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TENSORIAL VERSION OF THE CALCULUS OF VARIATIONS

by Biagio C. Casciaro and Jerzy J. Konderak

Abstract. We study the tensorial Euler-Lagrange equations and the tensorial Noether equations. 'The sufficient conditions' are obtained in a covariant form and by covariant steps for Variational Problems of any dimension. The obtained integro-differential equations can be applied to constrained and lower rank variational problems.

1. Introduction

This paper is a continuation of the papers [**2, 3**], where we give a tensorial, global and covariant version of the Calculus of Variations. In the seventies and eighties, many papers where published on this subjetct, and the problem is now considered solved; but there are at least five good reasons which support the point of view considered here.

In order to explain these reasons, we need the general scheme in which we shall work. We consider three C^∞-differentiable manifolds S, M and N, and fix a suitable tensor field g of type (2,2) on the Cartesian product $M \times N$. We also fix a family $(f_s)_{s\in S}$ of C^∞-differentiable mappings $f_s : M \to N$, depending differentially on $s \in S$, and a family $(\Omega_s)_{s\in S}$ of n-dimensional submanifolds with boundary of M, depending differentially on $s \in S$, and having a regular enough boundary $\partial\Omega_s$. Finally, M is supposed to be orientable and a volume form dv is fixed on it. By this scheme, both the Euler–Lagrange equation and the Noether equation can be obtained from the total differential $d\rho$ of the

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action functional $\rho : S \to \mathbb{R}$ defined by:

$$\rho(s) = \int_{\Omega_s} g\left(\frac{\partial X_{f_s}}{\partial x}, \frac{\partial X_{f_s}}{\partial x}\right) dv.$$

In fact, when a suitable expression of $d\rho$ is obtained (cf. section 6.1), the Noether theorem follows by requiring that $(d\rho)_{s_0}$, being $s_0 \in S$ and $(f_s)_{s\in S}$ suitably chosen (cf. section 6.3); while the same requirement, when S coincides with an open interval $]-\varepsilon, \varepsilon[$ of $\mathbb{R}$, with $\varepsilon > 0$, and on S the standard chart (S, id) is fixed, gives the Euler–Lagrange equation, being $s_0 = 0$ in this case (cf. section 6.2).

Apparently, this case is simpler than the problems considered in the previously quoted papers. It is not the case, since in a future paper we are able to prove that any variational problem of the first order can be suitably reduced to the case considered here. We consider this approach to the problem, because it simplifies the matters, it allows a better understanding of the involved calculations and it also allows us to obtain remarkable applications without the use of long calculations.

We suppose that all the geometric objects involved are C^∞-differentiable, for the sake of simplicity. We can always lower the degree of differentiability and find its standard lower bound by standard methods, cf. [**25, 21**], without this bound the methods and problems are quite different, cf. [**8**].

Now we are ready to turn to our reasons. The first reason is historical in nature. In fact, in most papers published on the subject, e.g. cf. [**1, 27, 13**] and others, the authors determine the bundles in which the standard geometric objects used in the Calculus of Variations are globally defined (generally jet-bundles and affine bundles), instead of finding covariant tensorial objects associated with the considered problem. We do not think this solution was what the mathematicians of the past had in mind when they considered the problem of covariance and globalization. Almost certainly, they were thinking of tensor fields and covariant derivative, and in this paper we use them exclusively. In other words we completely restore the correspondence between Euclidean spaces and differentiable manifolds in the Calculus of Variations, by determining the rules by which 'partial derivatives' can be replaced by 'covariant derivatives'.

The second reason is the extreme simplicity of the method, which, as just said, consists in the extension of the *formal derivatives* of the involved geometric objects to connections. This seems to be a fair task in itself, since, roughly speaking, the set of partial derivatives is nothing but the covariant differential with respect to the standard connection of the Euclidean space.

In order to compute $d\rho$ we fix three connections $\tilde{\nabla}$, ∇_1 and ∇_2 on S, M and N, respectively and consider the connection $\nabla = \tilde{\nabla} \times \nabla_1 \times \nabla_2$ on the Cartesian product $S \times M \times N$. The connection ∇ determines the covariant differential ∇H for any tensor field $H \in \mathcal{I}^r_s(S \times M \times N)$ of type (r, s). Then ∇H allows to obtain the covariant derivative $\nabla_T H$ of H with respect to any tensor field $T \in \mathcal{I}^1_k(S \times M \times N)$, by the obvious construction. These kinds of covariant derivative were proposed by E. Bompiani as locally defined geometric objects and studied by O.M. Amici and one of the authors, cf. [**2, 3**], as global covariant derivatives (in more general case which can be also used for the Calculus of Variations). It is immediately evident that these covariant derivatives obey the same rules as the standard covariant derivatives, with the exception of a minor change in the Leibniz rule. This exception disappears when the local expression of $\nabla_T H$ is considered (cf. section 3). Among the rules satisfied by these covariant derivatives, the possibility of restricting them to the closed submanifolds of $S \times M \times N$ plays an important role. In fact, one can use it in order to consider the geometric objects tangent to the family (f_s) as tensor fields of type (1.1), defined on the graph $\mathbf{G}_f$ of the mapping $f : S \times M \to N$ obtained by putting $f(s, x) = f_s(x)$, for each $(s, x) \in S \times M$ (cf. section 2). Then the differential operator induced by on $\mathbf{G}_f$ by ∇ via the previous definition of covariant derivative allows us to compute the covariant derivatives of the 'tangent' (1,1) tensor fields and the derivatives of any tensor fields with respect to them. These results correspond to the possibility of computing all the necessary 'formal derivatives' of the functions $h(s, x) = g\big(\frac{\partial X_{f_s}}{\partial x}, \frac{\partial X_{f_s}}{\partial x}\big)_x$ and of the vector field $X(s, x) = g\big(\frac{\partial X_{f_s}}{\partial s}, \frac{\partial X_{f_s}}{\partial x}\big)_x$ defined on $\mathbf{G}_f$.

The only reason for us to use local coordinates is the previously quoted diversity of the Leibnitz rule, which requires the use of some permutation morphisms, cf. [**2, 3**].

The third reason is a practical one. In fact, it is evident that in the case $S =]-\varepsilon, \varepsilon[$ the obtained first variation is exactly the classical one. Moreover, the second variation of ρ can be easily computed, cf. [**2, 3**].

The fourth reason is our attempt to free the Calculus of Variation from its dependence from the geometry of M and N (from the classical point of view, it depends on the Euclidean geometry via the classical derivative and the volume form). One can suppose that these geometries are determined by ∇_1, dv and ∇_2, respectively, so that if $\breve{\nabla}_1$, $d\breve{v}$ and $\breve{\nabla}_2$ determine different geometries on M and N, the tensor field $\Pi = \nabla - \breve{\nabla}$, being $\breve{\nabla} = \tilde{\nabla} \times \breve{\nabla}_1 \times \breve{\nabla}_2$ and the function $k : M \to \mathbb{R}$ defined by $dv = k d\breve{v}$, play the same role as the one of the functions which determine the local change of frames. Moreover, from this point of view, our method allows one to compute in an extremely simple way all the quantities related to any Variational Problem directly by means of connections and volume forms, previously fixed, Legendre transformations and symplectic form included. Hence, the natural question which arises here is why so much effort and so much time was spent solving so trivial problem.

Recall that the problem was posed by Einstein in the terms for the action of the Hilbert integral, cf. [**24**], and that the classical Calculus of Variations quitted to play a central role, because of its supposed lack of covariance, until the jet–bundles techniques were developed. We collect all the needed definitions and their properties we are going to use in sections 2, 3, 4 and 5, in order to give them a unitary and tensorial formulation; while section 6 is devoted to the Euler–Lagrange equation and to the Noether Theorem.

But first, let us observe that our equations and formulas have an unusual feature. This depends on our choices and not on our methods. In fact, we add the function ν which is generally not used. Moreover we use completely general connections, since our purpose is to show that the determination of the properties of the variational problem does not depend on the connection itself, as is generally thought, cf. e.g. [**17, 18**] for the Riemannian variational problem and [**26**] for the Finslerian case. The use of torsion–free connections drastically simplifies formulae. We also stop the determination of whole equation at the level of an integro-differential equation, because the theory of 'equivalent integrals' is not trivial for $\dim M > 1$ (see, e.g., [**3**], where the relations between this problem and suitable cohomological groups of the Vinogradov type are proved) even if g has maximal rank and there are no constrains. In particular this observation is also true for the 'Noether preserved currents' when the group of infinitesimal isometrics is fixed. Then it seems likely that there is the necessity of a tensorial theory of 'equivalent integrals'. We also use one variation (f_s) of the function f_{s_0}. These possibilities result from the fact that our point of view simplifies all the calculations involved in the subject, hence we can take into account more aspects of the theory of the Calculus of Variations. As an example, from our point of view it is also evident that Euler–Lagrange equations and Noether equations have the same origin. We can also restore the old definition of invariant Lagrangian. In recent years, the requirement for the Lagrangians to be invariant under some subgroup of the group of diffeomorphisms of M or N has been used more frequently because of physical motivations. The old definition requires the invariance with respect to subgroups of diffeomorphisms of $M \times N$, cf. [**14**]. Furthermore, the old definition contains a mistake. We show it by an example and give the correct definition.

