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HOMOTOPICAL DYNAMICS

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Abstract. In this paper we give a review of some recent results of study the dynamics of a map f with use of topological invariants. We restrict our consideration to these invariants which are determined by the homotopy class of f. Our main object of interest is the set of homotopy minimal periods of f, i.e., the set of natural numbers which are minimal periods for all maps which are homotopic to f. We also show that the same tools are useful in the study of minimal periods of a map of the sphere which commutes with a free homeomorphism and in establishing that the logarithm of spectral radius of a map of compact nilmanifold is a lower bound for the topological entropy.

1. Homotopical Dynamics. Let $f : X \to X$ be a self-map of a compact connected polyhedron X.

Definition 1.1. We say that an invariant of f describes the dynamics of f if it depends on $\{f^n\}_{n\in\mathbb{N}}$.

Example 1.2. Below we include a list of invariants describing the dynamics.

Let $P^m(f) = \mathrm{Fix}\,(f^m)$ be the set of points of period m, and $P_m(f) = P^m(f) \setminus \bigcup_{k<m} P^k(f)$ be the set of points for which m is the minimal period. Then the set of all periodic points $P(f) := \bigcup_m \mathrm{Fix}\,(f^m) = \bigcup_m P_m(f)$ is such an invariant.

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Also the set of all minimal periods, denoted by $\mathrm{Per}(f)$, the topological entropy, denoted by $\mathbf{h}(f)$ (see [**9**] for a definition of entropy), the spectral radius of the induced map $H_*(f;\mathbb{R})$ on the real cohomology spaces, denoted by $\mathrm{sp}\,(f)$ (cf. [**39**] and Section 4), are invariants describing the dynamics.

Also the asymptotical Lefschetz and Nielsen numbers, denoted by $L^\infty(f)$ and respectively $N^\infty(f)$ (cf. [**11, 19**]), give such exemplifications as well. Other invariants which describe the dynamics of f are the set of non-wandering points (cf. [**9**]), denoted by $\Omega(f)$, and discrete Conley index, denoted by $\mathrm{Con}(f)$ (cf. [**33**]).

Definition 1.3. We say that an invariant characterizes the homotopy dynamics of f if it has the same value for every g homotopic to f.

It follows from the definition that $\mathrm{sp}\,(f)$, $L^\infty(f)$, $N^\infty(f)$, and $\mathrm{Con}(f)$ are homotopy dynamics invariants. On the other hand, $P(f)$ and $\mathrm{Per}(f)$ are not homotopy invariants (cf. Examples 1.5, 4.4). Neither is $\mathbf{h}(f)$, because it is not continuous with respect to C^0–topology of $\mathrm{Map}(X,\, X)$ (cf. [**9**]).

Definition 1.4. Define the set of homotopy minimal periods as the set

$$\mathrm{HPer}(f) \;:=\; \bigcap_{g\simeq f} \mathrm{Per}(g)\,,$$

i.e., $m\in\mathbb{N}$ is a homotopy minimal period of f if it is the minimal period for every g homotopic to f.

By definition $\mathrm{HPer}(f)\subset\mathrm{Per}(f)$ and the inclusion is proper in general.

Example 1.5. Let us take $f=\mathrm{id}_{S^1}$. Of course $f\simeq g_\alpha$, where $g=g_\alpha$ is the rotation by α, a small irrational angle. Then $\mathrm{Per}(g_\alpha)=\emptyset$, and $\mathrm{HPer}(f)\underset{\neq}{\subset}\mathrm{Per}(g)$.

For a map of a smooth manifold any homotopy dynamics invariant provides an information about the rigid part of dynamics, because a small perturbation of a map f is homotopic to it. In particular, the following is true.

Remark 1.6. If X is a smooth manifold then $\mathrm{HPer}(f)=\mathrm{HPer}(h)$ for any small perturbation h of f.

For given $r\in\mathbb{Z}$, let $z\mapsto z^r$ be a map of S^1. We have $\mathrm{HPer}(f)=\emptyset$ for $r=1$; $\mathrm{HPer}(f)=\{1\}$ for $r=0$. For $r=-1$, let us consider the map f which is the composition of the map $z\mapsto\overline{z}$ and a homeomorphism $h: S^1\to S^1$ which has two fixed points $\{-1,\,1\}$ and pushes points of the upper hemisphere from $\{1\}$ towards $\{-1\}$, and points of the lower hemisphere from $\{-1\}$, towards $\{1\}$. Since h is homotopic to the identity, $\deg f=-1$. On the other hand, $P(f)=\{-1,1\}$, thus $\mathrm{Per}=\{1\}$, which shows that $\mathrm{HPer}(f)=\{1\}$, because $L(f)\neq 0$ for every map of $\deg(f)=-1$.

The set of homotopy minimal periods (under another name) was first studied for selfmaps of the circle $M = S^1$ by S. Efremova in [**7**] and L. Block, J. Guckenheimer, M. Misiurewicz, L. S. Young in [**4**]. They wanted to prove an analogue of the Šarkovskii theorem of [**37**] for the circle maps. As the result they got the following theorem.

THEOREM 1.7. *Let* $f : S^1 \to S^1$ *be a map of the circle and* $A_f = r \in \mathbb{Z} = \mathcal{M}_{1\times 1}(\mathbb{Z})$ *the degree of* f.

There are three types of the sets of minimal homotopy periods of f :

(E) $\mathrm{HPer}(f) = \emptyset$ *if and only if* $r = 1$.

(F) $\mathrm{HPer}(f) \neq \emptyset$ *and is finite if and only if* $r = -1$ *or* $r = 0$. *Then* $\mathrm{HPer}(f) = \{1\}$.

(G) $\mathrm{HPer}(f) = \mathbb{N}$ *for the remaining* r, *i.e.* $|r| > 1$, *with except one* *special case* $r = -2$ *when* $\mathrm{HPer}(f) = \mathbb{N} \setminus \{2\}$. □

Next L. Alsedá, S. Baldwin, J. Llibre, R. Swanson, and W. Szlenk examined the case $M = T^2$ in [**1**]. To give a description of the set of the homotopy minimal periods (which they called "the minimal set of periods") they used the Nielsen theory. Their main theorem, after a reformulation in our terms, may be stated as follows.

THEOREM 1.8. *Let* $f : T^2 \to T^2$ *be a map of the torus,* $A \in \mathcal{M}_{2\times 2}(\mathbb{Z})$ *the linearization of* f, *and* $\chi_A(t) = t^2 - at + b$ *be its characteristic polynomial.*

There are three types of the sets of minimal homotopy periods of f :

(E) $\mathrm{HPer}(f) = \emptyset$ *if and only if* $-a + b + 1 = 0$.

