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ON LEBESGUE MEASURABILITY OF HENSTOCK–KURZWEIL INTEGRABLE FUNCTIONS

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Abstract. Every Henstock–Kurzweil integrable function on a compact interval in $\mathbb{R}$ is Lebesgue measurable. We give a new elementary proof.

1. Introduction. The following result is well-known.

Theorem 1.1. *Every function that is Henstock–Kurzweil integrable on a compact interval in $\mathbb{R}$ is also Lebesgue measurable.*

Standard proofs of this result use advanced tools, like the Vitali Covering Theorem and the Fundamental Theorem of Calculus (see for example [**1, 3**]). We shall prove Theorem 1.1 using definition of Henstock–Kurzweil integrability and some basic properties of the Lebesgue measure only. A proof of a more general theorem can be found in [**2**].

2. Some definitions and notations. Let $\mathcal{L}^*$ denote the outer Lebesgue measure in $\mathbb{R}$ and let $\mathcal{L} : \mathfrak{L} \longrightarrow [0, +\infty]$ be the Lebesgue measure, where $\mathfrak{L}$ denotes the σ-algebra of Lebesgue measurable sets in $\mathbb{R}$. We consider a closed interval $P = [a, b] \subset \mathbb{R}$, $a < b$, and $\mathcal{M}(P, \mathcal{L})$, the collection of all Lebesgue measurable functions $f \colon P \longrightarrow \mathbb{R}$.

In the rest of this note we use the following notation. If $P = \bigcup\limits_{i=1}^{n} P_i$, where $P_i = [x_{i-1}, x_i]$, $a = x_0 < x_1 < \ldots < x_n = b$, then we say that $\{P_i\}_{i=1}^n$ is a *partition of* P. If $\xi_i \in P_i$, $i = 1, \ldots, n$, then the set of ordered pairs $\{(P_i, \xi_i) : \ i = 1, \ldots, n\} = (P_i, \xi_i)_{i=1}^n$ is called a *tagged partition of* P. We denote by $\mathcal{T}(P)$ the collection of all tagged partitions of P.

For any function $\delta \colon P \longrightarrow \mathbb{R}_{>0}$, let

$$\mathcal{S}(\delta) := \{(P_i, \xi_i)_{i=1}^n \in \mathcal{T}(P) : |P_i| := x_i - x_{i-1} \leq \delta(\xi_i), \ i = 1, \ldots, n\}.$$

Definition 2.1. We say that a function $f\colon P \longrightarrow \mathbb{R}$ is Henstock–Kurzweil integrable on P if there exists a number $I \in \mathbb{R}$ such that for every $\varepsilon > 0$ there exists a function $\delta\colon P \longrightarrow \mathbb{R}_{>0}$ such that if $(P_i, \xi_i)_{i=1}^n \in \mathcal{S}(\delta)$, then

$$\Big| \sum_{i=1}^n f(\xi_i)|P_i| - I \Big| \le \varepsilon.$$

In this case we say that I is the Henstock–Kurzweil integral of f and put $\int_P f := I$. We denote by $\mathcal{HK}(P)$ the collection of all Henstock–Kurzweil integrable functions on P.

3. Proof of Theorem 1.1. First we shall show that if the following lemma holds true, then we can prove Theorem 1.1.

Lemma 3.1. *If $f \notin \mathcal{M}(P, \mathcal{L})$, then there exist an $A \in \mathfrak{L}$ such that $0 < \mathcal{L}(A) < \infty$ and numbers $\alpha < \beta$ such that $\mathcal{L}^*(A \cap \{f \le \alpha\}) = \mathcal{L}^*(A \cap \{f \ge \beta\}) = \mathcal{L}(A)$.*

Indeed, assume for a while that Lemma 3.1 holds.

Proof of Theorem 1.1. Without loss of generality, assume that $P = [0, 1]$. Fix an $f \in \mathcal{HK}(P)$. Suppose that $f \notin \mathcal{M}(P, \mathcal{L})$. Then, by Lemma 3.1, we find an $A \in \mathfrak{L}$ such that $0 < \mathcal{L}(A) < \infty$, and numbers $\alpha < \beta$ such that $\mathcal{L}^*(A \cap \{f \le \alpha\}) = \mathcal{L}^*(A \cap \{f \ge \beta\}) = \mathcal{L}(A)$.

