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GENERALIZED HÉNON DIFFERENCE EQUATIONS WITH DELAY

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Abstract. Charles Conley once said his goal was to reveal the discrete in the continuous. The idea here of using discrete cohomology to elicit the behavior of continuous dynamical systems was central to his program. We combine this idea with our idea of "expanders" to investigate a difference equation of the form $x_n = F(x_{n-1}, \dots, x_{n-m})$ when F has a special form. Recall that the equation $x_n = q(x_{n-1})$ is chaotic for continuous real-valued q that satisfies $q(0) < 0$, $q(1/2) > 1$, and $q(1) < 0$. For such a q, it is also easy to analyze $x_n = q(x_{n-k})$ where $k > 1$. But when a small perturbation $g(x_{n-1}, \dots x_{n-m})$ is added, the equation

$$x_n = q(x_{n-k}) + g(x_{n-1}, \dots, x_{n-m})$$

(where $1 < k < m$) is far harder to analyze and appears to require degree theory of some sort. We use k-dimensional cohomology to show that this equation has a 2-shift in the dynamics when g is sufficiently small.

1. Introduction

Charles Conley once said his goal was to reveal the discrete in the continuous. The idea here of using discrete cohomology to elicit the behavior of continuous dynamical systems was central to his program. We combine this idea with our idea of "expanders" to investigate a difference equation.

For a continuous map $f : \mathbf{R}^m \to \mathbf{R}^m$ there is often a need to show there is a trajectory following a particular "itinerary". An *itinerary* is a sequence (X_i)

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of compact sets for i a positive or nonnegative integer (a *forward itinerary*) or i an integer (a *two-sided intinerary*). A *trajectory* ($x_{i+1} = f(x_i)$) follows the sequence (X_i) of sets if $x_i \in X_i$ for all i.

In this introduction we write y in $\mathbf{R}^m$ as $(x_{-1}, x_{-2}, \dots, x_{-m})$ with negative subscripts to simplify the conversion of the maps in the abstract to maps in $\mathbf{R}^m$. Let the map in the abstract be of the form

$$F(x_{-1}, x_{-2}, \dots, x_{-m}) = q(x_{-k}) + g(x_{-1}, \dots, x_{-m}).$$

The difference equation can also be viewed as a map $f : \mathbf{R}^m \to \mathbf{R}^m$ given by

$$x = \begin{pmatrix} x_{n-1} \\ \vdots \\ x_{n-m} \end{pmatrix} \overset{f}{\to} \begin{pmatrix} F(x) \\ x_{n-1} \\ \vdots \\ x_{n-m-1} \end{pmatrix}.$$

Let $J = [0, 1]$. Given two disjoint intervals I_1 and I_2 in J, we define the sets $\tilde{I}_i = J^{k-1} \times I_i \times J^{m-k}$ for $i = 1, 2$. For carefully chosen I_i, these sets $\tilde{I}_i$ are called symbol sets in [**2**] and play a pivotal role.

Our main conclusion is that for appropriately chosen $\tilde{I}_1$ and $\tilde{I}_2$, there is a compact invariant set Q in J^m for the map f such that for every itinerary $\pi : \mathbf{Z} \to \{1, 2\}$ (where $\mathbf{Z}$ denotes the integers) there is at least one trajectory (x_n) such that

$$y_n := (x_{n-1}, \dots, x_{n-m}) \in Q$$

for all n, and y_n follows the specified itinerary, i.e., $y_n \in \tilde{I}_{\pi(n)}$ for all n. Furthermore, when the dynamics are restricted to Q, every trajectory in Q has sensitive dependence on initial data (as defined in [**2**]). More generally the existence of such trajectories can often be guaranteed if $f(x_i)$ "crosses" X_{i+1} in some particular fashion that is uniform for all i.

To formalize and give a variety of examples of this idea we assume the following:

(1) $(X_i), (Y_i)$ are sequences of compact sets in $\mathbf{R}^m$; $B_i := X_i \cap Y_i$, $Z_i := X_i \cup Y_i$; Z_i and X_i are rectangles (products of intervals). (See Figures 1, 2 and 3.)

(2) For some $k \le n$, B_i is homeomorphic to $S^{k-1} \times R_i$ (where R_i is a rectangle) and is the union of some or all of the faces of X_i.

(3) $f(B_i) \subset Y_{i+1}$, $f(X_i) \subset Z_{i+1}$.

2. Background and notation

In the paper, $\mathbf{Z}$ denotes the set of integers, $\mathbf{N}$ denotes the positive integers, $\tilde{\mathbf{N}}$ denotes the nonnegative integers, and $\mathbf{R}$ denotes the real numbers. If A is a subset of $\mathbf{R}^m$, then $D_\epsilon(A) = \{x \in \mathbf{R}^m : d(x, y) < \epsilon \text{ for some } y \in A\}$. We

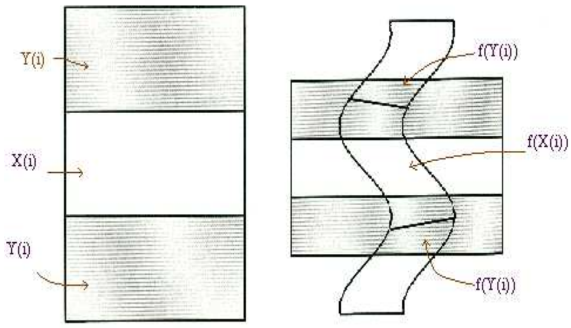

FIGURE 1. A case in which $H^1(Z_i, Y_i)$ would be appropriate.

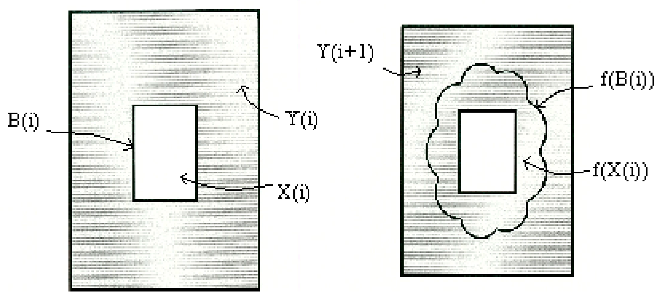

FIGURE 2. A case where $H^2(Z_i, B_i)$ would be used.

denote points in $\mathbf{R}^m$ as both row vectors and column vectors, and switch freely between the two, as is convenient. In particular, we find understanding the behavior of a map for high dimension m easier when the points are written as column vectors.

2.1. Cohomology. In writing this paper, we assume the reader has studied some cohomology theory, though not necessarily recently. We could have used homology theory but we prefer Čech–Alexander–Spanier cohomology theory (as presented by Spanier [**4**] and Eilenberg and Steenrod [**1**]) because of its stronger properties and have chosen to use it here.

We will say (A, B) is a *pair* if A and B are compact and $B \subset A$. If (C, D) is a pair we write $f : (A, B) \to (C, D)$ to mean A is the domain of f and

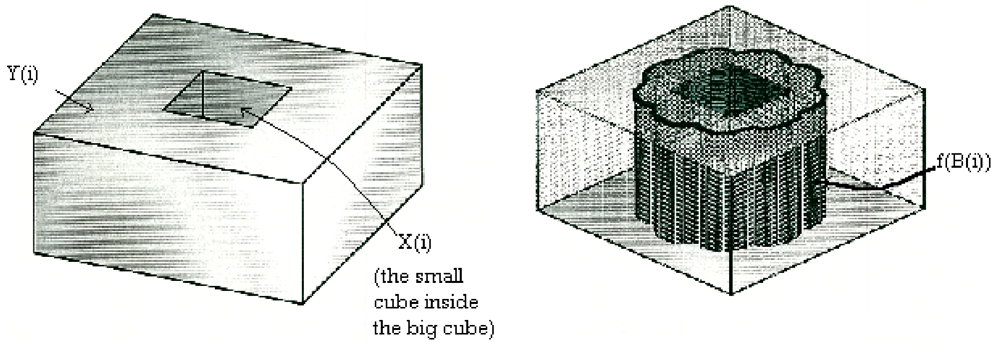

FIGURE 3. Here we would need $H^2(Z_i, Y_i)$ again.