(F) $\mathrm{HPer}(f)$ *is nonempty and finite for 6 cases corresponding to one of the six pairs* (a, b) *listed below*

$$(0,0),\ (-1,0),\ (-2,1),\ (0,1),\ (-1,1),\ (1,1).$$

Then $\mathrm{HPer}(f) \subset \{1, 2, 3\}$. *Moreover, the sets* T_A *and* $\mathrm{HPer}(f)$ *are as follows:*

Cases of Type (F)

$(a,\ b)$	T_A	$\mathrm{HPer}(f)$
$(0,\ 0)$	$\mathbb{N}$	$\{1\}$
$(0,\ 1)$	$\mathbb{N} \setminus 4\mathbb{N}$	$\{1, 2\}$
$(-1,\ 0)$	$\mathbb{N} \setminus 2\mathbb{N}$	$\{1\}$
$(-1,\ 1)$	$\mathbb{N} \setminus 3\mathbb{N}$	$\{1\}$
$(-2,\ 1)$	$\mathbb{N} \setminus 2\mathbb{N}$	$\{1\}$
$(1,\ 1)$	$\mathbb{N} \setminus 6\mathbb{N}$	$\{1,2,3\}$

(G) $\mathrm{HPer}(f)$ *is infinite for the remaining* a, *and* b. *Furthermore,* $\mathrm{HPer}(f)$ *is equal to* $\mathbb{N}$ *for all pairs* $(a, b) \in \mathbb{Z}^2$ *with the exception of the following special cases listed below. We say that a pair* $(a, b) \in \mathbb{Z}^2$ *satisfies*

condition 1^0 *if* $a \neq 0$ *and* $a+b+1=0$,
condition 2^0 *if* $a+b=0$,
condition 3^0 *if* $a+b+2=0$ *respectively,*

and (a, b) *is not one of the pairs of case* (E) *and* (F).

The table below sets forth the special cases.

(a, b)	T_A	$\mathrm{HPer}(f)$
$(-2, 2)$	$\mathbb{N}$	$\mathbb{N} \setminus \{2, 3\}$
$(-1, 2)$	$\mathbb{N}$	$\mathbb{N} \setminus \{3\}$
$(0, 2)$	$\mathbb{N}$	$\mathbb{N} \setminus \{4\}$
(a, b) : (a, b) *satisfies* 1^0	$\mathbb{N} \setminus 2\mathbb{N}$	$\mathbb{N} \setminus 2\mathbb{N}$
(a, b) : (a, b) *satisfies* 2^0	$\mathbb{N}$	$\mathbb{N} \setminus \{2\}$
(a, b) : (a, b) *satisfies* 3^0	$\mathbb{N}$	$\mathbb{N} \setminus \{2\}$

□

A qualitative progress in methods had been made by B. Jiang and J. Llibre who gave a description of the set of homotopy minimal periods for the torus $M = T^d$, with any $d \in \mathbb{N}$ ([**20**]). To prove a general theorem (cf. Theorem 3.2), they applied a fine combinatorics argument and a deep algebraic number theory theorem they proved, but also used a topological result of You ([**43, 44**]) on the periodic points on tori. The mentioned number theory theorem is close to A. Schinzel's theorem on prime divisors (cf. [**38**]).

It was natural to ask whether this theorem can be extended onto larger classes of compact manifolds containing the tori, namely: compact nilmanifolds, compact completely slovable solvmanifolds, exponential solvmanifolds. This paper gives a survey of recent results that include: a general description of the set of homotopy minimal periods for the maps of a compact nilmanifold (Theorem 3.2) and compact completely solvable solvmanifold, their exemplification for dimension 3 (Theorems 3.10, 3.14) with a detailed list, specification for homeomorphisms (Theorems 3.12,3.15) and consequences in the form of Šarkovskii type theorems (Theorems 3.11, 3.16).

We begin with a background on the above classes of manifolds and present a construction of the so-called linearization A_f of a map $X \to X$ of such a manifold (Definition 2.6). It is an integral $d \times d$ matrix, d the dimension of X, and is essential in formulation of the main theorems, thus in the description of the set of homotopy minimal periods. It properties can be also used for a proof of the Shub conjecture on an estimate of the topological entropy of a continuous map of a compact nilmanifold (Theorem 4.5). Finally, we display that a fine modification of Nielsen theory of periodic points can be used to prove the existence of infinitely many minimal periods (thus infinitely many periodic points) for a continuous map of the sphere provided it commutes with a free homeomorphism of a finite order (Theorems 5.7, 5.5).

2. Nilmanifolds and linearization of a map.

DEFINITION 2.1. Let $\Gamma \subset G$ be discrete, co-compact subgroup of a connected Lie group G of dimension d. (A co-compact subgroup is called *uniform*). We say that G is nilpotent if the central tower $G_i := [G_{i+1}, G]$, $\Gamma_i := G_i \cap \Gamma$, :

$$G_0 = \mathbf{e} < G_1 < G_2 < \cdots < G_{k-1} < G_k = G, \quad \text{and}$$

is finite. Then finite is also corresponding tower for Γ, $G_i := G_i \cap \Gamma$

$$\Gamma_0 = \mathbf{e} < \Gamma_1 \equiv \mathbb{Z}^{s_1} < \Gamma_2 < \cdots < \Gamma_{k-1} < \Gamma_k = \Gamma\,,$$

and each Γ_i is uniform in G_i (cf. [**3, 24, 35**]).

DEFINITION 2.2. We say that a compact manifold X of dimension d is a nilmanifold if it is the quotient space G/Γ of a simple-connected nilpotent group G by a uniform discrete subgroup $\Gamma \subset G$ ([**3, 24, 35**]).

EXAMPLE 2.3. $T^d := \mathbb{R}^d/\mathbb{Z}^d \equiv (S^1)^d$ is the torus.
If $d \leq 2$ then it is the only example.