Fix an $\varepsilon > 0$ satisfying $\mathcal{L}(A) > \frac{2\varepsilon}{\beta-\alpha} + 2\varepsilon$. Let $\delta\colon P \longrightarrow \mathbb{R}_{>0}$ be such that

$$\Big| \sum_{i=1}^n f(\xi_i)|P_i| - \int_P f \Big| \le \varepsilon \quad \text{for every} \quad (P_i, \xi_i)_{i=1}^n \in \mathcal{S}(\delta).$$

Define

$$\begin{aligned} A_m &:= A \cap \{f \le \alpha\} \cap \{x \in P : \tfrac{1}{m} \le \tfrac{\delta(x)}{2}\}, \\ B_m &:= A \cap \{f \ge \beta\} \cap \{x \in P : \tfrac{1}{m} \le \tfrac{\delta(x)}{2}\}, \qquad m \in \mathbb{N},\ m \ne 0. \end{aligned}$$

Then there exists an $m_0 \ge 1$ such that $\mathcal{L}^*(A_{m_0}) \ge \mathcal{L}(A) - \varepsilon$ and $\mathcal{L}^*(B_{m_0}) \ge \mathcal{L}(A) - \varepsilon$.

Indeed, suppose that $\mathcal{L}^*(A_m) < \mathcal{L}(A) - \varepsilon$ for every $m \in \mathbb{N}$, $m \ne 0$. Thanks to the regularity of $\mathcal{L}$, for every $m \in \mathbb{N}$, $m \ne 0$, there exists a $C_m \in \mathfrak{L}$ such that $\mathcal{L}(C_m) = \mathcal{L}^*(A_m)$ and $A_m \subset C_m \subset A$. We may assume that the sequence $\{C_m\}_{m=1}^\infty$ is increasing.

To see this, for given $C_1, C_2, \dots$ define $D_j^m := C_m \setminus C_j$, $m \in \mathbb{N}$, $m \ne 0$, $j \ge m+1$, and $C_m^{'} := C_m \setminus \bigcup\limits_{j=m+1}^{\infty} D_j^m$. Then $\{C_m^{'}\}_{m=1}^\infty$ is increasing and $A_m \subset C_m^{'} \subset A$, as well as $\mathcal{L}(C_m^{'}) = \mathcal{L}(C_m) = \mathcal{L}^*(A_m)$.

Finally we obtain $\mathcal{L}(\bigcup_{m=1}^{\infty} C_m) \leq \mathcal{L}(A) - \varepsilon$, but $A \cap \{f \leq \alpha\} \subset \bigcup_{m=1}^{\infty} C_m \subset A$, which implies $\mathcal{L}(A) = \mathcal{L}(\bigcup_{m=1}^{\infty} C_m)$; a contradiction. Thus, there exists an $m_1 \geq 1$ such that $\mathcal{L}^*(A_{m_1}) \geq \mathcal{L}(A) - \varepsilon$. Analogously, there exists an $m_2 \geq 1$ such that $\mathcal{L}^*(B_{m_2}) \geq \mathcal{L}(A) - \varepsilon$.
Then $m_0 := \max\{m_1, m_2\}$.

Define $\delta' := \min\{\delta, \frac{1}{2m_0}\}$. Fix an $(P_i, \xi_i)_{i=1}^n \in \mathcal{S}(\delta')$ and finally let

$$I_1 := \{i \in \{1,\dots,n\} : P_i \cap A_{m_0} = \varnothing\}, \qquad I_2 := \{i \in \{1,\dots,n\} : P_i \cap B_{m_0} = \varnothing\},$$

