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PRODUCT INTEGRAL IN A FRÉCHET ALGEBRA

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Abstract. Product integral of locally summable function in a Fréchet algebra is defined and some of its properties are proved. The main tool is the Arens-Michael representation of Fréchet algebra which allows us to extend the notion of the product integral from a Banach to Fréchet algebra.

1. Introduction. Let $\mathcal{X}$ be a Fréchet algebra, i.e. $\mathcal{X}$ is a complex algebra which is a Fréchet space with topology induced by an increasing sequence of seminorms $(p_n)_{n\in\mathbb{N}}$ such that

$$p_n(xy) \le p_n(x)p_n(y) \qquad \forall n \in \mathbb{N}\ \forall x, y \in \mathcal{X}.$$

and $\mathcal{X}$ contains a unit $\mathbb{I}$ such that $p_n(\mathbb{I}) = 1 \qquad \forall n \in \mathbb{N}$.

A linear mapping $h : \mathcal{X} \to \mathcal{Y}$, where $\mathcal{X}$, $\mathcal{Y}$ are Fréchet algebras, is homomorphism of algebras iff $h(xy) = h(x)h(y) \quad \forall x, y \in \mathcal{X}$. Obviously, $h(\mathbb{I}_{\mathcal{X}}) = \mathbb{I}_{\mathcal{Y}}$, where $\mathbb{I}_{\mathcal{X}}$, $\mathbb{I}_{\mathcal{Y}}$ are units in $\mathcal{X}$, $\mathcal{Y}$, respectively.

For example, if $\Omega \in \operatorname{top} \mathbb{C}^n$, B is a complex Banach space, $End\,B$ denotes the algebra of linear continuous mappings $B \to B$, then $\mathcal{H}ol(\Omega, End\,B)$, the space of all $End\,B$-valued holomorphic functions on Ω, is a Fréchet algebra. In fact, there is a sequence $K_1, K_2, \dots$ of compact subsets of Ω such that for each $s \in \mathbb{N}$: $K_s \subset \operatorname{int} K_{s+1}$ and $\bigcup_{s=1}^{\infty} K_s = \Omega$. For any $h \in \mathcal{H}ol(\Omega, End\,B)$, set

$$p_s(h) := \sup\{\|h(z)\|_{End\,B} :\ z \in K_s\}.$$

Then $(p_s)_{s\in\mathbb{N}}$ is an increasing sequence of seminorms that defines local uniform convergence topology in $\mathcal{H}ol(\Omega, End\,B)$.

Setting, for any $h, f \in \mathcal{H}ol(\Omega, End\,B)$:

$$h \cdot f : \Omega \ni z \overset{\text{def}}{\longmapsto} h(z) \circ f(z) \in End\,B,$$

from the continuity of multiplication in the Banach algebra $End\,B$, we get that the seminorms p_s are submultiplicative.

Obviously, if $End\,B$ is not commutative then $\mathcal{H}ol(\Omega, End\,B)$ is a noncommutative Fréchet algebra.

The aim of this paper is to extend the notion of the product integral to Fréchet algebras. This kind of integrals have naturally arisen in the algebra $\mathcal{H}ol(\mathbb{C}^n, End\,B)$ while solving distributional Cauchy problem

$$\begin{cases} \frac{d}{dt}u(t) &= \sum_\alpha A_\alpha(t) \circ D^\alpha u(t) + f(t) \\ u(t_0) &= u_0 \end{cases}$$

(given $A_\alpha \in End\,B$, $f : J \to \mathcal{D}'_{temp}(\mathbb{R}^n; B)$, $u_0 \in \mathcal{D}'_{temp}(\mathbb{R}^n; B)$ – the space of tempered distributions). Details of this application will be included in a forthcoming publication.

The basic tool in the study of Fréchet algebras is a representation of $\mathcal{X}$ as an inverse limit of Banach algebras. In order to establish notation, we shall recall fundamental theorem on Arens-Michael representation of a Fréchet algebra, see **[2]**, **[5]**, **[1]** .