The last reason is that our method encompasses the jet bundle method and can be extended to more general differential operators. In fact, in our paper we use globally defined linear connections only to simplify the language, but linear connection can be replaced by any family of local connection. As an example, by taking in each chart of M the linear connection induced by the standard connection on $\mathbb{R}^m$, one obtains the theories based on jet-bundles. One can also use a linear connection of the first order and type $((h, h'), (k, k'))$, cf. [**2**], in such a way the Euler–Lagrange equation is determined by tensor fields of an arbitrary order, etc.

We conclude observing that in order to avoid quite a long list, references relate to the papers and books directly herein quoted only, and by a short recapitulation of the results contained therein.

In section 2, we study some properties of the graph $\mathbf{G}_f$ of a map $f : M \to N$. In section 3, the properties of the used covariant derivative are introduced. In section 4 properties related to the volume forms are collected. In section 5 we consider the Jacobian function which is induced by a diffeomorphism. With this Jacobian we associate some tensor fields to obtain the Reynolds transport theorem in a covariant form, cf. [**27**]. The results related to the Calculus of Variations are contained in section 6. It is well known that the calculations needed in the case considered here are quite long. Hence we do not include applications of the technique presented here. We shall present some applications and examples in our forthcoming paper, cf. [**9**].

2. Differential structures induced on a graph of a map

2.1. Decomposition of tensors. Let M^m, N^n be differentiable manifolds of dimension m and n, respectively. In the present paper, differentiability of manifolds and maps will always mean of C^∞–differentiability. Let (U, ϕ) be a chart on M. Then we put $\phi = (x^1, \dots, x^m)$. Similarly by (V, ψ) we shall denote a chart on N where $\psi = (y^1, \dots, y^n)$. The indices i, j will vary from 1 to m and α, β will vary from 1 to n. On the Cartesian product of manifolds $M \times N$, there exists a canonical structure of the diffrentiable manifold with the charts defined as $(U \times V, \varphi \times \psi)$. We shall use all the standard notations of this case.

Let $p_1 : M \times N \to M$ and $p_2 : M \times N \to N$ be the canonical projection on the first and the second factor of the Cartesian product, respectively. Let $\mathcal{X}_1$ be the set of sections of the pull-back bundle $(p_1)^{-1}TM$ and $\mathcal{X}_1$ be the set of sections of the pull-back bundle $(p_2)^{-1}TN$. The spaces $\mathcal{X}_1$, $\mathcal{X}_2$ may be regarded as $C^\infty(M \times N)$-submodules of $\mathcal{X}(M \times N)$ and we have the following isomorphism of $C^\infty(M \times N)$-modules $\mathcal{X}(M \times N) \cong \mathcal{X}_1 \oplus \mathcal{X}_2$. The decomposition of the tangent bundle of $M \times N$ induces the decomposition of the 1-forms on $M \times N$. Hence we have $\mathcal{I}^0_1(M \times N) = \mathcal{I}^0_{1,1} \oplus \mathcal{I}^0_{1,2}$ where $\mathcal{I}^0_{1,1}$ is the set of the sections of the pull-back bundle $p_1^{-1}(T^*M)$ and $\mathcal{I}^0_{1,2}$ is the set of sections of the pull-back bundle $p_2^{-1}(T^*N)$.

By $\mathcal{I}^p_q(M \times N)$ we shall denote the space of tensors of the type (p, q) on $M \times N$. Using the decomposition of $T(M \times N)$ and $T^*(M \times N)$, we may decompose the tensor algebra over manifold $M \times N$ into the direct sum of subbundles. We are interested in the following subspaces of $T^1_1(M\times N)$:

$$P_1 \in P_{1(x,y)}(M\times N) \iff P_1 = P^i_{1j}\frac{\partial}{\partial x^i}\otimes dx^j \tag{2.1}$$

$$P_2 \in P_{2(x,y)}(M\times N) \iff P_2 = P^\alpha_{2j}\frac{\partial}{\partial y^\alpha}\otimes dx^j \tag{2.2}$$

with the obvious meaning of the symbols used.

We shall denote by $P_1(M\times N)$, $P_2(M\times N)$ and $P(M\times N)$ the subbundle of $T^1_1(M\times N)$ having as a fiber over $(x,y)\in M\times N$ the direct sum of the real vector space $P_{1(x,y)}(M\times N)$, $P_{2(x,y)}(M\times N)$ and

$$P_{(x,y)}(M\times N) = P_{1(x,y)}(M\times N)\oplus P_{2(x,y)}(M\times N)$$

respectively. Moreover we denote by $\mathcal{P}_1(M\times N)$, $\mathcal{P}_2(M\times N)$ and $\mathcal{P}(M\times N)$ the $C^\infty(M\times N)$-module of the sections on the previous respective bundle.

Let us consider the splitting of $\mathcal{I}^2_2(M\times N)$. We are particularly interested in the tensors of the type $(2,2)$, which may be expressed as follows:

$$g = g^{ij}_{\alpha\beta}\frac{\partial}{\partial x^i}\otimes\frac{\partial}{\partial x^j}\otimes dy^\alpha\otimes dy^\beta\ ,\ g^{ij}_{\alpha\beta} = g^{ji}_{\beta\alpha}. \tag{2.3}$$

2.2. Tensors on a graph. Let $f : M \to N$ be a differentiable map. The graph $\mathbf{G}_f = \big\{(x,y)\in M\times N | f(x) = y\big\}$ of f is a regular differentiable submanifold of $M\times N$ of the dimension m.

There are the following natural maps: $\widetilde{f} : M \ni x \mapsto (x, f(x)) \in \mathbf{G}_f$ and $\widehat{f} : \mathbf{G}_f \ni (x, f(x)) \mapsto x \in M$. Using $\widetilde{f}$ and $\widehat{f}$ as identification maps and the standard notation, a simple calculus in local coordinates shows that the elements

$$\frac{\partial}{\partial x^i} + (\partial_i f^\alpha)\frac{\partial}{\partial y^\alpha} \quad\text{and}\quad dx^i$$

for all $i = 1,\ldots,m$ being $\partial_i f^\alpha = \frac{\partial X_{fa}}{\partial x^i}$, determine a local base of $T_{(x,f(x))}\mathbf{G}_f$ and $T^*_{(x,f(x))}\mathbf{G}_f$, respectively.

It is also trivial that the family $(\frac{\partial X_{f\alpha}}{\partial x^i})_{\varphi(x)}$ has the same transformation rules with respect to the change of the coordinates as the elements of $P_{2(x,f(x))}(M\times N)$ restricted to the graph $\mathbf{G}_f$. Hence it follows that

$$\big(\frac{\partial X_f}{\partial x}\big)_x := \big(\partial_i f^\alpha\big)_{\varphi(x)}\big(\frac{\partial}{\partial y^\alpha}\big)_{f(x)}\otimes(dx^i)_x \in P_{2(x,f(x))}(M\times N)$$

for all $x\in U\cap f^{-1}(V)$, when $U\cap f^{-1}(V)$ is not empty, defines globally a tensor field on $\mathbf{G}_f$ which belongs to $\mathcal{P}_2(\mathbf{G}_f)$.

Let K_M be the Kronecker tensor field on M. Analogously, we can consider the tensor field $P_f := K_M + \frac{\partial X_f}{\partial x}\in\mathcal{P}(\mathbf{G}_f)$ has the following local expression

$$P_f = \delta^i_j\big(\frac{\partial}{\partial x^i} + \partial_i f^\alpha\frac{\partial}{\partial y^\alpha}\big)\otimes dx^j. \tag{2.4}$$

The transformation laws of g allow us to consider the map $g(\frac{\partial X_f}{\partial x}, \frac{\partial X_f}{\partial x}) : \mathbf{G}_f \to \mathbb{R}$ such that $g(\frac{\partial X_f}{\partial x}, \frac{\partial X_f}{\partial x})_{(x,f(x))} = g^{ij}_{\alpha\beta(x,f(x))}(\partial_i f^\alpha)_x(\partial_j f^\beta)_x$. Suppose that a volume form dv is given on M. Then the tensor g defines a natural functional on the space of differentiable mappings from M to N. Namely we have:

$$E_{\Omega,g}(f) := \int_\Omega g\left(\frac{\partial X_f}{\partial x}, \frac{\partial X_f}{\partial x}\right) dv$$

where Ω is a subset of M such that the integral is well-defined. Generally we shall suppose that Ω is a compact domain. The quantity $E_{\Omega,g}(f)$ shall be called *the energy of f in Ω with respect to g.*

Now we consider a further C^∞-differentiable manifold S with the $\dim S = r$. Let (W,τ) be a local chart on S and let $\tau = (s^1,\dots,s^r)$. Moreover we suppose that $A, B, \dots$ run from 1 to r. Let $f : M \times S \to N$ be a differentiable mapping. Then using a similar procedure as before we get the following tensors $\frac{\partial X_f}{\partial x} = (\partial_i f^\alpha)\frac{\partial}{\partial y^\alpha} \otimes dx^i$ and $\frac{\partial X_f}{\partial s} = (\partial_A f^\alpha)\frac{\partial}{\partial y^\alpha} \otimes ds^A$ with the obvious meaning of the used symbols. For these tensors the previous considerations hold. Let K_S be the Kronecker tensor of S. Then we define two tensor fields on $\mathbf{G}_f$ in the following way: $P^1_f := K_M + \frac{\partial X_f}{\partial x}$, $P^2_f := K_S + \frac{\partial X_f}{\partial s}$.

