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DIFFERENCE METHOD FOR AN ELLIPTIC SYSTEM OF NON-LINEAR DIFFERENTIAL-FUNCTIONAL EQUATIONS

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Abstract. We consider a weakly coupled system of second order differential-functional equations of elliptic type with boundary conditions of the Dirichlet type. The system is in the general nonlinear form

$$f_l(x, u(x), (u_l)_x(x), (u_l)_{xx}(x), u) = 0,$$

where $l = 1, \dots, p$. We propose an implicite difference scheme for this problem and under certain assumptions we show the convergence, stability, existence and uniqueness of the solution for this scheme. An error estimations are also given.

1. Introduction. Let $D = [0, X]^n \subset R^n$, $X < +\infty$. We consider the system of second order differential-functional equations of elliptic type

$$f_l\left(x, u(x), (u_l)_x(x), (u_l)_{xx}(x), u\right) = 0 \quad \text{for } x \in \text{int}D \ \ (l = 1, \dots, p) \tag{1.1}$$

with boundary conditions of the Dirichlet type

$$u_l(x) = \varphi_l(x) \ \text{ for } x \in \partial D \ \ (l = 1, \dots, p), \tag{1.2}$$

where $x = (x_1, \dots, x_n)$, $u = \left(u_1, \dots, u_p\right)$, $(u_l)_x = \left(\frac{\partial u_l}{\partial x_1}, \dots, \frac{\partial u_l}{\partial x_n}\right)$, $(u_l)_{xx} = [\frac{\partial^2 u_l}{\partial x_i \partial x_j}]_{i,j=1,\dots,n}$ $(l = 1, \dots, p)$.

The general form of (1.1) admits strong nonlinearity in all arguments of f_l $(l = 1, \dots, p)$. It is known that assumptions under which we can obtain convergence theorems for nonlinear equations of form (1.1) depend mostly on the second order difference expressions used. In [**1**], [**6**]–[**10**], a seven-point scheme is used, in [**3**] – the nine-point scheme and the other one is used in

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[**4**]–[**5**]. Each time the scheme forces different assumptions about left side of differential equation(s). More discussion on the subject can be found for example in [**5**].

In our paper we use difference expressions from [**4**]–[**5**] and generalize the results of J. Kaczmarczyk in [**5**], where one elliptic differential equation was considered, to nonlinear system of elliptic differential equations which may contain functional parts. We define an implicit difference scheme for problem (1.1), (1.2) and under certain assumptions concerning the functions $f_l,\ u_l\ (l=1,\dots,p)$ and the mesh size h we show the stability, uniqueness and convergence of the solution to this scheme.

We use the well-known notation introduced in papers [**1**]–[**10**] and numerous others.

2. Assumptions A. We assume that

(**A1**) Scalar functions $f_l : E \ni (x,y,q,w,z) \longrightarrow f_l\,(x,y,q,w,z) \in R$ $(l=1,\dots,p)$, where $E := D \times R^p \times R^n \times R^{n^2} \times B(D)$, $x=(x_1,\dots,x_n)$, $y=\left(y_1,\dots,y_p\right)$, $q=(q_1,\dots,q_n)$, $w=[w_{ij}]_{i,j=1..n}$,

$$B(D) := \{z=(z_1,z_2,\dots,z_p) : \underset{k\in N}{\exists}\ \underset{D_1,\dots,D_k}{\exists} D_1\cup\ldots\cup D_k = D,$$
$$D_i\cap D_j=\varnothing \text{ for } i\neq j \text{ and restriction } z_l|_{D_i}\in C(D_i)$$
$$(l=1,\dots,p;\ i,j=1,\dots,k)\}$$

(all functions piecewise continuous on D), are such that

$$f_l\,(x,y,q,w,z) - f_l\,(x,\bar{y},\bar{q},\bar{w},z) = \sum_{\mu=1}^{p}\alpha_{l\mu}\,(y_\mu-\bar{y}_\mu) + \sum_{i=1}^{n}\beta_{li}\,(q_i-\bar{q}_i) + \sum_{i,j=1}^{n}\gamma_{lij}\,(w_{ij}-\bar{w}_{ij}), \tag{2.1a}$$

$$|f_l(x,y,q,w,z) - f_l(x,y,q,w,\bar{z})| \le \chi_l||z-\bar{z}|| \quad (l=1,\dots,p) \tag{2.1b}$$

for any $x\in D,\ y,\bar{y}\in R^p,\ q,\bar{q}\in R^n,\ w,\bar{w}\in R^{n^2},\ z,\bar{z}\in B(D)$.
Functions $\alpha_{l\mu},\beta_{li},\gamma_{lij}$ are defined in E, χ_l are constants $(l,\mu=1,\dots,p;$ $i,j=1,\dots,n)$ and

$$||z|| := \max_{l=1,\dots,p}\{\sup_{x\in D}|z_l(x)|\} \quad \text{for } z\in B(D). \tag{2.2}$$

(**A2**) There exist constants Γ, L such that functions $\alpha_{l\mu},\beta_{li},\gamma_{lij}$ $(l,\mu=1,\dots,p;$ $i,j=1,\dots,n)$ satisfy in E following conditions

$$0\le\alpha_{l\mu}\ (l\neq\mu),\quad \alpha_{ll}+\chi_l+\sum_{\substack{\mu=1\\ \mu\neq l}}^{p}\alpha_{l\mu}\le -L<0 \quad (l=1,\dots,p), \tag{2.3}$$

$$|\beta_{li}|\le\Gamma. \tag{2.4}$$

There exist bounded in E and symmetric matrices $[G_{lij}(x)]_{i,j=1..n}$ $(l = 1,\dots,p)$ and constant g such that

$$\gamma_{lii}(x,y,q,w,z) - \sum_{\substack{i,j=1\\ i\neq j}}^{n} G_{lij}(x) \geq g > 0\,, \quad |\gamma_{lij}| \leq G_{lij}\ (i \neq j), \tag{2.5}$$

$$\gamma_{lij} = \gamma_{lji} \ \text{ in } E\ (i,j = 1,\dots,n;\ j = 1,\dots,p). \tag{2.6}$$

(**A3**) There exists a solution $u(x)$ of problem (1.1), (1.2) such that $u \in C^2(D)$.

3. Discretization. We introduce the uniform net in the cube D. If a sequence of indices $M = (m_1,\dots,m_n)$, $m_i = 0,1,\dots,N$ $(i = 1,\dots,n)$ is given, then by x^M we denote the nodal point with the coordinates $x^M = (x_1^{m_1},\dots,x_n^{m_n})$, where $x_i^{m_i} = m_i h$ $(i = 1,\dots,n)$, $0 < h = X/N$ and $N \geq 2$.