For a ring $\mathcal{R}$ with unity (e.g. $\mathcal{R} = \mathbb{R}$, $\mathcal{R} = \mathbb{C}$) let $\mathcal{N}_n(\mathcal{R})$ denote the group of all unipotent upper triangular matrices whose entries are elements of the ring $\mathcal{R}$, i.e.

$$\begin{bmatrix} 1 & r_{12} & \cdot & \cdots & \cdot & r_{1n} \\ 0 & 1 & r_{23} & \cdot & \cdots & r_{2n} \\ 0 & 0 & 1 & r_{34} & \cdots & r_{3n} \\ \cdot & \cdot & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdots & 1 & r_{n-1\,n} \\ 0 & 0 & \cdot & \cdots & 0 & 1 \end{bmatrix}.$$

— *Iwasawa manifolds*: $\mathcal{N}_n(\mathbb{R})/\mathcal{N}_n(\mathbb{Z})$ and $\mathcal{N}_n(\mathbb{C})/\mathcal{N}_n(\mathbb{Z}[\imath])$, where $\mathbb{Z}[\imath]$ is the ring of Gaussian integers are examples of nilmanifolds of dimension 3 not diffeomorphic to the torus. They are called Heisenberg manifolds. The Iwasawa 3-manifold $\mathcal{N}_3(\mathbb{R})/\mathcal{N}_3(\mathbb{Z})$, is called "Baby Nil"

It is known ([**3**]) that for $d = 3$ every nilmanifold is, up to diffeomorphism, of the form $\mathcal{N}_3(\mathbb{R})/\Gamma_{p,q,r}$, where the subgroup $\Gamma_{p,q,r}$, with fixed $p,\, q,\, r \in \mathbb{N}$ consists of all matrices of the form

$$\begin{bmatrix} 1 & \frac{k}{p} & \frac{m}{p\cdot q\cdot r} \\ 0 & 1 & \frac{l}{q} \\ 0 & 0 & 1 \end{bmatrix}, \quad \text{where} \quad k,\, l,\, m \in \mathbb{Z}.$$

Moreover, as a complete set of representatives of the diffeomorphims classes of such manifolds we can take the family $\{\mathcal{N}_3(\mathbb{R})/\Gamma_{1,1,r}\}$, $r \in \mathbb{N}$.

Fadell and Husseini in [**8**] showed that the category of compact nilmanifolds is equivalent to the category of manifolds of so called nilpotent class defined below by an inductive procedure.

Definition 2.4. Let $\mathcal{N}$ denote a class of compact connected manifolds satisfying the following two conditions:

N.1 $\mathcal{N}$ contains all tori (products of circles)

N.2 $\mathcal{N}$ contains also total spaces X of fibrations: Given any map $g : X \to X$, where $X \in \mathcal{N}$ is not a torus, there is a fiber map f

$$\begin{array}{ccc} T & \xrightarrow{f_1} & T \\ \downarrow & & \downarrow \\ X & \xrightarrow{f} & X \\ \downarrow & & \downarrow \\ B & \xrightarrow{\bar{f}} & B\,, \end{array}$$

where p is a principal T–fibration, T a torus, $B \in \mathcal{N}$ and $f \simeq g$. □

Note that the exact sequence of homotopy of $(F \subset X, p, B)$, $F = T^s$, $s < d$ yields that the group $\pi_1(X) = \Gamma$ is an extension of the form

$$\pi_1(T^s) = \mathbb{Z}^s \subset \pi_1(X) = \Gamma \to \pi_1(B) = \Gamma/\mathbb{Z}^s\,.$$

Moreover, the Fadell–Husseini fibration allows us to associate with f a $d \times d$ integral matrix $A = A_f$ by inductive procedure described below.

Definition 2.5. For $X = T^d$ we define A_f as the matrix of homomorphism induced by f on $\pi_1(T^d) \simeq H_1(T^d;\mathbb{Z}) \simeq H^1(T^d;\mathbb{Z}) \simeq \mathbb{Z}^d$. Note that this matrix defines a linear map of $\mathbb{R}^d$ preserving $\mathbb{Z}^d$ and such that the induced map $[A_f] : T^d \to T^d$ is homotopic to f.

In general, we define the linearization matrix by induction on the dimension of nilmanifold.

Definition 2.6. Let $f : X \to X$, $X \in \mathcal{N}$, $\dim X = d$, be given as a fiber map of the Fadell–Husseini fibration $(f_1, f, \tilde{f})$ of (T^{d_1}, X, B), $B \in \mathcal{N}$, $\dim B = \tilde{d}$ with A_{f_1}, $A_{\tilde{f}}$ the linearizations of f_1 and $\tilde{f}$, respectively. We call the integral $d \times d$ matrix

$$A = A_f \in \mathcal{M}_{d\times d}(\mathbb{Z}), \quad A_f := A_{f_1} \oplus A_{\tilde{f}}$$

the linearization of f.

It is easy to show that the matrix A_f express the Lefschetz number of f by the following formula (cf. [**10, 14**]).

THEOREM 2.7. *Let* $f : X \to X$, $\dim X = d$ *be a map of nilmanifold and* $A = A_f$ *the linearization of matrix of* f . *Then*

$$L(f^m) \ = \ \det(\mathrm{I} \ - A^m)$$

for every $m \in \mathbb{N}$. □

The important property of a self-map f of compact nilmanifold relates the Lefschetz number of f with the Nielsen number making the latter effectively computable.

THEOREM 2.8. [ANOSOV THEOREM] *For a map* f *of nilmanifold we have*

$$N(f) \ = \ |L(f)| \ .$$

For a proof see [**2**], [**8**]. The formula was first proved for a torus map in [**6**]. □

By the definition (Def. 2.2) a compact nilmanifold x is so-called Eilenberg–Moore $K(\Gamma, 1)$–space (cf. [**41**]), because $\pi_1(X) = \Gamma$ and $\pi_k(X) = 0$ for $k > 1$. Consequently the set $[X,\, X]$ of homotopy classes of self-maps of M is in one-to-one correspondence with the set of all homomorphisms of the fundamental group $\pi_1(X) \ = \ \Gamma$ ([**41**]). The uniform and discrete subgroups of nilpotent groups have additionally a property of the extension of a homomorphism, called the rigidity property.

PROPOSITION 2.9. *Let* $X = G/\Gamma$ *be a compact nilmanifold and* $\pi_1(X) = \Gamma$ *its fundamental group. Then* $[X,\, X] \ \longleftrightarrow \ \mathrm{Hom}(\Gamma,\, \Gamma)$. *Moreover, every homomorphism* $\phi : \Gamma \to \Gamma$ *has a unique extension* $\Phi : G \to G$.

For a proof see [**35**]. □

The linearization matrix A_f (cf. Definitions 2.5 and 2.6) is also well-defined if f is a self-map of an NR–manifold X (cf. [**22**]). The later is a subclass of all compact solvmanifolds (cf. Def. 2.10) containing so called exponential solvmanifolds (see Def. 2.12, also [**3, 22**], [**17**] for the definition and more details on exponential manifolds). It is remarkable that A_f has a nice analytical description if X is so called completely solvable solvmanifold (cf. Theorem 2.13).