3. Connections on a graph of a map

3.1. Bompiani's operators. Let ∇_1, ∇_2 be two connections on M and N respectively, having Γ^{1i}_{jk} and $\Gamma^{2\alpha}_{\beta\gamma}$ as respective Christoffel symbols. We consider the canonically induced connection $\nabla = \nabla_1 \times \nabla_2$ on $M \times N$. Let $P \in \mathcal{I}^1_1(M \times N)$ and $T \in \mathcal{I}^p_q(M \times N)$. Then a generalized covariant derivative is defined in the following way:

$$\nabla_P T = C^1_{p+1}\big((\nabla T) \otimes P\big), \tag{3.1}$$

cf. [**2**]. It is possible to consider more general operators of the above type. Such operators were introduced by Bompiani. Properties of such operators were studied in [**2**]. For applications of these operators to variational problems, see [**25**, 5.120, p.291]. Here we consider only the one defined by (3.1) for the sake of brevity. Let us consider $P \in \mathcal{P}(M \times N)$, $Q \in \mathcal{P}_2(M \times N)$ and $g \in \mathcal{I}^2_2(M \times N)$ verifying (2.3).

Observation 3.1. By using the local expression of the covariant differential with respect to ∇ in the natural chart $(U \times V, \varphi \times \psi)$ and by (2.1), (2.2), (2.3) and (3.1) we get:

$$(3.2)\quad \nabla_P Q = (P_r^{1j}\partial_j Q_t^\alpha + P_r^{2\beta}\partial_\beta Q_t^\alpha + P_r^{2\beta}\Gamma_{\beta\gamma}^{2\alpha}Q_t^\gamma - P_r^{1j}\Gamma_{jt}^{1k}Q_k^\alpha)\frac{\partial}{\partial y^\alpha}\otimes \otimes dx^t \otimes dx^r$$

$$(3.3)\quad \nabla_P g = \big(P_k^{1r}\partial_r g_{\alpha\beta}^{ij} + P_k^{1r}\Gamma_{rt}^{1i}g_{\alpha\beta}^{tj} + P_k^{1r}\Gamma_{rt}^{1j}g_{\alpha\beta}^{it} + P_k^{2\gamma}\partial_\gamma g_{\alpha\beta}^{ij} - P_k^{2\gamma}\Gamma_{\gamma\alpha}^{2\sigma}g_{\sigma\beta}^{ij} - P_k^{2\gamma}\Gamma_{\gamma\beta}^{2\sigma}g_{\alpha\sigma}^{ij}\big)\frac{\partial}{\partial x^i}\otimes\frac{\partial}{\partial x^j}\otimes dy^\alpha\otimes dy^\beta\otimes dx^k.$$

Remark 3.1. The operator (3.1) needs a permutation diffeomorphism in order to preserve the derivation rules with respect to the tensor product, cf. [**2**]. For the sake of brevity we here prefer to describe this morphism implicitly by writing the local expressions of the derivatives as in Observation 3.1.

3.2. Local expressions. Let us consider S as in section 2.2. Suppose that a connection $\widetilde{\nabla}$ is given on S, with Christoffel symbols $\widetilde{\Gamma}_{BC}^A$. Then, on $M\times S$, we can consider the connection $\check{\nabla} = \nabla_1 \times \widetilde{\nabla}$, and on $M\times S\times N$, the connection $\nabla = \nabla_1\times\widetilde{\nabla}\times\nabla_2$. Let $f : M\times S \to N$ be a differentiable mapping. Then from Observation 3.1 follows:

Observation 3.2. Under the above assumptions, with respect to the canonical chart $(U\times W\times V, \varphi\times\tau\times\psi)$ the following holds

$$\nabla_{P_f^1}P_f^2 = \big[\partial_{iA}^2 f^\alpha + \Gamma_{\beta\gamma}^{2\alpha}\partial_i f^\beta \partial_A f^\gamma\big]\frac{\partial}{\partial y^\alpha}\otimes ds^A\otimes dx^i$$

$$\nabla_{P_f^2}P_f^1 = \big[\partial_{iA}^2 f^\alpha + \Gamma_{\beta\gamma}^{2\alpha}\partial_A f^\beta \partial_i f^\gamma\big]\frac{\partial}{\partial y^\alpha}\otimes dx^i\otimes ds^A$$

$$\nabla_{P_f^1}P_f^1 = \big[\partial_{ij}^2 f^\alpha + \Gamma_{\beta\gamma}^{2\alpha}\partial_j f^\alpha \partial_i f^\beta - \Gamma_{ij}^{1t}\partial_t f^\alpha\big]\frac{\partial}{\partial y^\alpha}\otimes dx^i\otimes dx^j.$$

Let $\sigma : \mathcal{I}_2^1(M\times S\times N) \to \mathcal{I}_2^1(M\times S\times N)$ be a $C^\infty(M\times S\times N)$-linear map such that $\sigma(\frac{\partial}{\partial y^\alpha}\otimes ds^A\otimes dx^i) := \frac{\partial}{\partial y^\alpha}\otimes dx^i\otimes ds^A$, $\sigma(\frac{\partial}{\partial y^\alpha}\otimes dx^i\otimes ds^A) := \frac{\partial}{\partial y^\alpha}\otimes ds^A\otimes dx^i$ for $i = 1,\dots,m$, $A = 1,\dots,r$ and σ fixes the remaining elements of the local base. Then we can define the torsion of the previous differential operators by

$$\widetilde{T}(P_f^1, P_f^2) := \nabla_{P_f^1}P_f^2 - \sigma(\nabla_{P_f^2}P_f^1) = T_{\beta\gamma}^{2\alpha}\partial_i f^\beta\partial_A f^\gamma\frac{\partial}{\partial y^\alpha}\otimes dx^i\otimes ds^A = T^2(\frac{\partial X_f}{\partial x}, \frac{\partial X_f}{\partial s})$$

where T^2 is the torsion tensor field of ∇_2, and the used contractions are given by the local expression. Then we consider $\nabla_{P_f^1}g$ and $\nabla_{P_f^2}g$ where g is defined by (2.3). From Observation 3.1 there follows

OBSERVATION 3.3. We have

$$\begin{aligned}\nabla_{P_f^2} g &= (\partial_A f^\gamma \partial_\gamma g^{ij}_{\alpha\beta} - \partial_A f^\gamma \Gamma^{2\sigma}_{\gamma\alpha} g^{ij}_{\sigma\beta} - \partial_A f^\gamma \Gamma^{2\sigma}_{\gamma\beta} g^{ij}_{\alpha\sigma}) \frac{\partial}{\partial x^i} \otimes \frac{\partial}{\partial x^j} \otimes \\ &\otimes dy^\alpha \otimes dy^\beta \otimes ds^A \\ \nabla_{P_f^1} g &= (\partial_k g^{ij}_{\alpha\beta} + \Gamma^{1i}_{kt} g^{tj}_{\alpha\beta} + \Gamma^{1j}_{kt} g^{it}_{\alpha\beta} + \partial_k f^\gamma \partial_\gamma g^{ij}_{\alpha\beta} - \partial_k f^\gamma \Gamma^{2\sigma}_{\gamma\alpha} g^{ij}_{\sigma\beta} \\ &\quad - \partial_k f^\gamma \Gamma^{2\sigma}_{\gamma\beta} g^{ij}_{\alpha\sigma}) \frac{\partial}{\partial x^i} \otimes \frac{\partial}{\partial x^j} \otimes dy^\alpha \otimes dy^\beta \otimes dx^k.\end{aligned}$$

We also consider the mapping $g(\frac{\partial X_f}{\partial x}, \frac{\partial X_f}{\partial x}) : \mathbf{G}_f \to \mathbb{R}$ defined as a family of maps $\{g(\frac{\partial X_f}{\partial x}, \frac{\partial X_f}{\partial x})_x\}_{x \in M}$ from S into $\mathbb{R}$ such that

$$g\left(\frac{\partial X_f}{\partial x}, \frac{\partial X_f}{\partial x}\right)_x (s) = g\left(\frac{\partial X_f}{\partial x}, \frac{\partial X_f}{\partial x}\right)_{(x, f(x,s))}$$

for all $s \in S$. Hence we obtain a family of 1-forms on M depending differentiably on $x \in M$, by considering the total differential of $g(\frac{\partial X_f}{\partial x}, \frac{\partial X_f}{\partial x})_x$ for each $x \in M$. We denote this family by $d^S g(\frac{\partial X_f}{\partial x}, \frac{\partial X_f}{\partial x})$. Then from Observations 3.2 and 3.3 we get

$$d^S g(\frac{\partial X_f}{\partial x}, \frac{\partial X_f}{\partial x}) = (\nabla_{P_f^2} g)(\frac{\partial X_f}{\partial x}, \frac{\partial X_f}{\partial x}) + 2g(\frac{\partial X_f}{\partial x}, \nabla_{P_f^2} P_f^1) \tag{3.4}$$

where the obvious contractions are used. Analogously, from Observations 3.2 and 3.3, by a simple computation, we obtain:

