y) = 1$. Hence H must have elements of order 6 and so with x and U as above, $U \neq 1$. Indeed some chief H-factor of S, say V, is a permutation module for $\overline{H}$ with $|C_V(x)| = 4$. Thus $|U| \geq 4$. But $|\widehat{C}| = 2$ and so $\widehat{C}$ does not act transitively on $Ux \setminus \{x\}$, contrary to Lemma 3.7. □

Lemma 3.9 *$\overline{H}$ is not isomorphic to $PSL(2,7)$.*

Proof Suppose that $\overline{H} = PSL(2,7)$. As $\hat{G}$ is a 2-group acting transitively on the involutions of U, $|U| \le 2$. According to [23, 2.8.10], the nontrivial irreducible $GL(3,2)$-modules are the standard 3-dimensional module V, its dual V^* and the Steinberg module, which is the nontrivial constituent of $V \otimes V^*$. As a 3-element of $GL(3,2)$ has 1-dimensional fixed point space on V and V^* and 2-dimensional fixed point space on the Steinberg module, it follows that S has a unique irreducible composition factor and this has dimension 3, i.e. $S = V$ or V^* as $\overline{H}$-module. But then, as $C_G(S) = S$ and $\mathrm{Aut}(S) = H/S$, we obtain that $G = H$, a contradiction. □

Lemma 3.10 *$\overline{H}$ is not isomorphic to $PSL(2,8)$.*

Proof Suppose $\overline{H} = PSL(2,8)$. Let $y \in G \setminus H$ be an element of order 3. Then yS lies in a complement of a Frobenius subgroup of G/S of order 21. Hence there exists $ty \in Hy$ of order 6 with $(ty)^3 \in S$. However there also exists $sy \in Hy$ of order 6 with $(sy)^3 \in H \setminus S$, a contradiction. □

Lemma 3.11 *$\overline{H}$ is not isomorphic to $PSL(2,9)$ or $PSL(3,4)$.*

Proof Suppose $\overline{H} = PSL(2,9)$ or $PSL(3,4)$. Then either $\overline{G} = \mathrm{Aut}(A_6)$ or $\overline{H} = PSL(3,4)$ with $|G : H| = 2$. In either case, $|\hat{G}| = 6$. Again as $\hat{G}$ acts transitively on the involutions of U, we conclude that $|U| \le 4$. Let E be a Sylow 3-subgroup of H. Then $E = \mathbb{Z}_3 \times \mathbb{Z}_3$ and $N_G(E)$ transitively permutes the elements of E of order 3. Hence $|C_T(y)| \le 4$ for all $y \in E \setminus \{1\}$ and so, by Theorem 2.3, $|S| \le 2^8$.

Suppose that $\overline{H} = PSL(3,4)$. Then $G \setminus H$ contains an element y such that $\overline{y}$ has order 2 and centralizes $N_{\overline{H}}(\overline{E})$ and $N_{\overline{G}}(\overline{E}) = \overline{EQ} \times \langle \overline{y} \rangle$ for some quaternion group Q of order 8 transitively permuting the nonidentity elements of E. Now $\overline{EQ}$ acts faithfully on $C_S(y)$ by Thompson's $A \times B$ Lemma (see [22, 11.7]). However, by Clifford Theory, a faithful $\overline{EQ}$ module must have dimension at least 8. As $\dim S \le 8$, this would force $C_S(y) = S$, which is absurd. Hence $\overline{G}$ is isomorphic to $\mathrm{Aut}(A_6)$. Again $N_{\overline{G}}(\overline{E})$ contains a subgroup $\overline{EQ}$ with $\overline{Q} = Q_8$, as above. Therefore $\dim(S) = 8$ and $C_S(E) = 1$, and S is a faithful irreducible $N_G(E)$-module. In particular, E acts nontrivially on $U = C_T(x) = \mathbb{Z}_2 \times \mathbb{Z}_2$ and so UE is isomorphic to $A_4 \times \mathbb{Z}_3$. As $\overline{G}$ contains a subgroup isomorphic to S_6, by inspection we see that $N_G(E)$ contains an involution t centralizing x such that $E\langle t \rangle$ is isomorphic to $S_3 \times \mathbb{Z}_3$. Then $UE\langle t \rangle = S_4 \times \mathbb{Z}_3$ with $x \in Z(UE\langle t \rangle)$. But then the coset Ht contains elements of order 6 and 12, whence $a_G > b_{G/H}$, contrary to assumption. □

Completion 3.12 In the case where $H = G^{\mathrm{sol}} < G$, we have studied G/S, where S is the solvable radical of G. Having exhausted all possible structures for G/S, we conclude that $S = 1$ and Proposition 3.1 may be readily verified. In fact, as $S = 1$, the possibilities for G are enumerated in Lemma 3.2. In the case where $H = PSL(2,5)$, $PSL(2,7)$, $PSL(2,8)$, $PSL(2,9)$ or $PSL(3,4)$, every element of H has prime power order. Hence, $b_{G/H} \leq 1$ for every G in Lemma 3.2, unless $G = \mathrm{Aut}(A_6)$ or $P\Sigma L(2,27)$. As the two exceptional groups are covered by the comments following the statement of Proposition 3.1, we have completed the proof of Proposition 3.1. □

Now, by using the Laitinen number a_G, we are able to determine completely the cases where $IO(G,G^{\mathrm{sol}}) \neq 0$ for finite nonsolvable groups G.

