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DISCRIMINANT AND THE ŁOJASIEWICZ EXPONENT

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Abstract. An effective formula for the Łojasiewicz exponent of a plane curve singularity is given.

1. Introduction. Suppose that a polynomial $f = f(x, y)$ in two complex variables defines an isolated singularity at the origin $\mathbf{0} \in \mathbf{C}^2$, i.e., $f(\mathbf{0}) = 0$ and the gradient $\nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right)$ has an isolated zero at $\mathbf{0} \in \mathbf{C}^2$. The Łojasiewicz exponent $\mathcal{L}_0(f)$ of f at $\mathbf{0} \in \mathbf{C}^2$ is by definition, the smallest $\theta > 0$ such that there exist a neighborhood U of $\mathbf{0} \in \mathbf{C}^2$ and a constant $c > 0$ such that

$$|\nabla f(z)| \geq c|z|^\theta \quad \text{for each } z \in U.$$

For the basic properties of the Łojasiewicz exponent we refer to Teissier's paper [**11**] (see also [**8**]). In [**3**] Chądzyński and Krasiński gave a formula for the Łojasiewicz exponent (in a more general setting of polynomial mappings of $\mathbf{C}^2$), which allows to calculate $\mathcal{L}_0(f)$ by using the resultant $R(x, u, v) = \mathrm{res}_y(\frac{\partial f}{\partial x} - u, \frac{\partial f}{\partial y} - v)$ depending on two parameters u, v. The aim of this note is to calculate $\mathcal{L}_0(f)$ in terms of the discriminant $\Delta(x, t) = \mathrm{disc}_y(f(x, y) - t)$ which depends on one parameter t. Our proof is based on the fact discovered independently by Bogusławska [**1**] and Kuo and Parusiński [**7**] that the Łojasiewicz exponent $\mathcal{L}_0(f)$ is attained along the polar curve $\alpha\frac{\partial f}{\partial x} + \beta\frac{\partial f}{\partial y} = 0$, provided that the line $\beta x - \alpha y = 0$ is not tangent to the curve $f = 0$.

2. Preliminaries. In the sequel, we need some basic properties of plane curve singularities. Our main reference is [**4**] (Chapter I, Section 3). By an algebraic curve we mean a nonconstant polynomial f up to multiplication by a nonzero scalar. If $f = 0$ is an algebraic curve passing through $\mathbf{0} \in \mathbf{C}^2$, then by $\mu_0(f)$ and $\mathrm{ord}_0 f$ we denote the Milnor number and the order (the multiplicity) of $f = 0$ at the origin, respectively. If $f = 0$ and $g = 0$ are two algebraic curves then $i_0(f,g)$ is the intersection number of $f=0$ and $g=0$ at $\mathbf{0}\in\mathbf{C}^2$. Recall that $\mu_0(f)=i_0(\frac{\partial f}{\partial x},\frac{\partial f}{\partial y})$ and $i_0(f,\frac{\partial f}{\partial y})=\mu_0(f)+i_0(f,x)-1$ (Teissier's formula).

Let $\mathbf{C}\{x\}$ be the ring of convergent power series. We extend it to the ring of convergent fractional power series $\mathbf{C}\{x\}^*=\bigcup_{p\ge 1}\mathbf{C}\{x^{1/p}\}$. For any $y(x)\in\mathbf{C}\{x\}^*$, $\operatorname{ord} y(x)$ and $y(0)$ are defined. There is $\operatorname{ord} y(x)>0$ if and only if $y(0)=0$. Now let $f(x,y)=a_0(x)y^N+a_1(x)y^{N-1}+\cdots+a_N(x)$, $N>0$, be a polynomial such that $a_0(0)\neq 0$, so that $\frac{1}{a_0(x)}$ exists in $\mathbf{C}\{x\}$. Then $f(x,y)=a_0(x)\prod_{i=1}^N(y-y_i(x))$, where $y_i(x)\in\mathbf{C}\{x\}^*$ (Puiseux Theorem). Let $I=\{\,i:\operatorname{ord} y_i(x)>0\,\}$. Then for any polynomial $g=g(x,y)$, Zeuthen's rule holds: $i_0(f,g)=\sum_{i\in I}\operatorname{ord} g(x,y_i(x))$.