$$I_3 := \{1,\dots,n\} \setminus (I_1 \cup I_2).$$

We see that $\mathcal{L}(A \cap \bigcup_{i \in I_1} P_i) \leq \mathcal{L}(A) - \mathcal{L}(A \cap (\bigcup_{i \in I_1} P_i)^c) \leq \mathcal{L}(A) - \mathcal{L}^*(A_{m_0}) \leq \varepsilon$ and similarly $\mathcal{L}(A \cap \bigcup_{i \in I_2} P_i) \leq \varepsilon$. For $i \in I_3$, there is $\sum_{i \in I_3} |P_i| \geq \mathcal{L}(A) - 2\varepsilon$ and $P_i \cap A_{m_0} \neq \varnothing$, $P_i \cap B_{m_0} \neq \varnothing$, so. Thus, for $i \in I_3$, we find $\xi_i' \in P_i \cap A_{m_0}$, $\xi_i'' \in P_i \cap B_{m_0}$. For $i \notin I_3$, let $\xi_i = \xi_i' = \xi_i''$. Consider tagged partitions $(P_i, \xi_i')_{i=1}^n$, $(P_i, \xi_i'')_{i=1}^n$. We observe that $(P_i, \xi_i')_{i=1}^n$, $(P_i, \xi_i'')_{i=1}^n \in \mathcal{S}(\delta)$ ([1]). Now, since f is integrable, we get

$$\Big| \sum_{i=1}^n f(\xi_i')|P_i| - \sum_{i=1}^n f(\xi_i'')|P_i| \Big| \leq 2\varepsilon.$$

On the other hand, $\Big| \sum_{i=1}^n f(\xi_i')|P_i| - \sum_{i=1}^n f(\xi_i'')|P_i| \Big| = \sum_{i \in I_3} (f(\xi_i'') - f(\xi_i'))|P_i|$ and

$$\sum_{i \in I_3} (f(\xi_i'') - f(\xi_i'))|P_i| \geq (\beta - \alpha) \sum_{i \in I_3} |P_i| \geq (\beta - \alpha)(\mathcal{L}(A) - 2\varepsilon) > 2\varepsilon,$$

which is a contradiction. □

To prove Lemma 3.1, suppose for a while that the following lemma holds.

Lemma 3.2. *For every $A \notin \mathfrak{L}$ there exists a $B \in \mathfrak{L}$ such that $A \subset B$ and $\mathcal{L}^*(A \cap C) = \mathcal{L}(B \cap C)$ for every $C \in \mathfrak{L}$ (then we will write $B \in A^{\mathfrak{L}}$).*

Proof of Lemma 3.1. Step 1. If $B \notin \mathfrak{L}$, then there exists a $D \in \mathfrak{L}$ such that $0 < \mathcal{L}(D) < \infty$ and $\mathcal{L}^*(D \cap B) = \mathcal{L}^*(D \setminus B) = \mathcal{L}(D)$.

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[1]For example: since $|P_i| \leq \delta'(\xi_i)$, then $|P_i| \leq \frac{1}{2m_0}$, $i = 1,\dots,n$. For $i \in I_3$, there is $\frac{1}{m_0} \leq \frac{\delta(\xi_i')}{2}$. Therefore, $|P_i| \leq \delta(\xi_i')$ and $(P_i, \xi_i')_{i=1}^n \in \mathcal{S}(\delta)$.

Indeed, there exists a $C \in \mathfrak{L}$ of finite measure such that $B \cap C \notin \mathfrak{L}$. Thanks to Lemma 3.2, we find $A_1, A_2 \in \mathfrak{L}$ such that $A_1 \in (B \cap C)^{\mathfrak{L}}$, $A_2 \in (C \setminus B)^{\mathfrak{L}}$. Then

$$C \setminus A_2 \subset C \cap B \subset C \cap A_1. \tag{1}$$

Consider $D := (C \cap A_1) \setminus (C \setminus A_2) = C \cap A_1 \cap A_2 \in \mathfrak{L}$. Then (1) and the fact that the Lebesgue measure is complete implies $\mathcal{L}(D) > 0$. Also, $\mathcal{L}(D) < \infty$, because $D \subset C$. Finally, $\mathcal{L}^*(D \cap B) = \mathcal{L}^*(D \cap C \cap B) = \mathcal{L}(D \cap A_1) = \mathcal{L}(D)$ and $\mathcal{L}^*(D \setminus B) = \mathcal{L}^*(D \cap (C \setminus B)) = \mathcal{L}(D \cap A_2) = \mathcal{L}(D)$.