$f(A) \subset C$ and $f(B) \subset D$. Note that (3) above says that f *maps* (X_i, B_i) *into* (Z_{i+1}, Y_{i+1}).

It is perhaps easiest to think about the cohomology of a pair (A, B) as the cohomology of the pair that results if the set B is collapsed to a point. Hence, if $A = [0, 1]$ and B is $\{0, 1\}$, identifying 0 with 1 results topologically in a circle or rather the pair $(S^1, \{b\})$ where $b \in S^1$.

If A, B are compact and $B \supset A$, the corresponding *inclusion* map (for A and B) is denoted $i : A \to B$, and is defined by $i(a) = a$ for all $a \in A$. Similarly, a *(pair) inclusion* $i : (A, B) \to (A', B')$ is defined if $A \subset A'$ and $B \subset B'$. We use cohomology groups with coefficients in $\mathbf{Z}$. We also use the symbol j to denote inclusion maps, as is customary, and in case several inclusion maps are being considered, we use subscripts (e.g., i_1 or j_2) to avoid confusion.

An *upper sequence* of groups is a sequence (G^i, ϕ^i) where for each i, G^i is a group and $\phi^i : G^i \to G^{i+1}$ is a homomorphism. An upper sequence is *exact* if for each integer i, $\phi^i(G^i)$ is the kernel of G^{i+1}. The sequence is of *order* 2 if the composition of any two successive homomorphisms of the sequence yields the trivial homomorphism.

If X is a space, define $(A, B) \times X := (A \times X, B \times X)$. Let I denote the unit interval $[0, 1]$. Two maps $f, g : (A, B) \to (C, D)$ are said to be *homotopic* if there is a map $H : (A, B) \times I \to (C, D)$ such that $f(x) = H(x, 0)$ and $g(x) = H(x, 1)$ for each $x \in A$. For $t \in I$, H_t denotes the map defined by $H_t(x) = H(x, t)$ for $x \in A$. A pair (A, B) contained in a pair (C, D) is called a *retract* of (C, D) if there exists a map $r : (C, D) \to (A, B)$ such that $r(x) = x$ for each x in A. The map r is called a *retraction.* The pair (A, B) is a *deformation retract* of (C, D) if there is a retraction $r : (C, D) \to$

(A, B) and the composition $r \circ i$, where $i : (A, B) \to (C, D)$ is the inclusion, is homotopic to the identity map $(A, B) \to (A, B)$. The pair (C, D) is a *strong deformation retract* of (A, B) if the latter homotopy can be chosen to leave each point of B fixed (i.e., $H(x, t) = x$ for $x \in B$). The pairs (A, B) and (C, D) are *homotopically equivalent* if there exist maps $f : (A, B) \to (C, D)$ and $g : (C, D) \to (A, B)$ such that $f \circ g$ is homotopic to the identity on (C, D) and $g \circ f$ is homotopic to the identity on (A, B).

For convenience, we list the axioms of cohomology and some other facts that we use ([**1**] and [**4**]): Suppose (X, A), (Y, B), and (Z, C) are compact pairs. If $f : (X, A) \to (Y, B)$ is continuous, then for each integer k, f induces a homomorphism $f_k^* : H^k(Y, B) \to H^k(X, A)$. As is customary, we depend on context to tell which of the homomorphisms induced by f is intended, and write only $f^* : H^k(Y, B) \to H^k(X, A)$. For the pair (X, A), and integer k, $H^q(X, A)$ is the *q–dimensional relative cohomology group* of X mod A. Cohomology groups are abelian groups; our coefficient group is the group of integers $\mathbf{Z}$ (thus this is also suppressed in the notation).

Axiom 1c. If f is the identity function on (X, A), then f^* is the identity isomorphism.

Axiom 2c. If $f : (X, A) \to (Y, B)$ and $g : (Y, B) \to (Z, C)$, then $(g \circ f)^* = f^* \circ g^*$.

Axiom 3c. The boundary operator, denoted by δ, is a homomorphism from $H^{k-1}(A)$ to $H^k(X, A)$ with the property that $\delta \circ (f \mid A)^* = f^* \circ \delta$. (Again, the notation is ambiguous, and we rely on context to determine which groups and which homomorphism is intended.)

Axiom 4c. (*Partial exactness.*) If $i : A \to X, j : X \to (X, A)$ are inclusion maps, then the upper sequence of groups and homomorphisms

$$\cdots \xrightarrow{i^*} H^{k-1}(A) \xrightarrow{\delta} H^k(X, A) \xrightarrow{j^*} H^k(X) \xrightarrow{i^*} H^k(A) \xrightarrow{\delta} \cdots$$

is of order 2. If (X, A) is triangulable, the sequence is exact. This upper sequence is called the cohomology sequence of the pair (X, A).

Axiom 5c. If the maps f, g are homotopic maps from (X, A) into (Y, B), then $f^* = g^*$.

Axiom 6c. (**The excision axiom**.) If U is open in X, and $\overline{U}$ is contained in the interior of A, then the inclusion map $i : (X \backslash U, A \backslash U) \to (X, A)$ induces isomorphisms, i.e., $H^k(X, A) \cong H^k(X \backslash U, A \backslash U)$ for all k.

Axiom 7c. If p is a point, then $H^k(\{p\}) = \{0\}$ for $k \neq 0$.

Theorem [**1**] Suppose $f : (X, A) \to (Y, B)$ and $g : (Y, B) \to (X, A)$. If f and g are homotopy equivalent, then f and g induce isomorphisms $f^* : H^k(Y, B) \to H^k(X, A)$ and $g^* : H^k(X, A) \to H^k(Y, B)$ with $(f^*)^{-1} = g^*$.

Theorem [**1**] If (X', A') is a deformation retract of (X, A), then the inclusion map $i : (X', A') \to (X, A)$ induces isomorphisms $i^* : H^k(X, A) \to H^k(X'A')$. Furthermore, if $r : (X, A) \to (X', A')$ is the associated retract, then $(i^*)^{-1} = r^*$.

In addition to the usual cohomology axioms and theorems above, Čech–Alexander–Spanier cohomology satisfies the following *strong excision property* and *weak continuity property*:

Theorem [**4**] (**Strong excision property**.) Let (X, A) and (Y, B) be pairs, with X and Y paracompact Hausdorff and A and B closed. Let $f : (X, A) \to (Y, B)$ be a closed continuous map such that f induces a one-to-one map of $X \backslash A$ onto $Y \backslash B$. Then, for all k, $f^* : H^k(Y, B) \to H^k(X, A)$ is an isomorphism.

Theorem [**4**] (**Weak continuity property.**) Let $\{(X_\alpha, A_\alpha)\}_\alpha$ be a family of compact Hausdorff pairs in some space, directed downward by inclusion, and let $(X, A) = (\cap_{\alpha \in A} X_\alpha, \cap_{\alpha \in A} A_\alpha)$. The inclusion maps $i_\alpha : (X, A) \subset (X_\alpha, A_\alpha)$ induce an isomorphism

$$\{i^*_\alpha\} : \lim_{\rightarrow} H^\kappa(X_\alpha, A_\alpha) \to H^k(X, A).$$

Dynamical considerations often require us to consider pairs of pairs which are rather similar. If $P_1 = (A, B)$ and $P_2 = (C, D)$ are pairs such that $A \subset C$ and $B \subset D$, $A \backslash B = C \backslash D$, and (A, B) is a deformation retract of (C, D), then we say P_2 is an *expansion* of P_1. This could be the case in the above example if $C = [-1, 2]$ and $D = [-1, 0] \cup [1, 2]$. Note that if D is identified to a point, the fact that D is larger than B makes negligible difference.