DEFINITION 2.10. The compact quotient G/Γ , G of solvable Lie group by a uniform closed subgroup Γ is called solvmanifold. If Γ is a discrete subgroup, then the solvmanifold is called special.

For a given Lie group G let $\mathcal{G}$ denote its Lie algebra, which is isomorphic, as the vector space, to the tangent space T_e at the neutral element of G .

Definition 2.11. A solvable Lie group G is called completely solvable if for every $X \in \mathcal{G}$ and the adjoint (linear) map $\mathrm{ad}_X : \mathcal{G} \to \mathcal{G}$ we have $\sigma(\mathrm{ad}_X) \subset \mathbb{R}$, where $\sigma(A)$ denotes the spectrum of the quadratic matrix A.

Definition 2.12. A Lie group G is called exponential if $\exp : T_eG = \mathcal{G} \to G$ is onto $\Longleftrightarrow$ $\sigma(\mathrm{ad}_X) \cap \imath\mathbb{R} = \emptyset$,

For a map of a compact completely solvable manifold the matrix A_f of linearization of f can be defined in an analytic way described below (cf. [**17**]). This definition is based on the following theorem.

Theorem 2.13. (Hattori) *Let $(\Lambda^*\mathcal{G}^*, \delta)$ be the Chevalley–Eilenberg differential complex of invariant exterior forms associated with the Lie algebra $\mathcal{G}$ of a simply connected, completely solvable Lie group G. If $\Gamma \subset G$ is a uniform discrete subgroup, then*

$$H^*(G/\Gamma; \mathbb{R}) \cong H^*(\Lambda^*\mathcal{G}^*, \delta).$$

This theorem was first showed for a compact nilmanifold by Nomizu (cf. [**34**]). For a proof of the version stated here see [**36**]. □

Consequently, we can construct a $d \times d$ matrix by the following procedure: First we derive the homomorphism f_* induced on the fundamental group $\Gamma = \pi_1(X)$, $X = G/\Gamma$ of X:

$$f \mapsto (\phi := f_*) : \Gamma \to \Gamma .$$

Next, we extend this homomorphism ϕ to a unique continuous homomorphism Φ of G, by Proposition 2.9:

$$\phi \mapsto \Phi : G \to G .$$

Finally, we take the derivative of the homomorphism (a continuous homomorphism of a Lie group is analytic) at the neutral element of G:

$$\Phi \mapsto A := D\Phi_e \ : \ (\mathcal{G} = T_eG) \to \mathcal{G} .$$

In this way we get a matrix A, in general different than A_f of Definition 2.6 but still having the property $L(f) = \det(\mathrm{I} - A)$ (cf. [**17**]).

3. Main theorems.

Definition 3.1. Let $f : X \to X$ be a map of compact nilmanifold and A its linearization (Def. 2.6). Put

$$\mathbb{N} \supset T_A := \{m \,; \det(\mathrm{I} - A^m) \neq 0\} .$$

We call T_A set of algebraic minimal homotopy periods.

By an obvious reason if $m \notin T_A$ then $m \notin \mathrm{HPer}(f)$, because $L(f^m) = N(f^m) = 0$ (cf. [**20**], [**14**]). Our main theorem gives a characterization of the set of minimal homotopy periods for a map of a compact nilmanifold, or compact completely solvable solvmanifold.

THEOREM 3.2. *Let* $f : X \to X$ *be a map of a compact nilmanifold, or compact completely solvable solvmanifold, of dimension* d, $A = A_f$ *its linearization, and* $T_A \subset \mathbb{N}$ *as above.*

Then $\mathrm{HPer}(f) \subset T_A$ *and it is in one of the following three (mutually exclusive) types:*

(E) $\mathrm{HPer}(f) = \emptyset \iff N(f) = L(f) = 0 \iff 1 \in \sigma(A)$;
(F) $\mathrm{HPer}(f)$ *is nonempty and finite* $\iff$ *all eigenvalues of* A *are either zero or roots of unity;*
(G) $\mathrm{HPer}(f)$ *is infinite and* $T_A \setminus \mathrm{HPer}(f)$ *is finite.*

Moreover, for all d *there are finite sets* $P(d)$, $Q(d)$ *of* $\mathbb{N}$ *such that* $\mathrm{HPer}(f) \subset P(d)$ *in Type F and* $T_A \setminus \mathrm{HPer}(f) \subset Q(d)$ *in Type* (G).

It is worth of pointing out that the statement of this theorem is the same as that of its correspondent in the case where X is the torus T^d (cf. [**20**]).

It was noted in [**14**] that the combinatorics and number theory argument of Jiang, Llibre proof for torus map carries over the nilmanifold case. However, a proof required a new topological assertion — a partial Wecken theorem for periodic points. To formulate it we need the notion of the "periodic" Nielsen numbers introduced by Boju Jiang in [**18**].

DEFINITION 3.3. Let $f : X \to X$ be a map of a finite polyhedron X. Then one can define two numeric homotopy invariants of f, called Nielsen n–periodic numbers, and denoted by $NP_n(f)$, and $NF_n(f)$ such that

$$NP_n(f) \leq \#P_n(f) \qquad NF_n(f) \leq \#P^n(f) = \mathrm{Fix}\,(f^n)\,.$$

A definition is a little bit technical so we refer to [**18**] for it.

The main topological component of the proof of Theorem 3.2 was the following fact shown in ([**14**] Th. B).

THEOREM 3.4. *Let* $f : X \to X$ *be a selfmap of a compact nilmanifold* X. *If* $NP_n(f) = 0$ *then* $f \simeq g$, *for some* $g : X \to X$ *such that* $P_n(g) = \emptyset$.

If $X = T^d$ is a torus the corresponding result was already known ([**43, 44**]) and used by Jiang and Llibre in [**20**]. As a matter of fact, the quoted result of You [**44**] was even in a stronger form that corresponds to Theorem 3.6 stated below. □

The original proof of Theorem 3.4 essentially used the fact that X is a nilmanifold, but very soon Jezierski noted that it is a fact of general theory of periodic points, i.e. does not depend on a special structure of manifold (cf. [**12**].

THEOREM 3.5. *Let* $f : X \to X$, *be a map of a* PL*–manifold with* $\dim X \geq 4$, *and* $N\Phi_n(f) = 0$ *then* $f \simeq g$ *such that* $\mathrm{Fix}\,(g^n) = \emptyset$. □

Next Jezierski studied this problem in full generality proving a correspondent of the Wecken theorem for periodic points which was conjectured by Halpern in early eighties and called "Halpern conjecture" in the literature (cf. [**13**]).

