

## NOTE ON REFLEXIVITY AND INVARIANT MEANS

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**Abstract.** Applying Šmulian Theorem, we show that if a Banach space  $X$  is not reflexive then the space of bounded functions from  $\mathbb{Z}$  with values in  $X$  does not admit an invariant mean.

Invariant means on amenable groups are an important tool in many parts of mathematics, especially in harmonic analysis (see [5, 6]). For basic properties of invariant means, we refer the reader to [5]. We would only like to mention that a large class of “reasonable” groups is amenable, including abelian, solvable and finite.

Invariant means and their generalizations for vector-valued functions play also an important role in the stability of functional equations and selections of set-valued functions (see, for example, [8, 3, 4, 7, 1]). Thus it seems natural to ask what are possible limitations of the use of invariant means. We will show that invariant means are, in some sense, naturally restricted to reflexive Banach spaces.

Let  $G$  be a group and  $X$  be a Banach space. The space of all bounded functions from  $G$  into  $X$  is denoted by  $B(G, X)$ .

We are now ready to quote from [3] the definition of the generalization of invariant mean for vector-valued functions.

DEFINITION 1. We say that a linear function  $m : B(G, X) \rightarrow X$  is an *invariant mean* if the following conditions hold:

- (i) for every  $f \in B(G, X)$  and  $a \in G$  there is

$$m_x(f(a+x)) = m_x(f(x+a)) = m(f),$$

where the subscript  $x$  next to  $m$  implies that the mean is taken with respect to the variable  $x$ .

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- (ii) for every  $f \in B(G, X)$  and closed convex bounded subset  $V$  of  $X$ , if  $\text{im}(f) \subset V$ , then

$$m(f) \in V.$$

The first condition says that the mean is translation invariant, and the second that if the function is “contained” in a closed convex set then its mean belongs to the same set.

One can easily notice that there exists a mean  $m : B(G, \mathbb{R}) \rightarrow \mathbb{R}$  for a group  $G$  if and only if  $G$  is amenable.

There arises a natural question if for a given amenable group  $G$  and Banach space  $X$ , the space  $B(G, X)$  admits an invariant mean. Z. Gajda shows ([3], Theorem 2.3) that if a Banach space  $X$  is reflexive, this is really the case.

**THEOREM G.** *Let  $G$  be an amenable group, and let  $X$  be a reflexive Banach space. Then  $B(G, X)$  admits an invariant mean.*

In the proof of our main results we will need the following property of invariant means.

**LEMMA 1.** *Let  $X$  be a Banach space and let  $m : B(\mathbb{Z}, X) \rightarrow X$  be an invariant mean.*

*Let  $f \in B(\mathbb{Z}, X)$  be arbitrary and let  $V$  be a closed convex bounded subset of  $X$ . If there exists an  $n \in \mathbb{N}$  such that*

$$f(\mathbb{Z} \setminus [-n, n]) \subset V,$$

*then  $m(f) \in V$ .*

**PROOF.** Let  $v \in V$  be arbitrarily fixed and let  $g \in B(\mathbb{Z}, X)$  be defined by

$$g(z) := \begin{cases} f(z) & \text{for } z \in \mathbb{Z} \setminus [-n, n], \\ v & \text{for } z \in \mathbb{Z} \cap [-n, n]. \end{cases}$$

Clearly  $m(g) \in V$ , as  $\text{im}(g) \subset V$ . To show that  $m(f) \in V$ , it is enough to prove that  $m(f) = m(g)$

Since  $f - g \in B(\mathbb{Z}, X)$ , there exists an  $r \in \mathbb{R}$  such that  $\text{im}(f - g) \subset B(0, r)$ , where  $B(0, r)$  denotes the closed ball with the center at zero and radius  $r$ . As  $(f - g)|_{\mathbb{Z} \setminus [-n, n]} = 0$ , applying the fact that the mean is translation invariant and linear, we obtain that

$$\begin{aligned} m(f - g) &= \frac{1}{k+1} \sum_{i=0}^k m_x((f - g)(x)) \\ (1) \qquad &= \frac{1}{k+1} \sum_{i=0}^k m_x((f - g)(x + (2n+1)i)) \\ &= \frac{1}{k+1} m_x \left( \sum_{i=0}^k (f - g)(x + (2n+1)i) \right), \end{aligned}$$

for all  $k \in \mathbb{N}$ . However, the image of the function  $\sum_{i=0}^k (f-g)(x+(2n+1)i)$  is contained in the set  $\frac{1}{k+1}B(0, r)$ . This and (1) imply that  $m(f-g) \subset \frac{1}{k+1}B(0, r)$ . As  $k$  may be taken arbitrarily large, there follows that  $m(f-g) = 0$ .  $\square$

Before going to our main result, we first need to quote Šmulian Theorem (see [2], Theorem 2, V.6.2).

**ŠMULIAN THEOREM.** *A convex subset  $K$  of a Banach space  $X$  is weakly compact if and only if every decreasing sequence of non-void closed convex subsets of  $K$  has a non-empty intersection.*

Now we are ready to prove that invariant mean cannot be constructed for functions with values in non-reflexive Banach spaces.

**THEOREM 1.** *Let  $X$  be an arbitrary non-reflexive Banach space. Then  $B(\mathbb{Z}, X)$  does not admit an invariant mean.*

**PROOF.** Let  $B(0, 1)$  denote the closed unit ball in  $X$ . Since  $X$  is not reflexive,  $B(0, 1)$  is not compact in the weak topology (Theorem 7, V.4.6 [2]). Therefore, by the Šmulian Theorem, there exists a decreasing sequence  $\{V_n\}_{n \in \mathbb{N}}$  of non-void closed convex subsets of  $B(0, 1)$  such that

$$(2) \quad \bigcap_{n \in \mathbb{N}} V_n = \emptyset.$$

We choose a function  $f : G \rightarrow X$  in such a way that

$$f(n) \in V_{|n|} \quad \text{for } n \in \mathbb{N}.$$

Suppose, for contradiction, that there exists an invariant mean  $m$  on  $B(\mathbb{Z}, X)$ . As  $f(\mathbb{Z} \setminus [-n, n]) \subset V_n$  for every  $n \in \mathbb{N}$ , by Lemma 1 and (2) there is

$$m(f) \in \bigcap_{l=1}^{\infty} V_n = \emptyset,$$

a contradiction.  $\square$

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