When P_2 is an expansion of P_1, the pair inclusion map $j : P_1 \to P_2$ induces a map on the cohomology groups and that map is an isomorphism. Note that P_1 is a deformation retract of P_2.

Proposition 1. [**1**] *When P_2 is a deformation retract of P_1, $j^* : H^n(P_1) \to H^n(P_2)$ is an isomorphism for all n. Thus, when P_2 is an expansion of P_1, $j^* : H^n(P_2) \to H^n(P_1)$ is an isomorphism for all n.*

Each B_i has the cohomology of a $(k-1)$-sphere, and (X_i, B_i) has the cohomology of (D^n, S^{n-1}), where $D^n = \{x \in R^n : d(x, \mathbf{0}) \leq 1\}$ and $S^{n-1} = \{x \in R^n : d(x, \mathbf{0}) = 1\}$ ($\mathbf{0}$ denotes the origin).

For k a positive integer, the cohomology groups we need are

(a) $H^0(S^k) = \mathbf{Z}$, $H^0(S^0) = \mathbf{Z} \oplus \mathbf{Z}$, $H^k(S^k) = \mathbf{Z}$, and $H^n(S^k) = \{0\}$ for $n \neq k$;

(b) $H^k(D^k, S^{k-1}) = H^{k-1}(S^{k-1}) = \mathbf{Z}$;

(c) $H^0(D^k) = \mathbf{Z}$, and $H^n(D^k) = \{0\}$ for $n \neq 0$.

Some of the properties of cohomology are illustrated when soap bubbles are created on a more or less circular frame Y. Some bubbles will exist independent of the frame while other soap surfaces exist because of the frame. If E is the latter type, it has a boundary $E \cap Y$ in Y, a boundary that contains a topological circle that runs around Y. This may be stated in the language of cohomology by saying that E has nonzero 2–dimensional cohomology stemming from Y, and we write that the coboundary operator

$$\delta : H^1(E \cap Y) \to H^2(E, E \cap Y)$$

has nonzero range. We will restrict attention to those E that lie in some compact set $X \cup Y$.

2.2. Chaos and the two-shift. Suppose that X is a metric space and Q is a compact subset of X. A finite collection $\mathcal{S} = \{S_1, S_2, \ldots, S_p\}$ of mutually disjoint sets is a *collection of symbol sets*, and each S_i is a *symbol set.* Recall that a sequence $\mathbb{S} := (S_{i_0}, S_{i_1}, \ldots, S_{i_n}, \ldots)$, each member of which is a member of $\mathcal{S}$, is a *forward itinerary.* If $f : Q \to X$ is continuous, and $x \in Q$ such that for each nonnegative integer n, $f^n(x) \in S_{i_n}$ for all $n = 0, 1, 2, \ldots$, where $f^n(x) = f(f^{n-1}(x))$ for $n \in \mathbf{N}$ and $f^0(x) = x$, we say the point x *follows the forward itinerary* $\mathbb{S}$. Next, when $\mathcal{E}$ is a nonempty family of nonempty closed subsets of Q such that for each $E \in \mathcal{E}$ and each $S_i \in \mathcal{S}$, there is a compact subset $D_i \subset E \cap S_i$ such that $f(D_i) \in \mathcal{E}$ (that is, $f(D_i)$ expands D_i to a member of $\mathcal{E}$), we call $\mathcal{E}$ a *family of expanders* for $\mathcal{S}$, and each member E of $\mathcal{E}$ an *expander.*

A closed subset Q^* of Q is *invariant* under f if $f(Q^*) = Q^*$. If Q^* is an invariant set for f, and $x \in Q^*$, then $f^n(x) \in Q^*$, and is thus defined, for all $n \in \tilde{\mathbf{N}}$. In addition to "one-sided" sequences of points or sets (such as $\mathbb{S} := (S_{i_0}, S_{i_1}, \ldots, S_{i_n}, \ldots)$ above), we may also discuss "two-sided" sequences of points or sets. The former case means that subscripts are in $\tilde{\mathbf{N}}$, and the latter that subscripts are in $\mathbf{Z}$. Given a collection of sets $\mathcal{S} = \{S_1, S_2, \ldots, S_p\}$, we say sequence (one-sided or two-sided) is an itinerary (in $\mathcal{S}$) if each $S_{i_n} \in \mathcal{S}$. A *trajectory* in a set Q^* is a sequence (x_n) for n either in $\tilde{\mathbf{N}}$ (the one-sided case) or $\mathbf{Z}$ (the two-sided case) such that $x_{n+1} = f(x_n)$ for all n. We say that an itinerary (S_{i_n}) (either one-sided or two-sided) is *followed* in Q^* if there is a trajectory (x_n) (one-sided or two-sided, respectively) in Q^* such that $x_n \in S_{i_n}$

for each n. If $x \in Q^*$ and $x_i \in Q^*$ for all $i \in \tilde{\mathbf{N}}$, then the *sequence* (x_i) *separates from* x (or, more precisely, the *trajectories of* x_i *separate from the trajectory of* x) if $x_i \to x$ as $i \to \infty$, and there is a $\delta > 0$ such that for all $i > 0$ there is an $m = m(i) \in \mathbf{N}$ such that $d(f^m(x_i), f^m(x)) \geq \delta$ for all i. A point x having such a sequence (x_i) with all x_i in Q^*, is called *sensitive to initial data* (in Q^*). We say a set Q^* is *chaotic* if it is nonempty, invariant, has a trajectory whose positive limit set is Q^*, and every $x \in Q^*$ is sensitive to initial data in Q^*.

Lemma 2. (The Chaos Lemma [**2**].) *Suppose that X is a metric space, Q is a compact subset of X, $f : Q \to X$ is continuous, $\mathcal{S} = \{S_1, S_2, \ldots, S_p\}$ is a collection of symbol sets, with $p \geq 2$, associated with the map f, and $\mathcal{E}$ is an associated family of expanders for $\mathcal{S}$. Then there is a closed, chaotic, invariant subset Q^* of Q such that for every two-sided itinerary $\mathbb{S} = (S_{i_n})_{n=-\infty}^{\infty}$ of members of $\mathcal{S}$, there is a two-sided trajectory in Q^* that follows it.*

Suppose M is a positive integer greater than 1. Then $\sum_M$ denotes the set of all bi-infinite sequences $s = (\ldots s_{-1} \bullet s_0 s_1 \ldots)$ such that $s_i \in \{1, 2, \ldots, M\}$. If for $s = (\ldots s_{-1} \bullet s_0 s_1 \ldots)$ and $t = (\ldots t_{-1} \bullet t_0 t_1 \ldots)$ in $\sum_M$, we define $d(s,t) = \sum_{i=0}^{\infty} \frac{|s_i - t_i|}{2^i}$, then d is a distance function on $\sum_M$. The topological space $\sum_M$ generated by the metric function d is a Cantor set. A natural homeomorphism on the space $\sum_M$ is the *shift homeomorphism* σ defined by $\sigma(s) = \sigma(\ldots s_{-1} \bullet s_0 s_1 \ldots) = (\ldots s_{-1} s_0 \bullet s_1 \ldots) = s'$ for $s = (\ldots s_{-1} \bullet s_0 s_1 \ldots) \in \sum_M$, i.e., $\sigma(s) = s'$, where $s'_i = s_{i+1}$. More specifically, the map σ is called the *shift on M symbols.*

Proposition 3. *Suppose that X is a metric space, Q is a compact subset of X, $f : Q \to X$ is continuous, $\mathcal{S} = \{S_1, S_2, \ldots, S_p\}$ is a collection of symbol sets, with $p \geq 2$, associated with the map f, and $\mathcal{E}$ is an associated family of expanders for $\mathcal{S}$. Then if Q^* is the closed, chaotic, invariant subset of Q guaranteed by the Chaos Lemma, there is a continuous map $\phi : Q^* \to \sum_2$ such that $\phi \circ (f \mid Q^*) = \sigma \circ \phi$. In other words, the dynamics of f on Q^* factors over the dynamics of the shift on 2 symbols.*

2.3. Hénon-like maps and difference equations. For $a, b \in \mathbf{R}$, present day authors generally write maps $H_{a,b}$ in the Hénon family as

$$H_{a,b}(x, y) = (a - by - x^2, x)$$

with the corresponding difference equation being

$$x_{n+1} = a - bx_{n-1} - x_n^2.$$

Usually, b is "small", and the $-bx_{n-1}$ term can be taken to represent the presence of some noise in the system. The dynamics producing term in the difference equation is $-x_n^2$. One could also consider the family $\tilde{H}_{a,b}$ of maps defined by

$$\tilde{H}_{a,b}(x,y) = (a - bx - y^2, x)$$

with the corresponding difference equation being

$$x_{n+1} = a - bx_n - x_{n-1}^2.$$

Now the dynamics producing term is $-x_{n-1}^2$, with $-bx_n$ contributing only noise. Thus, a delay has been introduced.