$$\begin{aligned}&(\nabla_{P_f^1} g)(\frac{\partial X_f}{\partial s}, \frac{\partial X_f}{\partial x}) + g\big(\nabla_{P_f^1} P_f^2, \frac{\partial X_f}{\partial x}\big) + g\big(\frac{\partial X_f}{\partial s}, \nabla_{P_f^1} P_f^1\big) \\ &= \left[\partial_k \big(g^{ij}_{\alpha\beta} \frac{\partial X_{f\alpha}}{\partial s^A} \frac{\partial X_{f\beta}}{\partial x^j}\big) + g^{tj}_{\alpha\beta} \frac{\partial X_{f\alpha}}{\partial s^A} \frac{\partial X_{f\beta}}{\partial x^j} \Gamma^{1i}_{kt}\right] \frac{\partial}{\partial x^i} \otimes dx^k \otimes ds^A \\ &= \nabla\left[g\big(\frac{\partial X_f}{\partial s}, \frac{\partial X_f}{\partial x}\big)\right]\end{aligned}$$

where instead of introducing the list of contractions and permutations which we have used, cf. [**2**], we prefer to calculate the local expressions. Let C denote the contraction of $\frac{\partial}{\partial x^i}$ with dx^i; then the above equation becomes

$$\begin{aligned}d^S g(\frac{\partial X_f}{\partial x}, \frac{\partial X_f}{\partial x}) &= (\nabla_{P_f^2} g)(\frac{\partial X_f}{\partial x}, \frac{\partial X_f}{\partial x}) + 2g(\frac{\partial X_f}{\partial x}, T^2(\frac{\partial X_f}{\partial s}, \frac{\partial X_f}{\partial x})) \\ &\quad + 2C\Big(\nabla g(\frac{\partial X_f}{\partial s}, \frac{\partial X_f}{\partial x})) - 2C((\nabla_{P_f^1} g)(\frac{\partial X_f}{\partial s}, \frac{\partial X_f}{\partial x})\Big) \\ &\quad - 2Cg(\frac{\partial X_f}{\partial s}, \nabla_{P_f^1} P_f^1).\end{aligned}$$

In what follows, C will not be written. In the last equation the following observation was used: from the local point of view, when we contract

$g^{ij}_{\alpha\beta}\frac{\partial X_{f\alpha}}{\partial x^i}\frac{\partial}{\partial x^j}\otimes dy^\beta$ with $(\nabla_{P^1_f}P^2_f)^\beta_{jA}\frac{\partial}{\partial y^\beta}\otimes dx^j\otimes ds^A$, in order to obtain a 1-form we have only the choice $g^{ij}_{\alpha\beta}\frac{\partial X_{f\alpha}}{\partial x^i}(\nabla_{P^1_f}P^2_f)^\beta_{jA}ds^A$, independently of the position of ds^A in the tensorial product. Consequently, we can delete the isomorphism σ.

4. Volume forms

4.1. Densities 1. We suppose M orientable and consider a volume element dv on M. Let (U,φ) be a chart on M then

$$dv = \lambda dx^1\wedge\cdots\wedge dx^m \tag{4.1}$$

where $\lambda : U\to\mathbb{R}^+$ is a scalar density (we consider oriented charts only). We recall that the charts (U',φ') of M such that $dv = dx'^1\wedge\cdots\wedge dx'^m$ determine an atlas of M, cf. [**23**, p.195]. The charts of this atlas will be called *charts adapted to the volume form dv*. Let (U,φ) and (U',φ') be two charts on M, with $\varphi = (x^1,\dots,x^n)$ and $\varphi' = (x'^1,\dots,x'^n)$. By considering the $n\times n$ matrices $\Theta = (\Theta^h_{h'}) = (\frac{\partial X_{x^h}}{\partial x'^{h'}})$ we have

$$\partial_k \det\Theta = \Theta^h_{h'}\Theta^{h'}_{hk} \tag{4.2}$$

where $\Theta^h_{h'}$ is the algebraic complement of $\Theta^{h'}_h$ in Θ. Hence, $\partial_k\log\det\Theta = \Theta'^h_{h'}\cdot\Theta^{h'}_{hk}$. Moreover, putting $dv = \lambda' dx'^1\wedge\cdots\wedge dx'^m$ we get

$$\lambda'(x) = \lambda(x)\det(\Theta^{-1}(x)) \quad \text{for each} \quad x\in U\cap U'. \tag{4.3}$$

We recall that dv determines a bijection between $C^\infty(M)$ and $\wedge^n(M)$ such that $C^\infty(M)\ni\xi\to\xi dv\in\wedge^n(M)$. Moreover, dv induces a $C^\infty(M)$-isomorphism of modules $\mathcal{X}(M)$ and $\wedge^{n-1}(M)$ such that

$$\mathrm{i}_v : \mathcal{X}(M)\ni X\mapsto \mathrm{i}_X dv\in\wedge^{n-1}(M),$$

cf. [**18**]. Suppose that (U,φ) is a chart on M, not necessary adapted to the volume form. Then for $X = X^i\frac{\partial}{\partial x^i}$ we can put

$$d\mathrm{i}_v(X) := d\big(\mathrm{i}_v(X)\big) = \big(\partial_i X^i + X^i\partial_i\log\lambda\big)dv \tag{4.4}$$

where the last identity follows from simple computations.

Corollary 4.1. *If (U,φ) is an adapted chart for the volume form dv then $di_v(X) = \partial_i X^i dv$, cf.* [**23**, p.195].

Let $X\in\mathcal{I}^1_1(M\times S)$ and let us suppose $X = X^i_A\frac{\partial}{\partial x^i}\otimes ds^A$, with respect to the chart $(U\times W,\varphi\times\tau)$, cf. section 2.2. Then one can consider

$$\mathrm{i}_v(X) := \sum_{i=1}^n(-1)^{i-1}X^i_A\lambda dx^1\wedge\cdots\wedge\widehat{dx^i}\wedge\cdots\wedge dx^n\otimes ds^A.$$

The changes of charts affect the components of X like

$$X'^{i'}_{A'}(x,s) = X^i_A(x,s)\Theta^{i'}_i(x)(\frac{\partial X_{s^A}}{\partial s'^{A'}})_s.$$

Hence we can put

$$d\mathrm{i}_v(X) := \Big(\frac{\partial}{\partial x^i}(X^i_A) + X^i_A\frac{\partial}{\partial x^i}(\log\lambda)\Big)dv\otimes ds^A.$$

Moreover, if (U,φ) is a chart adapted to the volume form dv, then $d\mathrm{i}_v(X) = (\frac{\partial}{\partial x^i}(X^i_A))dv\otimes ds^A$.

4.2. Densities 2. Let $\mathcal{D}(M)$ denote the set of all connections on M. Hence there exists a map $\mathcal{C}_v : \mathcal{D}(M) \to \mathcal{I}^0_1(M)$ defined locally in the following way: if (U,φ) is a chart on M and ∇ is a connection with Christoffel symbols Γ^i_{jk} then

$$C_v(\nabla) := (\Gamma^i_{ik} - \partial_k\log\lambda)dx^k \tag{4.5}$$

where (4.1) is used. Suppose that (U',φ') is another chart on M such that $U'\cap U\neq\emptyset$ and $\Gamma'^{i'}_{j'k'}$ are the Christoffel symbols induced by the chart (U',φ'). Then using (4.2) and the natural relations between the Christoffel symbols of ∇ with respect to the above charts, it is easily seen that $\mathcal{C}_v(\nabla)$ is a globally defined 1-form on M. Moreover, it follows from (4.5) that if (U,φ) is an adapted chart then $\omega_k = \Gamma^i_{ik}$.

We suppose that dv is the volume element associated with a pseudo-Riemannian metric g. In such a case $\lambda = |\det(g_{ij})|^{1/2}$ where (g_{ij}) are components of the metric g with respect to a chart (U,φ). Let ∇ be the Levi-Civita connection of g. Since we have the following explicit identity for Christoffel symbols of the pseudo-Riemannian connection, we may compute

$$\begin{aligned}\Gamma^i_{ik} &= \frac{1}{2}g^{it}\big(\partial_i g_{tk} + \partial_k g_{ti} - \partial_t g_{ki}\big)\\ &= \frac{1}{2}\partial_k\log\big(|\det(g)|\big).\end{aligned} \tag{4.6}$$

where, abusing language, we denote the $n\times n$ matrix (g_{ij}) by g. Then we apply (4.6) and obtain $\omega_k = 0$ for $k = 1,\dots,m$. Hence:

Corollary 4.2. *If ∇ is a Levi-Civita connection and dv is the volume element of the same pseudo-Riemannian metric then: $\mathcal{C}_v(\nabla) = 0$, cf.* [**25**, p.380, (3.2)].

Now we come back to the general case when ∇ is an arbitrary connection. Let X be a vector field on M. Then, denoted by C the obvious contraction and by using (4.4) and (4.5), we have: $C(\nabla X)dv = d\big(i_v(X)\big) + \big(\mathcal{C}_v\nabla\big)(X)dv$.

