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DEPENDENCE OF A WEAK SOLUTION OF THE FIRST ORDER DIFFERENTIAL EQUATION ON A PARAMETER

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Abstract. The purpose of this paper is to present some theorems on differentiability with respect to h of a weak solution of the evolution equation $\dot{u}_h(t) = A_h u_h(t) + f_h(t)$, $u_h(0) = u_h^0$, with a parameter $h \in [a, b] \subset \mathbb{R}$ and with a variable operator A_h.

Introduction. We consider the abstract first-order initial value problem

$$\frac{d}{dt}u(t) = Au(t) + f(t) \quad \text{for} \quad t \in (0, \tau], \tag{1}$$

$$u(0) = x, \tag{2}$$

where A is a densely defined, closed linear operator on a Banach space X, $x \in X$ and $f \in L^1(0, \tau; X)$ (see [**2**, III.3.1], [**3**, Appendix C5]). For a Banach space X, X^*, $B(X)$, $C(X)$ will denote its dual space, the set of bounded li-near operators and the set of closed linear operators from X into itself, respectively. Let $\langle \cdot, \cdot \rangle : X \times X^* \longrightarrow \mathbb{K}$ be the duality pairing. For an operator A, $D(A)$, $\varrho(A)$, $R(\lambda, A)$ and A^* will denote its domain, resolvent set, resolvent and adjoint, respectively.

Definition 1. (see [**1**]) A function $u \in C([0, \tau];\ X)$ is a weak solution of (1) on $[0, \tau]$ if and only if for every $v \in D(A^*)$ the function $\langle u(t), v \rangle$ is absolutely continuous on $[0, \tau]$ and

$$\frac{d}{dt}\langle u(t), v \rangle = \langle u(t), A^* v \rangle + \langle f(t), v \rangle \ a.e. \text{ on } [0, \tau].$$

J. M. Ball in [**1**] proved that

Theorem 1. *For each $x \in X$, there exists a unique weak solution u of problem* (1)–(2) *if and only if A is the infinitesimal generator of a C_0 semigroup*

$\{T(t)\}_{t\geq 0}$ *of bounded linear operators on* X, *and in this case* u *is given by*

$$u(t) = T(t)x + \int_0^t T(t-s)f(s)ds \quad t \in [0,\tau]. \tag{3}$$

The main object of this paper is to present some theorems on differentiability (with respect to a parameter $h \in [a,b]$) of the weak solution of the first order initial value problem with A, $D(A)$, f and the initial value dependent on h. Most of the results concerning the dependence of the weak solution of the problem (1)–(2) on a parameter have been obtained under the assumption that the operators $\{A_h\}_{h\in[a,b]}$, of a given family of linear, closed operators

$$A_h : X \supset D_h \longrightarrow X$$

with domains $D_h \subset X$, have domains independent of h (see, e.g., [**5, 6**]). In this paper we assume that $D(A_h) = D_h$ depends on h and for each $h \in [a,b]$ $\overline{D_h} = X$ (Theorem 9). One of possible ways of handling some problems concerning operators $\{A_h\}_{h\in[a,b]}$ with domains $D_h \subset X$ depending on h is to find a sufficiently regular family $\{B_h\}_{h\in[a,b]}$ of automorphisms of the Banach space X such that $B_h(D_h) = D$, where D is a fixed linear subspace of X (Theorem 6).

1. Preliminaries. For the reader's convenience, we recall some theorems concerning the operator calculus for unbounded operators and the theory of semigroups of operators (see, e.g., [**2, 3, 4, 7, 8, 9**]). Let A be a generator of a strongly continuous semigroup $\{T(t)\}_{t\geq 0}$.

PROPOSITION 1. *The following statements are true:*

(i) $\exists M \geq 1 \quad \exists \beta \geq 0 : \quad \|T(t)\| \leq Me^{\beta t}$,
(ii) $\forall x \in D(A) \quad T(t)x \in D(A) : \quad \frac{d}{dt}T(t)x = AT(t)x = T(t)Ax$,
(iii) $\forall x \in X \quad \forall v \in D(A^*) : \quad \frac{d}{dt}\langle T(t)x, v\rangle = \langle T(t)x, A^*v\rangle$,
(iv) $\forall v \in D(A^*) : \quad T^*(t)v \in D(A^*) \quad A^*T^*(t)v = T^*(t)A^*v$.
(v) $\forall f \in L^1(0,\tau;X) \quad \forall v \in X^* \quad \forall t \in (0,\tau]$: *the function*

$$[0,t] \ni s \to \langle f(s), T^*(t-s)v\rangle$$

is integrable and $\int_0^t \langle f(s), T^*(t-s)v\rangle ds = \langle \int_0^t T(t-s)f(s)ds, v\rangle$.