3. Main result. Let $f=f(x,y)$ be a polynomial defining an isolated singularity at the origin. We say that the line $x=0$ is in general position with respect to the curve $f=0$ if the following conditions are satisfied

(i) $\deg_y f=\deg f(0,y)$,
(ii) $\operatorname{ord}_0 f=\operatorname{ord} f(0,y)$,
(iii) all nonzero roots of the equation $f(0,y)=0$ are simple.

Assumptions (i)–(iii) have simple geometrical interpretation. Let $\mathbf{x}_\infty$ be the point at infinity of the line $x=0$. Then condition (i) means that the line $x=0$ is not tangent at $\mathbf{x}_\infty$ to the projective closure of the curve $f(x,y)=0$, while (ii) means that $x=0$ is not tangent to $f(x,y)=0$ at $\mathbf{0}$. Finally, (iii) means that the line $x=0$ and the curve $f(x,y)=0$ are transverse at points of $\mathbf{C}^2$ different from $\mathbf{0}$. We can always obtain conditions (i)–(iii) by using a linear automorphism of $\mathbf{C}^2$, provided that f has no multiple factor.

Let t be a new variable and let $\Delta(x,t)=\operatorname{disc}_y(f(x,y)-t)$ be the discriminant of the polynomial $f(x,y)-t\in\mathbf{C}[x,t][y]$. The main result of this note is

THEOREM 3.1. *Suppose that the line $x=0$ is in general position with respect to the curve $f=0$ and let $\Delta(x,t)=\sum_{i\ge 0}\Delta_i(t)x^i$. Then $p=\operatorname{ord}\Delta(x,0)$ is finite and*

$$\mathcal{L}_0(f)=\left(\min_{i=0}^{p-1}\left\{\frac{\operatorname{ord}\Delta_i}{p-i}\right\}\right)^{-1}-1.$$

Moreover, $p=\mu_0(f)+\operatorname{ord}_0 f-1$.

The proof of Theorem 3.1 is given in Section 6 of this note. Now let us compare our theorem with the main result of Chądzyński and Krasiński paper [**3**]. Let u, v be two new variables and let $R(x,u,v)=\operatorname{res}_y\left(\frac{\partial f}{\partial x}-u,\frac{\partial f}{\partial x}-v\right)$ be the y-resultant of the polynomials $\frac{\partial f}{\partial x}-u$, $\frac{\partial f}{\partial y}-v\in\mathbf{C}[x,u,v][y]$. Using Theorem 3.1 of [**3**], we get

THEOREM 3.2. (*cf.* [**3**]) *Let f be a polynomial with a finite number of critical points. Suppose that the axes are in general position with respect to the curve $f = 0$ and that the line $x = 0$ intersects the set of critical points of f at the origin only. Let $R(x, u, v) = \sum_{i \geq 0} R_i(u, v)x^i$. Then $m = \operatorname{ord} R(x, 0, 0)$ is finite and*

$$\mathcal{L}_0(f) = \left(\min_{i=0}^{m-1} \left\{ \frac{\operatorname{ord}_0 R_i}{m - i} \right\} \right)^{-1}.$$

Moreover, $m = \mu_0(f)$.

4. Discriminant. Let $f = f(x, y)$ be a polynomial defining an isolated singularity at the origin. We assume that the line $x = 0$ is in general position with respect to the curve $f = 0$ and set $\Delta(x, t) = \operatorname{disc}_y(f(x, y) - t) = a_0(x)^{-1} \operatorname{res}_y(f(x, y) - t, \frac{\partial f}{\partial y}(x, y))$. Let $N = \deg_y f$ and $n = \operatorname{ord}_0 f$. We may write

$$f(x, y) = a_0(x)y^N + a_1(x)y^{N-1} + \cdots + a_N(x),$$

where $a_0(0)a_{N-n}(0) \neq 0$ and $a_j(0) = 0$ for $j > N - n$.