Corollary 3.13 *Let G be a finite nonsolvable group. Then*

(1) $IO(G,G^{\mathrm{sol}}) = 0$ *for* $a_G \leq 1$,

(2) $IO(G,G^{\mathrm{sol}}) \neq 0$ *for* $a_G \geq 2$, *except when* $G = \mathrm{Aut}(A_6)$ *or* $P\Sigma L(2,27)$,

(3) $IO(G,G^{\mathrm{sol}}) = 0$ *and* $a_G = 2$ *when* $G = \mathrm{Aut}(A_6)$ *or* $P\Sigma L(2,27)$.

Proof Set $H = G^{\mathrm{sol}}$. By the Second Rank Lemma in Section 0.4, we know that $\mathrm{rk}\, IO(G,H) = a_G - b_{G/H}$. If $a_G \leq 1$, then $a_G = b_{G/H}$, and thus $IO(G,H) = 0$. In turn, if $a_G \geq 2$, then except when $G = \mathrm{Aut}(A_6)$ or $P\Sigma L(2,27)$, $a_G > b_{G/H}$ by Proposition 3.1, and thus $IO(G,H) \neq 0$. In the exceptional cases, we know that $a_G = b_{G/H} = 2$, and thus $IO(G,H) = 0$. □

Proof of Theorem A3 Let G be a finite nonsolvable group. We shall prove that $LO(G) = 0$ for $a_G \leq 1$, and $LO(G) \neq 0$ for $a_G \geq 2$, except when $G = \mathrm{Aut}(A_6)$ or $P\Sigma L(2,27)$, and in the exceptional cases, we shall prove that $LO(G) = 0$ (we already know that $a_G = 2$).

By the Subgroup Lemma in Section 0.4, the following holds:

$$IO(G,G^{\mathrm{sol}}) \subseteq LO(G) \subseteq IO(G,O^p(G)) \subseteq IO(G,G)$$

for any prime p. If $a_G \leq 1$, then $IO(G,G) = 0$ by the First Rank Lemma in Section 0.1, and thus $LO(G) = 0$. If $a_G \geq 2$, then except when $G = \mathrm{Aut}(A_6)$ or $P\Sigma L(2,27)$, Corollary 3.13 asserts that $IO(G,G^{\mathrm{sol}}) \neq 0$, and thus $LO(G) \neq 0$.

For $G = \mathrm{Aut}(A_6)$, $O^2(G) = A_6 = G^{\mathrm{sol}}$ (and $O^p(G) = G$ for any prime $p \neq 2$). Hence $IO(G,O^2(G)) = 0$ by Corollary 3.13, and thus $LO(G) = 0$.

For $G = P\Sigma L(2,27)$, $O^3(G) = PSL(2,27) = G^{\mathrm{sol}}$ (and $O^p(G) = G$ for any prime $p \neq 3$). Hence $IO(G,O^3(G)) = 0$ by Corollary 3.13, and thus again $LO(G) = 0$, completing the proof. □

4 Proof of the Realization Theorem

In this section, we shall prove the Realization Theorem stated in Section 0.2; i.e., we shall prove that $\mathcal{LO}(G) \subseteq \mathcal{LS}m(G)$ for any finite Oliver gap group G. The proof follows from a number of results which we collect below. The key results are obtained in Theorems 4.3 and 4.4.

Let G be a finite group. Following [32], consider the real G-module

$$V(G) = (\mathbb{R}[G] - \mathbb{R}) - \bigoplus_{p \mid |G|} (\mathbb{R}[G]^{O^p(G)} - \mathbb{R})$$

where $\mathbb{R}[G]$ denotes the real regular G-module, $\mathbb{R}[G]^{O^p(G)}$ has the canonical action of G, and G acts trivially the subtracted summands $\mathbb{R}$. The family of the isotropy subgroups in $V(G) \setminus \{0\}$ consists of subgroups H of G such that H is not large in G; i.e., $H \notin \mathcal{L}(G)$ (cf. [32]). In particular, $V(G)$ is $\mathcal{L}$-free.

