UNIVERSITATIS IAGELLONICAE ACTA MATHEMATICA, FASCICULUS XLII
2004

ON SPECIAL VALUES FOR PENCILS OF PLANE CURVE SINGULARITIES

by Arkadiusz Płoski

Abstract. Let $(F_t : t \in \mathbf{P}^1)$ be a pencil of plane curve singularities and let μ_0^t be the Milnor number of the fiber F_t. We prove a formula for the jumps $\mu_0^t - \inf\{\mu_0^t : t \in \mathbf{P}^1\}$. As an application, we give a description of the special values of the pencil $(F_t : t \in \mathbf{P}^1)$.

Introduction. Let $(F_t : t \in \mathbf{P}^1)$, $\mathbf{P}^1 = \mathbf{C} \cup \{\infty\}$ be a pencil of plane curve singularities defined by two coprime power series $f, g \in \mathbf{C}\{X, Y\}$ without constant term. That is $F_t = f - tg$ for $t \in \mathbf{C}$ and $F_\infty = g$. Let μ_0^t be the Milnor number of the fiber F_t and let

$$\mu_0^{\min} = \inf\{\mu_0^t : t \in \mathbf{P}^1\}.$$

Our aim is to give a formula for the jumps $\mu_0^t - \mu_0^{\min}$ by means of the meromorphic fraction f/g considered on the branches of the Jacobian curve

$$j(F) = \frac{\partial f}{\partial X}\frac{\partial g}{\partial Y} - \frac{\partial f}{\partial Y}\frac{\partial g}{\partial X} = 0.$$

Roughly speaking we will show that $\mu_0^t - \mu_0^{\min} =$ the number of zeros of $f/g - t$ if $t \in \mathbf{C}$ and $\mu_0^\infty - \mu_0^{\min} =$ the number of poles of f/g on the branches of the Jacobian curve $j(F) = 0$ provided that $\mu_0^t \neq +\infty$ (resp. $\mu_0^\infty \neq +\infty$). Then we prove a known result on the special values of the pencil $(F_t : t \in \mathbf{P}^1)$.

2000 *Mathematics Subject Classification.* 14H20, 32S10.
Key words and phrases. Pencil of plane curve singularities, Milnor number, special value.
Supported in part by the KBN grant No 2 P03A 022 15.

1. Preliminaries. Let $f \in \mathbf{C}\{X,Y\}$ be a non-zero power series without constant term. We say that the curve $f=0$ is singular if $\partial f/\partial X(0,0) = \partial f/\partial Y(0,0) = 0$. A branch P is a prime ideal of $\mathbf{C}\{X,Y\}$ generated by an irreducible power series p. Let $\mathcal{B}$ be the set of all branches. For any curve $f=0$, we put

$$\mathcal{B}(f) = \{P \in \mathcal{B} :\ f \equiv 0 (\mathrm{mod} P)\}\ .$$

We put by definition $(f,g)_0 = \dim_{\mathbf{C}} \mathbf{C}\{X,Y\}/(f,g)$, the intersection multiplicity of f and g. Note that $(f,g)_0 = +\infty$ if and only if f and g have a common factor. The Milnor number $\mu_0(f)$ is defined to be $\mu_0(f) = (\partial f/\partial X, \partial f/\partial Y)_0$. Then $\mu_0(f) = +\infty$ if and only if the curve $f=0$ is not reduced (i.e. the power series f has a multiple factor).

The following lemma is well known (see [**3**] and [**6**]).

Lemma 1.1. *Let $f=0$ and $g=0$ be two curves without a common branch. Let $j(f,g) = (\partial f/\partial X)(\partial g/\partial Y) - (\partial f/\partial Y)(\partial g/\partial X)$. Then*

$$(f, j(f,g))_0 = \mu_0(f) + (f,g)_0 - 1.$$

In particular, the curve $f=0$ is not reduced if and only if the curves $f=0$ and $j(f,g)=0$ share a common branch.

