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PLURISUBHARMONIC EXTENSION IN NON-DEGENERATE ANALYTIC POLYHEDRA

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Abstract. Let $\mathcal{P}$ be a connected, non-degenerate analytic polyhedron in $\mathbb{C}^n$, $n \geq 2$. In this note we characterize those continuous functions $\partial\mathcal{P} \to \mathbb{R}$ which can be extended to a plurisubharmonic function in $\mathcal{P}$.

1. Introduction. Let D be a bounded, connected and open set in $\mathbb{C}^n$, $n \geq 1$, and let $f_1, f_2, \ldots, f_N$, $(N \geq n)$ be holomorphic on D. The set

$$\mathcal{P} = \{z \in D : |f_i(z)| < 1,\ i = 1, \ldots, N\}\,,$$

is called an *analytic polyhedron* if $\mathcal{P}$ is relatively compact in D. The functions f_i will be called *defining functions for* $\mathcal{P}$. For every increasing multi-index I, $1 \leq i_1 < \cdots < i_k \leq N$, of length $k \in \{1, \ldots, n\}$ let

$$\sigma_I = \{z \in \mathbb{C}^n : |f_i(z)| = 1,\ i \in I\} \cap \bar{\mathcal{P}}\,.$$

The *distinguished boundary* of an analytic polyhedron is the part of the topological boundary that is given by

$$\Gamma_n = \bigcup_{|I|=n} \sigma_I\,.$$

In this note we will assume, in addition, that the analytic polyhedron $\mathcal{P}$ is *non-degenerate*, i.e., for every increasing multi-index $1 \leq i_1 < \cdots < i_k \leq N$ of length $k \in \{1, \ldots, n\}$ we have that $\partial f_{i_1} \wedge \cdots \wedge \partial f_{i_k} \neq 0$ on σ_I. In particular, this assumption implies that σ_I is an orientable $(n-k)$-dimensional complex

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submanifold of $\mathbb{C}^n$ with the natural orientation induced by $\mathbb{C}^n$. A polyhedron is not necessarily connected, but in this setting it has at most a finite number of connected components (see e.g. [**1**]).

Throughout this note, we assume that $\mathcal{P}$ is a connected, non-degenerate analytic polyhedron in $\mathbb{C}^n$, $n \geq 2$. As in [**5**], consider the bounded plurisubharmonic function $\varphi : \mathcal{P} \to (-\infty, 0)$ defined by

$$\varphi(z) = \max\{\log|f_1(z)|, \ldots, \log|f_N(z)|, -1\}.$$

The set $\{z \in \mathcal{P} : \varphi(z) < c\}$ is relatively compact in $\mathcal{P}$ for every $c \in (-\infty, 0)$ and therefore $\mathcal{P}$ is a so called *hyperconvex domain.* Recall that a hyperconvex domain is regular with respect to the Laplace operator, i.e., if Ω is hyperconvex, then every continuous real-valued function defined on $\partial\Omega$ has a harmonic extension to Ω and this extension can be taken continuous on $\bar{\Omega}$ (see e.g. [**7**]).

Given a continuous, real-valued function f on $\partial\mathcal{P}$, it is not always possible to find a plurisubharmonic extension of f. However, if f has a plurisubharmonic extension, then it is possible to solve certain non-homogenous complex Monge–Ampère equations (see [**2, 3**]). On the other hand, if we prescribe the function on a smaller part of the topological boundary, more precisely on Γ_n, it was proved in [**5**] that we can always find a plurisubharmonic function with *essentially* the given boundary values. This was later generalized by Górski [**10**] using the same techniques. Bremermann's result in [**5**] was also the starting point in Levenberg and Okada [**12**], where the authors consider the non-homogenous complex Monge–Ampère equation in the bidisc. In this note our main goal is to characterize those continuous functions $\partial\mathcal{P} \to \mathbb{R}$ which can be extended to a plurisubharmonic function in $\mathcal{P}$. Our main result is:

Theorem A. *Assume that $\mathcal{P}$ is a connected, non-degenerate analytic polyhedron in $\mathbb{C}^n$, $n \geq 2$. If $f : \partial\mathcal{P} \to \mathbb{R}$ is a real-valued continuous function, then the following assertions are equivalent:*

(1) there exists a function s which is plurisubharmonic on $\mathcal{P}$, continuous on $\bar{\mathcal{P}}$ and $s|_{\partial\mathcal{P}} = f$,
(2) for every increasing multi-index I of length $k \in \{1, \ldots, n-1\}$ the function f is plurisubharmonic on the $(n-k)$-dimensional complex manifold σ_I,
(3) the function f is subharmonic on every analytic disc d embedded in $\partial\mathcal{P}$, i.e., the function $f \circ d$ is subharmonic on the unit disc Δ in $\mathbb{C}$ for every injective, holomorphic function $d : \Delta \to \bar{\mathcal{P}}$ with $d(\Delta) \subseteq \partial\mathcal{P}$,