LEMMA 4.1. $\operatorname{ord} \Delta(x, 0) = \mu_0(f) + n - 1, \quad \operatorname{ord} \Delta(0, t) = n - 1.$

PROOF. By Puiseux' Theorem, we get $\frac{\partial f}{\partial y}(x, y) = Na_0(x) \prod_{i=1}^{N-1}(y - z_i(x))$ with $z_i(x) \in \mathbf{C}\{x\}^*$ for $i = 1, \ldots, N - 1$.

Since $\operatorname{ord} \frac{\partial f}{\partial y}(0, y) = \operatorname{ord} f(0, y) - 1 = n - 1$, we may assume that $z_1(0) = \cdots = z_{n-1}(0) = 0$ and $z_j(0) \neq 0$ for $j \geq n$. Moreover, $f(0, z_j(0)) \neq 0$ for $j \geq n$, because the nonzero roots of the equation $f(0, y) = 0$ are simple.

By the classical formula for the resultant, we get

$$\Delta(x, t) = N^n a_0(x)^{N-1}(f(x, z_1(x)) - t) \cdots (f(x, z_{N-1}(x)) - t).$$

Thus we get $\Delta(0, t) = t^{n-1}\epsilon(t)$, where $\epsilon(0) \neq 0$ and $\operatorname{ord} \Delta(0, t) = n-1$. On the other hand, $\Delta(x, 0) = f(x, z_1(x)) \cdots f(x, z_{n-1}(x))\eta(x)$, $\eta(0) \neq 0$ in $\mathbf{C}\{x\}^*$ and $\operatorname{ord} \Delta(x, 0) = \sum_{j=1}^{n-1} \operatorname{ord} f(x, z_j(x)) = i_0(f, \frac{\partial f}{\partial y}) = \mu_0(f) + i_0(f, x) - 1 = \mu_0(f) + n - 1$ by Zeuthen's rule and Teissier's formula. $\square$

REMARK 4.2. Usually, Lemma 4.1 is stated for the local case, i.e., when Δ is the discriminant of a distinguished polynomial f.

5. Newton polygon. Let $f = f(x, y)$ be a polynomial (a convergent power series) such that $f(0, 0) = 0$. We call f convenient if $f(x, 0)f(0, y) \neq 0$. For any convenient power series f, by $\mathcal{N}(f)$ we denote the set of all compact segments of the Newton polygon of f (see [**12**], p. 16 for the detailed description of the Newton polygon). For any $S \in \mathcal{N}(f)$, by $i(S)$ we denote the inclination of S defined to be the negative of the reciprocal of its slope.

PROPOSITION 5.1 (Newton–Puiseux Theorem). *Let $m = \operatorname{ord} f(x, 0)$ and $n = \operatorname{ord} f(0, y)$.*

I. *If* $y_1(x), \dots, y_n(x)$ *are Puiseux roots of order* > 0 *of the equation* $f(x,y) = 0$ *with unknown* y *then* $\{\operatorname{ord} y_1(x), \dots, \operatorname{ord} y_n(x)\} = \{i(S) : S \in \mathcal{N}(f)\}$.

II. *If* $x_1(y), \dots, x_m(y)$ *are Puiseux roots of order* > 0 *of the equation* $f(x,y) = 0$ *with unknown* x *then* $\{\operatorname{ord} x_1(y), \dots, \operatorname{ord} x_m(y)\} = \{i(S)^{-1} : S \in \mathcal{N}(f)\}$.

PROOF. Part I follows from [**12**], Lemma 2.4.4. Since the Newton polygons of the polynomials (power series) $f(x,y)$ and $f(y,x)$ are symmetrical with respect to the diagonal of the first quarter, Part II follows from Part I. $\square$

6. Proof of the Main Theorem. To prove Theorem 3.1 we need two lemmas.

LEMMA 6.1. *Let* $\Delta(x,t)$ *be a polynomial such that* $p = \operatorname{ord}\Delta(x,0)$ *and* $q = \operatorname{ord}\Delta(0,t)$ *are positive and finite. Let* $t_1(x), \dots, t_q(x)$ *be the Puiseux roots of order* > 0 *of the equation* $\Delta(x,t) = 0$ *with unknown* t. *Let* $\Delta(x,t) = \sum_{i\geq 0} \Delta_i(t)x^i$. *Then*

$$\max_{k=1}^{q}\{\operatorname{ord} t_k(x)\} = \left(\min_{i=0}^{p-1}\left\{\frac{\operatorname{ord}\Delta_i}{p-i}\right\}\right)^{-1}.$$