Let $G(M,\beta) := \{A \in C(X) \colon \overline{D(A)} = X,\ (\beta,+\infty) \subset \varrho(A)$ and $\|R(\xi,A)\| \leq M(\xi-\beta)^{-k}$ for $\xi > \beta$ and $k = 1,2,\dots\}$.

Now we recall a well-known theorem.

THEOREM 2. *A linear operator* A *is a generator of a strongly continuous semigroup iff* $A \in G(M,\beta)$, *for some* M, β. *Then* $\|T(t)\| \leq Me^{\beta t}$.

Let $\Omega = [a,b] \subset \mathbb{R}$, where $a < b$. In [**7**] (IX.2.16) it is established that if $\{R(\lambda, A_h)\}_{h\in\Omega}$ is a family strongly continuous at λ_0 for some $\lambda_0 > \beta$, and $\forall h \in \Omega \;\; A_h \in G(M,\beta)$ then $\forall x \in X \;\; \{T_h(t)x\}_{h\in\Omega}$ continuously depend on h, so it is easy to prove the next theorem.

THEOREM 3. *Suppose that*

(a) $\{A_h\}_{h\in\Omega} \subset G(M,\beta)$,
(b) $\exists \lambda > \beta \quad \forall x \in X: \quad \Omega \ni h \longrightarrow R(\lambda, A_h)x \in X$ *is continuous,*
(c) mappings $\Omega \ni h \longrightarrow u_h^0 \in X$ *and* $\Omega \ni h \longrightarrow f_h \in L^1(0,\tau;X)$ *are continuous.*

Then for each $h \in \Omega$ *there exists exactly one weak solution of the problem*

$$\frac{d}{dt}u(t) = A_h u(t) + f_h(t) \quad \text{for} \quad t \in (0,\tau], \tag{4}$$

$$u(0) = u_h^0 \tag{5}$$

given by

$$u_h(t) = T_h(t)u_h^0 + \int_0^t T_h(t-s)f_h(s)ds \quad t \in [0,\tau], \tag{6}$$

and

$$\lim_{h\to h_0} u_h(t) = u_{h_0}(t)$$

uniformly with respect to $t \in [0,\tau]$ *for each* $h_0 \in \Omega$.

2. Families of linear operators. Let $\{B_h\}_{h\in\Omega}$ be a family of linear, bounded operators with domains $D(B_h) = X$.

DEFINITION 2. We call the family $\{B_h\}_{h\in\Omega}$ weakly continuous (weakly differentiable) if for any $x \in X$ the mapping

$$\Omega \ni h \longrightarrow B_h x \in X$$

is weakly continuous (weakly differentiable).

DEFINITION 3. We say that the family $\{B_h\}_{h\in\Omega} \subset B(X)$ has weakly continuous weak derivative if there exists a weakly continuous family of linear operators $\{B_h'\}_{h\in\Omega}$ such that for each $x \in X$ and each $v \in X^*$

$$\frac{d}{dh}\langle B_h x, v\rangle = \langle B_h' x, v\rangle.$$

THEOREM 4. *Assume that the family* $\{B_h\}_{h\in\Omega} \subset B(X)$ *has weakly continuous weak derivative. Then*

(i) $\forall h \in \Omega: \quad B_h' \in B(X)$,
(ii) the family $\{B_h'\}_{h\in\Omega}$ *is uniformly bounded,*

(iii) the family $\{B_h^*\}_{h\in\Omega}$ *is* w^**-differentiable and*

$$[B_h^{'}]^* = [B_h^*]^{'}.$$

Proof. Fix $v \in X^*$. A weakly*-convergent sequence converges to an element of X^*, so there exists $w \in X^*$ for which

$$\langle B_h^{'}x, v\rangle = \lim_{k\to 0}\left\langle x, \frac{B_{h+k}^* - B_h^*}{k}v\right\rangle = \langle x, w\rangle.$$

Setting $(B_h^*)^{'}v := w$, we see that

$$\forall x \in X \quad \forall v \in X^* : \quad \langle B_h^{'}x, v\rangle = \langle x, (B_h^*)^{'}v\rangle.$$

This implies that

$$D((B_h^{'})^*) = X^* \quad \text{and} \quad (B_h^{'})^* = (B_h^*)^{'}.$$

By the closed graph theorem, $(B_h^{'})^*$ is bounded; there follows that the operator $B_h^{'}$ is bounded (see [**4**, Theorem 2.12.4]). This proves (i).