By arguing as in [40, the proof of Theorem 0.3] in the case G is an Oliver group, we obtain the following theorem which allows us to construct Oliver equivalent real $\mathcal{L}$-free G-modules (cf. [45, Theorem 0.4]).

Theorem 4.1 (cf. [40]) *Let G be a finite Oliver group. Let $V_1, \dots, V_k$ be real $\mathcal{L}$-free G-modules all of dimension $d \geq 0$, such that $V_i - V_j \in IO(G)$ for all $1 \leq i, j \leq k$. Set $n = d + \ell \dim V(G)$ for an integer ℓ. If ℓ is sufficiently large, there exists a smooth action of G on the n-disk D with $D^G = \{x_1, \dots, x_k\}$ and $T_{x_i}(D) = V_i \oplus \ell V(G)$ for all $1 \leq i \leq k$.*

By using equivariant surgery developed in [2], [3], [31], [32], [36]–[38], so called "deleting–inserting" theorems are obtained in [31, Theorem 2.2] for any finite nonsolvable group G, and in [38, Theorems 0.1 and 4.1] for any finite Oliver group G. Under suitable conditions, these theorems allow us to modify a given smooth action of G on a sphere S (resp., disk D) with fixed point set F, in such a way that the resulting smooth action of G on S (resp., D) has a fixed point set obtained from F by deleting or inserting a number of connected components of F. We restate only the "deleting part" of [38, Theorem 0.1] in a modified form presented in [41, Theorem 18], where the G-orientation condition of [38] is replaced by the weaker $\mathcal{P}$-orientation condition of [41].

Let G be a finite group. Then a real G-module V is called *G-oriented* if V^H is oriented for each $H \leq G$, and the transformation $g\colon V^H \to V^H$ is orientation preserving for each $g \in N_G(H)$. More generally, a real G-module V is called *$\mathcal{P}$-oriented* if V^P is oriented for each $P \in \mathcal{P}(G)$, and also the transformation $g\colon V^P \to V^P$ is orientation preserving for each $g \in N_G(P)$.

For example, the realification $r(U)$ of a complex G-module U is G-oriented. If V is a real G-module, then the G-module $2V = V \oplus V$ is the realification of the complexification of V, and thus $2V$ is G-oriented.

For a smooth manifold F with the trivial action of G, a real G-vector bundle ξ over F is called *$\mathcal{L}$-free* if each fiber of ξ is $\mathcal{L}$-free (as a real G-module).

Let M be a smooth G-manifold. We denote by $\mathcal{F}_{\text{iso}}(G;M)$ the family of the isotropy subgroups G_x of G occurring at points $x \in M$. For $H \subseteq G$, the set M^H (resp., $M^{=H}$) consists of points $x \in M$ with $G_x \supseteq H$ (resp., $G_x = H$). In general, M^H (resp., $M^{=H}$) may have connected components of different dimensions. Henceforth, by $\dim M^H$ (resp., $\dim M^{=H}$) we mean the maximum of the dimensions of the connected components of M^H (resp., $M^{=H}$).

We denote by $\mathcal{PC}(G)$ the family of subgroups H of G such that H/P is cyclic for some $P \trianglelefteq H$ with $P \in \mathcal{P}(G)$. Clearly, $\mathcal{P}(G) \subseteq \mathcal{PC}(G)$. Moreover, if G is a finite Oliver group, then $\mathcal{PC}(G) \cap \mathcal{L}(G) = \varnothing$, and thus $\mathcal{P}(G) \cap \mathcal{L}(G) = \varnothing$ (cf. [32]). The family $\mathcal{PC}(G)$ was considered for the first time by Oliver [43], and it was denoted by $\mathcal{G}^1(G)$.

Now, we state an equivariant surgery result which allows us to construct smooth actions of G on spheres with prescribed fixed point sets. The result is a special case of [41, Theorem 18] (cf. [41, Theorem 36]).

Theorem 4.2 (cf. [41, Theorem 18]) *Let G be a finite Oliver group acting smoothly on a homotopy sphere Σ. Let F be a union of connected components of the fixed point set Σ^G. Suppose that the following five conditions hold.*

(1) $\dim \Sigma^P > 2\dim \Sigma^H$ *for all subgroups $P < H \subseteq G$ with $P \in \mathcal{P}(G)$.*

(2) $\dim \Sigma^P \geq 5$ *and* $\dim \Sigma^{=H} \geq 2$ *for any $P \in \mathcal{P}(G)$ and $H \in \mathcal{PC}(G)$.*

(3) *Σ^P is simply connected for any $P \in \mathcal{P}(G)$.*

(4) *The tangent G-module $T_x(\Sigma)$ is $\mathcal{P}$-oriented for some $x \in F$.*

(5) *The equivariant normal bundle $\nu_{F \subset \Sigma}$ is $\mathcal{L}$-free.*