Let $\mathcal{X}$ be a Fréchet algebra with the increasing sequence $(p_n)_{n\in\mathbb{N}}$ of submultiplicative seminorms. Let $\pi_n : \mathcal{X} \to X_n := \mathcal{X}/_{\{p_n=0\}}$ be the quotient map for each $n \in \mathbb{N}$. $(X_n)_{n\in\mathbb{N}}$ is a family of normed algebras. Let us consider the family of its completions $(B_n)_{n\in\mathbb{N}}$ (Banach algebras) and the family of isometric homomorphisms $J_n : X_n \to B_n$ $(n \in \mathbb{N})$ with dense images. For any $n \in \mathbb{N}$, there is the unique mapping $\eta_n : X_n \to X_{n-1}$ that commutes the diagram

(1)
$$\begin{tikzcd} \mathcal{X} \arrow[r, "\pi_n"] \arrow[dr, "\pi_{n-1}"'] & X_n \arrow[d, "\eta_n"] \\ & X_{n-1} \end{tikzcd} ;$$

η_n is a continuous homomorphism of algebras and $|\eta_n| \le 1$. Since $J_n(X_n)$ is dense in B_n for any $n \in \mathbb{N}$, there is the unique mapping $\overline{\eta_n} : B_n \to B_{n-1}$ that commutes the diagram

(2)
$$\begin{tikzcd} X_n \arrow[r, "J_n"] \arrow[d, "\eta_n"'] & B_n \arrow[d, "\bar{\eta}_n"] \\ X_{n-1} \arrow[r, "J_{n-1}"] & B_{n-1} \end{tikzcd}$$

and such that $\overline{\eta_n}$ is a homomorphism of algebras and $|\overline{\eta_n}| \le 1$.

THEOREM 3. *The set*

$$B := \{(b_n)_{n\in\mathbb{N}} \in \bigtimes_{n\in\mathbb{N}} B_n : \quad \overline{\eta_n}(b_n) = b_{n-1} \quad \forall n \in \mathbb{N}\}$$

is a Fréchet algebra with topology induced by the increasing sequence of seminorms

$$q_k((b_n)_{n\in\mathbb{N}}) := |b_k|_{B_k} \qquad \text{for } k \in \mathbb{N}.$$

Moreover, the mapping

$$\iota : \mathcal{X} \ni x \overset{\text{def}}{\longmapsto} ((J_n \circ \pi_n)(x))_{n\in\mathbb{N}} \in B$$

is a topological isomorphism of Fréchet algebras.

REMARK 4. If $\mathcal{X}$ is a Fréchet algebra, then there is a countable family of continuous homomorphisms of algebras $\mathcal{H} \subset \bigcup_{\mathcal{Y}} \mathcal{H}om\,(\mathcal{X}, \mathcal{Y})$, $\mathcal{Y}$ being Banach algebras, that separates points on $\mathcal{X}$.

PROOF. Obviously, $(\pi_n)_{n\in\mathbb{N}}$ separates points on $\mathcal{X}$. The family of continuous homomorphisms $(J_n \circ \pi_n)_{n\in\mathbb{N}}$ separates points on $\mathcal{X}$. □

In the case that a Fréchet space X does not have the structure of algebra, theorem 3 is still true, but $(B_n)_{n\in\mathbb{N}}$ denotes a family of Banach spaces and $J_n : X_n \to B_n\,(n \in \mathbb{N})$ a family of linear isometries with dense images. In such a case mappings η_n, $\bar{\eta}_n$ are linear continuous, B is a Fréchet space, ι is a topological isomorphism.

Obviously, when X is a Fréchet space, there is a countable family $\mathcal{L} \subset \bigcup_{Y} \mathcal{L}(X, Y)$, Y being Banach spaces, that separates points on X, for example $(J_n \circ \pi_n)_{n\in\mathbb{N}}$ separates points on X.

In the sequel, we will use the notions of the absolutely continuous function with values in a Fréchet space and of its integral. For the reader's convenience, we enclose definitions of those notions and some of their properties. The integral is understood in the generalized Bochner sense. Thanks to K. Holly [**4**], differently from [**6**], we use Arens-Michael representation in our definitions.

2. Absolute continuity and integration in a Fréchet space. If X be a Fréchet space, let us consider an increasing sequence of seminorms $(p_n)_{n\in\mathbb{N}}$ that induces the topology $\operatorname{top} \mathrm{X}$. Let $\pi_n : X \to X_n := X/_{\{p_n=0\}}$ be the quotient maps and $(B_n)_{n\in\mathbb{N}}$ the family of Banach spaces with corresponding family of linear isometries $J_n : X_n \to B_n$ having dense images in B_n.