Using well-known results we achieve that the above conditions are also equivalent to:

(4) *for every* $p \in \partial\mathcal{P}$ *and every Jensen measure* μ, *defined on* $\bar{\mathcal{P}}$ *with barycenter* p *it holds that*

$$f(p) \le \int_{\partial\mathcal{P}} f \, d\mu$$

(see Section 2 for the definition of Jensen measures),

(5) *the Perron–Bremermann envelope* PB_f *is a maximal plurisubharmonic function on* $\mathcal{P}$, *continuous on* $\bar{\mathcal{P}}$, *and* $\lim_{z\to\xi} PB_f(z) = f(\xi)$ *for every* $\xi \in \partial\mathcal{P}$ *(see Section 2 for the definition of the Perron–Bremermann envelope).*

The equivalence between (1) and (4) was proved by the fourth-named author in [**15**] and the equivalence between (1) and (5) follows by Walsh's theorem (see Section 2 or [**14**]). For the case $n = 1$, properties (2) and (3) do not make sense and (1), (4) and (5) are valid for every continuous function on $\partial\mathcal{P}$. If $n \ge 2$, then the unit polydisc $\Delta^n \subseteq \mathbb{C}^n$, is an example of a connected, non-degenerate analytic polyhedron given by the n coordinate functions as defining functions. In the polydisc case the equivalence between property (1) and (3) was shown by Sadullaev [**13**] for $n = 2$ and later this was generalized by Błocki in [**4**] to $n \ge 2$ (see also Example 3.6 in [**15**]).

2. The boundaries of a polyhedra and preliminaries. In this section we will first recall some different notions of boundaries. Let $\mathcal{P}$ be an analytic polyhedron in $\mathbb{C}^n$, $n \ge 1$, without any extra assumption. The *topological boundary* $\partial\mathcal{P}$ of the polyhedron is given by

$$\partial\mathcal{P} = \bigcup_{\substack{|I|=k \\ 1\le k\le n}} \sigma_I \,,$$

where σ_I is defined as in the introduction by

$$\sigma_I = \{z \in \mathbb{C}^n : |f_i(z)| = 1,\ i \in I\} \cap \bar{\mathcal{P}} \,.$$

Let $A(P)$ denote the algebra of continuous functions on $\bar{\mathcal{P}}$ that are holomorphic on $\mathcal{P}$. The *Šilov boundary* with respect to the algebra $A(P)$ is defined as the smallest closed set $S(P) \subseteq \bar{\mathcal{P}}$ with the property that the modulus of any function in $A(P)$ attains its maximum on $S(P)$. The distinguished boundary is in general larger than the Šilov boundary with respect to $A(P)$. It was proved in [**11**] that under the assumption that $\mathcal{P}$ is non-degenerate, the distinguished boundary and Šilov boundary with respect to $A(P)$ are the same. Moreover, they coincide with the set of points in $\partial\mathcal{P}$ through which there is no germ of an analytic variety of positive dimension contained in $\partial\mathcal{P}$. From these considerations, Proposition 2.1 follows. We refer to [**9**] for more information about the function algebraic viewpoint of boundaries.

Proposition 2.1. *Assume that $\mathcal{P}$ is a non-degenerate analytic polyhedron in $\mathbb{C}^n$, $n \geq 2$. Let $\Delta \subseteq \mathbb{C}$ be the open unit disc and assume that $d : \Delta \to \bar{\mathcal{P}}$ is a holomorphic function, then the following holds:*

a) if $d(\Delta) \subseteq \Gamma_n$, then d is constant.
b) if $d(\Delta) \subseteq \partial\mathcal{P}$ is non-constant, then there exists an increasing multi-index I of length $k \in \{1, \ldots, n\}$, such that $d(\Delta) \subseteq \sigma_I$.

Remark. Without the assumption of non-degeneracy of $\mathcal{P}$, there can be analytic structure in the Šilov boundary (see e.g. [**11**]).

Definition 2.2. Let $\Omega \subseteq \mathbb{C}^n$ be a bounded domain and let μ be a non-negative, regular Borel measure supported on $\bar{\Omega}$. The measure μ is a *Jensen measure with barycenter at $z \in \bar{\Omega}$* if

$$u(z) \leq \int_{\bar{\Omega}} u \, d\mu \,,$$

for every continuous function $u : \bar{\Omega} \to [-\infty, \infty)$, not identically $-\infty$, such that $u \in \mathcal{PSH}(\Omega)$. The set of all Jensen measures with barycenter at z will be denoted by $\mathcal{J}_z^c(\bar{\Omega})$.