THEOREM 3.6. *Let* $f : X \to X$ *be a map of a PL–manifold with* $\dim X \geq 4$ *and* $N\Phi_n(f) = q$ *, then* $f \simeq g$ *such that* $\#\mathrm{Fix}\,(g^n) = q$.

Another tools for the proof of Theorem 3.2, besides already mentioned Theorem 3.4, are Proposition 2.7, Theorem 2.8 and the following Möbius' formula:

$$N(f^m) = \sum_{k|m} NP_k(f) \iff NP_m(f) = \sum_{k|m} \mu(m/k) N(f^k)\,.$$

A description of $\mathrm{HPer}(f)$ is attainable due to an observation established by Boju Jiang and Llibre by a fine combinatorial argument (cf. [**20**]).

THEOREM 3.7. *Let* $f : X \to X$ *be a map of a compact nilmanifold. Then* $m \notin \mathrm{HPer}(f)$ *if and only if either* $N(f) = 0$ *or* $N(f^m) = N(f^{m/p})$ *for some prime factor* p *of* m .

Finally, we show that there exists $P(d)$ such that $N(f^m) > N(f^{m/p})$ for all $m \in T_A$ (i.e. $N(f^m) \neq 0$), $m > P(d)$. A nontrivial estimate from below of the rate of convergence of an algebraic number of module 1 is necessary. We need a new notation. Let α be an algebraic number of degree d and $w(x) = a_0x^d + a_1x^{d-1} + + \cdots + a_d$ its minimal polynomial with roots $\alpha_1, \ldots, \alpha_d$. The measure of α is defined as

$$M(\alpha) \ := \ a_0 \prod_{i=1}^{d} \max\{1,\, |a_i|\}\,.$$

The crucial is a characterization of an algebraic number: B. Jiang–Llibre, also Mignotte ([**20**]):

THEOREM 3.8. *For every algebraic number* α *of degree d and every* $m \in \mathbb{N}$ *such that* $\alpha^m \neq 1$ *, we have*

$$|1 - \alpha^m| \ > \ \frac{1}{2} e^{-9\alpha H^2}, \qquad \text{where}$$

$$a \ = \ \max\bigl\{20,\, 12.85|\log \alpha| + \frac{1}{2}\log M(\alpha)\bigr\},$$

$$H \ = \ \max\bigl\{17,\, \frac{d}{2}\log m + 0.66d + 3.25\bigr\}\,.$$

□

The below inequality was obtained as a consequence of Theorem 3.8 in [**20**]. It is used to prove the last part of Theorem 3.2.

Corollary 3.9. *Let* $\rho := \mathrm{sp}\,(A)$. *Then*

$$\frac{N(f^m)}{N(f^n)} > \frac{\rho^{m/2}-1}{e^{9d(41.4+\frac{d}{2}\log\rho)(d\log m)^2}}\,.$$

□

Recently a computer program in "Mathematica", deriving $\mathrm{HPer}(A_f)$ for a given A was written by Komendarczyk and the author in [**23**]. The dependence of the length of interval $[1,\,P(d)]$ and $[1, Q(d)]$ of the statement of Theorem 3.2 only on the dimension d is not necessary for such a program, because we work with a fixed A. Moreover, under additional assumption that $\sigma(A)\cap\{z=1\}$ consists only of nontrivial roots from unity, a modification the inequality 3.9 drastically cuts the interval of searching.

3.1. *Lower dimensions — a complete description.* It is natural to give a complete list of all sets of homotopy minimal periods in the case if the dimension of a manifold is small. Note that proofs of the corresponding theorems for $X = S^1$, and $X = T^2$ (Theorems 1.7 and 1.8 — cf. [**4, 1**]) already contain such a list in their formulations. In the paper [**20**] Jiang and Llibre gave such a list for maps of $M = T^3$ including a separate table for homeomorphisms. An aim of the work [**15**] was to give such a list for a three dimensional nilmanifold not homeomorphic to the torus. The corresponding theorem says the following.

Theorem 3.10. *Let* $f : X \to X$ *be a map of three-dimensional compact nilmanifold* X *not diffeomorphic to* T^3. *Let* $A = A_1 \oplus \bar{A} \in \mathcal{M}_{3\times 3}(\mathbb{Z})$ *be the matrix induced by the fibre map* $f = (f_1,\,\bar{f})$ *and* $\chi_A(t) = \chi_{A_1}(t)\cdot\chi_{\bar{A}}(t) = (t-d)(t^2-at+b)$ *be its characteristic polynomial. Then* $d = b$ *and there are three types for the minimal homotopy periods of* f:

(E) $\mathrm{HPer}(f) = \emptyset$ *if and only if* $d = 1$ *or* $-a+d+1=0$.

(F) $\mathrm{HPer}(f)$ *is nonempty and finite only for* 2 *cases corresponding to* $d = 0$ *combined with one of the two pairs* $(a,\,b)$

$$(0,\,0)\,,\ \textit{and}\ (-1,\,0)\,.$$

Then we have $\mathrm{HPer}(f) = \{1\}$. *Moreover, the sets* T_A *and* $\mathrm{HPer}(f)$ *are the following:*

$(d,\ a,\ b)$	T_A	$\mathrm{HPer}(f)$
$(0,\ 0,\ 0)$	$\mathbb{N}$	$\{1\}$
$(0,\ -1,\ 0)$	$\mathbb{N}\setminus 2\mathbb{N}$	$\{1\}$

(G) $\mathrm{HPer}(f)$ *is infinite for the remaining* $(d,\,a,\,b=d)$.

Furthermore, $\mathrm{HPer}(f)$ *is equal to* $\mathbb{N}$ *for all triples* $(d,\,a,\,b=d)\in\mathbb{Z}^3$ *except the following special cases listed below.*

Special Cases of Type (G)

(d, a, b)	T_A	HPer(f)
$a+d+1=0$, *with* $a \neq 0$, *and* $d \notin \{-2, -1, 0, 1\}$	$\mathbb{N} \setminus 2\mathbb{N}$	$\mathbb{N} \setminus 2\mathbb{N}$
(0, −2, 0)	$\mathbb{N}$	$\mathbb{N} \setminus \{2\}$
(−1, 1, −1)	$\mathbb{N} \setminus 2\mathbb{N}$	$\mathbb{N} \setminus 2\mathbb{N}$
(−1, −1, −1)	$\mathbb{N} \setminus 2\mathbb{N}$	$\mathbb{N} \setminus 2\mathbb{N}$
(−2, 1, −2)	$\mathbb{N} \setminus 2\mathbb{N}$	$\mathbb{N} \setminus 2\mathbb{N}$
(−2, 0, −2)	$\mathbb{N}$	$\mathbb{N} \setminus \{2\}$
(−2, 2, −2)	$\mathbb{N}$	$\mathbb{N} \setminus \{2\}$

Moreover, for every pair subset $\mathcal{S}_1 \subset \mathcal{S}_2 \subset \mathbb{N}$, *appearing as* $\mathrm{HPer}(f)$ *and* T_A *listed above there exists a map* $f : X \to X$ *such that* $\mathrm{HPer}(f) = \mathcal{S}_1$ *and* $T_A = \mathcal{S}_2$.