Proof. ([**5**], [**10**], Prop. 4.1). We may assume that $f(0,Y) \neq 0$. Using Delgado's formula ([**1**], Prop. 7.4.1) we get

$$(f, j(f,g))_0 = (f, \partial f/\partial Y)_0 + (f,g)_0 - (f,X)_0. \tag{1}$$

On the other hand, by Teissier's formula ([**11**], Chap. II, Prop. 1.2), one can write

$$(f, \partial f/\partial Y)_0 = \mu_0(f) + (f,X)_0 - 1 \tag{2}$$

and the lemma follows. □

For every branch P we denote by $\mathcal{M}_P$ the field of fractions of the ring $\mathbf{C}\{X,Y\}/P$. Let $f,g \in \mathbf{C}\{X,Y\}$ be coprime power series. Put

$$\mathcal{D}(f/g) = \{P \in \mathcal{B} :\ g \not\equiv 0 (\mathrm{mod}\ P)\}.$$

Then for every $P \in \mathcal{D}(f/g)$ the fraction f/g defines an element of $\mathcal{M}_P$ which we also denote by f/g. We put, for $P \in \mathcal{D}(f/g)$:

$$\mathrm{ord}_P(f/g) = (f,p)_0 - (g,p)_0,$$

where p is a generator of P. Clearly, ord_P is a valuation of the field $\mathcal{M}_P$.

Let $\mathbf{P}^1 = \mathbf{C} \cup \{\infty\}$ and let us denote by $f/g \mapsto (f/g)(P) \in \mathbf{P}^1$ the place associated with ord_P.

Recall that $(f/g)(P) = \infty$ if $\mathrm{ord}_P f < \mathrm{ord}_P g < +\infty$ and $(f/g)(P) = 0$ if $\mathrm{ord}_P g < \mathrm{ord}_P f$.

Lemma 1.2. *Suppose that* $\operatorname{ord}_P f/g \geq 0$ *for a* $P \in \mathcal{D}(f/g)$ *and let* $t_0 = (f/g)(P)$. *Then* $\operatorname{ord}_P(f - tg) = \operatorname{ord}_P g$ *if* $t \neq t_0$ *and* $\operatorname{ord}_P(f - t_0 g) > \operatorname{ord}_P g$.

Proof. Obvious. □

2. Result. Let $f = 0$ and $g = 0$ be two curves without a common branch. We put $F = (f, g)$ and $j(F) = j(f, g)$. Let us consider the pencil $(F_t : t \in \mathbf{P}^1)$ where $F_t = f - tg$ for $t \in \mathbf{C}$ and $F_\infty = g$. Let $\mu_0^t = \mu_0(F_t)$. If $j(F)(0, 0) \neq 0$ then $\mu_0^t = 0$ for all $t \in \mathbf{P}^1$. In the sequel we assume that $j(F)(0, 0) = 0$.

Proposition 2.1. *Let* $t \in \mathbf{P}^1$. *Then the following two conditions are equivalent:*

(i) $\mu_0^t = +\infty$,

(ii) *the curves* $j(F) = 0$ *and* $F_t = 0$ *share a common branch.*

Proof. We use Lemma 1.1 to power series F_t, g if $t \in \mathbf{C}$ and to power series $F_\infty = g$ and f if $t = \infty$.

Proposition 2.1 implies Bertini's theorem "the set $\{t \in \mathbf{P}^1 \colon F_t$ is not reduced$\}$ is finite". Indeed, it is easy to check that

$$\#\{t \in \mathbf{P}^1 : \mu_0^t = +\infty\} \leq \#\mathcal{B}(j(F)).$$

Let $\mu_0^{\min} = \inf\{\mu_o^t : t \in \mathbf{P}^1\}$. By Bertini's theorem, $\mu_0^{\min}$ is an integer.

Let us put

$$\mathcal{U}(F) = \{P \in \mathcal{B}(j(F)) : \operatorname{ord}_P f \geq \operatorname{ord}_P g\}$$

and

$$\mathcal{U}(F)^c = \{P \in \mathcal{B}(j(F)) : \operatorname{ord}_P f < \operatorname{ord}_P g\}.$$

Thus $\mathcal{U}(F) \subset \mathcal{D}(f/g)$ and $\mathcal{U}(F)^c \subset \mathcal{D}(f/g)$ provided that $\mu_0^\infty < +\infty$.

For every branch P of the Jacobian curve $j(F) = 0$ we denote by $m(P)$ the multiplicity of P, i.e., the greatest integer $m > 0$ such that $j(F) \equiv 0 (\operatorname{mod} P^m)$. By convention, a sum extended over an empty set equals zero.