We extend this idea to maps on arbitrarily high, but finite-dimensional spaces $\mathbf{R}^m$. We call a difference equation $F : \mathbf{R}^m \to \mathbf{R}$ *Hénon-like with delay* k (where $1 \le k \le m$) if there are m-dimensional cubes C, C_1, and C_2 in $\mathbf{R}^m$ with $C_1 \cup C_2 \subset C$, $\epsilon > 0$, and maps $\Phi : \mathbf{R}^m \to \mathbf{R}$, and $\Psi : \mathbf{R} \to \mathbf{R}$ such that

(1) if $\Pi_k : \mathbf{R}^m \to \mathbf{R}$ denotes the projection to the kth coordinate, and $\Pi_k(C_i) = I_i$ for $i = 1, 2$, then $\Pi_k(C) \supset \Psi(I_i) \supset D_\epsilon(I_1) \cup D_\epsilon(I_2)$,

(2) $|\Phi(x)| < \epsilon$ for $x \in C$,

(3) $\min\{d(\Psi(x), y) : x \in I_1 \cup I_2, y \notin \Pi_k(C)\} > \epsilon$, and

(4) $F(x) = \Psi(x_k) + \Phi(x)$ for $x \in C$.

The function F gives the form of the difference equation that interests us, and we can write $x_n = F(x_{n-1}, \ldots, x_{n-m})$. However, we study F via its m–dimensional dynamical system counterpart, namely,

$$f : \mathbf{R}^m \to \mathbf{R}^m,$$

is defined by

$$f(u) = f \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_m \end{pmatrix} = \begin{pmatrix} F(u) \\ u_1 \\ \vdots \\ u_{m-1} \end{pmatrix}.$$

In our earlier paper [**3**], we considered low-dimensional Hénon-like maps (where the tools of algebraic topology are not needed). Two examples from that paper follow, and provide easy-to-understand examples of what we wish to achieve in higher dimensions:

Example 4. ($k = 1$ example in the plane.) Let $C = [-1, 1] \times [-1, 1]$, and $-1 < a < b < c < d < 1$. Let $C_1 = [a, b] \times [-1, 1]$, $C_2 = [c, d] \times [-1, 1]$. If F is a Hénon-like difference equation on $\mathbf{R}^2$ with delay $k = 1$ (with associated $\epsilon > 0$ and 2–cubes C, C_1, and C_2), and f is the associated dynamical system on $\mathbf{R}^2$, then $\mathcal{S} = \{C_1, C_2\}$ is a collection of symbol sets for the associated map $f : \mathbf{R}^2 \to \mathbf{R}^2$, $\mathcal{E} = \{E : E$ is a path in C that intersects both $\{a\} \times [-1, 1]$ and $\{d\} \times [-1, 1]\}$ is a family of expanders for $\mathcal{S}$, and we can conclude that there is a closed, invariant, chaotic subset C^* of C such that $f \mid C^*$ factors over the shift on 2–symbols. Note that if $E \in \mathcal{E}$, E contains subpaths $E_1 \subset C_1$ and $E_2 \subset C_2$ such that $f(E_i) \in \mathcal{E}$. (In fact, "f stretches $C_i \cap E$ across $[a, d] \times [-1, 1]$ in the sense that $f(C_i \cap E)$ must contain a path that extends from $\{a\} \times [-1, 1]$ to $\{d\} \times [-1, 1]$ in C".) See Figure 4.

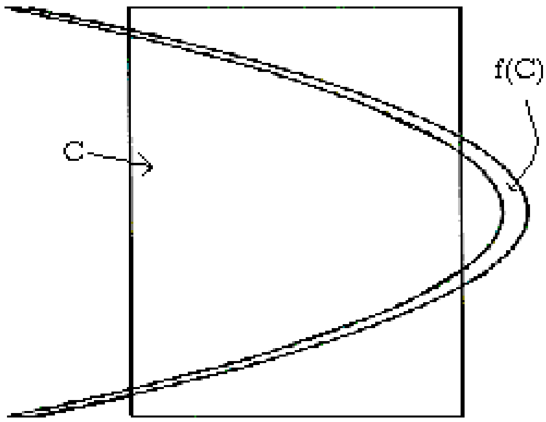

FIGURE 4. The set C and its image $f(C)$ as they might look.

EXAMPLE 5. ($k = 2$ EXAMPLE IN THE PLANE) Let $C = [-1, 1] \times [-1, 1]$, and $-1 < a < b < c < d < 1$. Let $C_1 = [a, d] \times [a, b]$, $C_2 = [a, d] \times [c, d]$. If F is a Hénon-like difference equation on $\mathbf{R}^2$ with delay $k = 2$ (with associated $\epsilon > 0$ and 2–cubes C, C_1, and C_2), and f is the associated dynamical system on R^2, then $\mathcal{S} = \{C_1, C_2\}$ is a collection of symbol sets for the associated map $f : \mathbf{R}^2 \to \mathbf{R}^2$, $\mathcal{E} = \{C_1 \cup C_2\}$ is a family of expanders for $\mathcal{S}$, and we can conclude that there is a closed, invariant, chaotic subset C^* of C such that $f \mid C^*$ factors over the shift on 2–symbols. Note that for $i = 1, 2$, $f(C_i) \supset C_1 \cup C_2$. (This time,"$f$ stretches C_i completely across $[a, d] \times [a, d]$" in the sense that it covers the entire rim of $[a, d] \times [a, d]$.)

In these two examples, each set Z_i (from the introduction) is C, and C_1 and C_2 *both* correspond to X_i. It is this idea of f "stretching" each of two smaller cubes C_1 and C_2 "across" the larger containing cube C that we must make precise.

3. Preliminary results

Recall that we have assumed the following:

(1) $(X_i), (Y_i)$ are sequences of compact sets in $\mathbf{R}^m$; $B_i := X_i \cap Y_i$, $Z_i := X_i \cup Y_i$; Z_i and X_i are rectangles (products of intervals).

(2) For some $k \leq n$, B_i is homeomorphic to $S^{k-1} \times R_i$ (where R_i is a rectangle) and is the union of some or all of the faces of X_i.

(3) $f(B_i) \subset Y_{i+1}$, $f(X_i) \subset Z_{i+1}$.

We now assume a stronger version of (3):

(3*) f is a continuous map from $\mathbf{R}^m$ to $\mathbf{R}^m$, $f \mid X_i : (X_i, B_i) \to (Z_{i+1}, Y_{i+1})$ induces an isomorphism from $H^k(Z_{i+1}, Y_{i+1})$ onto $H^k(X_i, B_i)$, and for some $\epsilon > 0$, $f \mid (\overline{D_\epsilon(X_i)} \cap Z_i)$ maps $(Z_i \cap \overline{D_\epsilon(X_i)}, Y_i \cap \overline{D_\epsilon(X_i)})$ into (Z_{i+1}, Y_{i+1}).

Definition. We say E *k–crosses* the pair (Z, Y) if E is compact, $E \subset Z$, and the inclusion map $(E, E \cap Y) \to (Z, Y)$ induces an isomorphism from $H^k(Z, Y)$ onto $H^k(E, E \cap Y)$.