A proof is based on a classification of all homomorphisms of the nilpontent group $\Gamma_{1,1,r}$ (cf. 2.9). What is remarkable that a condition on an integer 3×3 matrix A for being a linearization of a map does not depend on r, and consequently relies upon a condition on a matrix for being a homomorphism of the Heisenberg algebra (cf. Example 2.3). Note also that an algebraic condition on linearization for $M \not\cong T^3$ is more restrictive since $\chi(A) = \chi(A_1)\,\chi(\overline{A}) = (t-d)(t^2 - at + b)$ is the product here. But additionally the topology yields that $r = \deg f_1 = \deg \bar{f} = \det \bar{A} = b$ (cf. [**15**]). □

As a consequence of the derived list of all sets of homotopy minimal periods we got the following Šarkovskii type theorem for a map of compact three dimensional nilmanifold.

Corollary 3.11. *If a self map of a 3-nilmanifold different than* 3*-torus is such that* $3 \in \mathrm{HPer}(f)$ *then* $\mathbb{N} \setminus 2\mathbb{N} \subset \mathrm{HPer}(f) \subset \mathrm{Per}(f)$. *If* $2 \in \mathrm{HPer}(f)$ *then* $\mathbb{N} = \mathrm{HPer}(f) = \mathrm{Per}(f)$. *In particular, the first assumption is satisfied if* $L(f^3) \neq L(f)$ *and the second if* $L(f^2) \neq L(f)$. □

As for the torus case (cf. [**20**]) we specified the classification for homeomorphisms.

Theorem 3.12. *Let* $f : X \to X$ *be a homeomorphism of three-dimensional compact nilmanifold* X *not diffeomorphic to* T^3. *Let* $A = A_1 \oplus \bar{A} \in \mathcal{M}_{3\times 3}(\mathbb{Z})$ *be the linearization matrix and* $\chi_A(t) = (t-d)(t^2 - at + b)$ *its characteristic polynomial.*

Then $d = b = \pm 1$ *and consequently* $\mathrm{HPer}(f) = \emptyset$ *iff* $d = 1$ *(i.e. if* f *preserves the orientation), or* $d = -1$ *and* $a = 0$. *In particular,* $\mathrm{HPer}(f) = \emptyset$ *for every preserving orientation homeomorphism. For* $d = -1$ *(i.e. if* f *reverses the orientation) and the remaining* a *we have* $\mathrm{HPer}(f) = \mathbb{N}$ *with the* *only two exceptions occuring for* $a = 1$ *or* $a = -1$. *For these special cases* $T_A = \mathrm{HPer}(f) = \mathbb{N} \setminus 2\mathbb{N}$.

The statement follows from Theorem 3.10 and the fact that $d = \pm 1$ (cf. [**15**]). □

It is worth of pointing out that the proof of Theorem 3.2 for the completely solvable solvmanifolds followed the argument of [**15**] and did not depend on Theorem 3.6. Consequently, the use of Theorem 3.6 cuts it essentially and let us to extend it onto maps of NR–manifold. On the other side, the supposed structure of the completely solvable solvmanifolds is of importance if we wish to identify $[X, X]$ with $\mathrm{Hom}(\Gamma, \Gamma]$, in respect of the rigidity property 2.9. A direct computation shows (cf. [**17**]).

LEMMA 3.13. *Let* $A : \mathcal{G} \to \mathcal{G}$ *be any endomorphism of Lie algebra of a three-dimensional connected, solvable, completely solvable group* G. *Then it has the following form with respect to the basis* $\{\mathcal{X}, \mathcal{Y}, \mathcal{Z}\}$

$$A = \begin{bmatrix} a & 0 & 0 \\ b & r & s \\ c & u & v \end{bmatrix},$$

where the coefficients satisfy the following conditions:
either $r = v = s = u = 0$ *and* $a \in \mathbb{Z}$ *is an arbitrary integer,*
or there exists a coefficient r, u, s, v *different from* 0 *and then* $a \in \{-1, 1\}$.
Moreover we have:

1. *if* $a = -1$ *then* $r = v = 0$;
2. *if* $a = \ \ 1$ *then* $s = u = 0$. □

It led to the correspondent classification theorem for three dimensional completely solvable solvmanifolds (cf. [**17**]).

THEOREM 3.14. *Let* $f : X \to X$ *be a map of a compact three dimensional completely solvable special solvmanifold which is not diffeomorphic to a nilmanifold. Let next* $\mathcal{M}_{3\times 3}(\mathbb{Z})$ *be the linearization. Then we have three mutually disjoint cases:*

(E) $\mathrm{HPer}(f) = \emptyset$ *iff* $L(f) = 0$ *iff* $a = 1$ *or (* $a = -1$ *and* $s\,u = 1$ *).*
(G) $\mathrm{HPer}(f) = \mathbb{N}$ *iff* $a \neq \{-2, -1, 0 + 1\}$ *and* $r = s = u = v = 0$;
$\mathrm{HPer}(f) = \mathbb{N} \setminus \{2\}$ *iff* $a = -2 \quad r = s = u = v = 0$;
$\mathrm{HPer}(f) = \mathbb{N} \setminus 2\mathbb{N}$ *iff* $a = -1$, $|s\,u| \geq 2$ *and* $r = v = 0$.
(F) $\mathrm{HPer}(f) = \{1\}$ *in the remaining cases.* □

It is worth to emphasize that there exists a countable family $\{X_n = G/\Gamma_n\}$ of not diffeomorphic three–dimensional, completely solvable special solvmanifolds, where G is the unique connected, simple–connected completely solvable Lie group corresponding to the Lie algebra of Lemma 3.13.

As in the case of nilmanifolds we specified this theorem for homeomorphisms and got a Šarkovskii type theorem ([**17**]).