To prove (ii), fix $x \in X$. A function

$$\Omega \ni h \longrightarrow B_h^{'}x \in X$$

is weakly continuous, so it is bounded. There exists $M = M(x)$ such that for each $h \in \Omega$ $\left\|B_h^{'}x\right\| \le M(x)$. By the Banach–Steinhaus Theorem, there exists $C > 0$ that $\forall h \in \Omega$ $\left\|B_h^{'}\right\| \le C$.

One easily verifies that (iii) holds. $\square$

Let us consider densely defined linear operators A and B with domains $D(A)$ and $D(B)$, respectively.

Theorem 5. *If* $\overline{D(A)} = \overline{D(B)} = X$ *and* $0 \in \rho(A) \cap \rho(B)$, *then the following properties are equivalent:*

(i) $D(A^*) = D(B^*)$,

(ii) $\exists M > 0 \quad \exists m > 0 \quad \forall x \in X : \quad m\left\|A^{-1}x\right\| \le \left\|B^{-1}x\right\| \le M\left\|A^{-1}x\right\|.$

If one of this properties holds, then the operator

$$A^{-1}B : D(B) \longrightarrow D(A)$$

is an isomorphism and $\overline{A^{-1}B} \in Aut(X)$.

Proof. $(i) \Rightarrow (ii)$

The linear operator $A^{-1}B : D(B) \longrightarrow D(A)$ is densely defined and bijective. The adjoint operator $(A^{-1}B)^* = B^*(A^*)^{-1}$ exists, is closed and by assumption (i), its domain $D(B^*(A^*)^{-1}) = X^*$. By the closed graph theorem, $(A^{-1}B)^*$ is bounded. So the operator $A^{-1}B$ is bounded, too (see [**2**]).

$$\forall y \in D(B) \quad \left\|A^{-1}By\right\| \le \left\|\overline{A^{-1}B}\right\| \|y\|.$$

Setting $x := By$ and $m := \left\|A^{-1}B\right\|^{-1}$, we get $m\left\|A^{-1}x\right\| \leq \|By\|$. Considering the operator $B^{-1}A$, we will get $\left\|B^{-1}x\right\| \leq M\|Ax\|$ for a suitably defined $M > 0$.

$(ii) \Rightarrow (i)$ Let $v \in D(A^*)$ be fixed. For $y \in D(B)$, the following is true

$$|\langle By, v\rangle| = \left|\langle AA^{-1}By, v\rangle\right| = \left|\langle A^{-1}By, A^*v\rangle\right| \leq \|A^*v\|\left\|A^{-1}By\right\|$$
$$\leq m^{-1}\|A^*v\|\left\|B^{-1}By\right\| = m^{-1}\|A^*v\|\|y\|.$$

The inequality $\langle By, v\rangle \leq C\|y\|$ implies the continuity of the linear mapping $y \to \langle By, v\rangle$ and it is equivalent to $v \in D(B^*)$. The theorem is proved. □

Now we consider a family $\{A_h\}_{h\in\Omega} \subset C(X)$ of densely defined operators. Assume that the domains $D(A_h^*) = D^*$ are independent of $h \in \Omega$ and suppose that $\forall h \in \Omega \quad 0 \in \rho(A_h)$. By Theorem 5, for any $h, k \in \Omega$, $\overline{A_h^{-1}A_k} \in Aut(X)$.

$$B(h,k) := \overline{A_h^{-1}A_k}.$$

It is easy to see that for any $h, k, l \in \Omega$:

(a) $B(h,h) = I$,
(b) $B(h,k)B(k,l) = B(h,l)$,
(c) $[B(h,k)]^{-1} = B(k,h)$,
(d) $A_h^{-1} = B(h,k)A_k^{-1}$.

Theorem 6. *Suppose that for each* $h \in \Omega$:

(a) $A_h \in C(X)$ *and* $\overline{D(A_h)} = X$,
(b) $0 \in \rho(A_h)$,
(c) mapping $\Omega \ni k \to B(k,h) \in Aut(X)$ *is continuous in* $k = h$,

then

(i) $\forall h \in \Omega$: *mappings* $k \to B(k,h)$ *and* $k \longrightarrow B(h,k)$ *are continuous in* Ω,
(ii) mapping $\Omega \ni h \longrightarrow A_h^{-1} \in B(X)$ *is continuous,*
(iii) $\exists M, m > 0\ \forall h, k \in \Omega\ \forall x \in X :\ m\left\|A_h^{-1}x\right\| \leq \left\|A_k^{-1}x\right\| \leq M\left\|A_h^{-1}x\right\|.$

Proof. It is easy to see (i). To prove (ii), we notice that

$$\left\|A_h^{-1} - A_k^{-1}\right\| = \left\|B(h,k)A_k^{-1} - A_k^{-1}\right\| \leq \|B(h,k) - I\|\left\|A_k^{-1}\right\| \to 0$$

as $h \to k$.