Lemma 6. *Assume E k–crosses (Z_i, Y_i) and assume (1),(2), and (3*). Then there is a compact set $\widehat{E} \subset E$ such that $f \mid \widehat{E}$ induces an isomorphism from $H^k(Z_{i+1}, Y_{i+1})$ onto $H^k(\hat{E}, \hat{E} \cap Y_i)$.*

Proof. There is $\epsilon > 0$ such that $f \mid (\overline{D_\epsilon(X_i)} \cap Z_i)$ maps $(Z_i \cap \overline{D_\epsilon(X_i)}, Y_i \cap \overline{D_\epsilon(X_i)})$ into (Z_{i+1}, Y_{i+1}). Let $U = Z_i \backslash \overline{D_\epsilon(X_i)}$. Then $\overline{U} \subset Y_i$, and, by excision, the inclusion $i_1 : (E \backslash U, (E \cap Y_i) \backslash U) \to (E, (E \cap Y_i))$ induces an isomorphism $i_1^* : H^k(E, (E \cap Y_i)) \to H^k(E \backslash U, (E \cap Y_i) \backslash U)$. Likewise, if i_2 denotes the inclusion from $(Z_i \backslash U, Y_i \backslash U)$ into (Z_i, Y_i), i_2^* is an isomorphism. By assumption, the inclusion $i_4 : (E, (E \cap Y_i)) \to (Z_i, Y_i)$, induces an isomorphism $i_4^* : H^k(Z_i, Y_i) \to H^k(E, E \cap Y_i)$. Let i_3 denote the inclusion from $(E \backslash U, (E \cap Y_i) \backslash U)$ into $(Z_i \backslash U, Y_i \backslash U)$. Since $i_4 \circ i_1 = i_2 \circ i_3$, and $(i_4 \circ i_1)^*$ and i_2^* are isomorphisms, so is ${i_3}^*$. Thus, i_3^* is an isomorphism from $H^k(Z_i \backslash U, Y_i \backslash U)$ onto $H^k(E \backslash U, (E \cap Y_i) \backslash U)$.

Suppose i_5 denotes the inclusion map from (X_i, B_i) into $(Z_i \backslash U, Y_i \backslash U)$. Since $(Z_i \backslash U, Y_i \backslash U)$ is an expansion of (X_i, B_i), $i_5^* : H^k(Z_i \backslash U, Y_i \backslash U) \to H^k(X_i, B_i)$ is an isomorphism. Furthermore, $f \mid X_i = (f \mid (Z_i \backslash U)) \circ i_5$, and since $(f \mid X_i)^*$ and i_5^* are isomorphisms, so is $(f \mid (Z_i \backslash U))^*$. Then $(f \mid (Z_i \backslash U)) \circ i_3 : (E \backslash U, (E \cap Y_i) \backslash U) \to (Z_{i+1}, Y_{i+1})$ induces an isomorphism, and $(f \mid (Z_i \backslash U)) \circ i_3 = f \mid (E \backslash U)$. Let $\hat{E} = E \backslash U$. Then $(f \mid \hat{E})^* : H^k(Z_{i+1}, Y_{i+1}) \to H^k(\hat{E}, \hat{E} \cap Y_i)$ is an isomorphism. □

Lemma 7. *Suppose the pairs (Z, Y) and (X, B) satisfy conditions (1) and (2). Then if E k–crosses (Z, Y), $E \cap X$ k–crosses (X, B).*

PROOF. Note that $X \cup Y = Z$ and $X \cap Y = B$. For each positive integer n, let $U_n = Z \backslash \overline{D_{1/n}(X)}$. Then U_n is open in Z, and $\overline{U_n} \subset Int_Z Y$ and $\overline{U_n} \cap X = \varnothing$. By assumption, the inclusion $j : (E, E \cap Y) \to (Z, Y)$ induces an isomorphism $j^* : H^k(Z, Y) \to H^k(E, E \cap Y)$. By excision, for each n, if $j_n : (E \backslash U_n, (E \cap Y) \backslash U_n) \to (Z \backslash U_n, Y \backslash U_n)$ denotes the inclusion, j_n induces an isomorphism $j_n^* : H^k(Z \backslash U_n, Y \backslash U_n) \to H^k(E \backslash U_n, (E \cap Y) \backslash U_n)$. Then applying the weak continuity property to the associated intersection of pairs and associated direct limit of cohomology groups, it follows that if $j_E : (E \cap X, E \cap B) \to (X, B)$ is the inclusion, $j_E^* : H^k(X, B) \to H^k(E \cap X, E \cap B)$ is an isomorphism. Hence, $E \cap X$ k–crosses (X, B). □

LEMMA 8. *Suppose (Z, Y) and (Z, Y') are pairs that satisfy conditions (1) and (2), and $Y' \supset Y$. Then the inclusion $i : (Z, Y) \to (Z, Y')$ induces an isomorphism $i^* : H^k(Z, Y') \to H^k(Z, Y)$.*

PROOF. There is a map $\beta : (Z, Y') \to (Z, Y)$ such that $\beta \circ i$ is homotopic to the identity on (Z, Y), and if $\Lambda = \overline{Z \backslash Y'}$, $\beta | \Lambda$ is one to one. By the strong excision property, β^* is an isomorphism. Since $\beta \circ i$ is homotopic to $id_{(Z,Y)}$, $(\beta \circ i)^* = i^* \circ \beta^*$ is the identity isomorphism. Then i^* is an isomorphism. □

LEMMA 9. *Suppose (Z, Y) and (Z, Y') are pairs that satisfy conditions (1) and (2), and $Y' \supset Y$. Let $X = \overline{Z \backslash Y}$, $X' = \overline{Z \backslash Y'}, B = Z \cap Y$ and $B' = Z \cap Y'$. Suppose E k–crosses (Z, Y). Then E k–crosses (Z, Y') and $E \cap X'$ k–crosses (X', B').*

PROOF. Let $\Lambda = \overline{Z \backslash Y}, \Lambda' = \Lambda \backslash (Z \backslash Y), \Gamma = \overline{Z \backslash Y'}, \Gamma' = \Gamma \backslash (Z \backslash Y')$. Since Λ and Γ are m-dimensional rectangles, there is a homeomorphism $\beta : (\Lambda, \Lambda') \to (\Gamma, \Gamma')$. Also, $E \cap \Lambda$ k–crosses (Λ, Λ').

Let $i : (Z, Y) \to (Z, Y'), i_1 : (E, E \cap Y) \to (Z, Y)$, $i_2 : (E, E \cap Y) \to (E, E \cap Y')$, and $i_3 : (E, E \cap Y) \to (E, E \cap Y')$ denote the respective inclusions. Then i_1 and i induce isomorphisms, and $i \circ i_1 = i_2 \circ i_3$. Thus, $(i_2 \circ i_3)^* = i_3^* \circ i_2^*$ is an isomorphism.

Let $j : (E \cap \Gamma, E \cap \Gamma') \to (\Gamma, \Gamma')$, and $j_1 : (\beta^{-1}(E \cap \Gamma), \beta^{-1}(E \cap \Gamma')) \to (\Lambda, \Lambda')$ denote the inclusions. Note that $\beta | \beta^{-1}(E \cap \Gamma)$ is a homeomorphism from $(\beta^{-1}(E \cap \Gamma), \beta^{-1}(E \cap \Gamma'))$ to $(E \cap \Gamma, E \cap \Gamma')$. Furthermore, $\beta \circ j_1 = j \circ \beta | \beta^{-1}(E \cap \Gamma)$, and $\beta | \beta^{-1}(E \cap \Gamma), \beta$, and j induce isomorphisms. Then j_1 also induces an isomorphism. Then $H^k(\beta^{-1}(E \cap \Gamma), \beta^{-1}(E \cap \Gamma'))$ and $H^k(E \cap \Gamma, E \cap \Gamma')$ are isomorphic to the integers. Then $H^k(E, E \cap Y')$ is also isomorphic to the integers, and it follows that i_3^* is an epimorphism from $H^k(Z, Y')$ to $H^k(E, E \cap Y')$, with both groups being isomorphic to the integers. Thus, i_2^* is an isomorphism, and E k–crosses (Z, Y'). That $E \cap X'$ k–crosses (X', B') follows from Lemma 4. □

4. The Difference Equation as an m–dimensional map

We can rewrite our difference equation in a slightly different form corresponding more closely to the properties we use.