In particular, if dv is a volume element associated with a pseudo-Riemannian metric g and ∇ is the Levi-Civita connection of g then

$$C\big(\nabla X\big)dv = di_v(X). \tag{4.7}$$

Then we consider the manifold S and we use the same conventions and notations as in section 2.2. Let $(\omega_x)_{x\in M}$ be a family of 1-forms on S depending differentiably on $x \in M$. Moreover let us suppose that $\omega_{(x,s)} = \omega_{A(x,s)}ds^A$ for each $x \in M$ and $s \in W$. Then we consider 1-form β defined on S by the local expression $\beta_s = \beta_A(s)ds^A$ with

$$\beta_A(s) = \int_\Omega \omega_{A(x,s)}dv$$

for $A = 1, \ldots, r$. In fact, the change of charts of S involves functions which are constant with respect to x. Hence we put

$$\beta := \int_\Omega \omega_x dv \tag{4.8}$$

Let $F : M \times S \to \mathbb{R}$ be a differentiable mapping; then for each $x \in M$ we may consider the function $F_x : S \to \mathbb{R}$ defined by $F_x(s) = F(x, s)$. In the following, we denote by $d^S F$ the family of 1-forms $\{dF_x\}_{x\in M}$. By (4.8) we may put

$$\int_\Omega d^S F dv := \int_\Omega dF_x dv. \tag{4.9}$$

It results in

$$d\int_\Omega F dv = \int_\Omega d^S F dv$$

for the well-known theorem of integration on manifolds, cf. [**22**].

Let N be a third manifold and let us denote by ∇ the connection on $M \times S \times N$ considered in section 3.2. Let $X \in \mathcal{I}^1_1(M \times S \times N)$ and let us suppose $X = X^i_A \frac{\partial}{\partial x^i} \otimes ds^A$ on the domain of the chart $(U \times W \times V, \varphi \times \tau \times \psi)$. By computing the local expression one gets $C(\nabla X) = (\frac{\partial}{\partial x^j}(X^j_A) + \Gamma^{1j}_{jk} X^k_A)ds^A$. Consequently by (4.7) we obtain

$$C(\nabla X)dv = di_v(X) + (C_v\nabla_1)(X)dv. \tag{4.10}$$

Hence by using the Stokes theorem, cf. [**22**], it follows that if Ω is a compact submanifold of M with a boundary and having the maximal dimension, then

$$\int_\Omega C\big(\nabla X\big)dv = \int_{\partial\Omega} i_v(X) + \int_\Omega \mathcal{C}_v\big(\nabla\big)(X)dv.$$

We conclude with the following observation. Let g be any pseudo-Riemannian metric having local components g_{ij}. Then there exists a function $\sigma : M \to \mathbb{R}$

having local expression $\sigma = \lambda^2/|\det(g_{ij})|$. Consequently we can consider the differentiable function $\widetilde{\sigma} = \sigma^{1/n} : M \to \mathbb{R}$ and the pseudo-Riemannian metric $\widetilde{g} = \widetilde{\sigma} g$. Then it follows that $|\det(\widetilde{g}_{ij})|^{1/2} = \lambda$. Consequently dv is the volume form of $\widetilde{g}$. Then if ∇ is the Levi-Civita connection of $\widetilde{g}$ it results $C_v(\nabla) = 0$.

5. Integration on manifolds

5.1. Jacobian function. Let $h : M \to N$ be a differentiable map, let (U, φ) and (V, ψ) be two charts of M and N, respectively, such that $U \cap h^{-1}(V) \neq \emptyset$. We set

$$\big(Th\big)(x) = (\frac{\partial X_h}{\partial x})_x \big(\partial_i h^\alpha\big)_x \big(\frac{\partial}{\partial y^\alpha}\big)_{h(x)} \otimes \big(dx^i\big)_x.$$

We suppose that $\dim(M) = \dim(N) = n$. Hence the indices of the charts vary over the same set of numbers. Therefore it make sense to define Jacobian

$$J_h(x) = \det\big(\partial_j h^i\big)_x. \tag{5.1}$$

We assume also that the volume form dv on M is given, with local expression given by (4.1). On N, there is given a volume form dw such that $dw|_V = \widetilde{\lambda} dy^1 \wedge \cdots \wedge dy^n$. We put

$$\widetilde{J}_h(x) := \frac{1}{\lambda(x)} J_h(x) \widetilde{\lambda}(h(x)) \tag{5.2}$$

for all $x \in U \cap h^{-1}(V)$. By using (4.3) and the transformation laws of J_h, it follows that the function $\widetilde{J}_h$ is defined globally on $\mathbf{G}_h$. The function $\widetilde{J}_h$ is called *the Jacobian function*, cf. [**23**, p.196, Def.22].

Let us suppose that h is a diffeomorphism and let Ω be a compact submanifold of M with boundary and $\dim \Omega = m$. Suppose also that $F : h(\Omega) \to \mathbb{R}$ is a differentiable map. Then by using the formula for the change of variables under the integral and standard arguments, cf. [**22**], we get

$$\int_{h(\Omega)} F(y) dw = \int_{\Omega} (F \circ h)(x) \widetilde{J}_h(x) dv. \tag{5.3}$$

Let us consider the inverse map $h^{-1} : N \to M$. It is clear that the graph of the map h is canonically diffeomorphic to the graph of the map h^{-1}. The diffeomorphism is given by $(x, h(x)) \to (h(x), x)$. Hence the tensor of type $(1,1)$ $T(h^{-1})$ defined on the graph of the map h^{-1} defines the tensor of the same type on the graph of $\mathbf{G}_h$ as follows

$$\big(T^{-1}h\big)_x := \Big(\partial_j (h^{-1})^i\Big)_{h(x)} \big(\frac{\partial}{\partial x^i}\big)_x \otimes \big(dy^j\big)_{h(x)}.$$

Let $x \in h^{-1}\big(h(U) \cap V\big) = U \cap h^{-1}(V)$. Then we consider the matrix $\big((\partial_j h^i)_x\big)$ for $i, j = 1, \dots, n$ and by $A^i_j(x)$ we denote the algebraic complement of the element $(\partial_j h^i)_x$ of this matrix. Now we put

$$\widetilde{A}^i_j(x) := \frac{1}{\lambda(x)} \widetilde{\lambda}(h(x)) A^i_j(x). \tag{5.4}$$

Therefore the following holds

$$\widetilde{A}^i_j(x) = \widetilde{J}_h(x) \Big(\partial_i (h^{-1})^j\Big)_{h(x)}. \tag{5.5}$$

Hence it follows that it makes sense to consider the following tensor field

$$\widetilde{A}(x) = \widetilde{A}^i_j(x) \big(\frac{\partial}{\partial x^i}\big)_x \otimes \big(dy^j\big)_{h(x)}$$

for all $x \in U \cap h^{-1}(V)$ which is defined globally on $\mathbf{G}_f$. In fact, by (5.5) the following relation holds: $\widetilde{A} = \widetilde{J}_h \cdot T^{-1}h$. Moreover it results

$$C^1_2(\widetilde{A} \otimes Th) = \widetilde{J}_h \cdot K_M. \tag{5.6}$$

5.2. Stokes theorem. Let S be C^∞-differentiable manifold and let us use all the definitions and notations of section 2.2. Moreover suppose that $(h_s)_{s \in S}$ is a family of diffeomorphisms onto $h_s : M \to N$ which depends differentiably on $s \in S$. Finally we assume that there are given two volume forms dv on M and dw on N. Since h_s is a diffeomorphism onto for all $s \in S$, we may define on N the tensor fields

$$V_s(y) = X_s\big(h_s^{-1}(y)\big) = \big(\frac{\partial X_h}{\partial s}\big)_{(h_s^{-1}(y), s)} \tag{5.7}$$

for all $y \in N$ and $s \in S$. We also consider the differentiable function $\widetilde{J}_h : M \times S \to \mathbb{R}$ defined by: $\widetilde{J}_h(x, s) = \widetilde{J}_{h_s}(x)$ for all $(x, s) \in M \times S$. Then using (4.1), (5.2), (5.4) and (5.6) one obtains

$$\Big(d^S \widetilde{J}_h\Big)_{(h_s^{-1}(y), s)} = \widetilde{J}_{h_s}\big(h_s^{-1}(y)\big) \big[d\big(\mathrm{i}_w(V_s)\big)\big]^*_y \tag{5.8}$$

where $(d\big(\mathrm{i}_w(V_s)\big)_y)^*$ is the 1-form along the canonical projection $pr_2 : N \times S \to S$ such that $d\big(\mathrm{i}_w(V_s)\big) = dw \otimes [d(\mathrm{i}_w(V_s))]^*$. We need a generalization of the transport Reynolds theorem, cf. [**27**]. For that purpose we put $\Omega_s = h_s(\Omega)$ for any $s \in S$. Let $F_s : \Omega_s \to \mathbb{R}$ be a family of maps depending differentiably on the parameter $s \in S$. The following identity is easily computable in a local coordinate system:

$$d^S\big(F_s \circ h_s\big)_x = \big(d^S F_s\big)_{h_s(x)} + (dF_s)_{h_s(x)}(X_s(x)).$$

Then from (4.9), (5.7) and (5.8) it follows that:

$$d\int\limits_{\Omega_s} F_s dw = \int\limits_{\Omega_s} d^S F_s dw + \int\limits_{\Omega_s} (dF_s)(V_s) dw + \int\limits_{\Omega_s} F_s d\mathrm{i}_w(V_s).$$

Since i_w is a $C^\infty(N)$-isomorphism of modules, $F_s\mathrm{i}_w(V_s) = \mathrm{i}_w(F_s(V_s))$ and from the Leibniz rule of dw we may conclude the following observation.