Let

$$\begin{aligned} Z &:= \{M : 0 \leq m_i \leq N, i = 1,\dots,n\},\\ Z_1 &:= \{M : 1 \leq m_i \leq N,\ i = 1,\dots,n\},\\ Z_2 &:= \{M : 0 \leq m_i \leq N-1,\ i = 1,\dots,n\}, \end{aligned} \tag{3.1}$$

and

$$\begin{aligned} -i(M) &:= (m_1,\dots,m_{i-1}, m_i - 1, m_{i+1},\dots,m_n) \quad (M \in Z_1),\\ i(M) &:= (m_1,\dots,m_{i-1}, m_i + 1, m_{i+1},\dots,m_n) \quad (M \in Z_2)\\ &\qquad (i = 1,\dots,n). \end{aligned}$$

For any net function $a_l : Z \ni M \longrightarrow a_l^M \in R$ $(l = 1,\dots,p)$, define the following operators:

$$a_l^{Mi} = 0.5h^{-1}\left(a_l^{i(M)} - a_l^{-i(M)}\right), \tag{3.2}$$

(3.3)

$$a_l^{Mij} = \begin{cases} h^{-2}\left(a_l^{i(M)} - 2a_l^M + a_l^{-i(M)}\right) & \text{for } i = j,\\ 0.25h^{-2}\left(a_l^{i(j(M))} - a_l^{-i(j(M))} - a_l^{i(-j(M))} + a_l^{-i(-j(M))}\right) & \text{for } i \neq j, \end{cases}$$
$$(M \in Z_1 \cap Z_2;\ i,j = 1,\dots,n).$$

$$\begin{aligned} a_l^{M+} := \sum_{\substack{i,j=1\\ i\neq j}}^{n} G_{lij}^M 0.25h^{-2}\cdot(a_l^{i(j(M))} + a_l^{-i(j(M))} + a_l^{i(-j(M))}\\ + a_l^{-i(-j(M))} - 4a_l^{i(M)} - 4a_l^{-i(M)} + 4a_l^M). \end{aligned} \tag{3.4}$$

Every function $b = (b_1, \ldots, b_p) \in B(D)$ is approximated by $b^* = \left(b_1^*, \ldots, b_p^*\right) \in B(D)$, where

$$b_\mu^*(x) := \sum_{M \in Z} \theta_M(x) b_\mu^M, \tag{3.5}$$

$$\theta_M(x) := \begin{cases} 0 & \text{for } x \notin I_M, \\ 1 & \text{for } x \in I_M, \end{cases} \tag{3.6}$$

$$I_M := \{x \in D : m_i h \le x_i < (m_i + 1)\, h,\ i = 1, \ldots, n\},$$

$$b_\mu^M := b_\mu(x^M) \quad (\mu = 1, \ldots, p;\ M \in Z). \tag{3.7}$$

In the same way, for every net function c: $Z \ni M \to c^M \in R^p$, we define c^* as

$$c_\mu^*(x) := \sum_{M \in Z} \theta_M(x) c_\mu^M \quad (\mu = 1, \ldots, p). \tag{3.8}$$

For differential-functional system (1.1), (1.2), let v^M $(M \in Z)$ be a solution of the system of difference equations

$$f_l\left(x^M, v^M, v_l^{MI}, v_l^{MIJ}, v^*\right) + v_l^{M+} = 0,\ (M \in Z_1 \cap Z_2;\ l = 1, \ldots, p) \tag{3.9}$$

with the boundary conditions

$$v_l^M = \varphi_l^M := \varphi_l(x^M),\ (M \in Z \setminus (Z_1 \cap Z_2);\ l = 1, \ldots, p), \tag{3.10}$$

where $v^M = (v_\mu^M)_{\mu=1..p}$, $v_l^{MI} = (v_l^{Mi})_{i=1..n}$, $v_l^{MIJ} = [v_l^{Mij}]_{i,j=1..n}$, $(l = 1, \ldots, p)$.

The operators $v_l^{Mi}, v_l^{Mij}, v_l^{M+} (l = 1, \ldots, p;\ i, j = 1, \ldots, n)$ and the function v^* are defined by (3.2)–(3.8).

System (3.9) and the boundary conditions (3.10) are generated by system (1.1) with the boundary conditions (1.2) with the terms v_l^{M+} $(l = 1, \ldots, p)$ added. Consider also a "perturbed" difference system

$$\begin{gathered} f_l\left(x^M, w^M, w_l^{MI}, w_l^{MIJ}, w^*\right) + w_l^{M+} = \varepsilon_l^M(h), \\ (M \in Z_1 \cap Z_2;\ l = 1, \ldots, p) \end{gathered} \tag{3.9'}$$

with the boundary conditions

$$w_l^M = \varphi_l^M,\ (M \in Z \setminus (Z_1 \cap Z_2);\ l = 1, \ldots, p). \tag{3.10'}$$

Let v and w be the solutions of problems (3.9)–(3.10) and (3.9')–(3.10'), respectively. Define

$$r^M := w^M - v^M = (r_\mu^M)_{\mu=1..p} \quad (M \in Z). \tag{3.11}$$

Then

$$\begin{gathered} r_\mu^{Mi} = w_\mu^{Mi} - v_\mu^{Mi}, \quad r_\mu^{Mij} = w_\mu^{Mij} - v_\mu^{Mij}, \quad r_\mu^* = w_\mu^* - v_\mu^* \\ (\mu = 1, \ldots, p;\ i, j = 1, \ldots, n;\ M \in Z_1 \cap Z_2). \end{gathered} \tag{3.12}$$

4. Stability theorem.

THEOREM 1 (Stability). *If the mesh size h satisfies the inequality*

$$-0.5\Gamma h + g \geq 0 \tag{4.1}$$

*and assumptions (**A1**)–(**A2**) hold then the following estimation is true:*

$$|r_l^M| \leq \frac{\varepsilon(h)}{L} \quad (M \in Z;\ l = 1, \dots, p), \tag{4.2}$$

where

$$\varepsilon(h) := \max_{\substack{M \in Z_1 \cap Z_2 \\ l=1,\dots,p}} |\varepsilon_l^M(h)|. \tag{4.3}$$

That means that difference scheme (3.9), (3.10) *is stable.*

PROOF. Since sets Z and $\{1, .., p\}$ are finite, there exist $A \in Z$ and $k \in \{1, .., p\}$ such that

$$|r_k^A| = \max_{\substack{M \in Z \\ l=1,\dots,p}} |r_l^M|, \tag{4.4}$$

where r is the function defined by (3.11).

If $A \in Z \setminus (Z_1 \cap Z_2)$, then $r_k^A = 0$, because of the boundary conditions (3.10) and (3.10'). In this case, (4.2) follows. Thus, we may assume $A \in Z_1 \cap Z_2$.