COROLLARY 3.15. *For any homeomorphism* $f : X \to X$ *of a compact special three dimensional completely solvable solvmanifold which is not diffeomorphic to a nilmanifold,* $\mathrm{HPer}(f)$ *is either empty or consists of the single number* 1. □

COROLLARY 3.16. *For a map as in Theorem 3.14 we have Šarkovski type implications:* $2 \in \mathrm{HPer}(f)$ *implies* $\mathrm{HPer}(f) = \mathbb{N}$. *If* $\mathrm{HPer}(f)$ *contains an even number then* $\mathbb{N}\setminus\{2\} \subset \mathrm{HPer}(f)$. *If* $\mathrm{HPer}(f)$ *contains at least two numbers then* $\mathbb{N} \setminus 2\mathbb{N} \subset \mathrm{HPer}(f)$. □

4. Topological entropy.

DEFINITION 4.1. Let X be a compact metric space, e.g. a compact manifold and $f : X \to X$ a self-map of X. We assign with f a real number $\mathbf{h}(f) \geq 0$, or ∞, called the topological entropy of f. Here we assume that X is a compact smooth manifold of dimension d.

For a given metric ρ, $n \in \mathbb{N}$, and a self-map $f : X \to X$ we define a new metric

$$\rho_n(x, y) := \max_{0 \leq i \leq n} \rho(f^i(x), f^i(y)).$$

For a given $\varepsilon > 0$ put

$$\mathrm{r}_n(f,\varepsilon) := \min \# \ \varepsilon\text{–net},$$

$$\mathrm{S}_n(f, \varepsilon) := \max \# \ \varepsilon\text{–separated set}.$$

$$\mathbf{h}(f) := \lim_{\varepsilon\to 0} \overline{\lim_{n\to\infty}} \frac{1}{n} \log \mathrm{r}_n(f, \varepsilon) = $$

$$= \lim_{\varepsilon\to 0} \overline{\lim_{n\to\infty}} \frac{1}{n} \log \mathrm{S}_n(f, \varepsilon).$$

For more details on the entropy see [**10**]. Roughly speaking, if $\mathbf{h}(f) > 0$ then the dynamics of f is complex (rich).

Let $H^*(f) : H^*(X; \mathbb{R}) \to H^*(X; \mathbb{R})$ be the linear map induced by f on the cohomology space

$$H^*(X; \mathbb{R}) := \bigoplus_0^d H^i(X; \mathbb{R}).$$

Recall that if X is a compact smooth manifold then the singular, Čech, simplicial, cellular, or de Rham cohomology theory are equivalent (cf. [**41**]). Denote by $\sigma(f)$ the spectrum of the linear map

$$H^*(f) : H^*(X;\mathbb{R}) \to H^*(X;\mathbb{R})$$

induced by $f : X \to X$. Next, by $\mathrm{sp}\,(f)$ we denote the spectral radius of the map $H^*(f)$.

Michael Shub ([**39**]) posed the following conjecture about the topological entropy.

CONJECTURE 4.2. *Let* $f : X \to X$ *be a* C^1*–map. Then*

$$\log \mathrm{sp}\,(f) \;\leq\; \mathbf{h}(f)\,.$$

Misiurewicz and Przytycki in [**32**] proved that the estimate (4.2) holds for a continuous map of the torus T^d. This led to conjecture posed by A. Katok that if we assume a special topological form of the manifold X then the estimate (4.2) holds for every continuous self-maps [**21**].

CONJECTURE 4.3. *Let* X *be a manifold with the universal cover homeomorphic to the Euclidean space* $\mathbb{R}^d$*. Then*

$$\forall \;\; C^0\text{–}map \quad f : X \to X\,, \quad \log \mathrm{sp}\,(f) \;\leq\; \mathbf{h}(f)\,.$$

Conjecture 4.2 was proved by Yomdin ([**42**]) if f is a $C^\infty-$map and for a few special cases under the general C^1 assumption (cf. [**10, 25, 26, 27**]). For example, Misiurewicz and Przytycki showed that $h(f) \geq \log|\deg(f)|$ ([**31**]).

Note that Conjecture 4.2 is not true for a C^0–map as follows from the following example given by Shub in [**39**].

EXAMPLE 4.4. Let $h_d : S^1 \to S^1$ be a map of the circle of degree $d \geq 2$, e.g. $h(z) := z^d$. Let next $\phi : [0,1] \to [0,1]$ be a map given as $\phi(t) := \sqrt{t}$. Representing S^2 as the suspension of S^1 i.e. $S^2 = S^1 \times [0,1]/\sim$ where $S^1 \times \{0\} \sim *$, $S^1 \times \{1\} \sim *$ and $(x,\, t) \sim (x,\, t)$ if $t \neq 0,\, 1$, we map

$$f([z,t]) \;:=\; [(h_d(z),\, \phi(t))]\,.$$

Then $\deg f = \deg h_d = d$. The set of non-wandering points of f (thus also periodic points) consists of two (fixed) points $[S^1 \times \{0\}]$ and $[S^1 \times \{1\}]$. The same holds for any n–dimensional sphere S^n.

Note that this map is locally near the South Pole equivalent to $2z^d\|z\|^{-1}$, thus not differentiable. Moreover, observe that every map $f : \mathcal{U} \to \mathbb{C}$ of a neighborhood $\mathcal{U}$ of 0 which in the polar coordinates is of the form $f(\theta,\, \rho) = (d \cdot \theta,\, \phi(\rho)$ $\phi(r) > r$, for $r > 0$, can not be smooth at 0. Indeed, then $|\det Df(0)| \geq 1$, because $D_r \subset f(D_r)$ for every disc D_r. But the later yields that f is a local diffeomorphism at 0, contrary to its form along the angle coordinate.

Recently an extension of the result of [**32**] onto larger class of manifolds than tori has been shown (cf. [**28**]).

THEOREM 4.5. **A.** *Let* $f : X \to X$ *be a continuous self-map of a compact nilmanifold* X *of dimension* d*. Then*

$$\log(\mathrm{sp}\,(f)) \;\leq\; \mathbf{h}(f)\,.$$

A step in the proof of Theorem 4.5 is the following fact.

PROPOSITION 4.6. *Let* $f : X \to X$ *be a self-map of a compact nilmanifold* X *of dimension* d *and* $A \in \mathcal{M}_{d\times d}$ *be the linearization of* f *and* $\wedge A := \bigoplus_0^d \wedge^l A$ *the sum of all skew-symmetric powers of* A. *Then* $\operatorname{sp}(f) \le \operatorname{sp}(\wedge A)$.