Then there exists a smooth action of G on the sphere S of the same dimension as Σ, and such that $S^G = F$ and $\nu_{F \subset S} \cong \nu_{F \subset \Sigma}$. Moreover, $\dim S^P = \dim \Sigma^P$ *for each $P \in \mathcal{P}(G)$.*

Let G be a finite group. Then a pair (P, H) of subgroups P and H of G is called *proper* if $P \in \mathcal{P}(G)$ and $P < H \subseteq G$. Following [42], for a real G-module V and a proper pair (P, H) of subgroups of G, we set

$$d_V(P, H) = \dim V^P - 2\dim V^H.$$

A real G-module V is called a *gap* G-module if $d_V(P,H) > 0$ for each proper pair (P,H) of subgroups of G. Therefore, by the definition of gap group recalled in Section 0.2, a finite group G is a gap group if and only if $\mathcal{P}(G) \cap \mathcal{L}(G) = \varnothing$ and G has a real $\mathcal{L}$-free gap G-module.

Now, by using Theorems 4.1 and 4.2, we obtain a result for actions on spheres similar to that one obtained in Theorem 4.1 for actions on disks.

Theorem 4.3 *Let G be a finite Oliver gap group. Let V be a real $\mathcal{P}$-oriented $\mathcal{L}$-free gap G-module containing $V(G)$ as a direct summand. Let $V_1,\dots,V_k$ be real $\mathcal{P}$-oriented $\mathcal{L}$-free G-modules all of dimension $d \geq 0$, and such that $V_i - V_j \in IO(G)$ for all $1 \leq i,j \leq k$. Set $n = d + \ell \dim V$ for some integer ℓ. If ℓ is sufficiently large, there exists a smooth action of G on the n-sphere S with $S^G = \{x_1,\dots,x_k\}$ and $T_{x_i}(S) = V_i \oplus \ell V$ for all $1 \leq i \leq k$.*

Proof Let $\mathcal{S}(G)$ be the family of all subgroups of G. By [32], we know that $\mathcal{F}_{\mathrm{iso}}(G, V(G) \smallsetminus \{0\}) = \mathcal{S}(G) \smallsetminus \mathcal{L}(G)$ and $\mathcal{PC}(G) \cap \mathcal{L}(G) = \varnothing$. Therefore

$$\mathcal{PC}(G) \subseteq \mathcal{F}_{\mathrm{iso}}(G, V(G) \smallsetminus \{0\}).$$

As V contains $V(G)$ as a direct summand, $\dim V^{=H} \geq \dim V(G)^{=H} \geq 1$ for each $H \in \mathcal{PC}(G)$. Now, for each $i = 1,\dots,k$, consider the invariant unit sphere

$$\Sigma_i = S(V_i \oplus \ell V \oplus \mathbb{R}),$$

where G acts trivially on $\mathbb{R}$. The fixed point set Σ_i^G consists of exactly two points, say a_i and b_i, at which $T_{a_i}(\Sigma_i) = T_{b_i}(\Sigma_i) = V_i \oplus \ell V$. Set $F_i = \{b_i\}$. We note that $n = d + \ell \dim V = \dim V_i + \ell \dim V = \dim \Sigma_i$.

We claim that the conditions (1)–(5) in Theorem 4.2 all hold for the sphere Σ_i, provided ℓ is sufficiently large. As $d_V(P,H) > 0$, we can choose ℓ so that

$$\ell d_V(P,H) > -d_{V_i}(P,H)$$

for each proper pair (P,H) of subgroups of G. Then

$$d_{V_i \oplus \ell V}(P,H) = d_{V_i}(P,H) + \ell d_V(P,H) > d_{V_i}(P,H) - d_{V_i}(P,H) = 0,$$

and thus $\dim \Sigma_i^P > 2\dim \Sigma_i^H$, proving that the condition (1) holds.

As $\dim V^{=H} \geq 1$ for each $H \in \mathcal{PC}(G)$, we see that the following holds:

$$\dim \Sigma_i^H \geq \dim(\ell V)^H = \ell \dim V^H \geq \ell \dim V^{=H} \geq \ell$$

and similarly $\dim \Sigma_i^{=H} \geq \ell \dim V^{=H} \geq \ell$. Hence, if $\ell \geq 5$, the condition (2) holds and the sphere Σ_i^P is simply connected for each $P \in \mathcal{P}(G)$, proving that the condition (3) also holds. As V_i and V are $\mathcal{P}$-oriented and $T_{b_i}(\Sigma_i) = V_i \oplus \ell V$, the condition (4) holds. Similarly, as V_i and V are L-free and $\nu_{F_i \subset \Sigma_i}$ has just one fiber $V_i \oplus \ell V$, the condition (5) holds. As a result, the conditions (1)–(5) in Theorem 4.2 all hold, proving the claim.