The difference equation F: Suppose $\epsilon > 0$, $0 < a_{11} < a_{12} < a_{21} < a_{22} < 1$, and k is an integer with $1 \leq k \leq m$. Let $J = [0,1]$, $I_1 = [a_{11}, a_{12}]$, and $I_2 = [a_{21}, a_{22}]$. Suppose $F : \mathbf{R}^m \to \mathbf{R}$, $\Phi : \mathbf{R}^m \to \mathbf{R}$, and $\Psi : \mathbf{R} \to \mathbf{R}$ such that

(1) for $i = 1, 2$, $\Psi(I_i) = [\epsilon, 1-\epsilon]$ with $\Psi(a_{11}) = \Psi(a_{22}) = \epsilon$ and $\Psi(a_{21}) = \Psi(a_{12}) = 1 - \epsilon$,

(2) $|\Phi(x)| < \epsilon$ for $x \in J^m$,

(3) $\min\{a_{11}, 1 - a_{22}\} > 2\epsilon$, and

(4) $F(x) = \Psi(x_k) + \Phi(x)$ for $x \in J^m$.

The function F gives the Hénon-like difference equation we are interested in, and we can write $x_n = F(x_{n-1}, \ldots, x_{n-m})$. However, we study F via its m–dimensional dynamical system counterpart, namely,

$$f : \mathbf{R}^m \to \mathbf{R}^m,$$

is defined by

$$f(u) = f\begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_m \end{pmatrix} = \begin{pmatrix} F(u) \\ u_1 \\ \vdots \\ u_{m-1} \end{pmatrix}$$

for $u \in \mathbf{R}^m$. (We write the points of $\mathbf{R}^m$ as m-dimensional column vectors for convenience.) We use several simplifications of f in order to prove that it has the properties we claim, and for those we need to define several new maps:

(1) Define $h : \mathbf{R}^m \to \mathbf{R}^m$ by

$$\begin{pmatrix} u_1 \\ \vdots \\ u_{k-1} \\ u_k \\ u_{k+1} \\ \vdots \\ u_m \end{pmatrix} \overset{h}{\to} \begin{pmatrix} u_k \\ u_1 \\ \vdots \\ u_{k-1} \\ u_{k+1} \\ \vdots \\ u_m \end{pmatrix}.$$

(2) Define $g : \mathbf{R}^m \to \mathbf{R}^m$ by

$$\begin{pmatrix} u_1 \\ \vdots \\ u_{k-1} \\ u_k \\ u_{k+1} \\ \vdots \\ u_m \end{pmatrix} \xrightarrow{g} \begin{pmatrix} u_1 \\ \vdots \\ u_{k-1} \\ F(u) \\ u_k \\ \vdots \\ u_{m-1} \end{pmatrix}.$$

(3) Define $T : \mathbf{R} \to \mathbf{R}$ by

$$T(x) = \begin{cases} 0 & \text{if } x \le a_{11} \\ \frac{x-a_{11}}{a_{12}-a_{11}} & \text{if } a_{11} \le x \le a_{12} \\ 1 & \text{if } a_{12} \le x \le a_{21} \\ \frac{a_{22}-x}{a_{22}-a_{21}} & \text{if } a_{21} \le x \le a_{22} \\ 0 & \text{if } x \ge a_{22} \end{cases}.$$

(See Figure 5.)

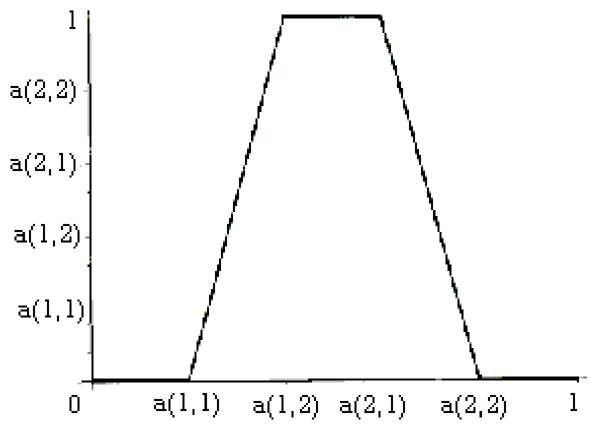

FIGURE 5. Graph of the map T.

(4) Define $g_0 : \mathbf{R}^m \to \mathbf{R}^m$ by

$$g_0(u) = g_0 \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_m \end{pmatrix} = \begin{pmatrix} u_1 \\ \vdots \\ u_{k-1} \\ F(u) \\ u_{k+1} \\ \vdots \\ u_m \end{pmatrix}.$$

(5) Note that $a_{11} > F(u) > 0$ if $u_k = a_{11}$ or $u_k = a_{22}$, and $a_{22} < F(u) < 1$ if $u_k = a_{12}$ or $u_k = a_{21}$. Define $g_{aff} : \mathbf{R}^m \to \mathbf{R}^m$ by

$$g_{aff}(u) = g_{aff} \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_m \end{pmatrix} = \begin{pmatrix} u_1 \\ \vdots \\ u_{k-1} \\ T(u_k) \\ u_{k+1} \\ \vdots \\ u_m \end{pmatrix}.$$

We are interested in the behavior of f on J^m only, so from now on, we consider only the behavior of f, g, h, g_0, and g_{aff} restricted to J^m. In section 2 we discussed how (X_i, B_i) is cohomologically identical to (Z_i, Y_i), the latter being an "expanded" version of the former. We need to consider a number of such pairs.

For $i, j = 1, 2$, define

(a) $\tilde{I}_i = J^{k-1} \times I_i \times J^{m-k}$,
(b) $\hat{I}_i = J^{k-2} \times I_i \times J^{m-k+1}$,
(c) $\tilde{I}_{i,j} = J^{k-2} \times I_j \times I_i \times J^{m-k}$,
(d) $R_{i,j} = \tilde{I}_i \backslash ((a_{11}, a_{22})^{k-2} \times (a_{j1}, a_{j2}) \times (a_{i1}, a_{i2}) \times J^{m-k})$,
(e) $\hat{R}_{i,j} = J^m \backslash ((a_{11}, a_{22})^{k-2} \times (a_{j1}, a_{j2}) \times (a_{i1}, a_{i2}) \times J^{m-k})$,
(f) $K_i = J^m \backslash ((a_{11}, a_{22})^{k-2} \times (a_{i1}, a_{i2}) \times (a_{11}, a_{22}) \times J^{m-k})$,
(g) $\tilde{K}_i = J^m \backslash ((a_{11}, a_{22})^{k-2} \times (a_{i1}, a_{i2}) \times (0, 1) \times J^{m-k})$,
(i) $L_i = J^m \backslash ((a_{11}, a_{22})^{k-1} \times (a_{i1}, a_{i2}) \times J^{m-k})$,
(j) $\hat{P}_i = (J^m, L_i)$,
(k) $P_i = (J^{k-1} \times I_i, \partial(J^{k-1} \times I_i)) \times J^{m-k}$,

(l) $Q_i = (J^m, K_i)$,

(m) $O_{i,j} = (\tilde{I}_i, R_{i,j})$, and

(n) $\hat{P}_{i,j} = (J^m, \hat{R}_{i,j})$.

In our paper [**3**] we considered the case $m = k$ and in effect showed that both $\tilde{I}_1$ and $\tilde{I}_2$ are subsets of each $f(\tilde{I}_i)$ for both $i = 1$ and 2. That is a special case of $\tilde{I}_1$ and $\tilde{I}_2$ being mapped by f across both $\tilde{I}_1$ and $\tilde{I}_2$. When $m > k$, a more general example would be to picture intuitively the image of $\tilde{I}_i$ as a k-dimensional surface with boundary, with the surface stretching across the boundary $\partial(J^{k-1} \times I_i) \times J^{m-k}$.