Definition 2.2 is a slightly different definition of Jensen measures than the classical one, since it allows the measure to have support in $\bar{\Omega}$, and also since it considers Jensen measures for boundary points. It was proved in [**15**] (see also [**6**]) that if Ω is a bounded hyperconvex domain and $\mu \in \mathcal{J}_z^c(\bar{\Omega})$, $z \in \partial\Omega$, then the support of μ is contained in $\partial\Omega$. In the case of an analytic polyhedron we can gain more detailed information on the support of a Jensen measure with barycenter at a boundary point.

Proposition 2.3. *Let $\mathcal{P}$ be an analytic polyhedron in $\mathbb{C}^n$, $n \geq 2$, and let $p \in \partial\Omega$. Define $J = \{j \in I : |f_j(p)| = 1\}$ and $\Sigma_p = \cap_{j \in J} f_j^{-1}(f_j(p))$. If $\mu \in \mathcal{J}_p^c(\bar{\mathcal{P}})$, then the support of μ is contained in Σ_p and $\mu \in \mathcal{J}_z^c(\bar{\Sigma}_p)$. Conversely, if μ is a probability measure defined on $\mathcal{P}$ with support in $\bar{\Sigma}_p$ such that $\mu \in \mathcal{J}_p^c(\bar{\Sigma}_p)$, then $\mu \in \mathcal{J}_p^c(\bar{\mathcal{P}})$.*

Proof. Let u be a function defined on $\bar{\mathcal{P}}$ by

$$u(z) = \sum_{j \in J} \left(|f_j(z) + f_j(p)| - 2\right) \,.$$

Then $u \in \mathcal{PSH}(\mathcal{P}) \cap C(\bar{\mathcal{P}})$, $u \leq 0$ on $\bar{\mathcal{P}}$, and $u(z) = 0$ exactly for $z \in \Sigma_p$. Hence, if $\mu \in \mathcal{J}_p^c(\bar{\mathcal{P}})$, then

$$0 = u(p) \leq \int_{\bar{\mathcal{P}}} u \, d\mu \leq 0 \,,$$

which implies that $u = 0$ almost everywhere on $\bar{\mathcal{P}}$ with respect to μ, or in other words, that $\operatorname{supp}\mu \subseteq u^{-1}(0) = \Sigma_p$. For the other claim, first note that $\mathcal{PSH}(\mathbb{C}^n) \cap C(\mathbb{C}^n)|_{\Sigma_p}$ is dense in $\mathcal{PSH}(\Sigma_p) \cap C(\bar{\Sigma}_p)$, this follows from the fact that every $u \in \mathcal{PSH}(\Sigma_p) \cap C(\bar{\Sigma}_p)$ extends to a plurisubharmonic function on an open neighborhood of Σ_p in $\mathbb{C}^n$. Hence, any $v \in \mathcal{PSH}(\Sigma_p) \cap C(\bar{\Sigma}_p)$ can be uniformly approximated by functions in $\mathcal{PSH}(\mathbb{C}^n) \cap C(\mathbb{C}^n)$, so in particular, by functions in $\mathcal{PSH}(\mathcal{P}) \cap C(\bar{\mathcal{P}})$. Consequently, if $v \in \mathcal{PSH}(\Sigma_p) \cap C(\bar{\Sigma}_p)$, and $\mu \in \mathcal{J}_p^c(\bar{\mathcal{P}})$, then

$$v(p) \leq \int_{\bar{\Sigma}_p} v \, d\mu \,,$$

or, in other words, $\mu \in \mathcal{J}_p^c(\bar{\Sigma}_p)$. Conversely, if $u \in \mathcal{PSH}(\mathcal{P}) \cap C(\bar{\mathcal{P}})$, then $u|_{\Sigma_p} \in \mathcal{PSH}(\Sigma_p) \cap C(\bar{\Sigma}_p)$. Hence if $\mu \in \mathcal{J}_p^c(\bar{\Sigma}_p)$, then $\mu \in \mathcal{J}_p^c(\bar{\mathcal{P}})$. □

Let J and Σ_p be defined as in Proposition 2.3. From the fact that $\Sigma_p \subseteq \sigma_J$ and the additional assumption that the polyhedron is non-degenerate we obtain Corollary 2.4.

Corollary 2.4. *If $\mathcal{P}$ is a non-degenerate analytic polyhedron in $\mathbb{C}^n$, $n \geq 2$, then the following holds:*

(1) if $p \in \partial\mathcal{P}$ and $\mu \in \mathcal{J}_p^c(\bar{\mathcal{P}})$, then there exists an increasing multi-index $1 \leq i_1 < \cdots < i_k \leq N$ of length $k \in \{1, \ldots, n\}$ such that the support of μ is contained in σ_I,

(2) if $p \in \Gamma_n$, then $\mathcal{J}_p^c(\bar{\mathcal{P}}) = \{\delta_p\}$, where δ_p is the Dirac measure at p.