Our main result is the following.

Theorem 2.2. *With the notation introduced above*

(i) *if* $\mu_0^t \neq +\infty$ *for a* $t \in \mathbf{C}$, *then*

$$\mu_0^t - \mu_0^{\min} = \sum_{P \in \mathcal{U}(F)} m(P) \operatorname{ord}_P(f/g - t);$$

(ii) *if* $\mu_0^\infty \neq +\infty$, *then*

$$\mu_0^\infty - \mu_0^{\min} = -\sum_{P \in \mathcal{U}(F)^c} m(P) \operatorname{ord}_P(f/g).$$

Proof. Let us fix a $t \in \mathbf{C}$ such that $\mu_0^t \neq +\infty$. We have $j(F_t, g) = j(f, g)$ and $(F_t, g)_0 = (f, g)_0$. Applying Lemma 1.1 to F_t and g, we get

$$\mu_0^t = (F_t, j(f,g))_0 - (f,g)_0 + 1. \tag{3}$$

Let us write

$$j(f,g) = \prod_{i=1}^{k} p_i$$

with irreducible $p_i \in \mathbf{C}\{X, Y\}$ and let $P_i = (p_i)\mathbf{C}\{X, Y\}$. Therefore $(P_i)_{i=1,\dots,k}$ is a sequence of branches of $j(f, g) = 0$ counted with multiplicities. Let

$$I = \{i \in [1, k] : \operatorname{ord}_{P_i} f \geq \operatorname{ord}_{P_i} g\}$$

and observe that $\operatorname{ord}_{P_i} F_t = \operatorname{ord}_{P_i} f$ for $i \notin I$. Then

$$(F_t, j(f,g))_0 = \sum_{i=1}^{k} \operatorname{ord}_{P_i} F_t = \sum_{i \in I} \operatorname{ord}_{P_i} F_t + \sum_{i \notin I} \operatorname{ord}_{P_i} f$$

and by (3) we get

$$\mu_0^t = \sum_{i \in I} \operatorname{ord}_{P_i} F_t + \sum_{i \notin I} \operatorname{ord}_{P_i} f - (f,g)_0 + 1. \tag{4}$$

If $i \in I$ then by Lemma 1.2 we have $\operatorname{ord}_{P_i} F_t \geq \operatorname{ord}_{P_i} g$ with equality for $t \neq (f/g)(P_i)$. Using (4) we get

$$\mu_0^{\min} = \sum_{i \in I} \operatorname{ord}_{P_i} g + \sum_{i \notin I} \operatorname{ord}_{P_i} f - (f,g)_0 + 1. \tag{5}$$

and consequently

$$\begin{aligned} \mu_0^t - \mu_0^{\min} &= \sum_{i \in I} \operatorname{ord}_{P_i} F_t - \operatorname{ord}_{P_i} g = \sum_{i \in I} \operatorname{ord}_{P_i}(F_t/g) \\ &= \sum_{P \in \mathcal{U}(F)} m(P) \operatorname{ord}_P(f/g - t) \end{aligned}$$

for $F_t/g = f/g - t$. We have thus proved (i).

Let us suppose that $\mu_0^\infty \neq +\infty$. By Lemma 1.1 applied to g and f, we get

$$\mu_0^\infty = (g, j(f,g))_0 - (f,g)_0 + 1 = \sum_{i \in I} \operatorname{ord}_{P_i} g + \sum_{i \notin I} \operatorname{ord}_{P_i} g - (f,g)_0 + 1. \tag{6}$$

Now, by (5) and (6), we get

$$\begin{aligned} \mu_0^\infty - \mu_0^{\min} &= \sum_{i \notin I} \operatorname{ord}_{P_i} g - \operatorname{ord}_{P_i} f = -\sum_{i \notin I} \operatorname{ord}_{P_i}(f/g) \\ &= -\sum_{P \in \mathcal{U}(F)^c} m(P) \operatorname{ord}_P(f/g) \end{aligned}$$

which proves (ii). □

Remark 2.3. We can write (5) in the following form

$$\mu_0^{\min} = \sum_P \inf\{\operatorname{ord}_P f, \operatorname{ord}_P g\} - (f,g)_0 + 1.$$

3. Description of special values. Let

$$\Lambda(F) = \{t \in \mathbf{P}^1 : \ \mu_0^t > \mu_0^{\min}\}$$

be the set of special values of the pencil $(F_t : t \in \mathbf{P}^1)$ (see [**7**] and [**6**]). We put by convention $(f/g)(P) = \infty$ if $g \equiv 0 (\bmod\ P)$.