Let J be an interval in $\mathbb{R}$ and let $\operatorname{sn} \mathrm{X}$ denote the cone of all continuous seminorms on X.

DEFINITION 5. A function $f : J \to X$ is absolutely continuous iff for any $p \in \mathrm{sn}\,\mathrm{X}$ the function $p \circ f$ is absolutely continuous.

THEOREM 6. *The following conditions are equivalent:*

(i): $f : J \to X$ *is absolutely continuous;*
(ii): $\forall n \in \mathbb{N}\ p_n \circ f$ *is absolutely continuous;*
(iii): $\forall n \in \mathbb{N}\ J_n \circ \pi_n \circ f : J \to B_n$ *is absolutely continuous;*
(iv): *for any Banach space* Y *and any map* $L \in \mathcal{L}(X, Y)$*, the function* $L \circ f : J \to Y$ *is absolutely continuous.*

PROOF. It is sufficient to prove (iii) $\Rightarrow$ (iv). Let Y be a Banach space and $L \in \mathcal{L}(X, Y)$. There is $n \in \mathbb{N}$ such that L is p_n-continuous. Since J_n^{-1} is densely defined in B_n, there is the unique $\bar{L}_n \in \mathcal{L}(B_n, Y)$ such that the diagram:

$$\begin{tikzcd} X \arrow[r, "\pi_n"] \arrow[dr, "L"'] & X_n \arrow[r, "J_n"] \arrow[d] & B_n \arrow[dl, "\bar{L}_n"] \\ & Y & \end{tikzcd} \tag{7}$$

commutes and $L \circ f = \bar{L}_n \circ (J_n \circ \pi_n \circ f)$ is absolutely continuous. □

LEMMA 8. *Let* X*,* Y *be Fréchet spaces,* $L \in \mathcal{L}(X, Y)$*, and* $f : J \to X$ *absolutely continuous. Then the function* $L \circ f : J \to Y$ *is absolutely continuous.*

PROOF. Let $L \in \mathcal{L}(X, Y)$ and $q \in \mathrm{sn}\,\mathrm{Y}$. There are $p \in \mathrm{sn}\,\mathrm{X}$ and $C > 0$ such that $q(Lx) \le Cp(x)$ for all $x \in X$. Hence

$$q(L(f(t))) \le C \cdot p(f(t)) \qquad \text{for all } t \in J.$$

Since $p \circ f$ is absolutely continuous, so is $q \circ L \circ f$. This means that $L \circ f$ is absolutely continuous as $q \in \mathrm{sn}\,\mathrm{Y}$ was arbitrary. □

LEMMA 9. *Let* X*,* Y *be Fréchet spaces. Consider* $L \in \mathcal{L}(X, Y)$ *and a function* $f : J \to X$ *differentiable almost everywhere. Then the function* $L \circ f : J \to Y$ *is differentiable almost everywhere and* $(L \circ f)'(t) = L(f'(t))$ *for a.e.* $t \in J$*.*

PROOF. Let $t \in \mathrm{dom}\, f'$ and $q \in \mathrm{sn}\,\mathrm{Y}$. There are $p \in \mathrm{sn}\,\mathrm{X}$ and $C > 0$ such that

$$q\left(\frac{(L \circ f)(t+h) - (L \circ f)(t)}{h} - L(f'(t))\right) =$$
$$= q\left(L\left(\frac{f(t+h) - f(t)}{h} - f'(t)\right)\right) \le C \cdot p\left(\frac{f(t+h) - f(t)}{h} - f'(t)\right).$$

Letting $h \to 0$ we conclude that $L \circ f$ is differentiable in the set $\mathrm{dom} f'$ and $(L \circ f)'(t) = L(f'(t))$ for all $t \in \mathrm{dom} f'$. □

Lemma 10. *Let X, Y be Fréchet spaces. If $f : J \to X$ is absolutely continuous and $g : X \to Y$ satisfies the Lipschitz condition, then $g \circ f : J \to Y$ is absolutely continuous.*

Proof. Let $q \in \operatorname{sn} \mathrm{Y}$, $t, s \in J$. There are a seminorm $p \in \operatorname{sn} \mathrm{X}$ and a constant $L > 0$ such that for all $\tilde{x}, \tilde{y} \in X$

$$q(g(\tilde{x}) - g(\tilde{y})) \leq L \cdot p(\tilde{x} - \tilde{y}).$$

In particular, $q((g \circ f)(t) - (g \circ f)(s)) \leq L \cdot p(f(t) - f(s))$ and from the absolute continuity of f it follows that $g \circ f$ is absolutely continuous. □

Let X be a Fréchet space. Let us denote by M a measurable space with a nonnegative measure μ.