LEMMA 10. *For each* $i, j = 1, 2$, $(f \mid O_{ij})^*$ *maps* $H^k(\hat{P}_j)$ *isomorphically onto* $H^k(O_{ij})$.

PROOF. The proof requires a couple of steps. Fix $i \in \{1, 2\}$ and $j \in \{1, 2\}$. Note that $g_{aff}(\tilde{I}_i) = J^m$, $g_{aff}(\tilde{I}_{i,j}) = \hat{I}_j$, $g_{aff}(R_{i,j}) = \tilde{K}_j \subset K_j$, so $g_{aff} \mid O_{ij} : O_{ij} \to Q_j$. Likewise, $g_0 \mid O_{ij} : O_{ij} \to Q_j$ and $g \mid O_{ij} : O_{ij} \to Q_j$.

Since $g_{aff} \mid O_{i,j}$ can be viewed as both a map from $O_{i,j}$ to Q_j and as a map from $O_{i,j}$ to $(J^m, \tilde{K}_j)$, and we need to distinguish between these two, denote $g_{aff} \mid O_{i,j} : O_{i,j} \to (J^m, \tilde{K}_j)$ as $\tilde{g}_{aff}$, while continuing to denote $g_{aff} \mid O_{i,j} : O_{i,j} \to Q_j$ as $g_{aff} \mid O_{i,j}$. The map $\tilde{g}_{aff} : O_{ij} \to (J^m, \tilde{K}_j)$ is a homeomorphism, so ${\tilde{g}_{aff}}^* : H^k(J^m, \tilde{K}_j) \to H^k(O_{ij})$ is an isomorphism. If $i : (J^m, \tilde{K}_j) \to Q_j$ denotes the inclusion map, $i^* : H^k(Q_j) \to H^k(J^m, \tilde{K}_j)$ is an isomorphism by Lemma 5.

The map $g_0 \mid O_{ij}$ is homotopic to $g_{aff} \mid O_{ij}$: Define $H : O_{ij} \times [0, 1] \to Q_j$ by

$$(H(x,t))_k = t(g_0(x))_k + (1-t)(g_{aff}(x))_k,$$
$$(H(x,t))_l = x_l = (g_{aff}(x))_l = (g_0(x))_l \text{ for } k \neq l.$$

Thus, $(g_0 \mid O_{ij})^* : H^k(Q_j) \to H^k(O_{ij})$ is equal to $(g_{aff} \mid O_{ij})^* : H^k(Q_j) \to H^k(O_{ij})$.

The map $g_0 \mid O_{ij}$ is homotopic to the map $g \mid O_{ij}$: First define $\theta : O_{ij} \times [0, 1] \to O_{ij}$ by

$$\begin{aligned} \theta(x,t)_l &= x_l \text{ for } l < k, \\ \theta(x,t)_k &= x_k \text{ for } l = k, \\ \theta(x,t)_l &= t(x_{l-1}) + (1-t)(x_l) \text{ for } l > k. \end{aligned}$$

Define $\Upsilon : O_{ij} \times [0, 1] \to Q_j$ by

$$(\Upsilon(x,t))_l = x_l = (g(x))_l = (g_0(x))_l \text{ for } l < k,$$
$$(\Upsilon(x,t))_k = F(\theta_t(x)) \text{ for } l = k,$$
$$(\Upsilon(x,t))_l = t(x_{l-1}) + (1-t)(x_l) \text{ for } l > k.$$

Thus, $(g_0 \mid O_{ij})^* : H^k(Q_j) \to H^k(O_{ij})$ is equal to $(g \mid O_{ij})^* : H^k(Q_j) \to H^k(O_{ij})$.

We write f as a composition of maps: Note that $f = h \circ g$. The map h permutes the first k arguments, and is therefore a homeomorphism from Q_j to $\hat{P}_j$. Since h is a homeomorphism, h^* is an isomorphism. Hence to show f^* is an isomorphism, we need only show that g^* is an isomorphism. But, by the preceding arugument, it is. □

THEOREM 11. *Let $i, j \in \{1, 2\}$. Let E denote a compact set in $\mathbf{R}^m$ that k–crosses $\hat{P}_i$. Then E contains a closed subset $\hat{E}$ such that $f(\hat{E})$ k–crosses $\hat{P}_j$.*

PROOF. Note that, by Lemma 6, E k–crosses $\hat{P}_i$ means that $E \cap \tilde{I}_i$ k–crosses $O_{i,j}$. A consequence of Lemma 3 is that there is a compact set $\widehat{E} = E \cap \overline{D_\epsilon(\tilde{I}_{i,j})} \cap \tilde{I}_i \subset E \cap \tilde{I}_i$ such that $f \mid \widehat{E}$ induces an isomorphism from $H^k(\hat{P}_j)$ onto $H^k(\hat{E}, \hat{E} \cap \hat{R}_{i,j})$. We use the notation of the previous lemma.

Since g_{aff} is homotopic to g_0, and g_0 is homotopic to g, $g^*_{aff} = g^*_0 = g^*$. Further, $g_{aff} = i \circ \tilde{g}_{aff}$, where $\tilde{g}_{aff}$ is a homeomorphism from $(\tilde{I}_i, R_{ij})$ onto $(J^m, \tilde{K}_j)$ and $i : (J^m, \tilde{K}_j) \to (J^m, K_j)$ is the inclusion. Likewise, $g_0 = j \circ \tilde{g}_0$, where $\tilde{g}_0 : (\tilde{I}_i, R_{ij}) \to (g_0(\tilde{I}_i), g_0(R_{ij}))$ and $j : (g_0(\tilde{I}_i), g_0(R_{ij})) \to (J^m, K_j)$ is the inclusion, and $g = j_1 \circ \tilde{g}$, where $\tilde{g} : (\tilde{I}_i, R_{ij}) \to (g(\tilde{I}_i), g(R_{ij}))$ and $j_1 : (g(\tilde{I}_i), g(R_{ij})) \to (J^m, K_j)$ is the inclusion.

From the previous lemma, $g^*_{aff} = g^*_0 = g^*$ is an isomorphism, $\tilde{g}^*_{aff}$ is an isomorphism, and i^* is an isomorphism. Then j^* and j^*_1 must be epimorphisms. Note that $J^m \backslash \tilde{K}_j$ is homeomorphic to $g_0(\tilde{I}_i) \backslash g_0(R_{ij})$ (only the kth coordinate of any point is changed, and it is not changed much and is changed continuously). Then there is an isomorphism from $H^k(J^m, \tilde{K}_j)$ onto $H^k(g_0(\tilde{I}_i), g_0(R_{ij}))$, and $H^k(g_0(\tilde{I}_i), g_0(R_{ij}))$ must be isomorphic to the integers, as are $H^k(J^m, \tilde{K}_j)$, $H^k(J^m, K_j)$, and $H^k(\tilde{I}_i, R_{ij})$. Then $j^* : H^k(J^m, K_j) \to H^k(g_0(\tilde{I}_i), g_0(R_{ij}))$ is an epimorphism from between groups isomorphic to the integers, so j^* is an isomorphism. Thus, $(g_0(\tilde{I}_i), g_0(R_{ij}))$ k–crosses (J^m, K_j).