To obtain (iii), we infer from (i) that for a fixed $l \in \Omega$ there exist positive constants $M(l), m(l)$ such that for any $h, k \in \Omega$

$$\|B(h,l)\| \leq M(l) \quad \text{and} \quad \|B(l,k)\| \leq m(l).$$

Also

$$\left\|A_h^{-1}x\right\| = \left\|A_h^{-1}A_lA_l^{-1}A_kA_k^{-1}x\right\| \leq \|B(h,l)\|\|B(l,k)\|\left\|A_k^{-1}x\right\|$$
$$\leq M(l)m(l)\left\|A_k^{-1}x\right\|.$$

□

THEOREM 7. *Suppose that assumptions* (a)*,* (b)*,* (c) *of Theorem 6 are satisfied. If for each* $k \in \Omega$ *the family* $\{B(h,k)\}_{h\in\Omega}$ *has weakly continuous weak derivative* $\{\frac{\partial}{\partial h}B(h,k)\}_{h\in\Omega}$*, then*

(i) $\forall k \in \Omega \quad \exists C > 0 \quad \forall h \in \Omega : \quad h \neq k \Rightarrow \left\| \frac{B(h,k)-I}{h-k} \right\| \leq C,$

(ii) $\forall k, h \in \Omega :$ *the linear operator* $\frac{\partial}{\partial h}B(h,k)$ *is bounded,*

(iii) family $\{B^*(h,k)\}_{h\in\Omega}$ *is* w^**-differentiable and*

$$\frac{\partial}{\partial h}B^*(h,k) = \left[\frac{\partial}{\partial h}B(h,k)\right]^*,$$

(iv) family $\{B(k,h)\}_{h\in\Omega}$ *has weakly continuous weak derivative,*

(v) $\forall x \in X \quad \forall v \in D^* \quad \forall k \in \Omega :$

$$\frac{d}{dh}\langle x, A_h^* v\rangle \mid_{h=k} = \left\langle x, \left(\frac{\partial}{\partial h}B(k,h)|_{h=k}\right)^* A_k^* v\right\rangle.$$

PROOF. Let

$$\widetilde{B}(h,k) := \begin{cases} \frac{B(h,k)-I}{h-k} & \text{for } h \neq k \\ \frac{\partial}{\partial h}B(h,k)|_{h=k} & \text{for } h = k. \end{cases}$$

By assumption, the family $\{\widetilde{B}(h,k)\}_{h\in\Omega}$ is weakly continuous, so it is uniformly bounded.

(ii) and (iii) follow from Theorem 4.

To prove (iv), fix $k \in \Omega$ and $v \in X^*$. Let $h \in \Omega$ and $h \neq k$.

$$\left\langle \frac{B(k,h)-I}{h-k}x, v\right\rangle = \left\langle \frac{I-B(h,k)}{h-k}B(k,h)x, v\right\rangle$$

$$= \left\langle \frac{I-B(h,k)}{h-k}[B(k,h)-I]x, v\right\rangle + \left\langle \frac{I-B(h,k)}{h-k}x, v\right\rangle$$

$$\to -\left\langle \frac{\partial}{\partial h}B(h,k)_{|h=k}x, v\right\rangle,$$

when $h \to k$. The above relation follows from (i), norm continuity for the family $\{B(h,k)\}_{h\in\Omega}$ and

$$\left|\left\langle \frac{I-B(h,k)}{h-k}[B(k,h)-I]x, v\right\rangle\right| \leq \|v\| \left\|\frac{I-B(h,k)}{h-k}\right\| \|B(k,h)-I\| \|x\|$$

$$\leq C\|v\| \|x\| \|B(k,h)-I\| \to 0,$$

when $h \to k$.