The following description of the special values is due to different authors:

Theorem 3.1. (see [**9**], Théorème 1, [**8**], p. 410–411, [**1**], 7.4).
We have

$$\Lambda(F) = \{(f/g)(P) : \ P \in \mathcal{B}(j(F))\}.$$

Proof. First we prove the following:

$$\{t \in \mathbf{C} : \ \mu_0^t > \mu_0^{\min}\} = \{(f/g)(P) : \ P \in \mathcal{U}(F)\}. \tag{7}$$

Fix $t \in \mathbf{C}$. We will check that $\mu_o^t > \mu_0^{\min}$ if and only if there exists a $P \in \mathcal{U}(F)$ such that $(f/g)(P) = t$. If $\mu_0^t = +\infty$ then F_t has multiple factors. Thus there exists a branch P such that $F_t \equiv 0 (\bmod\ P^2)$. It is easy to check that $P \in \mathcal{U}(F)$ and $(f/g)(P) = t$.

Now suppose that $\mu_0^t < +\infty$. According to Theorem 2.2(i), the inequality $\mu_0^t > \mu_0^{\min}$ holds if and only if there exists $P \in \mathcal{U}(F)$ such that $\operatorname{ord}_P(f - tg) > \operatorname{ord}_P g$. The last inequality is equivalent to the condition $(f/g)(P) = t$. This proves (7).

Let us check the following property

$$\mu_0^{\infty} > \mu_0^{\min} \ \text{ if and only if } \ \mathcal{U}(F)^c \neq \emptyset. \tag{8}$$

Indeed, if $\mu_0^{\infty} = +\infty$, then there is a branch P such that $g \equiv 0 (\bmod\ P)$ and $P \in \mathcal{B}(j(F))$ by Proposition 2.1. Obviously $f \not\equiv 0 (\bmod\ P)$ and we get $\operatorname{ord}_P g = +\infty > \operatorname{ord}_P f$. Thus $P \in \mathcal{U}(F)^c$. If $\mu_0^{\infty} < +\infty$, then (8) follows from Theorem 2.2(ii). Theorem 3.1 follows from (7) and (8) for $(f/g)(P) = \infty$ if $P \in \mathcal{U}(F)^c$. □

When studying the singularities at infinity of a polynomial in two complex variables of degree $N > 1$, one considers the pencil defined by $F = (f, l^N)$, where $l = 0$ is a smooth curve which is not a component of the curve $f = 0$. Clearly, $\mu_0^{\infty} = +\infty$. Using Theorem 3.1, we get

Corollary 3.2. (see [**4**], Proposition 2.2).

$$\Lambda(F) \cap \mathbf{C} = \{(f/l^N)(P) : \ P \in \mathcal{B}(j(f,l)) \ \textit{and} \ \operatorname{ord}_P f / \operatorname{ord}_P l \geq N\}.$$

4. Special values and the discriminant curve. Let U, V be variables. For every branch P of $\mathbf{C}\{X,Y\}$ we define

$$F(P) = \{\Phi(U,V) \in \mathbf{C}\{U,V\} : \Phi(f(X,Y), g(X,Y)) \equiv 0 (\mathrm{mod}\ P)\}.$$

Thus $F(P)$ is a branch of $\mathbf{C}\{U,V\}$. Let $L = (U,V)$. By definition, we have $L_t = U - tV$ for $t \in \mathbf{C}$ and $L_\infty = V$.

The (reduced) discriminant curve $\Delta_F = 0$ is the curve with branches $F(P)$, where P runs over branches of the Jacobian curve $j(F) = 0$. The description of special values by means of the discriminant is due to Lê Dũng Tráng [**6**] (see also [**2**]). Both authors use topological methods.