Observation 5.1. We have the following identity:

$$d\int\limits_{\Omega_s} F_s dw \;=\; \int\limits_{\Omega_s} d^S F_s dw + \int\limits_{\Omega_s} d\mathrm{i}_w(F_s V_s) = \int\limits_{\Omega_s} d^S F_s dw + \int\limits_{\partial\Omega_s} F_s \mathrm{i}_w(V_s).$$

We need to explain the integral $\int_{\partial\Omega_s} F_s\mathrm{i}_w(V_s)$ in Observation 5.1. For this purpose we shall consider manifolds M, N of dimension m and $m-1$ respectively. We assume that dv, dw are two volume forms on M and N. Suppose that X is a vector field on N. Then there is defined a $(m-1)$-form $\mathrm{i}_w(X)$. Let $h : M \to N$ be a differentiable map. Then we have the induced map $h^* : \wedge(N) \to \wedge(M)$ which is a morphism of vector bundles. Hence $h^*\big(\mathrm{i}_w(X)\big) \in \wedge(M)$, and

$$h^*(dy^1 \wedge \cdots \wedge \widehat{dy^\alpha} \wedge \cdots \wedge dy^m) = J_{h,\alpha} dx^1 \wedge \cdots \wedge dx^{m-1}$$

where $J_{h,\alpha}$ is the determinant of the matrix obtained from the Jacobian matrix of h by eliminating α–th column. Hence we get the following local identity

$$h^*\big(\mathrm{i}_w(X)\big) = (-1)^{\alpha-1} X^\alpha J_{h,\alpha} \widetilde{\lambda}\big(h(x)\big) dx^1 \wedge \cdots \wedge dx^{m-1}. \tag{5.9}$$

We suppose that a compact submanifold Ω with boundary of N is given and there exists a submanifold Ω' on M such that $h(\Omega') = \partial\Omega$. Then for a given $F : N \to \mathbb{R}$, we define

$$\int\limits_{\partial\Omega} F\mathrm{i}_w(X) := \int\limits_{\Omega'} \big(F \circ h\big) h^*\big(\mathrm{i}_w(X)\big).$$

We define $J_{h,X} := (-1)^{\alpha-1} X^\alpha J_{h,\alpha}$ and observe that

$$J_{h,X} \;=\; \det \begin{pmatrix} X^1 & \dots & \dots & X^m \\ \frac{\partial X_{h^1}}{\partial x^1} & \dots & \dots & \frac{\partial X_{h^1}}{\partial x^{m-1}} \\ \vdots & & & \vdots \\ \vdots & & & \vdots \\ \frac{\partial X_{h^m}}{\partial x^1} & \dots & \dots & \frac{\partial X_{h^1}}{\partial x^{m-1}} \end{pmatrix}.$$

Hence from (5.9) we get $h^*\big(\mathrm{i}_w(X)\big) = (1/\lambda) J_{h,X} \widetilde{\lambda} \circ h dv$. This implies that

$$\widetilde{J}_{h,X}(x) := \frac{1}{\lambda(x)} J_{h,X}(x) \widetilde{\lambda}\big(h(x)\big)$$

is a well-defined global map on M. In fact, $h^*\big(\mathrm{i}_w(X)\big)$ and dv are both defined globally and we have $h^*\big(\mathrm{i}_w(X)\big) = \widetilde{J}_{h,X}dv$. In the rest of this section we assume that: $h : M \times [0, s_1] \to N$ is a differentiable map such that $h|_{M\times]0,s_1]}$ is a diffeomorphism onto $h(M \times [0, s_1])$ which is a submanifold of N. In this case, cf. (5.7), V_s is defined on $h(M \times \{s\}) = M_s$ for each $s \in [0, s_1]$ and it gives the tangent vector to the curve $\gamma_p : [0, s_1] \to N$ defined as $\gamma_p(s) = h(p, s)$ for each $p \in M$. Hence from (5.1), (5.2) and (5.7) we get that $J_{h_s,V_s} = J_h$ and $\widetilde{J}_{h_s,V_s} = \widetilde{J}_h$, where $\widetilde{J}_h$ denotes the Jacobian function of h calculated with respect to the volume elements dw on N and $dv \wedge ds$ on $M \times [0, s_1]$.

Let $F : h(\Omega \times [0, s_1]) \to \mathbb{R}$ be a differentiable map. We put $F_s = F|_{h(\Omega\times\{s\})}$ then

$$\begin{aligned}
\int_0^{s_1}\Big(\int_{h_s(\Omega)} F_s \circ \mathrm{i}_w(V_s)\Big)ds &= \int_0^{s_1}\Big(\int_{\Omega} F_s \circ h_s \widetilde{J}_{h_s,V_s} dv\Big)ds \\
&= \int_{\Omega\times[0,s_1]} F \circ h \widetilde{J}_h dw \wedge ds \\
&= \int_{h(\Omega\times[0,s_1])} F dw.
\end{aligned}$$

The last equality in the identities above is not directly true because h is not a diffeomorphism. However from (5.3) there follows that such equality holds for each domain of the type $\Omega \times [\varepsilon, s_1]$ where $\varepsilon \in]0, s_1]$. Hence our identity may be obtained in the limit when ε tends to zero.

6. Conditions for minimum and Noether theorem

6.1. Intermediate identity. Let M and N be two differentiable manifolds of dimension m and n respectively. Let Ω be a compact submanifold of dimension m with a boundary. We assume that a tensor field $g \in \mathcal{I}_2^2(M \times N)$ is given, verifying (2.3). Let S be an r-dimensional differentiable manifold. Suppose also that $f_s : M \to N$ is a family of differentiable maps depending differentiably on $s \in S$ and $h_s : M \to M$ is a family of diffeomorphisms depending differentiably on $s \in S$. Moreover, let us consider a differentiable function $\nu : \mathbb{R} \to \mathbb{R}$. Then we put

$$\rho(s) = E_{\Omega_s,g,\nu}(f_s) = \int_{\Omega_s} \nu\big(g(\frac{\partial X_{f_s}}{\partial x}, \frac{\partial X_{f_s}}{\partial x})\big)dv \tag{6.1}$$

where $\Omega_s = h_s(\Omega)$, with Ω a compact submanifold of M with a boundary of maximum dimension. We need to compute $d\rho$. At first we denote by ρ_s° :

$M \to \mathbb{R}$ the function defined by $\rho_s^\circ(x) = g_{(x,f_s(x))}\big((\frac{\partial X_{f_s}}{\partial x})_x, (\frac{\partial X_{f_s}}{\partial x})_x\big)$, and by $\dot\nu : \mathbb{R} \to \mathbb{R}$, the first derivative of ν. Then from Observation 5.1 and (6.1) it follows that:

$$d\rho = \int\limits_{\Omega_s} \dot\nu(\rho_s^\circ)\big(d^S g(\frac{\partial X_{f_s}}{\partial x}, \frac{\partial X_{f_s}}{\partial x})\big)dv + \int\limits_{\Omega_s} \mathrm{di}_v\big(\nu(g(\frac{\partial X_{f_s}}{\partial x}, \frac{\partial X_{f_s}}{\partial x}))V_s\big) \tag{6.2}$$

where the notations of section 5.2 are used. Moreover by an analogous computation as for (3.4) we get

$$d\dot\nu(\rho_s^\circ) = \ddot\nu(\rho_s^\circ)(\nabla_{P_f^1} g)(\frac{\partial X_{f_s}}{\partial x}, \frac{\partial X_{f_s}}{\partial x}) + 2\ddot\nu(\rho_s^\circ) g(\frac{\partial X_{f_s}}{\partial x}, \nabla_{P_f^1} P_f^1) \tag{6.3}$$

where $\ddot\nu$ is the second derivative of ν. Finally, using the linearity of dv, the Leibniz rule (6.3), (3.5) and (4.10) we may conclude the following observation concering (6.2)

Observation 6.1. The following holds true

$$\begin{aligned}
d\rho \ =\ & \int\limits_{\Omega_s} \dot\nu(\rho_s^\circ)(\nabla_{P_f^2} g)(\frac{\partial X_{f_s}}{\partial x}, \frac{\partial X_{f_s}}{\partial x})dv - 2\int\limits_{\Omega_s} \dot\nu(\rho_s^\circ)(\nabla_{P_f^1} g)(\frac{\partial X_{f_s}}{\partial s}, \frac{\partial X_{f_s}}{\partial x})dv \\
& - 2\int\limits_{\Omega_s} \ddot\nu(\rho_s^\circ)\Big((\nabla_{P_f^1} g)(\frac{\partial X_{f_s}}{\partial x}, \frac{\partial X_{f_s}}{\partial x})\Big) g(\frac{\partial X_{f_s}}{\partial s}, \frac{\partial X_{f_s}}{\partial x})dv \\
& +2\int\limits_{\Omega_s} \dot\nu(\rho_s^\circ)(C_v\nabla_1) g(\frac{\partial X_{f_s}}{\partial s}, \frac{\partial X_{f_s}}{\partial x})dv \\
& +2\int\limits_{\Omega_s} \dot\nu(\rho_s^\circ) g(\frac{\partial X_{f_s}}{\partial x}, T^2(\frac{\partial X_{f_s}}{\partial s}, \frac{\partial X_{f_s}}{\partial x}))dv \\
& - 2\int\limits_{\Omega_s} \dot\nu(\rho_s^\circ) g(\frac{\partial X_{f_s}}{\partial s}, \nabla_{P_f^1} P_f^1)dv \\
& - 4\int\limits_{\Omega_s} \ddot\nu(\rho_s^\circ)\Big(g(\frac{\partial X_{f_s}}{\partial x}, \frac{\partial X_{f_s}}{\partial s})\Big)\Big(g(\frac{\partial X_{f_s}}{\partial x}, \nabla_{P_{f_s}^1} P_{f_s}^1)\Big)dv \\
& + 2\int\limits_{\Omega_s} \mathrm{di}_v\Big[\dot\nu(\rho_s^\circ) g(\frac{\partial X_{f_s}}{\partial s}, \frac{\partial X_{f_s}}{\partial x})\Big] + 2\int\limits_{\Omega_s} \mathrm{di}_v\big(\nu(\rho_s^\circ)V_s\big).
\end{aligned}$$

Formula in Observation 6.1 is the required intermediate one. We conclude this part of our paper with some remarks about the function ν.