PROOF. Set $\mathcal{N} = \mathcal{N}(\Delta)$. If $x_1(t), \dots, x_p(t)$ are the Puiseux roots of positive order of the equation $\Delta(x,t) = 0$ with unknown x, then by Proposition 5.1

$$\max_{k=1}^{q}\{\operatorname{ord} t_k(x)\} = \left(\min_{j=1}^{p}\{\operatorname{ord} x_i(t)\}\right)^{-1}.$$

Let $F \in \mathcal{N}$ be the first segment (with a vertex on the vertical axis) of the Newton polygon of Δ. Then again by Proposition 5.1, we get

$$\min_{j=1}^{p}\{\operatorname{ord} x_i(t)\} = i(F).$$

It suffices to observe that

$$i(F) = \min_{i=0}^{p-1}\left\{\frac{\operatorname{ord}\Delta_i}{p-i}\right\}$$

and the lemma follows. $\square$

LEMMA 6.2. *Suppose that the polynomial* f *defines an isolated singularity at the origin and that the line* $x = 0$ *is not tangent to the curve* $f = 0$ *at* $\mathbf{0} \in \mathbf{C}^2$. *Put* $n = \operatorname{ord}_0 f$ *and let* $z_1(x), \dots, z_{n-1}(x)$ *be the Puiseux roots of order* > 0 *of the equation* $\frac{\partial f}{\partial y} = 0$. *Then*

$$\mathcal{L}_0(f) = \max_{j=1}^{n-1}\{\operatorname{ord} f(x, z_j(x))\} - 1.$$

Proof. According to [**1**] and [**7**] (see also [**10**]), the Łojasiewicz exponent is attained along the curve $\frac{\partial f}{\partial y} = 0$, that is $\mathcal{L}_0(f)$ is the smallest $\theta > 0$ such that there exist a neighborhood U of $\mathbf{0} \in \mathbf{C}^2$ and a constant $c > 0$ such that $|\nabla f(z)| \geq c|z|^\theta$ for each $z \in U$ lying on the curve $\frac{\partial f}{\partial y} = 0$.

Let $r_j \geq 1$ be integers such that $z_j(u^{r_j}) \in \mathbf{C}\{u\}$ and let $p_j(u) = (u^{r_j}, z_j(u^{r_j}))$ for $j = 1, \ldots, n-1$. Then p_j are parameterizations of the branches of the curve $\frac{\partial f}{\partial y} = 0$ centered at $\mathbf{0} \in \mathbf{C}^2$ and it is easy to check (see [**10**], Lemma 2.1) that

$$\mathcal{L}_0(f) = \max_{j=1}^{n-1} \left\{ \frac{\operatorname{ord}(\nabla f \circ p_j)}{\operatorname{ord} p_j} \right\}.$$

where $\operatorname{ord}(\nabla f \circ p) = \inf\left\{ \operatorname{ord} \frac{\partial f}{\partial x} \circ p, \operatorname{ord} \frac{\partial f}{\partial y} \circ p \right\}$. Differentiating and taking orders show that $\operatorname{ord}(\nabla f \circ p_j) = \operatorname{ord}(f \circ p_j) - \operatorname{ord} p_j$ and we get

$$\mathcal{L}_0(f) = \max_{j=1}^{n-1} \left\{ \frac{\operatorname{ord}(f \circ p_j)}{\operatorname{ord} p_j} - 1 \right\} = \max_{j=1}^{n-1} \{ \operatorname{ord} f(x, z_j(x)) \} - 1.$$

□

Now we can give

Proof of Theorem 3.1. Let $t_j(x) = f(x, z_j(x))$ for $j = 1, \ldots, n-1$. Then $t_1(x), \ldots, t_{n-1}(x)$ is the sequence of Puiseux roots of order > 0 of the equation $\Delta(x,t) = 0$ (cf. proof of Lemma 4.1) and by Lemmas 6.2 and 6.1, we get $\mathcal{L}_0(f) = \max_{j=1}^{n-1}\{ \operatorname{ord} f(x, z_j(x)) \} - 1 = \max_{j=1}^{n-1}\{ \operatorname{ord} t_j(x) \} - 1 = \left(\min_{i=0}^{p-1} \left\{ \frac{\operatorname{ord} \Delta_i}{p-i} \right\}\right)^{-1} - 1.$ □

Remark 6.3. We could prove Lemma 6.2 by using properties of polar invariants (see [**11**] and [**9**]).