Now we show that the family $\{B(k,h)\}_{h\in\Omega}$ is weakly differentiable. Fix $r \in \Omega$.

$$\lim_{h\to r}\left\langle \frac{B(k,h)-B(k,r)}{h-r}x, v\right\rangle = \lim_{h\to r}\left\langle \frac{B(r,h)-I}{h-r}x, B^*(k,r)v\right\rangle$$
$$= -\left\langle \frac{\partial}{\partial h}B(h,r)_{|h=r}x, B^*(h,r)v\right\rangle.$$

To prove (v), let $x \in X$, $v \in D^*$ and $k \in \Omega$.

$$\left\langle x, \frac{A_h^* - A_k^*}{h-k}v\right\rangle = \left\langle \frac{B(k,h)-I}{h-k}x, A_k^* v\right\rangle$$
$$\to \left\langle \frac{\partial}{\partial h}B(k,h)_{|h=k}x, A_k^* v\right\rangle = \left\langle x, \left(\frac{\partial}{\partial h}B(k,h)_{|h=k}\right)^* A_k^* v\right\rangle,$$

when $h \to k$. □

3. Differentiability with respect to the parameter. In this section we will prove a theorem on differentiability of the weak solution with respect to a parameter, in the case when non-constant domains $D(A_h)$ are isomorphic. In this section we adopt the following.

Assumption A. Suppose that

(i) $\forall h \in \Omega$: a closed and densely defined operator A_h has a domain D_h,
(ii) for each $h \in \Omega$ the adjoint operator A_h^* has a domain $D(A_h^*) = D^*$,
(iii) $\exists M \geq 1,\ \beta \geq 0\ \forall h \in \Omega : \ A_h \in G(M,\beta)$,
(iv) $\forall h \in \Omega : \ 0 \in \varrho(A_h)$,
(v) $\forall k \in \Omega : \quad \Omega \ni h \longrightarrow \overline{A_k^{-1}A_h} \in Aut(X)$ is continuous in $h = k$,
(vi) $\forall k \in \Omega$: the family $\{\overline{A_k^{-1}A_h}\}_{h\in\Omega}$ has weakly continuous weak derivative.

To prove Theorem 9, we need the following theorem.

Theorem 8. *Suppose that for each* $h \in \Omega : \ u_h^0 \in X,\ f_h \in L^1(0,\tau;X)$ *and* u_h *is the weak solution of Cauchy problem* (4)–(5). *Then for each* $v \in D^*$, $\int_0^t \left\langle u_h(s), \frac{A_h^*-A_{h_0}^*}{h-h_0}T_{h_0}^*(t-s)v\right\rangle ds$ *exists and*

$$\left\langle \frac{u_h(t)-u_{h_0}(t)}{h-h_0}, v\right\rangle = \left\langle T_{h_0}(t)\frac{u_h^0 - u_{h_0}^0}{h-h_0}, v\right\rangle$$
$$(7)\qquad + \int_0^t \left\langle T_{h_0}(t-s)\frac{f_h(s)-f_{h_0}(s)}{h-h_0}, v\right\rangle ds$$
$$+ \int_0^t \left\langle u_h(s), \frac{A_h^*-A_{h_0}^*}{h-h_0}T_{h_0}^*(t-s)v\right\rangle ds \quad v \in D^*,\ h \neq h_0.$$

Proof. Fix $h, h_0 \in \Omega$. It follows from Proposition 1 that:

- $\frac{d}{dt}\langle u_h(t), v\rangle = \langle u_h(t), A_h^* v\rangle + \langle f_h(t), v\rangle$ for $v \in D^*$ a.e. on $[0,\tau]$,
- $T_h^*(t)v \in D^*$ and $A_h^* T_h^*(t)v = T_h^*(t)A_h^* v$ for $v \in D^*$,
- $\frac{d}{dt}\langle x, T_h^*(t)v\rangle = \frac{d}{dt}\langle T_h(t)x, v\rangle = \langle T_h(t)x, A_h^* v\rangle = \langle x, A_h^* T_h^*(t)v\rangle$ for $x \in X$, $v \in D^*$.

This implies that

$$\begin{aligned}\frac{d}{ds}\langle u_h(s), &T_{h_0}^*(t-s)v\rangle = \langle u_h(s), A_h^* T_{h_0}^*(t-s)v\rangle \\ &+ \langle f_h(s), T_{h_0}^*(t-s)v\rangle - \langle u_h(s), A_{h_0}^* T_{h_0}^*(t-s)v\rangle.\end{aligned} \tag{8}$$

Functions $s \to \frac{d}{ds}\langle u_h(s), T_{h_0}^*(t-s)v\rangle$ and $s \to \langle f_h(s), T_{h_0}^*(t-s)v\rangle$ are integrable. It is easy to see that $\langle u_h(s), A_{h_0}^* T_{h_0}^*(t-s)v\rangle = \langle T_{h_0}(t-s)u_h(s), A_{h_0}^* v\rangle$, so the function $s \to \langle u_h(s), A_{h_0}^* T_{h_0}^*(t-s)v\rangle$ is integrable. From this and (8) there follows that the function $s \to \langle u_h(s), A_h^* T_{h_0}^*(t-s)v\rangle$ is integrable in $[0,t]$.