Definition 11. A function $f : M \to X$ is measurable in the sense of Bochner iff $f(M)$ is separable in X and f is measurable.

Lemma 12. *Let us consider a function $f : M \to X$. The following conditions are equivalent:*

- **(i):** *$f : M \to X$ is measurable in the sense of Bochner;*
- **(ii):** *$\forall p \in \operatorname{sn} \mathrm{X}$ $f(M)$ is p-separable and f is p-measurable;*
- **(iii):** *$\forall n \in \mathbb{N}$ $f(M)$ is p_n-separable and f is p_n-measurable;*
- **(iv):** *if Y is a Banach space and $L \in \mathcal{L}(X, Y)$, then $L \circ f$ is measurable in the sense of Bochner.*

Proof. Implications (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) and (i) $\Rightarrow$ (iv) are obvious, so it suffices to show that (iii) $\Rightarrow$ (i) and (iv) $\Rightarrow$ (ii). To deal with the former, observe that for $n \in \mathbb{N}$ there exists an at most countable set $Q_n \subset f(M)$ which is p_n-dense in $f(M)$, so the set $Q := \bigcup_{n \in \mathbb{N}} Q_n$ is dense in $f(M)$.

Let $\mathcal{O} \in \operatorname{top} \mathrm{X}$. We prove that $f^{-1}(\mathcal{O})$ is measurable. Let $x \in \mathcal{O} \cap f(M)$. $\mathcal{O} - x$ is a neighbourhood of zero, hence there are $n \in \mathbb{N}$ and $r > 0$ such that $K_{p_n}(0, r) \subset \mathcal{O} - x$. Therefore $(x + K_{p_n}(0, r)) \cap f(M) \in \operatorname{top}(\mathcal{O} \cap \mathrm{f(M)})$. Consequently, there is a sequence of sets $(E_k)_{k \in \mathbb{N}}$, of type $(x + K_{p_n}(0, r)) \cap f(M)$, such that $\displaystyle\bigcup_{k \in \mathbb{N}} E_k = \mathcal{O} \cap f(M)$. The function f is p_n-measurable, thus $f^{-1}(x + K_{p_n}(0, r))$ is measurable for all $n \in \mathbb{N}$ and $r > 0$. Finally,

$$f^{-1}(\mathcal{O}) = f^{-1}(\mathcal{O} \cap f(M)) = f^{-1}\left(\bigcup_{k \in \mathbb{N}} E_k\right) = \bigcup_{k \in \mathbb{N}} f^{-1}(E_k)$$

is a measurable set. Therefore $f : M \to X$ is measurable.

We now turn to (iv) $\Rightarrow$ (ii). Let $p \in \operatorname{sn} \mathrm{X}$. Let $\pi : X \to X/_{\{p=0\}}$ denote the quotient map. Let us consider a Banach space Y and an isometry $\mathcal{J} : X/_{\{p=0\}} \to Y$. Obviously, $L := \mathcal{J} \circ \pi \in \mathcal{L}(X, Y)$, so by (iv) $(L \circ f)(M)$ is separable in Y and $L \circ f : M \to Y$ is measurable. Thus $L \circ f : M \to \operatorname{im} \mathcal{J}$ is measurable, thence $\pi \circ f = \mathcal{J}^{-1} \circ L \circ f : M \to X/_{\{p=0\}}$ is measurable.

Let $\mathcal{O} \in \operatorname{top}(\mathrm{X}, \mathrm{p})$. Then $\pi(\mathcal{O}) \in \operatorname{top} \mathrm{X}/_{\{\mathrm{p}=0\}}$ and the set $(\pi \circ f)^{-1}(\pi(\mathcal{O}))$ is measurable. But

$$(\pi \circ f)^{-1}(\pi(\mathcal{O})) = f^{-1}(\mathcal{O}),$$

hence $f : M \to X$ is p-measurable.