From (3.9), (3.9'), (2.1a), (2.1b) and formulas (3.12) we derive

(4.5)

$$\begin{aligned}
\varepsilon_l^M(h) =& f_l\left(x^M, w^M, w_l^{MI}, w_l^{MIJ}, w^*\right) + w_l^{M+} \\
=& f_l\left(x^M, w^M, w_l^{MI}, w_l^{MIJ}, w^*\right) - f_l\left(x^M, v^M, v_l^{MI}, v_l^{MIJ}, v^*\right) + r_l^{M+} \\
=& f_l(x^M, w^M, w_l^{MI}, w_l^{MIJ}, w^*) - f_l(x^M, v^M, v_l^{MI}, v_l^{MIJ}, w^*) \\
&+ f_l(x^M, v^M, v_l^{MI}, v_l^{MIJ}, w^*) - f_l(x^M, v^M, v_l^{MI}, v_l^{MIJ}, v^*) + r_l^{M+} \\
=& \sum_{\mu=1}^{p} \alpha_{l\mu}^M r_\mu^M + \sum_{i=1}^{n} \beta_{li}^M r_l^{Mi} + \sum_{i,j=1}^{n} \gamma_{lij}^M r_l^{Mij} + r_l^{M+} \\
&+ f_l(x^M, v^M, v_l^{ML}, v_l^{MIJ}, w^*) - f_l(x^M, v^M, v_l^{MI}, v_l^{MIJ}, v^*) = \epsilon_l^M(h)
\end{aligned}$$

and

$$\sum_{\mu=1}^{p} \alpha_{l\mu}^M r_\mu^M + \sum_{i=1}^{n} \beta_{li}^M r_l^{Mi} + \sum_{i,j=1}^{n} \gamma_{lij}^M r_l^{Mij} + r_l^{M+} - \chi_l ||r^*|| \leq \epsilon_l^M(h),$$

$$\sum_{\mu=1}^{p} \alpha_{l\mu}^M r_\mu^M + \sum_{i=1}^{n} \beta_{li}^M r_l^{Mi} + \sum_{i,j=1}^{n} \gamma_{lij}^M r_l^{Mij} + r_l^{M+} + \chi_l ||r^*|| \geq \epsilon_l^M(h).$$

By (2.2) there is

$$(4.6)\qquad ||r^*|| = \max_{l=1,\dots,p}\{\sup_{x\in D}|\sum_{M\in Z}\theta_M(x)r_l^M|\} = \max_{l=1,\dots,p}\{\max_{M\in Z}|r_l^M|\} = |r_k^A|,$$

which leads to inequalities

$$(4.7)\qquad \begin{gathered}\sum_{\mu=1}^{p}\alpha_{l\mu}^M r_\mu^M + \sum_{i=1}^{n}\beta_{li}^M r_l^{Mi} + \sum_{i,j=1}^{n}\gamma_{lij}^M r_l^{Mij} + r_l^{M+} - \chi_l|r_k^A| \le \epsilon_l^M(h),\\ \sum_{\mu=1}^{p}\alpha_{l\mu}^M r_\mu^M + \sum_{i=1}^{n}\beta_{li}^M r_l^{Mi} + \sum_{i,j=1}^{n}\gamma_{lij}^M r_l^{Mij} + r_l^{M+} + \chi_l|r_k^A| \ge \epsilon_l^M(h)\\ (M\in Z_1\cap Z_2;\ l=1,\dots,p).\end{gathered}$$

For $M = A$ and $l = k$, from (4.7) and definition (4.2), we obtain the following two inequalities:

$$(4.8)\qquad \sum_{\mu=1}^{p}\alpha_{k\mu}^A r_\mu^A + \sum_{i=1}^{n}\beta_{ki}^A r_k^{Ai} + \sum_{i,j=1}^{n}\gamma_{kij}^A r_k^{Aij} + r_k^{A+} - \chi_k|r_k^A| \le \epsilon(h),$$

$$(4.9)\qquad \sum_{\mu=1}^{p}\alpha_{k\mu}^A r_\mu^A + \sum_{i=1}^{n}\beta_{ki}^A r_k^{Ai} + \sum_{i,j=1}^{n}\gamma_{kij}^A r_k^{Aij} + r_k^{A+} + \chi_k|r_k^A| \ge -\epsilon(h).$$

i) Assume $r_k^A \ge 0$. Then

$$(4.10)\qquad |r_k^A| = r_k^A \quad\text{and}\quad r_l^M \le r_k^A \ \ (l=1..p;\ M\in Z).$$

Now we shall prove that

$$(4.11)\qquad \sum_{i=1}^{n}\beta_{ki}^A r_k^{Ai} + \sum_{i,j=1}^{n}\gamma_{kij}^A r_k^{Aij} + r_k^{A+} \le 0.$$

We proceed as follows:

$$
\begin{aligned}
&\sum_{i=1}^{n} \beta_{ki}^{A} r_{k}^{Ai} + \sum_{i,j=1}^{n} \gamma_{kij}^{A} r_{k}^{Aij} + r_{l}^{A+} \\
&= \sum_{i=1}^{n} \beta_{ki}^{A} 0.5h^{-1}(r_{k}^{i(A)} - r_{k}^{-i(A)}) + \sum_{i=1}^{n} \gamma_{kii}^{A} h^{-2}(r_{k}^{i(A)} - 2r_{k}^{A} + r_{k}^{-i(A)}) \\
&\quad + \sum_{\substack{i,j=1 \\ i\neq j}}^{n} \gamma_{kij}^{A} 0.25h^{-2}(r_{k}^{i(j(A))} - r_{k}^{-i(j(A))} - r_{k}^{i(-j(A))} + r_{k}^{-i(-j(A))}) \\
&\quad + \sum_{\substack{i,j=1 \\ i\neq j}}^{n} G_{kij}^{A} 0.25h^{-2}\Big(r_{k}^{i(j(A))} + r_{k}^{-i(j(A))} + r_{k}^{i(-j(A))} + r_{k}^{-i(-j(A))} \\
&\qquad\qquad - 4r_{k}^{i(A)} - 4r_{k}^{-i(A)} + 4r_{k}^{A}\Big) \\
&= \sum_{i=1}^{n} \beta_{ki}^{A} 0.5h^{-1}(r_{k}^{i(A)} - r_{k}^{A}) + \sum_{i=1}^{n} \beta_{ki}^{A} 0.5h^{-1}(r_{k}^{A} - r_{k}^{-i(A)}) \\
&\quad + \sum_{i=1}^{n} \gamma_{kii}^{A} h^{-2}(r_{k}^{i(A)} - r_{k}^{A}) + \sum_{i=1}^{n} \gamma_{kii}^{A} h^{-2}(r_{k}^{-i(A)} - r_{k}^{A}) \\
&\quad + \sum_{\substack{i,j=1 \\ i\neq j}}^{n} \gamma_{kij}^{A} 0.25h^{-2}(r_{k}^{i(j(A))} - r_{k}^{A}) + \sum_{\substack{i,j=1 \\ i\neq j}}^{n} \gamma_{kij}^{A} 0.25h^{-2}(r_{k}^{A} - r_{k}^{-i(j(A))}) \\
&\quad + \sum_{\substack{i,j=1 \\ i\neq j}}^{n} \gamma_{kij}^{A} 0.25h^{-2}(r_{k}^{A} - r_{k}^{i(-j(A))}) + \sum_{\substack{i,j=1 \\ i\neq j}}^{n} \gamma_{kij}^{A} 0.25h^{-2}(r_{k}^{-i(-j(A))} - r_{k}^{A}) \\
&\quad + \sum_{\substack{i,j=1 \\ i\neq j}}^{n} G_{kij}^{A} 0.25h^{-2}(r_{k}^{i(j(A))} - r_{k}^{A}) + \sum_{\substack{i,j=1 \\ i\neq j}}^{n} G_{kij}^{A} 0.25h^{-2}(r_{k}^{-i(j(A))} - r_{k}^{A}) \\
&\quad + \sum_{\substack{i,j=1 \\ i\neq j}}^{n} G_{kij}^{A} 0.25h^{-2}(r_{k}^{i(-j(A))} - r_{k}^{A}) + \sum_{\substack{i,j=1 \\ i\neq j}}^{n} G_{kij}^{A} 0.25h^{-2}(r_{k}^{-i(-j(A))} - r_{k}^{A}) \\
&\quad + \sum_{\substack{i,j=1 \\ i\neq j}}^{n} G_{kij}^{A} h^{-2}(r_{k}^{A} - r_{k}^{i(A)}) + \sum_{\substack{i,j=1 \\ i\neq j}}^{n} G_{kij}^{A} h^{-2}(r_{k}^{A} - r_{k}^{-i(A)})
\end{aligned}
$$