Remark 6.1. Let $M = [a,b] \subset \mathbb{R}$. One may choose $g(\frac{\partial X_{f_s}}{\partial x}, \frac{\partial X_{f_s}}{\partial x})$ to be constant with respect to x, by parameterizing f_s by the arc–length and considering only regular curves. Consequently $\nu : \mathbb{R} \to \mathbb{R}$ may be chosen to be the identity map, being the critical points of $E_{\Omega,g,\nu}$ independent from ν.

REMARK 6.2. If $\dim M > 1$ then it is impossible to say whether f_s may be chosen in such a way that $g(\frac{\partial X_{f_s}}{\partial x}, \frac{\partial X_{f_s}}{\partial x})$ depends on $x \in M$ or not. Such a possibility depends on the topology of M and on the degree of differentiability of the variational problem. Hence, ν can play an important rule as in the case $\nu(t) = |t|^{\frac{1}{2}}$, for any $t \in \mathbb{R}$.

REMARK 6.3. The obstructions to choose ρ° independent of $x \in M$ are topological. In the classical case these obstructions on M do not exist. In this case, it is usually assumed that M is an m-dimensional cube (in any case M has a 'simple' topology).

6.2. Euler–Lagrange equation. Now we suppose that $S =]-\varepsilon, \varepsilon[$, with ε a real positive number. We also fix the canonical atlas $\{(]-\varepsilon, \varepsilon[, id)\}$ where $id : S \to S$ is the identity map. In this case the 1-form $d\rho$ in Observation 6.1 becomes a function by skipping ds in its local expression with respect to the fixed chart. Consequently, we denote this function by $\frac{d}{ds}\rho$. We also suppose that $\Omega_s = \Omega$ for each $s \in S$ and that the identity map $id_M : M \to M$ coincides with h_s for each $s \in]-\varepsilon, \varepsilon[$. Hence the tensor field V_s defined by (5.7) is zero. Consequently it results that

$$\int\limits_{\Omega_s} \mathrm{di}_v(\nu(\rho_s^\circ)V_s) = 0. \tag{6.4}$$

We also suppose that the family $(f_s)_{s\in]-\varepsilon,\varepsilon[}$ has the property $f_s|_{\partial\Omega_s} = \varphi$, for each $s \in]-\varepsilon, \varepsilon[$. Then, $\frac{\partial X_{f_s}}{\partial s}$ is identically zero on the boundary $\partial\Omega$. Consequently it results that

$$\int\limits_{\Omega_s} \mathrm{di}_v\big(\dot{\nu}(\rho_s^\circ)g(\frac{\partial X_{f_s}}{\partial s}, \frac{\partial X_{f_s}}{\partial x}\big) = \int\limits_{\partial\Omega} \mathrm{i}_v\big(\dot{\nu}(\rho_s^\circ)g(\frac{\partial X_{f_s}}{\partial s}, \frac{\partial X_{f_s}}{\partial x})\big) = 0.$$

Finally we observe that if the mapping $f_0 : M \to N$ is a local minimum for the functional $E_{\Omega_s,g,\nu}$ then ρ has a local minimum at $s = 0$. Hence under the above assumptions we have the following theorem.

Theorem 6.1 (Euler–Lagrange). *If* $(\partial_s E_{\Omega_s,g,\nu}(f_0))_{s=0} = 0$, *then*

$$
\begin{aligned}
0 = &\int_\Omega \dot\nu(\rho_0^\circ)(\nabla_{P_Z} g)(\frac{\partial X_{f_0}}{\partial x}, \frac{\partial X_{f_0}}{\partial x})dv - 2\int_\Omega \dot\nu(\rho_0^\circ)(\nabla_{P_f} g)(Z, \frac{\partial X_{f_0}}{\partial x})dv \\
&- 2\int_\Omega \ddot\nu(\rho_0^\circ)(\nabla_{P_{f_0}} g)(\frac{\partial X_{f_0}}{\partial x}, \frac{\partial X_{f_0}}{\partial x})g(Z, \frac{\partial X_{f_0}}{\partial x})dv \\
&+ 2\int_\Omega \dot\nu(\rho_0^\circ)(C_v\nabla_1)g(Z, \frac{\partial X_{f_0}}{\partial x})dv + 2\int_\Omega \dot\nu(\rho_0^\circ)g(\frac{\partial X_{f_0}}{\partial x}, T(Z, \frac{\partial X_{f_0}}{\partial x}))dv \\
&- 2\int_\Omega \dot\nu(\rho_0^\circ)g(Z, \nabla_{P_{f_0}} P_{f_0})dv - 4\int_\Omega \ddot\nu(\rho_0^\circ)g(Z, \frac{\partial X_{f_0}}{\partial x})g(\frac{\partial X_{f_0}}{\partial x}, \nabla_{P_{f_0}} P_{f_0})dv
\end{aligned}
\tag{6.5}
$$

where $Z = (\frac{\partial X_{f_s}}{\partial s})_{s=0}$ *is the vector field defined on* $\mathbf{G}_{f_0}$ *obtained by dropping* ds *in the local expression of* $\frac{\partial X_{f_s}}{\partial s}$ *with respect to the fixed chart and* $P_Z^2 := \frac{\partial}{\partial s} + Z$. *Moreover we also used the identity* $P_{f_0} = P_{f_s}^1|_{s=0}$.

The above equation appears longer than the usual Euler–Lagrange equation, but it depends on our choices and not on our methods, as we stated in the introduction. Here are some remarks concerning Theorem 6.1.

Remark 6.4. If ∇_2 is a torsion-free connection, then $T^2(\frac{\partial X_{f_0}}{\partial s}, \frac{\partial X_{f_0}}{\partial x}) = 0$. Moreover dv may be obtained from a pseudo-Riemannian metric and we may choose a ∇_1 to be the Levi-Civita connection of this metric. Then $C_v(\nabla_1) = 0$. We prefer to choose ∇_1 and ∇_2 completely arbitrary to show that the properties of the solution of the variational problem are independent of both connections (compare with the Riemannian case and the Finslerian case, e.g. [**18, 20, 26**]).

Remark 6.5. We use one variation only of the function under consideration, because we think about an application to the *degenerate* case (the rank of g not maximum) and to the constrained case. Consequently we cannot delete the integration symbols. Moreover we cannot delete the integral symbols because the theory of 'equivalent integrals' is not trivial for $m > 1$, cf. [**25**].

Remark 6.6. We may choose ν as the identity map: then $\dot\nu(\rho_0^\circ) = 1$ and $\ddot\nu(\rho_0^\circ) = 0$. We prefer a different choice because of Remark 6.1.

Remark 6.7. When the choices in (6.4), (6.5) and (6.6) are made in the most favorable way (as in the classical case happens) and the non-triviality of 'equivalent integrals' is forgotten then the equation in Theorem 6.1 becomes $2\nabla_{P_f}\big(g(\frac{\partial X_f}{\partial x})\big) - (\nabla g)\frac{\partial X_f}{\partial x} = 0$ where $f = f_0$.

Remark 6.8. If ∇_1 and ∇_2 may be chosen to be the local connections induced by φ and ψ on U and W by the Euclidean connections of $\mathbb{R}^m$ and

$\mathbb{R}^n$, respectively, then Remark 6.7 gives exactly the classical Euler–Lagrange equation and has a local meaning only. Its global meaning can be restored by using the second order jet–bundle of $M \times N$. Hence, our general theory encompasses jet–bundles theory (at least in the second order case) as we stated in the Introduction.

REMARK 6.9. We also observe that when $M = \mathbb{R}$, the tensor field $\frac{\partial X_f}{\partial x}$ can be replaced by the vector field obtained by deleting dx in the local expression of $\frac{\partial X_f}{\partial x}$ with respect to the chart $(\mathbb{R}, id)$. Moreover the connection ∇_1 may be replaced by the standard derivative on $\mathbb{R}$. In this case the covariance with respect to the charts of M is lost as in the Riemannian case.

DEFINITION 6.1. We say that a mapping $f : M \to N$ is a *Z-weak critical point of* $E_{\Omega,g,\nu}$ if f verifies the equation in Theorem 6.1 for a vector field Z defined on the graph $\mathbf{G}_f$ of f.