Similarly, $(\pi \circ f)(M)$ is separable in $X/_{\{p=0\}}$. Therefore there exists an at most countable set $Q := \{q_1, q_2, \ldots\}$ which is dense in $(\pi \circ f)(M)$. For each $i \in \mathbb{N}$ there is $x_i \in f(M)$ such that $q_i = \pi(x_i)$. We see at once that $\{x_1, x_2, x_3, \ldots\}$ is p-dense in $f(M)$. □

Definition 13. A function $f : M \to X$ is summable iff f is measurable in the sense of Bochner and $\forall p \in \operatorname{sn} \mathrm{X} \displaystyle\int_M (p \circ f) d\mu < \infty$.

Lemma 14. *If X, Y are Fréchet spaces, $L \in \mathcal{L}(X, Y)$ and $f : M \to X$ is summable, then $L \circ f$ is summable.*

Proof. It is clear that $L \circ f$ is measurable in the sense of Bochner. Let $q \in \operatorname{sn} \mathrm{Y}$. Then $p := q \circ L \in \operatorname{sn} \mathrm{X}$ and

$$\infty > \int_M (p \circ f) d\mu = \int_M (q \circ (L \circ f)) d\mu.$$

Therefore $L \circ f$ is summable. □

Lemma 15. *Assume that $f : M \to X$ is measurable in the sense of Bochner. Then the following conditions are equivalent:*

(i): *f is summable;*
(ii): $\forall n \in \mathbb{N} \displaystyle\int_M (p_n \circ f) d\mu < \infty$;
(iii): *if Y is a Banach space and $L \in \mathcal{L}(X, Y)$, then $L \circ f$ is summable.*

Proof. From the definition it follows that (i) $\Rightarrow$ (ii). Conversely, for every $p \in \operatorname{sn} \mathrm{X}$, there are $n \in \mathbb{N}$ and $C > 0$ such that $p \leq C p_n$. Therefore $\displaystyle\int_M (p \circ f) d\mu \leq C \int_M (p_n \circ f) d\mu < \infty$.

By virtue of Lemma 12, (i) $\Rightarrow$ (iii). We will prove that (iii) $\Rightarrow$ (i). Let $p \in \operatorname{sn} \mathrm{X}$. As in the proof of Lemma 12 ((iv) $\Rightarrow$ (ii)), let us consider the quotient map π, a Banach space Y, an isometry $\mathcal{J}$, and $L := \mathcal{J} \circ \pi \in \mathcal{L}(X, Y)$. The function $L \circ f$ is summable. Thus

$$\begin{aligned}\infty > \int_M |(L \circ f)(t)|_Y d\mu(t) = & \int_M |\mathcal{J}((\pi \circ f)(t))|_Y d\mu(t) = \\ = & \int_M |\pi(f(t))|_{X/_{\{p=0\}}} d\mu(t) = \int_M p(f(t)) d\mu(t).\end{aligned}$$

□

Theorem 16. *Let $f : M \to X$ be summable. Then there is the unique vector in X, denoted by $\int_M f\, d\mu \in X$, such that if Y is a Banach space and $L \in \mathcal{L}(X, Y)$, then*

$$L\left(\int_M f d\mu\right) = \int_M (L \circ f) d\mu.$$

Proof. We shall use the symbols η_n, $\bar{\eta}_n$, B, ι from Theorem 3. Lemma 15 implies that $J_n \circ \pi_n \circ f$ is summable for all $n \in \mathbb{N}$. We claim that $\left(\int_M (J_n \circ \pi_n \circ f) d\mu\right)_{n \in \mathbb{N}} \in B$. Indeed, for any $n \in \mathbb{N}$

$$\bar{\eta}_n \left(\int_M (J_n \circ \pi_n \circ f) d\mu\right) = \int_M (\bar{\eta}_n \circ J_n \circ \pi_n \circ f) d\mu.$$

Since the diagram (2) commutes, there is $\bar{\eta}_n \circ J_n = J_{n-1} \circ \eta_n$. Moreover, the diagram (1) commutes, so $\eta_n \circ \pi_n = \pi_{n-1}$. Consequently,

$$\int_M (\bar{\eta}_n \circ J_n \circ \pi_n \circ f) d\mu = \int_M (J_{n-1} \circ \eta_n \circ \pi_n \circ f) d\mu = \int_M (J_{n-1} \circ \pi_{n-1} \circ f) d\mu,$$

which proves our claim.