$$
\begin{aligned}
&= \sum_{i=1}^{n} [0.5h^{-1}\beta_{ki}^{A} + h^{-2}\gamma_{kii}^{A}](r_k^{i(A)} - r_k^{A}) \\
&\quad + \sum_{i=1}^{n} [-0.5h^{-1}\beta_{ki}^{A} + h^{-2}\gamma_{kii}^{A}](r_k^{-i(A)} - r_k^{A}) \\
&\quad + \sum_{\substack{i,j=1\\ i\neq j}}^{n} G_{kij}^{A} h^{-2}(r_k^{A} - r_k^{i(A)}) + \sum_{\substack{i,j=1\\ i\neq j}}^{n} G_{kij}^{A} h^{-2}(r_k^{A} - r_k^{-i(A)}) \\
&\quad + \sum_{\substack{i,j=1\\ i\neq j}}^{n} 0.25h^{-2}[G_{kij}^{A} + \gamma_{kij}^{A}](r_k^{i(j(A))} - r_k^{A}) \\
&\quad + \sum_{\substack{i,j=1\\ i\neq j}}^{n} 0.25h^{-2}[G_{kij}^{A} - \gamma_{kij}^{A}](r_k^{-i(j(A))} - r_k^{A}) \\
&\quad + \sum_{\substack{i,j=1\\ i\neq j}}^{n} 0.25h^{-2}[G_{kij}^{A} - \gamma_{kij}^{A}](r_k^{i(-j(A))} - r_k^{A}) \\
&\quad + \sum_{\substack{i,j=1\\ i\neq j}}^{n} 0.25h^{-2}[G_{kij}^{A} + \gamma_{kij}^{A}](r_k^{-i(-j(A))} - r_k^{A}) \\
&= h^{-2}\sum_{i=1}^{n} [0.5h\beta_{ki}^{A} + \gamma_{kii}^{A}](r_k^{i(A)} - r_k^{A}) + h^{-2}\sum_{i=1}^{n} [-0.5h\beta_{ki}^{A} + \gamma_{kii}^{A}](r_k^{-i(A)} - r_k^{A}) \\
&\quad + \sum_{\substack{i,j=1\\ i\neq j}}^{n} G_{kij}^{A} h^{-2}(r_k^{A} - r_k^{i(A)}) + \sum_{\substack{i,j=1\\ i\neq j}}^{n} G_{kij}^{A} h^{-2}(r_k^{A} - r_k^{-i(A)}) \\
&\quad + \sum_{\substack{i,j=1\\ i\neq j}}^{n} 0.25h^{-2}[G_{kij}^{A} + \gamma_{kij}^{A}](r_k^{i(j(A))} - r_k^{A}) \\
&\quad + \sum_{\substack{i,j=1\\ i\neq j}}^{n} 0.25h^{-2}[G_{kij}^{A} - \gamma_{kij}^{A}](r_k^{-i(j(A))} - r_k^{A}) \\
&\quad + \sum_{\substack{i,j=1\\ i\neq j}}^{n} 0.25h^{-2}[G_{kij}^{A} - \gamma_{kij}^{A}](r_k^{i(-j(A))} - r_k^{A}) \\
&\quad + \sum_{\substack{i,j=1\\ i\neq j}}^{n} 0.25h^{-2}[G_{kij}^{A} + \gamma_{kij}^{A}](r_k^{-i(-j(A))} - r_k^{A})
\end{aligned}
$$

$$= h^{-2}\sum_{i=1}^{n}[0.5h\beta_{ki}^{A} + \gamma_{kii}^{A} - \sum_{\substack{i,j=1\\ i\neq j}}^{n} G_{kij}^{A}](r_k^{i(A)} - r_k^{A})$$
$$+ h^{-2}\sum_{i=1}^{n}[0.5h\beta_{ki}^{A} + \gamma_{kii}^{A} - \sum_{\substack{i,j=1\\ i\neq j}}^{n} G_{kij}^{A}](r_k^{-i(A)} - r_k^{A})$$
$$+ \sum_{\substack{i,j=1\\ i\neq j}}^{n} 0.25h^{-2}[G_{kij}^{A} + \gamma_{kij}^{A}](r_k^{i(j(A))} - r_k^{A})$$
$$+ \sum_{\substack{i,j=1\\ i\neq j}}^{n} 0.25h^{-2}[G_{kij}^{A} - \gamma_{kij}^{A}](r_k^{-i(j(A))} - r_k^{A})$$
$$+ \sum_{\substack{i,j=1\\ i\neq j}}^{n} 0.25h^{-2}[G_{kij}^{A} - \gamma_{kij}^{A}](r_k^{i(-j(A))} - r_k^{A})$$
$$+ \sum_{\substack{i,j=1\\ i\neq j}}^{n} 0.25h^{-2}[G_{kij}^{A} + \gamma_{kij}^{A}](r_k^{-i(-j(A))} - r_k^{A}). \tag{4.12}$$

Each term in the square brackets is nonnegative (due to (2.4)–(2.6), (4.1)) and terms in the parentheses are nonpositive (due to (4.10)). Thus (4.11) follows.

From (4.9) and (4.11) we derive

$$-\varepsilon(h) \le \sum_{\mu=1}^{p} \alpha_{k\mu}^{A} r_{\mu}^{A} + \chi_k r_k^{A} = \alpha_{kk}^{A} r_k^{A} + \sum_{\substack{\mu=1\\ \mu\neq k}}^{p} \alpha_{k\mu}^{A} r_{\mu}^{A} + \chi_k r_k^{A}$$
$$\le \alpha_{kk}^{A} r_k^{A} + \sum_{\substack{\mu=1\\ \mu\neq k}}^{p} \alpha_{k\mu}^{A} r_k^{A} + \chi_k r_k^{A} = (\alpha_{kk}^{A} + \chi_k + \sum_{\substack{\mu=1\\ \mu\neq k}}^{p} \alpha_{k\mu}^{A}) r_k^{A} \le -L r_k^{A} \tag{4.13}$$

due to (4.10) and (2.3). Hence, $-\varepsilon(h) \le -L r_k^{A}$ and

$$|r_k^{A}| = r_k^{A} \le \frac{\varepsilon(h)}{L}. \tag{4.14}$$

ii) Assume $r_k^A \le 0$. Then $|r_k^A| = -r_k^A$ and $r_l^M \ge r_k^A$ $(l = 1, \dots, p;\ M \in Z)$. Inequality (4.11) now takes the form

$$\sum_{i=1}^{n} \beta_{ki}^A r_k^{Ai} + \sum_{i,j=1}^{n} \gamma_{kij}^A r_k^{Aij} + r_k^{A+} \ge 0. \tag{4.15}$$