Suppose

$$w = \begin{pmatrix} w_{k+2} \\ \vdots \\ w_m \end{pmatrix} \in J^{m-k-1}.$$

Let

$$C = \left\{ \begin{pmatrix} x_1 \\ \vdots \\ F(\theta_1(x)) \\ x_k \\ w_{k+2} \\ \vdots \\ w_m \end{pmatrix} : g_0(x) \in g_0(\tilde{I}_i) \right\}$$

and

$$D = \left\{ \begin{pmatrix} x_1 \\ \vdots \\ F(\theta_1(x)) \\ x_k \\ w_{k+2} \\ \vdots \\ w_m \end{pmatrix} : g_0(x) \in g_0(R_{ij}) \right\}.$$

Then (C, D) is a deformation retract of $(g_0(\tilde{I}_i), g_0(R_{ij}))$, so $H^k(C, D) = H^k(g_0(\tilde{I}_i), g_0(R_{ij}))$. Likewise, let

$$C' = \left\{ \begin{pmatrix} x_1 \\ \vdots \\ F(\theta_1(x)) \\ x_k \\ w_{k+2} \\ \vdots \\ w_m \end{pmatrix} : g(x) \in g(\tilde{I}_i) \right\}$$

and

$$D' = \left\{ \begin{pmatrix} x_1 \\ \vdots \\ F(\theta_1(x)) \\ x_k \\ w_{k+2} \\ \vdots \\ w_m \end{pmatrix} : g(x) \in g(R_{ij}) \right\}.$$

Then (C', D') is a deformation retract of $(g(\tilde{I}_i), g(R_{ij}))$, so $H^k(C', D') = H^k(g(\tilde{I}_i), g(R_{ij}))$. Furthermore, $(C, D) = (C', D')$. Then $H^k(g(\tilde{I}_i), g(R_{ij})) =$

$H^k(g_0(\tilde{I}_i), g_0(R_{ij}))$, so $H^k(g(\tilde{I}_i), g(R_{ij}))$ is isomorphic to $\mathbf{Z}$. Then j_1 induces an isomorphism, and $g(\tilde{I}_i)$ k–crosses (J^m, K_j). Then, since h just permutes factors and is a homeomorphism, and $h \circ g = f$, $f(\tilde{I}_i)$ k–crosses $(J^m, L_j) = \hat{P}_j$.

Now suppose E k–crosses $\hat{P}_i$. Then $\hat{E} = E \cap \tilde{I}_i$ k–crosses $(\tilde{I}_i, R_{ij})$. Thus, if $i_3 : (\hat{E} \cap \tilde{I}_i, \hat{E} \cap R_{ij}) \to (\tilde{I}_i, R_{ij})$ is the inclusion, i_3^* is an isomorphism. Since $f : (\tilde{I}_i, R_{ij}) \to (J^m, L_j)$ induces an isomorphism, $\alpha := f \circ i_3$ induces an isomorphism from $H^k(J^m, L_j) = H^k(\hat{P}_j)$ onto $H^k(\hat{E} \cap \tilde{I}_i, \hat{E} \cap R_{ij})$. Furthermore, each of $H^k(\hat{E} \cap \tilde{I}_i, \hat{E} \cap R_{ij})$ and $H^k(J^m, L_j)$ is isomorphic to $\mathbf{Z}$. We can regard α as both a map from $(\hat{E} \cap \tilde{I}_i, \hat{E} \cap R_{ij})$ into $\hat{P}_j$ and as a map from $(\hat{E} \cap \tilde{I}_i, \hat{E} \cap R_{ij})$ onto $(f(\hat{E} \cap \tilde{I}_i), f(\hat{E} \cap R_{ij}))$, so to distinguish, we call the latter $\tilde{\alpha}$.

We need to backtrack:

(a) $\tilde{g}_{aff} \circ i_3$ can be regarded as a homeomorphism from $(\hat{E} \cap \tilde{I}_i, \hat{E} \cap R_{ij})$ onto $(g_{aff}(\hat{E} \cap \tilde{I}_i), g_{aff}(\hat{E} \cap R_{ij}))$, and it follows that $H^k(g_{aff}(\hat{E} \cap \tilde{I}_i), g_{aff}(\hat{E} \cap R_{ij}))$ is isomorphic to $\mathbf{Z}$.

(b) Let $\Lambda = \overline{J^m \backslash R_{ij}}$, $\Lambda' = \Lambda \backslash R_{ij}$. Then $E \cap \Lambda = \hat{E} \cap \Lambda$ k–crosses (Λ, Λ'). Since g_0, j, and $\tilde{g}_0$ induce isomorphisms, so does

$$\tilde{g}_0 | (E \cap \Lambda) : (E \cap \Lambda, E \cap \Lambda') \to (g_0(\tilde{I}_i), g_0(R_{ij}).$$

Let $\Gamma = \overline{J^m \backslash \tilde{K}_j}, \Gamma' = \Gamma \backslash \tilde{K}_j$, $\Omega = \overline{g_0(\Lambda) \backslash g_0(\Lambda')}, \Omega' = \Omega \backslash (g_0(\Lambda) \backslash g_0(\Lambda'))$. Since $\overline{J^m \backslash \tilde{K}_j}$ is homeomorphic to $\Omega = \overline{g_0(\tilde{I}_i) \backslash g_0(R_{ij})}$, there is a homeomorphism $\lambda : (\Omega, \Omega') \to (\Gamma, \Gamma')$. Let $\Delta = g_{aff}^{-1} \circ \lambda \circ g_0(E \cap \Lambda), \Delta' = g_{aff}^{-1} \circ \lambda \circ g_0(E \cap \Lambda')$. Note that $\Delta \subset \Lambda$ and $\Delta' \subset \Lambda'$. Define $\gamma : (E \cap \Lambda, E \cap \Lambda') \to (\Delta, \Delta')$ by $\gamma(x) = g_{aff}^{-1} \circ \lambda \circ g_0(x)$. Then γ is continuous and onto, and $\gamma = \tilde{g}_{aff}^{-1} \circ \lambda \circ (\tilde{g}_0 | (E \cap \Lambda))$. Since each of $\tilde{g}_{aff}^{-1}$, λ, and $\tilde{g}_0 | (E \cap \Lambda)$ induces an isomorphism, so does γ. It also follows that $H^k(g_0(E \cap \Lambda), g_0(E \cap \Lambda'))$ and $H^k(g_0(\hat{E} \cap \tilde{I}_i), g_0(\hat{E} \cap R_{ij}))$ are isomorphic to $\mathbf{Z}$. Then j_3^* is an isomorphism, with $j_3 : (g_0(\hat{E} \cap \tilde{I}_i), g_0(\hat{E} \cap R_{ij})) \to (J^m, K_j)$, and $(g_0(\hat{E} \cap \tilde{I}_i), g_0(\hat{E} \cap R_{ij}))$ k–crosses (J^m, K_j).

(c) That $(g(\hat{E} \cap \tilde{I}_i), g(\hat{E} \cap R_{ij}))$ k–crosses (J^m, K_j) follows from the argument that $(g(\tilde{I}_i), g(R_{ij}))$ k–crosses (J^m, K_j).

(d) Finally, since h is a homeomorphism, $(f(\tilde{I}_i), f(R_{ij}))$ k-crosses (J^m, K_j). □

5. Conclusion

We are ready then for our main conclusion. We use the notation of the previous section. Combining the results of the previous section with the Chaos Lemma, we have the following:

Theorem 12. *Let* $\mathcal{S} = \{\tilde{I}_1, \tilde{I}_2\}$ *be our collection of symbol sets. Let* $\mathcal{E} = \{E \subset J^m : E$ *is closed and* E *contains closed subsets* E_i *such that* E_i k*–crosses* $\hat{P}_i$ *for* $i = 1, 2\}$ *denote the associated collection of expanders. Then the map* f *is chaotic on a closed, invariant subset* Q^* *of* J^m *such that for every two-sided itinerary* $\mathbb{S} = (S_{i_n})_{n=-\infty}^{\infty}$ *of members of* $\mathcal{S}$*, there is a two-sided trajectory in* Q^* *that follows it. Furthermore, (1)* f *is sensitive to intial data on* Q^**, and (2) there is a continuous map* $\phi : Q^* \to \sum_2$ *such that* $\phi \circ (f \mid Q^*) = \sigma \circ \phi$*, i.e., the dynamics of* f *on* Q^* *factors over the dynamics of the shift on* 2 *symbols.*

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