6.3. Noether equation. Let $\triangle : M \times N \to M \times N$ be a differentiable mapping. Then we put

$$\triangle^1 = pr_1 \circ \triangle : M \times N \to N \quad \text{and} \quad \triangle^2 = pr_2 \circ \triangle : M \times N \to N$$

where pr_1 and pr_2 are the canonical projections on the first and second factor of the Cartesian product, respectively. Let $f_1, f_2 : M \to N$ be differentiable mapping.

DEFINITION 6.2. We say that f_2 is *$\triangle$–related to* f_1 iff

1. $f_2(\triangle^1(x, f_1(x)) = \triangle^2(x, f_1(x))$ for all $x \in M$
2. The mapping $\wedge : M \to N$ defined by $\wedge(x) = \triangle^1(x, f_1(x))$ for each $x \in M$ is a diffeomorphism from M into M. This map is not necessarily onto.

We would like to make an observation which is natural but we have not been able to find it in the literature.

OBSERVATION 6.2. Suppose that $M = N$. We consider the following diffeomorphism $\triangle : M \times N \to M \times N$ such that $\triangle(x, y) = (y, x)$. If the mappings $f_1, f_2 : M \to M$ are $\triangle$-related then

$$f_2(\triangle^1(x, f_1(x)) = \triangle^2(x, f_1(x)). \tag{6.6}$$

We observe also that (6.6) is equivalent to the property that $f_2(f_1(x)) = x$ for all $x \in M$. Hence under the above assumptions we get that $f_1 : M \to M$ is a diffeomorphism and the second condition of Definition 6.2 follows from the first one since $\wedge(x) = f_1(x)$ for all $x \in M$. As a consequence, f_1 has a $\triangle$–related map f_2 iff f_1 is a diffeomorphism. Hence, even when $\triangle$ is a diffeomorphism we can get differentiable mappings $f_1 : M \to N$ having no $\triangle$-related maps $f_2 : M \to N$.

Let $\triangle : M \times N \times S \to M \times N$ be a differentiable map. For each $s \in S$ we consider the differentiable map $\triangle_s : M \times N \to M \times N$ defined by $\triangle_s(x, y) = \triangle(x, y, s)$ for each $(x, y) \in M \times N$. Let $f : M \to N$ be a map.

Definition 6.3. We say that f *is* $(\triangle, S)$-*regular* iff for each $s \in S$ there exists a map $f_s : M \to N$ which is $\triangle_s$-related to f and which depends differentiably on $s \in S$. In such a case the map $\widehat{f} : M \times S \to N$ defined by $\widehat{f}(x, s) = f_s(x)$ is called $(\triangle, S)$-*variation of* f.

Suppose that conditions of Definitions 6.2 and 6.3 are satisfied. Then we put $\Omega_s = \wedge_s(\Omega)$. The family $(\Omega_s)_{s\in S}$ will be called $(\triangle, S, f)$-*variation of* Ω. Finally we say that g is $(\triangle, \nu, \varphi)$-*invariant in* f relatively to Ω iff

$$E_{\Omega_s,f_s,\nu} = E_{\Omega_s,f,\nu} + \int\limits_{\Omega_s} \varphi(x, f_s(x), s)dv \tag{6.7}$$

where $\varphi : M \times N \times S \to \mathbb{R}$ is a differentiable function such that $\widetilde{\varphi} : S \to \mathbb{R}$ defined by

$$\widetilde{\varphi}(s) := \int\limits_{\Omega_s} \varphi(x, f_s(x), s)dv$$

verifies the property

$$(d\widetilde{\varphi})_s = \int\limits_{\Omega_s} \mathrm{di}_v\big(\varphi(x, f_s(x), s)V_s\big)$$

for all $s \in S$ where V_s is defined by (5.8) using the diffeomorphism $\wedge_s$ for each $s \in S$; φ is called the *gauge function*, cf. [**31**]. With this type of generalization the Noether theorem takes the following form.

Theorem 6.2 (Noether). *Property (6.7) implies*

$$dE_{\Omega_s,g,\nu}(f_s) = \int\limits_{\Omega_s} di_v\big(\varphi(x, f_s(x), s)V_s\big). \tag{6.8}$$

The Noether equation is obtained simply by computation by (6.4) of the left hand side of (6.8) for the above particular case and the replacement of Ω_s by Ω using (5.3).

Before we proceed, we shall make the following observation.

Observation 6.3. Through all this paper we prefer to work on the most general case, expecting that this will help us to better understand the components which enter our problem. However this is not the case now because considering the most general case is almost equivalent to multiplying the terms of the right hand side of (6.8), and nothing more.

Hence we suppose that there exists $s_0 \in S$ such that $\triangle_{s_0}$ is the identity mapping. Then we get that $f_{s_0} = f$, $\wedge_{s_0} = id : M \to M$, $\Omega_{s_0} = \Omega$ where id is the identity map. Moreover a simple computation gives

$$(6.9)\qquad (\frac{\partial X_{f_{s_0}}}{\partial x})_{s_0} = \frac{\partial X_f}{\partial x}$$

$$(6.10)\quad \Big((\frac{\partial X_{f_s}}{\partial s})_x\Big)|_{s_0} = \Big((\frac{\partial X_{\wedge_s^2}}{\partial s})_{(x,f(x))}\Big)_{(x,f(x))} + \Big((\frac{\partial X_f}{\partial x})_x(\frac{\partial X_{\wedge_s^{-1}}}{\partial s})_x\Big)_{s_0}.$$

Furthermore from the second condition of Definition 6.2 it follows that

$$\Big((\frac{\partial X_{\wedge_x^{-1}}}{\partial x})_x\Big)_{s_0} = -\Big((\frac{\partial X_{\triangle_s^1}}{\partial s})_{(x,f(x))}\Big)_{s_0}.$$

Hence (6.10) becomes

$$(6.11)\quad \Big((\frac{\partial X_{f_s}}{\partial s})_x\Big)_{s_0} = \Big[(\frac{\partial X_{\triangle_s^2}}{\partial s})_{(x,f(x))} - (\frac{\partial X_f}{\partial x})_x(\frac{\partial X_{\triangle_s^1}}{\partial s})_{(x,f(x))}\Big]_{s_0} =: \eta(x)$$

which is a tensor field along f. Applying (6.4), (6.9) and (6.11) we get that for $s = s_0$ identity (6.8) becomes.

Observation 6.4.

$$\begin{aligned}
&\int_\Omega \dot\nu(\rho)(\nabla_{P_\eta^2} g)(\frac{\partial X_f}{\partial x}, \frac{\partial X_f}{\partial x})dv - 2\int_\Omega \dot\nu(\rho)(\nabla_{P_f^1} g)(\eta, \frac{\partial X_f}{\partial x})dv \\
&- 2\int_\Omega \ddot\nu(\rho)(\nabla_{P_f^1} g)(\frac{\partial X_f}{\partial x}, \frac{\partial X_f}{\partial x})g(\eta, \frac{\partial X_f}{\partial x})dv \\
&+2\int_\Omega \dot\nu(\rho)(C_v\nabla_1)g(\eta, \frac{\partial X_f}{\partial x})dv + 2\int_\Omega \dot\nu(\rho)g(\frac{\partial X_f}{\partial x}, T^2(\eta, \frac{\partial X_f}{\partial x}))dv \\
&- 2\int_\Omega \dot\nu(\rho)g(\eta, \nabla_{P_f^1} P_f^1)dv - 4\int_\Omega \ddot\nu(\rho)g(\eta, \frac{\partial X_f}{\partial x})g(\frac{\partial X_f}{\partial x}, \nabla_{P_f^1} P_f^1)dv \\
&+2\int_\Omega \mathrm{di}_v\Big(\dot\nu(\rho)g(\eta, \frac{\partial X_f}{\partial x})\Big) + 2\int_\Omega \mathrm{di}_v\Big(\nu(\rho)V_{s_0}\Big) \\
&= \int_\Omega \mathrm{di}_v\Big(\varphi(x, f(x)s_0)V_{s_o}\Big)
\end{aligned}$$

where

$$\rho(x) = g_{(x,f(x))}\big((\frac{\partial X_f}{\partial x})_x, (\frac{\partial X_f}{\partial x})_x\big)$$

for all $x \in M$.

The equation in the Observation 6.4 is called *Noether equation.*

Remark 6.10. Suppose that S is an r-dimensional Lie group and $\triangle : M \times M \times S \to M \times N$ is an action of S on $M \times N$ such that $\nu = id$ and $s_0 = 1_S$ i.e. the unity of S. If each $f : M \to N$ is $\triangle$-regular then Observation 6.4 is exactly the generalization given by H. Rund, cf. [**25**, p.294] of the Noether equation written in the covariant way. The original formulation of this equation is obtained by taking $\varphi = 0$. Moreover suppose that Obseravation 6.4 holds for each 'critical point' of $E_{\Omega,g,\nu}$ which is to be $\triangle$-regular. A critical point means here Z-critical for each Z coming from the set of homotopic variations admissible for the variational problem (we recall that in the classical problem one considers all homotopic variations). One obtains the so called 'conservation laws on shell', cf. [**12**].

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Università di Bari
Dipartimento di Matematica
Via Orabona 4
70125 Bari, Italy
e-mail: casciaro@dm.uniba.it
e-mail: konderak@dm.uniba.it