According to Theorem 3, ι is an isomorphism and for this reason there is $a \in X$ such that $(J_n \circ \pi_n)(a) = \int_M (J_n \circ \pi_n \circ f) d\mu$ for every $n \in \mathbb{N}$. Applying now the fact that the diagram (7) commutes, we obtain

$$L(a) = (\bar{L}_n \circ J_n \circ \pi_n)(a) = \bar{L}_n \left(\int_M (J_n \circ \pi_n \circ f) d\mu\right) = \int_M (L \circ f) d\mu,$$

which means that $a \in X$ is the desired vector.

To prove the uniqueness, it is sufficient to note that the family $\bigcup_Y \mathcal{L}(X, Y)$, Y being a Banach space, seperates points on X.

□

Lemma 17. *Let X, Y be Fréchet spaces and $L \in \mathcal{L}(X, Y)$. Assume that a function $f : M \to X$ is summable. Then $L\left(\int_M f d\mu\right) = \int_M (L \circ f) d\mu$.*

PROOF. If B is a Banach space and $T \in \mathcal{L}(Y, B)$, then $T \circ L \in \mathcal{L}(X, B)$. Setting $y := L\left(\int_M f d\mu\right)$, we have

$$T(y) = (T \circ L)\left(\int_M f d\mu\right) = \int_M (T \circ L \circ f) d\mu.$$

As B and T are arbitrary, Theorem 16 leads to $y = \displaystyle\int_M (L \circ f) d\mu$. □

LEMMA 18. *If $p \in \operatorname{sn} \mathrm{X}$ and a function $f : M \to X$ is summable, then*

$$p\left(\int_M f d\mu\right) \leq \int_M (p \circ f) d\mu.$$

PROOF. As in the proof of Lemma 12 ((iv)⇒(ii)), let $L := \mathcal{J} \circ \pi \in \mathcal{L}(X, Y)$; from Theorem 16

$$L\left(\int_M f d\mu\right) = \int_M (L \circ f) d\mu.$$

Obviously, $|L(x)|_Y = p(x)$ for every $x \in X$. In this way

$$\begin{aligned} p\left(\int_M f d\mu\right) &= \left|L\left(\int_M f d\mu\right)\right|_Y = \left|\int_M (L \circ f) d\mu\right|_Y \leq \\ &\leq \int_M |L(f(t))|_Y d\mu(t) = \int_M p(f(t)) d\mu(t). \end{aligned}$$

□

In consequence, we have

REMARK 19. If $f, g : M \to X$ are summable and equal a.e, then

$$\int_M f d\mu = \int_M g\, d\mu.$$

3. Product integral. Let $\mathcal{X}$ be a Fréchet algebra and let J be an interval in $\mathbb{R}$.

THEOREM 20. *Given $t_0 \in J$ and $\mathrm{x}_0 \in \mathcal{X}$, suppose that $A : J \to \mathcal{X}$ is locally summable. Then the problem*

$$\left\{ \begin{array}{rcl} \dot{x} &=& A(t)x \\ x(t_0) &=& \mathrm{x}_0 \end{array} \right. \tag{21}$$

has the unique (absolutely continuous) solution.

PROOF. Let $(p_n)_{n \in \mathbb{N}}$ be an increasing sequence of seminorms that induces the topology in $\mathcal{X}$. Let us consider the family $X_n := \mathcal{X}/_{\{p_n = 0\}}$ $(n \in \mathbb{N})$ of normed algebras and the family B_n of their respective completions. Consider the family of quotient maps $\pi_n : \mathcal{X} \to X_n$ $(n \in \mathbb{N})$ and the family of isometric homomorphisms $J_n : X_n \to B_n$ ($n \in \mathbb{N}$) with dense images. Let η_n, $\bar{\eta}_n$, ι be homomorphisms of algebras and let B be the Fréchet algebra introduced in Theorem 3.