We can see that calculations which were used to prove (4.11) may be repeated and in (4.12) each term in the square brackets and parentheses is nonnegative. Thus (4.15) follows. By (4.8) there is

$$\begin{aligned}\varepsilon(h) &\ge \sum_{\mu=1}^{p} \alpha_{k\mu}^A r_\mu^A - \chi_k(-r_k^A) = \alpha_{kk}^A r_k^A + \sum_{\substack{\mu=1\\ \mu\ne k}}^{p} \alpha_{k\mu}^A r_\mu^A + \chi_k r_k^A \\ &\ge \alpha_{kk}^A r_k^A + \sum_{\substack{\mu=1\\ \mu\ne k}}^{p} \alpha_{k\mu}^A r_k^A + \chi_k r_k^A = (\alpha_{kk}^A + \chi_k + \sum_{\substack{\mu=1\\ \mu\ne k}}^{p} \alpha_{k\mu}^A) r_k^A \ge -L r_k^A.\end{aligned}$$

Hence $\varepsilon(h) \ge -L r_k^A$ and

$$|r_k^A| = -r_k^A \le \frac{\varepsilon(h)}{L}. \tag{4.16}$$

From (4.14), (4.16), there always follows (4.2), which completes the proof of Theorem 1. □

5. Existence and uniqueness of difference solution.

Remark 5.1. Notice that (4.12) is true not only at certain nodal points as in the proof of Theorem 1. Let $z^M = (z_\mu^M)_{\mu=1..p}$ $(M \in Z)$ be an arbitrary net function. Then, for $l = 1, \dots, p;\ M \in Z_1 \cap Z_2$, equality (4.12) holds true in the set E :

$$\sum_{i=1}^{n}\beta_{li}^{M} z_{k}^{Mi} + \sum_{i,j=1}^{n} \gamma_{lij}^{M} z_{l}^{Mij} + z_{l}^{M+}$$

$$= h^{-2}\sum_{i=1}^{n}[0.5h\beta_{li}^{M} + \gamma_{lii}^{M} - \sum_{\substack{j=1\\ j\neq i}}^{n} G_{lij}^{M}](z_{l}^{i(M)} - z_{l}^{M})$$

$$+ h^{-2}\sum_{i=1}^{n}[0.5h\beta_{li}^{M} + \gamma_{lii}^{M} - \sum_{\substack{j=1\\ j\neq i}}^{n} G_{lij}^{M}](z_{l}^{-i(M)} - z_{l}^{M})$$

$$+ \sum_{\substack{i,j=1\\ i\neq j}}^{n} 0.25h^{-2}[G_{lij}^{M} + \gamma_{lij}^{M}](z_{l}^{i(j(M))} - z_{l}^{M})$$

$$+ \sum_{\substack{i,j=1\\ i\neq j}}^{n} 0.25h^{-2}[G_{lij}^{M} - \gamma_{lij}^{M}](z_{l}^{-i(j(M))} - z_{l}^{M})$$

$$+ \sum_{\substack{i,j=1\\ i\neq j}}^{n} 0.25h^{-2}[G_{lij}^{M} - \gamma_{lij}^{M}](z_{l}^{i(-j(M))} - z_{l}^{M})$$

$$+ \sum_{\substack{i,j=1\\ i\neq j}}^{n} 0.25h^{-2}[G_{lij}^{M} + \gamma_{lij}^{M}](z_{l}^{-i(-j(M))} - z_{l}^{M})$$

$$= h^{-2}\sum_{i=1}^{n}[0.5h\beta_{li}^{M} + \gamma_{lii}^{M} - \sum_{\substack{j=1\\ j\neq i}}^{n} G_{lij}^{M}]z_{l}^{i(M)}$$

$$+ h^{-2}\sum_{i=1}^{n}[0.5h\beta_{li}^{M} + \gamma_{lii}^{M} - \sum_{\substack{j=1\\ j\neq i}}^{n} G_{lij}^{M}]z_{l}^{-i(M)}$$

$$+ h^{-2}\sum_{\substack{i,j=1\\ i\neq j}}^{n} 0.25[G_{lij}^{M} + \gamma_{lij}^{M}]z_{l}^{i(j(M))} + h^{-2}\sum_{\substack{i,j=1\\ i\neq j}}^{n} 0.25[G_{lij}^{M} - \gamma_{lij}^{M}]z_{l}^{-i(j(M))}$$

$$+ h^{-2}\sum_{\substack{i,j=1\\ i\neq j}}^{n} 0.25[G_{lij}^{M} - \gamma_{lij}^{M}]z_{l}^{i(-j(M))} + h^{-2}\sum_{\substack{i,j=1\\ i\neq j}}^{n} 0.25[G_{lij}^{M} + \gamma_{lij}^{M}]z_{l}^{-i(-j(M))}$$

$$- 4h^{-2}\sum_{\substack{i,j=1\\ i\neq j}}^{n} 0.25G_{lij}^{M}z_{l}^{M} - 2h^{-2}\sum_{i=1}^{n}[0.5h\beta_{li}^{M} + \gamma_{lii}^{M} - \sum_{\substack{j=1\\ j\neq i}}^{n} G_{lij}^{M}]z_{l}^{M}.$$

Let

$$\delta_{li}^M := 0.5h\beta_{li}^M + \gamma_{lii}^M - \sum_{\substack{j=1\\ j\neq i}}^n G_{lij}^M,\ \ \delta_l^M := \sum_{i=1}^n \delta_{li}^M,\ \ \rho_l^M := \sum_{\substack{i,j=1\\ i\neq j}}^n G_{lij}^M \tag{5.1}$$
$$(l=1,\ldots,p;\ i=1,\ldots,n).$$

Then

(5.2)
$$\sum_{i=1}^n \beta_{li}^M z_k^{Mi} + \sum_{i,j=1}^n \gamma_{lij}^M z_l^{Mij} + z_l^{M+}$$
$$= h^{-2}\sum_{i=1}^n \delta_{li}^M z_l^{i(M)} + h^{-2}\sum_{i=1}^n \delta_{li}^M z_l^{-i(M)} + h^{-2}0.25 \sum_{\substack{i,j=1\\ i\neq j}}^n [G_{lij}^M + \gamma_{lij}^M] z_l^{i(j(M))}$$
$$+ h^{-2}0.25 \sum_{\substack{i,j=1\\ i\neq j}}^n [G_{lij}^M - \gamma_{lij}^M] z_l^{-i(j(M))} + h^{-2}0.25 \sum_{\substack{i,j=1\\ i\neq j}}^n [G_{lij}^M - \gamma_{lij}^M] z_l^{i(-j(M))}$$
$$+ h^{-2}0.25 \sum_{\substack{i,j=1\\ i\neq j}}^n [G_{lij}^M + \gamma_{lij}^M] z_l^{-i(-j(M))} - 4h^{-2}0.25\rho_l^M z_l^M - 2h^{-2}\sum_{i=1}^n \delta_{li}^M z_l^M.$$

THEOREM 2. *Let assumptions* (***A1***)–(***A2***) *are satisfied, there exist constants* L_1, K *such that inequalities*

$$-L_1 \le \alpha_{ll},\quad \gamma_{lii} \le K \quad (l=1,\ldots,p;\ i=1,\ldots,n) \tag{5.3}$$

hold true in the set E *and* $h>0$ *is so small that*

$$-0.5\Gamma h + g \ge 0. \tag{5.4}$$

Then there exists exactly one solution of differential problem (3.9)–(3.10).