Thus, we may apply Theorem 4.2 to obtain a smooth action of G on a copy S_i of the n-sphere such that $S_i^G = F_i = \{b_i\}$ and $T_{b_i}(S_i) = V_i \oplus \ell V$, provided ℓ is sufficiently large.

As $V_i - V_j \in IO(G)$ for all $1 \leq i, j \leq k$, Theorem 4.1 asserts that there exists a smooth action of G on the n-disk D_0 such that $D_0^G = \{x_1, \ldots, x_k\}$ and $T_{x_i}(D_0) = V_i \oplus \ell V$ for all $1 \leq i \leq k$, provided ℓ is sufficiently large.

The equivariant double $S_0 = \partial(D_0 \times [0,1])$ of D_0 is a copy of the n-sphere equipped with a smooth action of G such that $S_0^G = \{x_1, y_1, \ldots, x_k, y_k\}$ and

$$T_{x_i}(S_0) = T_{y_i}(S_0) = V_i \oplus \ell V = T_{b_i}(S_i)$$

for all $1 \leq i \leq k$. Now, consider the equivariant connected sum

$$S = S_0 \# S_1 \# \ldots \# S_k$$

of the n-spheres $S_0, S_1, \ldots, S_k$ formed by connecting sufficiently small invariant disk neighborhoods of the points $y_i \in S_0$ and $b_i \in S_i$ for all $1 \leq i \leq k$. Then S is the n-sphere with a smooth action of G such that $S^G = \{x_1, \ldots, x_k\}$ and $T_{x_i}(S) = V_i \oplus \ell V$ for all $1 \leq i \leq k$. □

We wish to remark that by using the methods of [38], [39], [40], [45], and [47], we can prove more general results than that presented in Theorem 4.3. In fact, the results of [41, Theorems 27 and 28] show that each isolated fixed point in Theorem 4.3 can be replaced by a smooth manifold which is simply connected or stably parallelizable. However, instead of using [41], we decided to give an independent proof of Theorem 4.3 due to simplifications which occur in the case where the fixed point set is a discrete space.

Let G be a finite group. A proper pair (P, H) of subgroups of G is called *odd* if $|H : P| = |H O^2(G) : P O^2(G)| = 2$ and $P O^p(G) = G$ for all odd primes p. Moreover, (P, H) is called *even* if (P, H) is not odd.

It follows from [32, Theorem 2.3] that for a proper pair (P, H) of subgroups of a finite group G, the following holds:

(1) $d_{V(G)}(P, H) = 0$ when (P, H) is odd, and

(2) $d_{V(G)}(P, H) > 0$ when (P, H) is even.

Recall that by definition a real G-module V is a gap G-module if $d_V(P,H) > 0$ for each proper pair (P,H) of subgroups of G. If $O^p(G) \neq G$ and $O^q(G) \neq G$ for two distinct odd primes p and q, or $O^2(G) = G$, then any proper pair (P,H) of subgroups of G is even by [42], and thus $V(G)$ is a gap G-module.

In order to ensure (stably) the $\mathcal{P}$-orientability of any real G-modules $V_1,\dots,V_k$ satisfying the condition that $V_i - V_j \in IO(G)$, we use the following lemma whose proof is given at the end of this section (cf. [41, Lemma 15]).

Key Lemma *Let G be a finite group. Let U and V be two real G-modules such that $U - V \in IO(G)$. Then the real G-module $U \oplus V$ is $\mathcal{P}$-oriented.*

The Key Lemma allows us to obtain the following modification of Theorem 4.3, which we will use to prove the Realization Theorem stated in Section 0.2.

Theorem 4.4 *Let G be a finite Oliver gap group and let $V_1,\dots,V_k$ be real $\mathcal{L}$-free G-modules with differences $V_i - V_j \in IO(G)$ for all $1 \leq i,j \leq k$. Then there exists a smooth action of G on a sphere S such that $S^G = \{x_1,\dots,x_k\}$ and $T_{x_i}(S) = V_i \oplus W$ for all $1 \leq i \leq k$ and some real $\mathcal{L}$-free G-module W. Moreover, S^P is connected for each $P \in \mathcal{P}(G)$.*

Proof As G is a gap group, there exists a real $\mathcal{L}$-free gap G-module U and so, in particular, $d_U(P,H) > 0$ for each proper pair (P,H) of subgroups of G. Set $V = 2U \oplus 2V(G)$. As $d_{V(G)}(P,H) \geq 0$ by [32, Theorem 2.3]

$$d_V(P,H) = d_{2U \oplus 2V(G)}(P,H) = 2d_U(P,H) + 2d_{V(G)}(P,H) > 0,$$

proving that V is a gap G-module. Clearly, V is $\mathcal{P}$-oriented, and V is $\mathcal{L}$-free as so are U and $V(G)$. Moreover, V contains $V(G)$ as a direct summand.