Proposition 4.6 can be derived from a linear algebra argument applied to the spectral sequence of the Fadell–Husseini fibration of Def. 2.4 or from a use of the de Rham complex of invariant forms on X (cf. [**28**]).

Next, it seems that one can modify the argument of [**32**] to show that

PROPOSITION 4.7. *We have* $\log \operatorname{sp}(\wedge A) \le \mathbf{h}(f)$.

Theorem 4.5 follows from Propositions 4.6 and 4.7 (cf. [**28**]). □

REMARK 4.8. Remark that if $L(f) = \det(\mathrm{I} - A) \neq 0$ then Theorem 4.5 follows directly from Proposition 4.6 and the Ivanov theorem ([**11**]).

Indeed, Ivanov (also Jiang [**19**]) showed that

$$\log N^\infty(f) := \limsup_n \frac{1}{n} \log N(f^n) \le \mathbf{h}(f). \tag{1}$$

On the other hand an elementary linear algebra argument shows that:

$$N^\infty(f) = \begin{cases} \operatorname{sp}(\wedge A) & \text{if } 1 \notin \sigma(A), \\ 0 & \text{if } 1 \in \sigma(A). \end{cases}$$

Note that and $N(f) = |L(f)|$ by the Anosov theorem. Consequently, $L(f) \neq 0 \iff N(f) \neq 0$, or equivalently $f \not\sim g$, where g is a fixed point free map then.

5. A symmetry that originates periodic points. There are many definitions of chaos. We shall use the following with a very week requirement on the map f.

DEFINITION 5.1. Let $f : X \to X$ be a map. We say that f originates <u>chaos</u> if either

$$\operatorname{Per}(f) \subset \mathbb{N} \qquad \underline{\text{is an infinite set}} \quad \text{or}$$

$$m \mapsto \#P_m(f) \qquad \underline{\text{is unbounded}}.$$

From Theorem 1.7 it follows that for a circle map if $|\deg f| > 1$ then f originates chaos in the above meaning. The same is not true for maps of S^n, $n \ge 2$, as follows from the Shub example (cf. Example 4.4).

One can ask what condition on $f : S^n \to S^n$, together with the necessary $|\deg f| > 1$ implies the existence of infinitely many periodic points. From the main theorem of [**40**] it follows that it is enough to assume that f is a C^1 map to have positive topological entropy, but it does not imply the existence of infinitely many periodic points in general. In [**16**] we showed that every continuous map $f : S^n \to S^n$, $n \geq 1$, of $\deg f = r$, where $|r| \geq 2$, originates chaos provided it commutes with a free homeomorphism $g : S^n \to S^n$ of a finite order. The sequence $\{\#\mathrm{Fix}\, f^k\}$ is unbounded and then $\mathrm{Per}(f)$ is infinite.

DEFINITION 5.2. Let X be a smooth manifold and $g : X \to X$ a homeomorphism of the finite order m. We say that g is free if for every $x \in X$ and $1 \leq k \leq m$, $g^k(x) = x$ implies $k = m$.

Equivalently, we say that an action of the cyclic group $\{g\} \simeq \mathbb{Z}_m$ on X is given by $(k, x) \mapsto g^k(x)$. If g is free then this action is called a free action.

DEFINITION 5.3. Let X be a smooth manifold with an action of a cyclic group $\mathbb{Z}_m$ defined by a homeomorphism $g : X \to X$. A map $f : X \to X$ is $\mathbb{Z}_m$–equivariant if $f\,\alpha = \alpha\, f$, for each $\alpha \in \mathbb{Z}_m$. Note that f is $\mathbb{Z}_m$–equivariant if it commutes with the generator of action, i.e. $f(gx) = gf(x)$.

A homotopy $H : X \times [0, 1] \to X$ is equivariant iff

$$z \in X, t \in [0,1], \alpha \in \mathbb{Z}_m \text{ implies } H(\alpha x, t) = \alpha H(x,t)$$

Suppose that we have a free action of $\mathbb{Z}_m$ on S^n, $n \geq 2$, i.e. given a free homeomorphism $g : S^n \to S^n$ of order m. To formulate our result we need a new notation.

DEFINITION 5.4. Let $m = p_1^{a_1} \ldots p_s^{a_s}$, $a_i > 0$, be the decomposition of m into prime powers. Let k be a natural number. We represent k as $k = p_1^{b_1} \ldots . p_s^{b_s} p_{s+1}^{a_{s+1}} \ldots p_r^{a_r}$, where $p_1, ..., p_r$ are distinct primes satisfying $p_i | m \iff i \leq s$, $b_i \geq 0$. Finally we put $k' := p_1^{b_1} \ldots . p_s^{b_s}$.

We are in position to formulate the main result of this section.

THEOREM 5.5. *Let $g : S^n \to S^n$ $n \geq 1$ be a free homeomorphism, and $f : S^n \to S^n$ a map commuting with g. Suppose that* $\deg f \neq -1, 0, 1$. *Then for every $k \in \mathbb{N}$ we have*

$$\#\mathrm{Fix}\, f^{mk} \geq m^2\, k',$$

where k' is defined above.

In particular, for $k = m^s$ we have

$$\#\mathrm{Fix}\, f^{m^{s+1}} \geq m^{s+2}.$$

To show this theorem we employed (see [**16**]) a fine modification of Nielsen number $NF_n(f)$ which estimates $\#\mathrm{Fix}\,(f^n)$ (cf. Definition 3.3). It can be applied to the map f/G induced by f on the quotient space X/G, which is then a generalized lens space. By this way we got an estimate of $\#\mathrm{Fix}\,((f/G)^n)$

(this estimate is not true if a map of X/G is not of the form f/G). Finally, by a geometrical reason these fixed points of $(f/G)^n$ give fixed points of f^{nm} . □

As a consequence we get:

Corollary 5.6. *Under the above assumptions*

$$\limsup_{l\to\infty} \frac{\#\mathrm{Fix}(f^l)}{l} \ \geq \ m\,.$$

□

For a cyclic group of prime order the method allows us also to estimate the number of m–periodic points of f , with m being the minimal period. Fix a prime number $p|m$ and restrict the action to $\mathbb{Z}_p \subset \mathbb{Z}_m$.

Theorem 5.7. *Let $f : S^n \to S^n$ be a continuous map which commutes with a free homeomorphism g of S^n of prime order p . If $\deg(f) \neq \pm 1$ then for each $s \in \mathbb{N}$ there exist at least $p-1$ mutually disjoint orbits of f of periodic points each of length p^s . Thus*

$$\#P_{p^s}(f) \geq (p-1)p^s\,.$$

The same is true for any map homotopic to f by equivariant maps. □

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