7. Discriminant and the Łojasiewicz exponent at infinity. Let $f : \mathbf{C}^2 \to \mathbf{C}$ be a polynomial with a finite number of critical points. By the Łojasiewicz exponent at infinity $\mathcal{L}_\infty(f)$, we mean the largest $\theta \in \mathbf{R}$ such that there exist a neighborhood V of infinity (i.e., an open subset V of $\mathbf{C}^2$ with compact complement $\mathbf{C}^2 \setminus V$) and a constant $C > 0$ such that

$$|\nabla f(z)| \geq C|z|^\theta \qquad \text{for all } z \in V.$$

By a result of Hà [**5**] (see also [**6**]), $\mathcal{L}_\infty(f) > -1$ if and only if f has no critical points at infinity.

Suppose that $\deg_y f = \deg f$ (this condition means that the line $x = 0$ does not intersect the curve $f = 0$ at infinity) and consider $\Delta(x,t) = \operatorname{disc}_y(f(x,y) - t) = \Delta_P(t)x^P + \Delta_{P-1}(t)x^{P-1} + \cdots + \Delta_0(t), \qquad \Delta_P(t) \neq 0.$

Then $\mathcal{L}_\infty(f) > -1$ if and only if $\Delta_P(t)$ is a constant (see [**2, 6**] and [**10**]). The following global counterpart of our main result holds.

Theorem 7.1. *Suppose that $\mathcal{L}_\infty(f) > -1$. Then*

$$\mathcal{L}_\infty(f) = \left(\max_{i=0}^{P-1}\left\{\frac{\deg \Delta_i}{P-i}\right\}\right)^{-1} - 1$$

and $P = \mu(f) + \deg f - 1$, where $\mu(f)$ is the total Milnor number (the sum of all Milnor numbers of all algebraic curves $f(x,y) - t = 0$, $t \in \mathbf{C}$).

We could prove Theorem 7.1 by modifying the proof of Theorem 3.1. The formula $P = \mu(f) + \deg f - 1$ follows from [**6**], Theorem 3.3(ii). Note also that Theorem 7.1 (as well as the formula for $\mathcal{L}_\infty(f)$ in the case of $\mathcal{L}_\infty(f) < -1$) can easily be obtained from [**10**], Theorem 1.2 and Proposition 3.4. The formulas for $\mathcal{L}_\infty(f)$ in terms of resultant, analogous to Theorem 3.2, are given in [**2**], Theorem 9.2.

An application of Theorems 3.1 and 7.1 follows.

Theorem 7.2. *Let $f : \mathbf{C}^2 \to \mathbf{C}$ be a polynomial with a finite number of critical points. Then*

$$\mathcal{L}_0(f) + 1 \geq (\mu(f) - \mu_0(f) + \deg f - \mathrm{ord}_0 f + 1)^{-1}(\mathcal{L}_\infty(f) + 1).$$

Proof. We may assume that $\mathcal{L}_\infty(f) + 1 > 0$ and that Theorems 3.1 and 7.1 apply to the polynomial f. By Theorem 3.1, $(\mathcal{L}_0(f) + 1)^{-1} = \frac{\mathrm{ord}\,\Delta_{i_0}}{p - i_0}$ for some $i_0 \in \{0, 1, \ldots, p-1\}$ and we get $(\mathcal{L}_0(f) + 1)^{-1} \leq \frac{\deg \Delta_{i_0}}{p - i_0} = \frac{P - i_0}{p - i_0}\frac{\deg \Delta_{i_0}}{P - i_0} \leq (P - p + 1)(\mathcal{L}_\infty(f) + 1)^{-1}$ by Theorem 7.1. This finishes the proof, since $P - p = \mu(f) - \mu_0(f) + \deg f - \mathrm{ord}_0 f$. □

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