Let V_0 be one of the G-modules $V_1,\dots,V_k$. So, by assumption, the difference $V_i - V_0$ is in $IO(G)$ for each $1 \leq i \leq k$, and thus $V_i \oplus V_0$ is $\mathcal{P}$-oriented by the Key Lemma. Clearly, each G-module $V_i \oplus V_0$ is $\mathcal{L}$-free. Again by assumption, $(V_i \oplus V_0) - (V_j \oplus V_0) \in IO(G)$ for all $1 \leq i,j \leq k$.

Now, we may apply Theorem 4.3 to conclude that there exists a smooth action of G on a sphere S such that $S^G = \{x_1,\dots,x_k\}$ and $T_{x_i}(S) = V_i \oplus V_0 \oplus \ell V$ for all $1 \leq i \leq k$, where ℓ is some sufficiently large integer. Set $W = V_0 \oplus \ell V$. Then W is $\mathcal{L}$-free. Moreover, $\dim W^P > 0$ for each $P \in \mathcal{P}(G)$, as W contains $V(G)$ as a direct summand. By Smith theory, S^P has $\mathbb{Z}_p$-homology of a sphere for any p-subgroup P of G. By the Slice Theorem, $\dim S^P \geq \dim W^P > 0$ and thus S^P is connected for each $P \in \mathcal{P}(G)$. □

Proof of the Realization Theorem Let G be a finite Oliver gap group. We shall prove that $LO(G) \subseteq LSm(G)$. So, take an element $U - V \in LO(G)$, the difference of two real L-free G-modules U and V with $U - V \in IO(G)$. Then Theorem 4.4 asserts that there exists a smooth action of G on a sphere S such that $S^G = \{x, y\}$ for two points x and y at which $T_x(S) = U \oplus W$ and $T_y(S) = V \oplus W$ for some real L-free G-module W, and S^P is connected for each $P \in \mathcal{P}(G)$. In particular, the action of G on S satisfies the 8-condition. Consequently, the G-modules $U \oplus W$ and $V \oplus W$ are Laitinen–Smith equivalent, and thus

$$U - V = (U \oplus W) - (V \oplus W) \in LSm(G),$$

completing the proof. □

In order to obtain Theorem 4.4 from Theorem 4.3 we have used the Key Lemma asserting that given two real G-modules U and V such that $U - V \in IO(G)$, the G-module $U \oplus V$ is $\mathcal{P}$-oriented, where G is an arbitrary finite group. By using some deep topological results about the existence of specific group actions, a proof of the assertion is presented in [41, Lemma 15]. In the remaining part of this section, we prove the Key Lemma using only algebraic arguments.

Lemma 4.5 *Let G be a finite group and let $T = \langle t \rangle$ be the cyclic subgroup of G generated by an element $t \in G$ of 2-power order. Let U and V be two real G-modules of the same dimension. If* $\dim U^T \equiv \dim V^T \pmod 2$, *then the determinants of the transformations $t : U \to U$ and $t : V \to V$ agree,* $\det(t|_U) = \det(t|_V)$.

Proof If W is a 2-dimensional irreducible real T-module, then the eigenvalues for t on W form a complex conjugate pair and so $\det(t|_W) = 1$.

Let m_U and m_V be the dimensions of the (-1)-eigenspace for t on U and V, respectively. Clearly, the hypothesis that $\dim U^T \equiv \dim V^T \pmod 2$ implies that $m_U \equiv m_V \pmod 2$. Therefore

$$\det(t|_U) = (-1)^{m_U} = (-1)^{m_V} = \det(t|_V),$$

as claimed. □

The next lemma is used in an inductive step of the proof of the Key Lemma.

Lemma 4.6 *Let G be a finite group such that $G = PT$ for some normal p-subgroup P (p odd) and some cyclic 2-subgroup $T = \langle t \rangle$. Let U and V be two non-zero real G-modules with $U^G = V^G = \{0\}$. If $U = V$ as P-modules, then the determinants of the transformations $t : U \to U$ and $t : V \to V$ agree,* $\det(t|_U) = \det(t|_V)$.

Proof We proceed by induction on $|P| + \dim U$. By assumption, $U = V$ as P-modules, and thus $\dim U = \dim V$. Therefore, by Lemma 4.5, it will suffice to prove that the congruence

$$\dim U^T \equiv \dim V^T \pmod 2$$

occurring in Lemma 4.5 holds. Clearly, if $P = 1$, then $\dim U^T = \dim V^T = 0$ by hypothesis, and we are done.