Step 2. Choose an $f \notin \mathcal{M}(P, \mathcal{L})$. Then there exists an $\alpha \in \mathbb{R}$ such that $\{f \le \alpha\} \notin \mathfrak{L}$. From Step 1, we find a $D \in \mathfrak{L}$ such that $0 < \mathcal{L}(D) < \infty$ and $\mathcal{L}^*(D \cap \{f \le \alpha\}) = \mathcal{L}^*(D \setminus \{f \le \alpha\}) = \mathcal{L}(D)$. Thus, $D \in (D \cap \{f \le \alpha\})^{\mathfrak{L}}$. Consider an increasing family of sets $\mathcal{A} := \{D \cap \{f \ge \alpha + \frac{1}{2^n}\}\}_{n \in \mathbb{N}}$. Then $\bigcup \mathcal{A} = D \setminus \{f \le \alpha\}$. Thus, there exists a $\beta > \alpha$ for which $\mathcal{L}^*(D \cap \{f \ge \beta\}) > 0$. We find a measurable set $A \subset D$ such that $D \cap \{f \ge \beta\} \subset A$ and $\mathcal{L}(A) = \mathcal{L}^*(D \cap \{f \ge \beta\})$. Finally, $\mathcal{L}^*(A \cap \{f \le \alpha\}) = \mathcal{L}^*(A \cap D \cap \{f \le \alpha\}) = \mathcal{L}(A \cap D) = \mathcal{L}(A)$ and $\mathcal{L}^*(A \cap \{f \ge \beta\}) = \mathcal{L}^*(D \cap \{f \ge \beta\}) = \mathcal{L}(A)$. $\square$

It remains to prove Lemma 3.2.

Proof of Lemma 3.2. Step 1. If $A \subset B \in \mathfrak{L}$, then $B \in A^{\mathfrak{L}}$ iff for every $D \in \mathfrak{L}$ satisfying $D \subset B \setminus A$ there is $\mathcal{L}(D) = 0$.

Assume first that $B \in A^{\mathfrak{L}}$ and fix a measurable set $D \subset B \setminus A$. Then $\mathcal{L}(D) = \mathcal{L}(B \cap D) = \mathcal{L}^*(A \cap D) = 0$.

Conversely, assume that for every measurable set $D \subset B \setminus A$ there holds $\mathcal{L}(D) = 0$. Suppose, seeking a contradiction, that $B \notin A^{\mathfrak{L}}$. Then there exists a $C \in \mathfrak{L}$ such that $\mathcal{L}^*(A \cap C) < \mathcal{L}(B \cap C)$. Choose a set $D \in \mathfrak{L}$ satisfying $A \cap C \subset D$ and $\mathcal{L}(D) = \mathcal{L}^*(A \cap C)$. Let $E := (B \cap C) \setminus D$. Then $\mathcal{L}(D) < \mathcal{L}(B \cap C)$, which implies $\mathcal{L}(E) > 0$. Obviously, $E \in \mathfrak{L}$. But we have also $E \subset B$ and $A \cap E \subset (A \cap C) \setminus D = \varnothing$; a contradiction.

Step 2. If $A \subset \bigcup_{n=1}^{\infty} A_n$, where $A_n \in \mathfrak{L}$ and $\mathcal{L}(A_n) < \infty$, $n = 1, 2, \ldots$, then there exists a set from $A^{\mathfrak{L}}$. Indeed, thanks to the regularity of $\mathcal{L}$, for every $n = 1, 2, \ldots$, there exists a $B_n \in \mathfrak{L}$ such that $A \cap A_n \subset B_n$ and $\mathcal{L}(B_n) = \mathcal{L}^*(A \cap A_n)$. In fact, $B_n \in (A \cap A_n)^{\mathfrak{L}}$.

To see this, choose a measurable set $D \subset B_n \setminus (A \cap A_n)$. Then $(A \cap A_n) \subset B_n \setminus D$, hence $\mathcal{L}(B_n \setminus D) = \mathcal{L}(B_n)$. But $\mathcal{L}(B_n) < \infty$, thus $\mathcal{L}(D) = 0$ and Step 1 implies $B_n \in (A \cap A_n)^{\mathfrak{L}}$.

Let $B = \bigcup_{n=1}^{\infty} B_n$. Obviously, $A \subset B$. Moreover, if we take a measurable set

$D \subset B \setminus A$, then $D \cap B_n \subset B_n \setminus (A \cap A_n)$. Therefore, from Step 1 there follows $\mathcal{L}(D \cap C_n) = 0$, so $D = \bigcup_{n=1}^{\infty} D \cap B_n$ is of measure zero and Step 1 completes the proof of Step 2.

Step 3. Every set in $\mathbb{R}$ satisfies the assumption of Step 2. $\square$

Theorem 1.1 is proved.

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