Integrating (8) over $[0,t]$, we obtain

$$\begin{aligned}\langle u_h(t), v\rangle - \langle u_h(0), T_{h_0}^*(t)v\rangle &= \int_0^t \langle f_h(s), T_{h_0}^*(t-s)v\rangle ds \\ &+ \int_0^t \langle u_h(s), [A_h^* - A_{h_0}^*]T_{h_0}^*(t-s)v\rangle ds.\end{aligned} \tag{9}$$

By (6) and (9),

$$\begin{aligned}\langle u_h(t) - u_{h_0}(t), v\rangle &= \int_0^t \langle u_h(s), [A_h^* - A_{h_0}^*]T_{h_0}^*(t-s)v\rangle ds \\ &+ \int_0^t \langle f_h(s), T_{h_0}^*(t-s)v\rangle ds + \langle u_h(0), T_{h_0}^*(t)v\rangle \\ &- \langle T_{h_0}(t)u_{h_0}(0), v\rangle - \int_0^t \langle T_{h_0}(t-s)f_{h_0}(s), v\rangle ds \\ &= \langle T_{h_0}(t)[u_h^0 - u_{h_0}^0], v\rangle + \int_0^t \langle T_{h_0}(t-s)[f_h(s) - f_{h_0}(s)], v\rangle ds \\ &+ \int_0^t \langle u_h(s), [A_h^* - A_{h_0}^*]T_{h_0}^*(t-s)v\rangle ds.\end{aligned} \tag{10}$$

The conclusion follows upon dividing (10) by $h - h_0$. □

Now we are able to prove the main theorem of this paper.

Theorem 9. *If the family $\{A_h\}_{h\in\Omega}$ satisfies Assumption A and*

(i) $\Omega \ni h \longrightarrow u_h^0 \in X$ is continuously differentiable,

(ii) $\Omega \ni h \longrightarrow f_h \in L^1(0,\tau;X)$ is continuously differentiable,

then for each $v \in D^*$ the function

$$\Omega \times [0,\tau] \ni (h,t) \longrightarrow \langle u_h(t), v\rangle \in \mathbb{R}$$

is differentiable with respect to h, *function* $[0,t] \ni s \longrightarrow \langle u_{h_0}(s), (A^*_{h_0})' T^*_{h_0}(t-s)v\rangle$ *is integrable in* $[0,t]$ *and*

$$\frac{\partial}{\partial h}\langle u_h(t), v\rangle_{|h=h_0} = \langle T_{h_0}(t)[(u^0_{h_0})'], v\rangle + \int_0^t \langle T_{h_0}(t-s) f'_{h_0}(s), v\rangle ds$$
$$+ \int_0^t \langle u_{h_0}(s), (A^*_{h_0})' T^*_{h_0}(t-s)v\rangle ds,$$

where "′" denotes differentiation with respect to h, *and*

$$\left(A^*_{h_0}\right)' := \left[\frac{\partial}{\partial h}\left(\overline{A^{-1}_{h_0} A_h}\right)_{|h=h_0}\right]^* A^*_{h_0}.$$

Proof. By previous Theorem 8, the function $t \to \frac{u_h(t)-u_{h_0}(t)}{h-h_0}$ satisfies equation (7). Denote

$$z_h(t) := T_{h_0}(t)\frac{u^0_h - u^0_{h_0}}{h-h_0} + \int_0^t T_{h_0}(t-s)\frac{f_h(s)-f_{h_0}(s)}{h-h_0} ds.$$

The function z_h is a weak solution of the Cauchy problem

$$\begin{cases} \frac{d}{dt} z_h(t) = A_{h_0} z_h(t) + F_h(t) \\ z_h(0) = z^0_h, \end{cases}$$

where

$$F_h(t) = \begin{cases} \frac{f_h - f_{h_0}}{h-h_0}(t) & \text{for } h \neq h_0 \\ f'_{h_0}(t) & \text{for } h = h_0 \end{cases}$$

and

$$z^0_h = \begin{cases} \frac{u_h - u_{h_0}}{h-h_0} & \text{for } h \neq h_0 \\ (u^0_{h_0})' & \text{for } h = h_0. \end{cases}$$

By Theorem 2,

$$\lim_{h\to h_0} z_h(t) = z_{h_0}(t)$$

uniformly with respect to $t \in [0,\tau]$, where

$$z_{h_0}(t) = T_{h_0}(t)[(u^0_{h_0})'] + \int_0^t T_{h_0}(t-s) f'_{h_0}(s) ds.$$