Suppose now that $P \neq 1$. Let K be the kernel of the P-action on U (and V). If $K \neq 1$, we are done by induction in G/K. Therefore we may assume that $K = 1$. Let E be a minimal normal subgroup of G with $E \leq P$. Suppose that $\dim U^E > 0$. Then $U^E = V^E$ and $U - U^E = V - V^E$ as P-modules and all four of these are G-modules. Hence induction yields that $\det(t|_{U^E}) = \det(t|_{V^E})$ and $\det(t|_{U-U^E}) = \det(t|_{V-V^E})$, and we are done.

Therefore we may assume that $\dim U^E = \dim V^E = 0$. Now, if $E \neq P$, we are done by induction in the group ET. As a result, we may assume that P is an elementary abelian p-group and that P is a minimal normal subgroup of G. Also, $\dim U^P = \dim V^P = 0$. If $t' \in T$, then the centralizer $C_P(t')$ is normal in G, hence is 1 or P. As a result, either $G = P \rtimes T$ is cyclic or the center $Z = Z(G)$ is a proper subgroup of T and the quotient G/Z is a Frobenius group with kernel PZ/Z and complement T/Z.

By [27, Chapter VII, Theorem 1.18], if W is an irreducible $\mathbb{R}[G]$-module, then there are the following two possibilities for the $\mathbb{C}[G]$-module $W \otimes_{\mathbb{R}} \mathbb{C}$:

(1) $W \otimes_{\mathbb{R}} \mathbb{C}$ is irreducible, and we say that W is *absolutely irreducible*, or

(2) $W \otimes_{\mathbb{R}} \mathbb{C} = W_1 \oplus W_2$, where W_1 and W_2 are irreducible $\mathbb{C}[G]$-modules which are complex conjugate (i.e., Galois conjugate).

If G is cyclic, the condition on U and V that $U^P = V^P = \{0\}$ ensures that

$$U \otimes_{\mathbb{R}} \mathbb{C} = U_1 \oplus U_2 \text{ and } V \otimes_{\mathbb{R}} \mathbb{C} = V_1 \oplus V_2$$

where U_2 (resp., V_2) is the complex conjugate module of U_1 (resp., V_1).

As $\dim(U_1)^T = \dim(U_2)^T$ and $\dim(V_1)^T = \dim(V_2)^T$, it follows that

$$\dim U^T \equiv 0 \equiv \dim V^T \pmod 2;$$

completing the case where G is cyclic. Therefore, we may assume that G/Z is a Frobenius group with $|T/Z| = 2^d$ for some integer $d \geq 1$.

By Clifford theory, we know that if W is an irreducible $\mathbb{C}[G]$-module whose kernel does not contain P, then $\dim W$ is divisible by 2^d. Thus in fact if W is any $\mathbb{C}[G]$-module with $W^P = \{0\}$, then $\dim W$ is divisible by 2^d.

Suppose that M is an absolutely irreducible $\mathbb{R}[G]$-module. Then the group Z maps into the group $\{I, -I\}$ of the real scalar transformations I and $-I$ of M. In fact, Z maps into the multiplicative group of the ring $\mathrm{End}_{\mathbb{R}[G]}(M) = \mathbb{R}$ of the endomorphisms of M, regarded as the ring of scalar linear transformations acting on M. Since Z is a 2-group, Z maps into the group of real 2^mth roots of 1, which is just $\{1, -1\}$. So we may assume that $|Z| \le 2$. If $|Z| = 1$, then we can replace G with a larger group, so that in fact we may assume without loss that $|Z| = 2$. We shall argue that Z acts trivially on M by computing the Frobenius–Schur indicator $\nu(\chi)$ of the character χ afforded by the absolutely irreducible $\mathbb{R}[G]$-module M. By definition,

$$\nu(\chi) = \frac{1}{|G|} \sum_{g \in G} \chi(g^2).$$

Note that $\chi = \mathrm{Ind}_{PZ}^{G}(\psi)$ for some irreducible character ψ of PZ such that $\mathrm{Res}_{P}^{PZ}(\psi) \neq 1_P$. Since PZ is a normal subgroup of G, thus $\chi(g) = 0$ for all $g \in G \setminus PZ$. Hence, in the displayed sum, all the terms are 0 except when $g^2 \in PZ$. Let $v \in T$ with $v^2 = z$. Then $g^2 \in PZ$ if and only if $g \in P\langle v\rangle$, which is a union of two cosets of PZ. Consider the squaring map on PZ. This is a two-to-one map of PZ onto P (if $x \in P$, then $x^2 = (xz)^2 \in P$). Since $P\langle v\rangle = PZ \cup PZv$, we have

$$\nu(\chi) = \frac{1}{|G|} \left(2 \sum_{g \in P} \chi(g) + \sum_{g \in PZv} \chi(g^2) \right).$$