PROOF. Let q be the cardinality of Z, $R^{q\times p} := \{V | V = (v_l^M)_{\substack{M\in Z\\ l=1,\ldots,p}}\}$ and h satisfy (5.4). Notice that from (2.3), (2.5) there follows

$$-L_1 \le -L < 0,\quad K \ge 0. \tag{5.5}$$

Define

$$F : R^{q\times p} \ni V \longrightarrow F(V) := C \in R^{q\times p},$$
$$C = (c_l^M),\qquad M\in Z, l=1,\ldots,p, \tag{5.6}$$

where

$$(5.7)\qquad c_l^M := \begin{cases} f_l\left(x^M, v^M, v_l^{MI}, v_l^{MIJ}, v^*\right) + v_l^{M+} & \text{for } M \in Z_1 \cap Z_2, \\ -Lv_l^M & \text{for } M \in Z \setminus Z_1 \cap Z_2 \end{cases}$$

$$(l = 1, \ldots, p),$$

$$(5.8)\qquad H := 1 - h^s L$$

and constant s is such that

$$(5.9)\qquad 1 - h^s(L_1 + h^{-2}(hn\Gamma + 2nK)) > 0.$$

Let $\Phi : R^{q\times p} \ni V \longrightarrow \Phi(V) := h^s F(V) + V \in R^{q\times p}$. We shall show that

$$(5.10)\qquad ||\Phi(V) - \Phi(W)||_1 \le H||V - W||_1 \text{ for all } V, W \in R^{q\times p},$$

where $||V||_1 := \max\limits_{\substack{M\in Z \\ l=1,\ldots,p}} |v_l^M|, \quad (V \in R^{q\times p})$.

There is $H < 1$ and

$$H := 1 - h^s L \ge 1 - h^s L_1 \ge 1 - h^s(L_1 + h^{-2}(hn\Gamma + 2nK)) > 0.$$

So

$$(5.11)\qquad H \in (0, 1).$$

Let us take two arbitrary elements $V, W \in R^{q\times p}$, $W = (w_l^M)_{\substack{M\in Z \\ l=1,\ldots,p}}$, $V = (v_l^M)_{\substack{M\in Z \\ l=1,\ldots,p}}$ and let

$$D := \Phi(V) - \Phi(W) = (d_l^M)_{\substack{M\in Z \\ l=1,\ldots,p}},$$
$$Y := W - V = (y_l^M)_{\substack{M\in Z \\ l=1,\ldots,p}} = (w_l^M - v_l^M)_{\substack{M\in Z \\ l=1,\ldots,p}}.$$

Then, for $M \in Z \setminus Z_1 \cap Z_2$, there is

$$\begin{aligned} d_l^M &= [h^s(-Lw_l^M) + w_l^M] - [h^s(-Lv_l^M) + v_l^M] \\ &= [h^s(-Ly_l^M) + y_l^M] = [-h^s L + 1]y_l^M = Hy_l^M \le H||Y||_1, \end{aligned}$$

hence

$$(5.12)\qquad d_l^M = Hy_l^M \le H||Y||_1.$$

For $M \in Z_1 \cap Z_2$ there follows

$$
\begin{aligned}
d_l^M &= h^s[f_l(x^M, w^M, w_l^{MI}, w_l^{MIJ}, w^*) - f_l(x^M, v^M, v_l^{MI}, v_l^{MIJ}, v^*) + y_l^{M+}] + y_l^M \\
&= h^s[f_l(x^M, w^M, w_l^{MI}, w_l^{MIJ}, w^*) - f_l(x^M, v^M, v_l^{MI}, v_l^{MIJ}, w^*) + y_l^{M+}] \\
&\quad + h^s[f_l(x^M, v^M, v_l^{MI}, v_l^{MIJ}, w^*) - f_l(x^M, v^M, v_l^{MI}, v_l^{MIJ}, v^*)] + y_l^M \\
&= h^s[\sum_{\mu=1}^{p} \alpha_{l\mu}^M y_\mu^M + \sum_{i=1}^{n} \beta_{li}^M y_l^{Mi} + \sum_{i,j=1}^{n} \gamma_{lij}^M y_l^{Mij} + y_l^{M+}] \\
&\quad + h^s[f_l(x^M, v^M, v_l^{ML}, v_l^{MIJ}, w^*) - f_l(x^M, v^M, v_l^{MI}, v_l^{MIJ}, v^*)] + y_l^M,
\end{aligned}
$$

and

(5.13a)

$$
h^s[\sum_{i=1}^{n} \beta_{li}^M y_l^{Mi} + \sum_{i,j=1}^{n} \gamma_{lij}^M y_l^{Mij} + y_l^{M+}] + h^s \sum_{\mu=1}^{p} \alpha_{l\mu}^M y_\mu^M - h^s \chi_l ||y^*|| + y_l^M \leq d_l^M,
$$

(5.13b)

$$
h^s[\sum_{i=1}^{n} \beta_{li}^M y_l^{Mi} + \sum_{i,j=1}^{n} \gamma_{lij}^M y_l^{Mij} + y_l^{M+}] + h^s \sum_{\mu=1}^{p} \alpha_{l\mu}^M y_\mu^M + h^s \chi_l ||y^*|| + y_l^M \geq d_l^M,
$$

where $||y^*|| = \max\limits_{l=1,\ldots,p} \{\sup\limits_{x \in D} |\sum\limits_{M \in Z} \theta_M(x) y_\mu^M|\} = \max\limits_{\substack{l=1,\ldots,p \\ M \in Z}} |y_l^M| \leq ||Y||_1.$

By (5.2), from (5.13b) there follows:

$$
\begin{aligned}
d_k^M &\leq h^s[\sum_{i=1}^{n} \beta_{li}^M y_l^{Mi} + \sum_{i,j=1}^{n} \gamma_{lij}^M y_l^{Mij} + y_l^{M+}] + h^s \sum_{\mu=1}^{p} \alpha_{l\mu}^M y_\mu^M + h^s \chi_l ||y^*|| + y_l^M \\
&= h^{s-2} \sum_{i=1}^{n} [\delta_{li}^M] y_l^{i(M)} + h^{s-2} \sum_{i=1}^{n} [\delta_{li}^M] y_l^{-i(M)} \\
&\quad + h^{s-2} 0.25 \sum_{\substack{i,j=1 \\ i \neq j}}^{n} [G_{lij}^M + \gamma_{lij}^M] y_l^{i(j(M))} + h^{s-2} 0.25 \sum_{\substack{i,j=1 \\ i \neq j}}^{n} [G_{lij}^M - \gamma_{lij}^M] y_l^{-i(j(M))} \\
&\quad + h^{s-2} 0.25 \sum_{\substack{i,j=1 \\ i \neq j}}^{n} [G_{lij}^M - \gamma_{lij}^M] y_l^{i(-j(M))} + h^{s-2} 0.25 \sum_{\substack{i,j=1 \\ i \neq j}}^{n} [G_{lij}^M + \gamma_{lij}^M] y_l^{-i(-j(M))} \\
&\quad - 4h^{s-2} 0.25 [\rho_l^M] y_l^M - 2h^{s-2} \sum_{i=1}^{n} [\delta_{li}] y_l^M + h^s \sum_{\mu=1}^{p} \alpha_{l\mu}^M y_\mu^M + h^s \chi_l ||y^*|| + y_l^M.
\end{aligned}
$$