Now we consider

$$\int_0^t \left\langle u_h(s), \frac{A^*_h - A^*_{h_0}}{h-h_0} T^*_{h_0}(t-s)v \right\rangle ds$$
$$= \int_0^t \left\langle u_h(s) - u_{h_0}(s), \frac{A^*_h - A^*_{h_0}}{h-h_0} T^*_{h_0}(t-s)v \right\rangle ds$$
$$+ \int_0^t \left\langle u_{h_0}(s), \frac{A^*_h - A^*_{h_0}}{h-h_0} T^*_{h_0}(t-s)v \right\rangle ds.$$

It is easy to see that

$$\frac{A_h^* - A_{h_0}^*}{h - h_0} = \frac{A_h^*(A_{h_0}^*)^{-1} - I^*}{h - h_0} A_{h_0}^* = \left(\frac{\overline{A_{h_0}^{-1} A_h} - I}{h - h_0} \right)^* A_{h_0}^*.$$

By Theorem 7 and Proposition 1,

$$\left\| \frac{A_h^* - A_{h_0}^*}{h - h_0} T_{h_0}^*(t-s)v \right\| \leq \left\| \frac{\overline{A_{h_0}^{-1} A_h} - I}{h - h_0} \right\| \left\| A_{h_0}^* T_{h_0}^*(t-s)v \right\|$$

$$\leq CMe^{\beta T} \left\| A_{h_0}^* v \right\|.$$

So, by the Lebesgue Theorem

$$\lim_{h \to h_0} \int_0^t \left\langle u_h(s) - u_{h_0}(s), \frac{A_h^* - A_{h_0}^*}{h - h_0} T_{h_0}^*(t-s)v \right\rangle ds = 0,$$

uniformly in $t \in [0, \tau]$.

Applying the Lebesgue Theorem again, we obtain

$$\int_0^t \left\langle u_{h_0}(s), \frac{A_h^* - A_{h_0}^*}{h - h_0} T_{h_0}^*(t-s)v \right\rangle ds \to \int_0^t \left\langle u_{h_0}(s), (A_{h_0}^*)' T_{h_0}^*(t-s)v \right\rangle ds,$$

when $h \to h_0$, where $\left(A_{h_0}^*\right)' = \left[\frac{\partial}{\partial h} \left(\overline{A_{h_0}^{-1} A_h} \right)_{|h=h_0} \right]^* A_{h_0}^*$. □

THEOREM 10. *Suppose that for each $h \in \Omega$ $A_h : X \longrightarrow X$ with $D(A_h) = D \subset X$ is a linear operator. If $0 \in \rho(A)$ and for each $x \in D$ the mapping*

$$\Omega \ni h \longrightarrow A_h x$$

is continuously differentiable, then the family $\{A_h A_k^{-1}\}_{h,k \in \Omega} \subset B(X)$ is continuous with respect to $(h, k) \in \Omega \times \Omega$.

PROOF. See [**8**] Lemma II.1.5. □

THEOREM 11. *Suppose that for each $h \in \Omega$, a closed and densely defined operator A_h has a bounded inverse and the family $\{A_h^*\}_{h \in \Omega}$, defined on a common domain $D^* := D(A_h^*)$, is continuously differentiable, i.e., for each $v \in D^*$ the function*

$$\Omega \ni h \longrightarrow A_h^* v \in X^*$$

is continuously differentiable, then the family $\{A_h\}_{h \in \Omega}$ has the following properties:

(1) for each $k \in \Omega$

$$\Omega \ni h \longrightarrow \overline{A_k^{-1} A_h} \in Aut(X)$$

is continuous in $h = k$,

(2) for each $k \in \Omega$ the family $\{\overline{A_k^{-1} A_h}\}_{h \in \Omega}$ is weakly differentiable.

Proof. By the above theorem, the family $\{A_h^*[A_k^*]^{-1}\}_{h,k\in\Omega} \subset B(X^*)$ is continuous with respect to $(h,k) \in \Omega \times \Omega$. It is easy to see that

$$A_h^*[A_k^*]^{-1} = \left(\overline{A_k^{-1}A_h}\right)^* .$$

The operator $\overline{A_k^{-1}A_h}$ is bounded and

$$\left\|A_h^*[A_k^*]^{-1}\right\| = \left\|\overline{A_k^{-1}A_h}\right\| .$$

This implies that the family $\left\{\overline{A_k^{-1}A_h}\right\}_{h,k\in\Omega}$ is continuous with respect to $(h,k) \in \Omega \times \Omega$.