Now $\frac{1}{|P|} \sum_{g \in P} \chi(g) = \langle \mathrm{Res}_P^G(\chi), 1_P \rangle$, the inner product of $\mathrm{Res}_P^G(\chi)$ and 1_P; i.e., it is the multiplicity of 1_P as a constituent of $\mathrm{Res}_P^G(\chi)$, which is exactly the dimension of M^P, which is 0 by assumption. So $2 \sum_{g \in P} \chi(g) = 0$ and

$$\nu(\chi) = \frac{1}{|G|} \sum_{g \in PZv} \chi(g^2).$$

Let $x \in P$. Then $vxv^{-1} = x^{-1}$. As $v^2 = z$, $vxvx = vxv^{-1}v^2x = x^{-1}v^2x = z$. Also $vxzvxz = vxvx = z$. So $g^2 = z$ for all $g \in PZv$. Thus

$$\nu(\chi) = \frac{|PZ|\,\chi(z)}{|G|} = \frac{\chi(z)}{\chi(1)}.$$

As χ is afforded by the absolutely irreducible $\mathbb{R}[G]$-module M, $\nu(\chi) = 1$ and so $\chi(z) = \chi(1)$, as claimed.

Suppose now that M is a sum of absolutely irreducible $\mathbb{R}[G]$-modules such that $M^P = \{0\}$. Then M may be regarded as a faithful $\mathbb{R}[G/Z]$-module, and thus M is a free $\mathbb{R}[T/Z]$-module by the representation theory of Frobenius groups.

Now consider the decomposition $U = M_U \oplus N_U$, where M_U is the sum of all the absolutely irreducible $\mathbb{R}[G]$-summands of U. Then, as $\mathbb{C}[G]$-modules,

$$N_U \otimes_{\mathbb{R}} \mathbb{C} = X_U \oplus Y_U$$

where Y_U is the complex conjugate module of X_U, so that in particular, we have $\dim(X_U)^T = \dim(Y_U)^T$. By the previous paragraph, M_U may be regarded as the sum of m_U free $\mathbb{R}[T/Z]$-modules for $m_U = \dim(M_U)^T$. As we know that $\dim(N_U)^T = 2\dim(X_U)^T$, it follows that

$$\dim U^T = \dim(M_U)^T + 2\dim(X_U)^T \equiv m_U \pmod 2.$$

Now we may do a similar analysis for $V = M_V \oplus N_V$ and $N_V \otimes_{\mathbb{R}} \mathbb{C} = X_V \oplus Y_V$ with obvious notations. Therefore, it suffices to show that $m_U \equiv m_V \pmod 2$ for $m_V = \dim(M_V)^T$. Note that

$$\dim U = 2^d m_U + 2\dim X_U = 2^d m_V + 2\dim X_V = \dim V.$$

By an earlier remark, both $\dim X_U$ and $\dim X_V$ are divisible by 2^d. So, dividing by 2^d, we see that $m_U \equiv m_V \pmod 2$, completing the proof. □

Proof of the Key Lemma Let G be a finite group. Let U and V be two real G-modules such that $U - V \in IO(G)$. We shall prove that the G-module $U \oplus V$ is $\mathcal{P}$-oriented. It suffices to show that for each $P \in \mathcal{P}(G)$ and each $g \in N_G(P)$, the determinants of the transformations $g: U^P \to U^P$ and $g: V^P \to V^P$ agree,

$$\det(g|_{U^P}) = \det(g|_{V^P}),$$

because then $\det(g|_{(U \oplus V)^P}) = 1$, as required.

Let $t \in G$ be an element of 2-power order. If $g = tx = xt$ for an element $x \in G$ of odd order, then $\det(x) = 1$, and therefore $\det(g) = \det(t)$. Thus it suffices to prove the claim for $g = t$. By induction on the order of G, we may assume that $G = PT$ for some normal p-subgroup P of G and some cyclic 2-subgroup T of G. Let t be a generator of T.

If $p = 2$, G is a 2-group and then by using the hypothesis that $U - V \in IO(G)$, we see that $U \cong V$ as G-modules. Therefore, the result is clear for $p = 2$.

Assume that p is odd. As $U - V \in IO(G)$, $U \cong V$ both as P-modules and T-modules. Write $U = U^P \oplus (U - U^P)$ and $V = V^P \oplus (V - V^P)$, and note that $\det(t|_{U^P}) = \det(t|_{V^P})$ if and only if $\det(t|_{U-U^P}) = \det(t|_{V-V^P})$. Since $U - U^P \cong V - V^P$ as P-modules, we may apply Lemma 4.6 to the G-modules $U - U^P$ and $V - V^P$ to conclude that $\det(t|_{U-U^P}) = \det(t|_{V-V^P})$, and thus $\det(t|_{U^P}) = \det(t|_{V^P})$, completing the proof. □

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