From (**A2**) there follows that each term in the square brackets is nonnegative, hence

$$
\begin{aligned}
d_l^M \le \{ & 2h^{s-2}\sum_{i=1}^{n}\delta_{li}^M + h^{s-2}0.25\sum_{\substack{i,j=1\\ i\neq j}}^{n}[G_{lij}^M+\gamma_{lij}^M] + h^{s-2}0.25\sum_{\substack{i,j=1\\ i\neq j}}^{n}[G_{lij}^M-\gamma_{lij}^M] \\
& + h^{s-2}0.25\sum_{\substack{i,j=1\\ i\neq j}}^{n}[G_{lij}^M-\gamma_{lij}^M] + h^{s-2}0.25\sum_{\substack{i,j=1\\ i\neq j}}^{n}[G_{lij}^M+\gamma_{lij}^M]\}||Y||_1 \\
& - \{4h^{s-2}0.25\rho_l^M + 2h^{s-2}\sum_{i=1}^{n}\delta_{li}^M\}y_l^M + h^s\alpha_{ll}^M y_l^M \\
& + h^s\sum_{\substack{\mu=1\\ \mu\neq l}}^{p}\alpha_{l\mu}^M y_\mu^M + h^s\chi_l||Y||_1 + y_l^M \\
\le \{ & 2h^{s-2}\delta_l^M + h^{s-2}0.25\sum_{\substack{i,j=1\\ i\neq j}}^{n}G_{lij}^M + h^{s-2}0.25\sum_{\substack{i,j=1\\ i\neq j}}^{n}G_{lij}^M \\
& + h^{s-2}0.25\sum_{\substack{i,j=1\\ i\neq j}}^{n}G_{lij}^M + h^{s-2}0.25\sum_{\substack{i,j=1\\ i\neq j}}^{n}G_{lij}^M\}||Y||_1 \\
& - \{h^{s-2}\rho_l^M + 2h^{s-2}\delta_l^M\}y_l^M + h^s\alpha_{ll}^M y_l^M + h^s\sum_{\substack{\mu=1\\ \mu\neq l}}^{p}\alpha_{l\mu}^M y_\mu^M + h^s\chi_l||Y||_1 + y_l^M \\
\le \{ & 2h^{s-2}\delta_l^M + h^{s-2}\rho_l^M\}||Y||_1 - \{h^{s-2}\rho_l^M + 2h^{s-2}\delta_l^M\}y_l^M + h^s\alpha_{ll}^M y_l^M \\
& + h^s(\sum_{\substack{\mu=1\\ \mu\neq l}}^{p}\alpha_{l\mu}^M + \chi_l)||Y||_1 + y_l^M \\
\le \{ & 2h^{s-2}\delta_l^M + h^{s-2}\rho_l^M\}||Y||_1 - \{h^{s-2}\rho_l^M + 2h^{s-2}\delta_l^M\}y_l^M + h^s\alpha_{ll}^M y_l^M + y_l^M \\
& + h^s(\chi_l + \sum_{\substack{\mu=1\\ \mu\neq l}}^{p}\alpha_{l\mu}^M)||Y||_1 \\
\le \{ & 2h^{s-2}\delta_l^M + h^{s-2}\rho_l^M\}||Y||_1 + \{-h^{s-2}\rho_l^M - 2h^{s-2}\delta_l^M + h^s\alpha_{ll}^M + 1\}y_l^M \\
& + h^s(\chi_l + \sum_{\substack{\mu=1\\ \mu\neq l}}^{p}\alpha_{l\mu}^M)||Y||_1
\end{aligned}
$$

$$\le \{2h^{s-2}\delta_l^M + h^{s-2}\rho_l^M\}||Y||_1 + |-h^{s-2}\rho_l^M - 2h^{s-2}\delta_l^M$$
$$+ h^s\alpha_{ll}^M + 1|||Y||_1 + h^s(\chi_l + \sum_{\substack{\mu=1\\ \mu\neq l}}^{p} \alpha_{l\mu}^M)||Y||_1$$
$$\le [2h^{s-2}\delta_l^M + h^{s-2}\rho_l^M + h^s(\chi_l + \sum_{\substack{\mu=1\\ \mu\neq l}}^{p} \alpha_{l\mu}^M)$$
$$+ |1 + h^s\alpha_{ll}^M - h^{s-2}\rho_l^M - 2h^{s-2}\delta_l^M|]||Y||_1.$$

From (5.4) and (**A2**) there follows $\rho_l^M \ge 0$, and by (5.3) there is

$$1 + h^s\alpha_{ll}^M - h^{s-2}\rho_l^M - 2h^{s-2}\delta_l^M$$
$$\ge 1 + h^s\alpha_{ll}^M - 2h^{s-2}\rho_l^M - 2h^{s-2}\delta_l^M \ge 1 - h^sL_1 - 2h^{s-2}(\rho_l^M + \delta_l^M)$$
$$= 1 - h^sL_1 - 2h^{s-2}(\sum_{\substack{i,j=1\\ i\neq j}}^{n} G_{lij}^M + \sum_{i=1}^{n}[0.5h\beta_{li}^M + \gamma_{lii}^M - \sum_{\substack{j=1\\ j\neq i}}^{n} G_{lij}^M])$$
$$= 1 - h^sL_1 - 2h^{s-2}\sum_{i=1}^{n}[0.5h\beta_{li}^M + \gamma_{lii}^M] \ge 1 - h^sL_1 - 2h^{s-2}\sum_{i=1}^{n}(0.5h\Gamma + K)$$
$$= 1 - h^sL_1 - 2h^{s-2}(0.5hn\Gamma + nK) = 1 - h^s(L_1 + h^{-2}(hn\Gamma + 2nK)) > 0$$

because of (5.9). So, from (2.3) there follows:

$$d_l^M \le [2h^{s-2}\delta_l^M + h^{s-2}\rho_l^M + h^s(\chi_l + \sum_{\substack{\mu=1\\ \mu\neq l}}^{p} \alpha_{l\mu}^M)$$
$$+ 1 + h^s\alpha_{ll}^M - h^{s-2}\rho_l^M - 2h^{s-2}\delta_l^M]||Y||_1$$
$$= [h^s(\chi_l + \sum_{\substack{\mu=1\\ \mu\neq l}}^{p} \alpha_{l\mu}^M) + 1 + h^s\alpha_{ll}^M||Y||_1 = [1 + h^s(\alpha_{ll}^M + \chi_l + \sum_{\substack{\mu=1\\ \mu\neq l}}^{p} \alpha_{l\mu}^M)||Y||_1$$
$$\le [1 - h^sL]||Y||_1 = H||Y||_1.$$

We conclude that

(5.14) $$d_l^M \le H||Y||_1 \text{ for } M \in Z, \quad l = 1, \ldots, p.$$

Similarly one can show that

(5.15) $$d_l^M \ge -H||Y||_1 \text{ for } M \in Z, \quad l = 1, \ldots, p.$$

So $|d_l^M| \le H||Y||_1$ for $M \in Z$, $l = 1, \ldots, p$. That means (5.10) is true.