Let $\Omega \subseteq \mathbb{C}^n$ be a bounded hyperconvex domain and let $f : \partial\Omega \to \mathbb{R}$ be a continuous function. The Perron–Bremermann envelope is defined by

$$PB_f(z) = \sup \left\{ w(z) : w \in \mathcal{PSH}(\Omega), \ \limsup_{\substack{\zeta \to \xi \\ \zeta \in \Omega}} w(\zeta) \leq f(\xi) \ \ \forall \xi \in \partial\Omega \right\}.$$

In this setting PB_f is always plurisubharmonic, but not necessarily continuous. In [**14**] Walsh proved that if

$$\liminf_{\substack{z \to \xi \\ z \in \Omega}} PB_f(z) = \limsup_{\substack{z \to \xi \\ z \in \Omega}} PB_f(z) = f(\xi)$$

for every $\xi \in \partial\Omega$, then $PB_f \in C(\bar{\Omega})$. We will refer to this result as Walsh's theorem.

3. Proof of Theorem A. The outline of the proof of Theorem A is as follows: We already know that (1), (4) and (5) are equivalent and therefore we start by proving the equivalence between (2) and (3). Then we proceed by proving that (2) implies (4) and finally that (4) implies (3).

Proof of Theorem A. Assume that f is plurisubharmonic on the $(n-k)$-dimensional complex manifold σ_I for every increasing multi-index I of length $k \in \{1, \ldots, n-1\}$ and let d be an analytic disc embedded in $\partial\mathcal{P}$, i.e., $d : \Delta \to \bar{\mathcal{P}}$ is an injective, holomorphic function with $d(\Delta) \subseteq \partial\mathcal{P}$. Proposition 2.1 implies that there exists an increasing multi-index I_0 of length $k_0 \in \{1, \ldots, n-1\}$ such that $d(\Delta) \subseteq \sigma_{I_0}$ and therefore (3) follows by Theorem 5.11 in [**8**]. For the converse assume that (3) holds and let $p \in \sigma_I$, where I is a fixed increasing multi-index I of length $k \in \{1, \ldots, n-1\}$. We now want to prove that f is plurisubharmonic in a neighbourhood of p in σ_I. The functions $f_{i_1}, \ldots, f_{i_k}$ which is a subset of the defining functions for $\mathcal{P}$, can be used as a start of a change of variables in a neighbourhood U of p in the following way: Let $w_1, \ldots, w_{n-k}$ be holomorphic functions on $\mathbb{C}^{n-k}$ such that

$$\partial w_1 \wedge \cdots \wedge \partial f_{i_1} \wedge \cdots \wedge \partial f_{i_k} \wedge \cdots \wedge \partial w_{n-k} \neq 0$$

on U. This is possible since $\mathcal{P}$ is non-degenerate by assumption. Moreover, we will consider the functions $w_1, \ldots, w_{n-k}$ as coordinates in $\mathbb{C}^{n-k}$. Let L be an arbitrary complex line through p in $\mathbb{C}^{n-k}$ and let $\tilde{d} : \mathbb{C} \to \mathbb{C}^{n-k}$ be a holomorphic function with $\tilde{d}(\mathbb{C}) = L$. The function $d : \Delta \to \mathcal{P}$ defined by $d = w^{-1} \circ \tilde{d}$ is an analytic disc with $p \in d(\Delta) \subseteq \sigma_I$ and, by assumption, $f \circ d$ is subharmonic. Hence, f is plurisubharmonic on σ_I, i.e., (2) holds.

Assume that (2) holds and let $p \in \partial\mathcal{P}$ and $\mu \in \mathcal{J}_p^c(\bar{\mathcal{P}})$, then by Proposition 2.3 it follows that $\operatorname{supp}\mu \subseteq \Sigma_p \subseteq \sigma_J$ and $\mu \in \mathcal{J}_p^c(\bar{\Sigma}_p)$ and therefore it follows by assumption that

$$f(p) \leq \int_{\Sigma_p} f \, d\mu = \int_{\bar{\mathcal{P}}} f \, d\mu \, .$$

Consequently, (4) holds.

To complete the proof, we must now show that (4) implies (3). Let $\phi : \Delta \to \partial\mathcal{P}$ be an analytic disc and let ds denote the Lebesgue measure on $\partial\Delta$. Then the push-forward of ds, $\phi_*(ds)$, is a Jensen measure defined on $\bar{\Sigma}_{\phi(0)}$ with barycenter $\phi(0)$ and therefore, by assumption and Proposition 2.3, it follows that u satisfies (3). □

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