Theorem 4.1. (see [**6**] Proposition 3.6.4, [**2**] Corollary 4.7) *Let $t_0 \in \mathbf{P}^1$. Then t_0 is a special value of the pencil $(F_t : t \in \mathbf{P}^1)$ if and only if L_{t_0} is a tangent to the discriminant curve $\Delta_F = 0$. Moreover, the fiber F_{t_0} is not reduced if and only if the line $L_{t_0} = 0$ is a branch of $\Delta_F = 0$.*

To prove Theorem 4.1 we need the following.

Lemma 4.2. *For every branch P of $\mathbf{C}\{X,Y\}$,*

$$\left(\frac{f}{g}\right)(P) = \left(\frac{U}{V}\right)(F(P)).$$

Proof. Let $(x(T), y(T)) \in \mathbf{C}\{T\}^2$, $x(0) = y(0) = 0$ be a parametrization of P. Therefore $P = \{h(X,Y) : h(x(T), y(T)) = 0\}$ and

$$\left(\frac{f}{g}\right)(P) = \left.\frac{f(x(T),y(T))}{g(x(T),y(T))}\right|_{T=0}.$$

To check (4.2) it suffices to observe that

$$F(x(T),y(T)) = (\, f(x(T),y(T)),\ g(x(T),y(T))\,)$$

is a parametrization of $F(P)$. □

Now we can give a proof.

Proof of Theorem 4.1. By Theorem 3.1 and Lemma 4.2, we get

$$\Lambda(F) = \left\{\left(\frac{U}{V}\right)(F(P)) :\ P \in \mathcal{B}(j(F))\right\} = \left\{\left(\frac{U}{V}\right)(Q) :\ Q \in \mathcal{B}(\Delta_F)\right\}.$$

On the other hand, it is very easy to see that $(U/V)(Q) = t$ if and only if the line $L_t = 0$ is tangent to the branch Q. This proves the first part of (4.1).

The second part of (4.1) follows from Proposition 2.1. Indeed, by (2.1) F_t is not reduced if and only if there is a branch $P \in \mathcal{B}(j(F))$ such that $F_t \equiv 0\,(\mathrm{mod}\ P)$ which is equivalent to $L_t \equiv 0\ (\mathrm{mod}\, F(P))$ that is to $F(P) = (L_t)$. □

References

1. Casas-Alvero E., *Singularities of Plane Curves*, London Math. Soc. Lecture Note Series, 276, Cambridge Univ. Press, 2000.
2. Caubel C., *Variation of the Milnor fibration in pencils of hypersurface singularities*, Proc. London Math. Soc., **(3) 83** (2001), 330–350.
3. Greuel G.-M., *Der Gauß–Manin Zusammenhang isolierter Singulatitäten von vollständingen Durchschuitten*, Math. Ann., **214** (1975), 235–266.
4. Garcia Barroso E., Płoski A., *Pinceaux de courbes planes et invariants polaires*, Ann. Polon. Math., **83**, **2** (2004), 113–128.
5. Gabrielov A., Lion J.M., Moussu R., *Ordre de contact de courbes intégral du plan*, C. R. Acad. Sci. Paris, t. **319**, Série I (1994), 219–221.
6. Lê Dũng Tráng, *Calculation of the Milnor number of isolated singularity of complete intersection*, Funct. Anal. Appl., **8** (1974), 127–131, Funkc. Anal. i Priloz., **8**, No. **2** (1974), 45–49.
7. Lê Dũng Tráng, Weber C., *Équisingularité dans les pinceaux de germes de courbes planes et C^0–suffisance*, L'Enseignement Mathématique, t. **43** (1997), 355–380.
8. Lê Văn Thành, Oka M., *Note on estimation of the number of the critical values at infinity*, Kodai Math. J., **17** (1994), 409–414.
9. Maugendre H., Michel F., *Fibration associée à un pinceau de courbes planes*, Annales de la Fac. des Sciences de Toulouse, Vol. **X**, N°**4** (2001), 745–777.
10. Płoski A., *The Milnor Number of a plane algebroid curve*, in: Materiały XVI Konferencji Szkoleniowej z Analizy i Geometrii Zespolonej, Łódź, 1995, 73–82.
11. Teissier B., *Cycles évanescentes, section planes et conditions de Whitney*, Astérisque, No. **7–8** (1973).

Received March 3, 2003

Technical University
Department of Mathematics
AL. 1000 L PP 7
25-314 Kielce
Poland
e-mail: matap@tu.kielce.pl