From (5.11) and the Banach theorem follows existence and uniqueness of solution to differential problem (3.9)–(3.10). This ends our proof. □

6. Convergence theorem.

THEOREM 3 (Convergence). *Let u be a solution of system* (1.1)–(1.2), $u \in C^2(D)$,

(6.1)
$$\eta_l^M(h) := f_l\left(x^M, u^M, u_l^{MI}, u_l^{MIJ}, u^*\right) + u_l^{M+} \quad (M \in Z_1 \cap Z_2;\ l = 1, \dots, p),$$
$$\eta(h) := \max_{\substack{M \in Z_1 \cap Z_2 \\ l=1,\dots,p}} |\eta_l^M(h)|, \quad u_l^M := u_l(x^M) \qquad (M \in Z;\ l = 1, \dots, p),$$

v be a solution of the system (3.9)–(3.10) *and assumptions* (***A1***)–(***A2***) *and* (4.1) *hold. Then*

$$\max_{\substack{M \in Z \\ l=1,\dots,p}} |u_l^M - v_l^M| \le \frac{\eta(h)}{L} \tag{6.2}$$

and

$$\lim_{h \to 0} \eta(h) = 0. \tag{6.3}$$

This means that difference solution v of (3.9)–(3.10) *converges to the differential solution u of* (1.1)–(1.2).

PROOF. For $r^M := u^M - v^M = (r_\mu^M)_{\mu=1\dots p}$ and $\varepsilon_l^M(h) := \eta_l^M(h)$ $(M \in Z_1 \cap Z_2)$, from Theorem 1 inequality (6.2) follows.

The proof of (6.3) is similar to that in [**5**]. We only show that $u_l^{M+} \longrightarrow 0$ when $h \longrightarrow 0$ $(M \in Z_1 \cap Z_2;\ l = 1, \dots, p)$. Using Taylor polynomials for u_l (for simplicity we omit lower index l in u) we obtain:

$$u^{i(j(M))} = u^M + u_{x_i}^M h + u_{x_j}^M h + \frac{1}{2} u_{x_i^2}^M h^2 + u_{x_i x_j}^M h^2 + \frac{1}{2} u_{x_j^2}^M h^2 + \delta_{ij}^{++}(x^M) h^2,$$
$$u^{-i(j(M))} = u^M - u_{x_i}^M h + u_{x_j}^M h + \frac{1}{2} u_{x_i^2}^M h^2 - u_{x_i x_j}^M h^2 + \frac{1}{2} u_{x_j^2}^M h^2 + \delta_{ij}^{-+}(x^M) h^2,$$
$$u^{i(-j(M))} = u^M + u_{x_i}^M h - u_{x_j}^M h + \frac{1}{2} u_{x_i^2}^M h^2 - u_{x_i x_j}^M h^2 + \frac{1}{2} u_{x_j^2}^M h^2 + \delta_{ij}^{+-}(x^M) h^2,$$
$$u^{-i(-j(M))} = u^M - u_{x_i}^M h - u_{x_j}^M h + \frac{1}{2} u_{x_i^2}^M h^2 + u_{x_i x_j}^M h^2 + \frac{1}{2} u_{x_j^2}^M h^2 + \delta_{ij}^{--}(x^M) h^2,$$
$$u^{i(M)} = u^M + u_{x_i}^M h + \frac{1}{2} u_{x_i^2}^M h^2 + \delta_{ij}^{+}(x^M) h^2,$$
$$u^{-i(M)} = u^M - u_{x_i}^M h + \frac{1}{2} u_{x_i^2}^M h^2 + \delta_{ij}^{-}(x^M) h^2,$$

where $u^M = u(x^M)$, $u^M_{x_i} = u_{x_i}(x^M)$ and so on, and for all x^M $(M \in Z_1 \cap Z_2)$ and $i, j = 1, \dots, n$ there exists function $\delta(h)$ such that

$$|\delta^{++}_{ij}(x^M)| \le \delta(h),\ |\delta^{-+}_{ij}(x^M)| \le \delta(h),\ |\delta^{+-}_{ij}(x^M)| \le \delta(h),\ |\delta^{--}_{ij}(x^M)| \le \delta(h),$$
$$|\delta^{+}_{ij}(x^M)| \le \delta(h),\ |\delta^{-}_{ij}(x^M)| \le \delta(h)$$

and $\delta(h) \longrightarrow 0$ as $h \longrightarrow 0$. Now we return to lower index l:

$$\begin{aligned}
u_l^{M+} = & \sum_{\substack{i,j=1\\ i\neq j}}^{n} G^M_{lij} 0.25 h^{-2}\big(u_l^{i(j(M))} + u_l^{-i(j(M))} \\
& + u_l^{i(-j(M))} + u_l^{-i(-j(M))} - 4u_l^{i(M)} - 4u_l^{-i(M)} + 4u_l^M\big) \\
= & \sum_{\substack{i,j=1\\ i\neq j}}^{n} G^M_{lij} 0.25 h^{-2}[4u^M + 2(u_l)^M_{x_i^2}h^2 + 2(u_l)^M_{x_j^2}h^2 - 8u^M \\
& - 4(u_l)^M_{x_i^2}h^2 + 4u^M + \delta^{++}_{ij}(x^M)h^2 + \delta^{-+}_{ij}(x^M)h^2 + \delta^{+-}_{ij}(x^M)h^2 \\
& + \delta^{--}_{ij}(x^M)h^2 - 4\delta^{+}_{ij}(x^M)h^2 - 4\delta^{-}_{ij}(x^M)h^2] \\
= & \sum_{\substack{i,j=1\\ i\neq j}}^{n} 0.25 G^M_{lij}[-2(u_l)^M_{x_i^2} + 2(u_l)^M_{x_j^2}] + \sum_{\substack{i,j=1\\ i\neq j}}^{n} 0.25 G^M_{lij}[\delta^{++}_{ij}(x^M) \\
& + \delta^{-+}_{ij}(x^M) + \delta^{+-}_{ij}(x^M) + \delta^{--}_{ij}(x^M) - 4\delta^{+}_{ij}(x^M) - 4\delta^{-}_{ij}(x^M)].
\end{aligned}$$

Matrices $G_l(x)$ are bounded and symmetric $(l = 1, \dots, p)$, so the second sum tends to 0 while $h \longrightarrow 0$ and the first sum is equal 0:

$$\begin{aligned}
& \sum_{\substack{i,j=1\\ i\neq j}}^{n} 0.25 G^M_{lij}[-2(u_l)^M_{x_i^2} + 2(u_l)^M_{x_j^2}] \\
& \qquad = -\sum_{\substack{i,j=1\\ i\neq j}}^{n} 0.5 G^M_{lij}[(u_l)^M_{x_i^2}] + \sum_{\substack{i,j=1\\ i\neq j}}^{n} 0.5 G^M_{lij}(u_l)^M_{x_j^2}] = 0.
\end{aligned}$$

This ends our proof. $\square$

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