Fix $v \in X^*$, $x \in X$ and $h_0 \in \Omega$. There exists exactly one $w \in D^*$ such that $A_{h_0}^* w = v$.

$$\left\langle \frac{\overline{A_{h_0}^{-1}A_h} - I}{h-h_0} x, v \right\rangle = \left\langle x, \left(\frac{\overline{A_{h_0}^{-1}A_h} - I}{h-h_0}\right)^* A_{h_0}^* w \right\rangle = \left\langle x, \frac{A_h^*[A_{h_0}^*]^{-1} - I^*}{h-h_0} A_{h_0}^* w \right\rangle$$

$$= \left\langle x, \frac{A_h^* w - A_{h_0}^* w}{h-h_0} \right\rangle \to \langle x, [A_h^* w]'_{|h=h_0} \rangle,$$

when $h \to h_0$. □

Example 1. (see [**10**, 2.1], [**10**, Example 2]) Let K be a bounded domain in $\mathbb{R}^2$ with boundary $S = \partial K$ of class C^2 and let $h \in [0,1]$.

The sets

$$D_h := \{u \in L^2(K) \ : \ u \in H^2(K) \ \text{ and } \ \frac{\partial u}{\partial n} + hu = 0 \ \text{ on } \ \partial K\},$$

$$D := \{u \in L^2(K) \ : \ u \in H^2(K) \ \text{ and } \ \frac{\partial u}{\partial n} = 0 \ \text{ on } \ \partial K\}$$

are dense linear subspaces of $L^2(K)$, where n is the interior unit normal vector field on S.

One can verify that $D_h \neq D_k$ for $h,k \in [0,1]$ and $h \neq k$.

Let $\phi : \overline{K} \times [0,1] \longrightarrow \mathbb{R}$ be a function of class C^1, such that

$$\frac{1}{2} \leq \phi_h(x) := \phi(x,h) \quad \text{for } \ x \in \overline{K}, \ h \in [0,1],$$

$$\phi_h(x) = 1 \ \text{ and } \ \frac{\partial \phi_h}{\partial n} = h \ \text{ for } \ x \in \partial K, \ h \in [0,1].$$

Let $\Phi_h : L^2(K) \longrightarrow L^2(K)$ be given by

$$\Phi_h(u) := \phi_h \cdot u \quad \text{for } \ u \in L^2(K), \ h \in [0,1].$$

One can verify that:

(1) $\Phi_h \in Aut(L^2(K))$,
(2) $\Phi_h(D_h) = D$,

(3) the mapping $[0,1] \ni h \longrightarrow \Phi_h \in B(L^2(K))$ is of class C^1.

Let $A := -\Delta + \lambda I : D \longrightarrow L^2(K)$. This operator is closed and, for λ large enough, it is onto and one-to-one. By the closed graph theorem, its inverse is bounded.

The family $A_h := A \circ \Phi_h : D_h \longrightarrow L^2(K)$ parametrized by $h \in [0,1]$ is a family of closed, densely defined linear operators with pairwise different domains. The domain $D(A_h^*) = D(A^*)$ is the same for all $h \in [0,1]$ and $\lim_{h\to k} \left\|A_h^{-1}A_k - I\right\| = 0$.

References

1. Ball J. M., *Strongly continuous semigroups, weak solutions, and the variation of constant formula*, Proc. Amer. Math. Soc., **63(2)** (1977), 370–373.
2. Dunford N., Schwartz T., *Linear Operators I. General Theory*, Interscience Publisher, 1958.
3. Engel K. J., Nagel R., *One-Parameter Semigroups for Linear Evolution Equations*, Springer-Verlag, 2000.
4. Hill E., Philips R. S., *Functional Analysis and Semi-Groups*, 1985.
5. Jużyniec M., *Weak solutions of evolution equations with parameter*, Univ. Iagel. Acta Math., **35** (1997), 189–204.
6. Jużyniec M., *Dependence on parameter of weak solution of an evolution equation*, Univ. Iagel. Acta Math., **37** (1999), 237–243.
7. Kato T., *Perturbation theory for linear operators*, Springer-Verlag, 1966.
8. Krein S. G., *Linear differential equations in Banach spaces*, Transl. Math. Monographs, Vol. **29**, Amer. Math. Soc., 1971.
9. Pazy A., *Semigroups of linear operators and applications to partial differential equations*, Springer-Verlag, 1983.
10. Winiarska T., Winiarski T., *Regularity of domains of parameterized families of closed linear operators*, Ann. Polon. Math., **80** (2003), 231–241.

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