Our proof starts with showing the existence of solution. By virtue of Theorem 18, the function $J_n \circ \pi_n \circ A : J \to B_n$ is locally summable for any $n \in \mathbb{N}$. Thus the problem

$$(21)_n \qquad \begin{cases} \dot{y} &= (J_n \circ \pi_n \circ A)(t)y \\ y(t_0) &= (J_n \circ \pi_n)(\mathrm{x}_0), \end{cases}$$

in Banach algebra B_n has the unique solution $y_n : J \to B_n$ for each $n \in \mathbb{N}$. Set $y := (y_n)_{n\in\mathbb{N}}$. We first check whether the function y has values in B.

Obviously, $\bar{\eta}_n \circ y_n : J \to B_{n-1}$ is absolutely continuous and differentiable a.e. Set $Z := \bigcup_{n=1}^{\infty} Z_n$, with Z_n the subset of J on which $\dot{y}_n$ does not exist. Clearly, $\mu(Z) = 0$. Fix $n \in \mathbb{N}$, $t \in J \setminus Z$. We have $\frac{d}{dt}(\bar{\eta}_n \circ y_n)(t) = \bar{\eta}_n(\dot{y}_n(t))$ since $\bar{\eta}_n \in \mathcal{L}(B_n, B_{n-1})$. On the other hand, y_n satisfies equation $(21)_n$, so

$$(22) \qquad \frac{d}{dt}(\bar{\eta}_n \circ y_n)(t) = (\bar{\eta}_n \circ J_n \circ \pi_n \circ A)(t) \cdot (\bar{\eta}_n \circ y_n)(t).$$

Since diagrams (2), (1) commute, (22) shows that

$$\frac{d}{dt}(\bar{\eta}_n \circ y_n)(t) = (J_{n-1} \circ \pi_{n-1} \circ A)(t) \cdot (\bar{\eta}_n \circ y_n)(t).$$

Similarly,

$$(\bar{\eta}_n \circ y_n)(t_0) = (\bar{\eta}_n \circ J_n \circ \pi_n)(\mathrm{x}_0) = (J_{n-1} \circ \pi_{n-1})(\mathrm{x}_0).$$

Therefore, both $\bar{\eta}_n \circ y_n : J \to B_{n-1}$ and y_{n-1} are solutions of $(21)_{n-1}$. But $(21)_{n-1}$ (a problem in Banach algebra B_n) has the unique solution, hence $\bar{\eta}_n \circ y_n = y_{n-1}$, and finally $y : J \to B$.

It is easy to check that $y : J \to B$ is absolutely continuous, differentiable a.e. and for $t \in J \setminus Z$, $\dot{y}(t) = (\dot{y}_n(t))_{n\in\mathbb{N}}$.

Set $x := \iota^{-1} \circ y$. Lemmas 8 and 9 easily imply that x is absolutely continuous, differentiable a.e. and $\dot{x} = \iota^{-1} \circ \dot{y}$ almost everywhere.

Fix $t \in J \setminus Z$.

$$\begin{aligned} \dot{x}(t) = {} & \iota^{-1}\left((\dot{y}_n(t))_{n\in\mathbb{N}}\right) = \iota^{-1}\left(((J_n \circ \pi_n \circ A(t)) \cdot y_n(t))_{n\in\mathbb{N}}\right) = \\ = {} & \iota^{-1}\left(((J_n \circ \pi_n \circ A(t))_{n\in\mathbb{N}} \cdot (y_n(t))_{n\in\mathbb{N}}\right) = A(t) \cdot \iota^{-1}(y(t)) = A(t)x(t). \end{aligned}$$

In the same manner, $x(t_0) = \mathrm{x}_0$. Therefore $x : J \to \mathcal{X}$ is a solution of (21).

To prove the uniqueness, assume that $x, \tilde{x} : J \to \mathcal{X}$ are two solutions of (21). Let us consider a Banach algebra $\mathcal{Y}$ and a homomorphism $h : \mathcal{X} \to \mathcal{Y}$.

Then $h \circ x$, $h \circ \tilde{x}$ are easily seen to be solutions of the problem

$$\begin{cases} \dot{y} &= (h \circ A(t)) \cdot y \\ y(t_0) &= h(\mathrm{x}_0) \end{cases}$$

in Banach algebra $\mathcal{Y}$. But this problem has the unique solution, thus $h \circ x = h \circ \tilde{x}$, and finally, by Remark 4, we conclude that $x = \tilde{x}$. □

THEOREM 23. *Fix $t_0 \in J$ and consider sequences $l_\nu \to l$ in $\mathcal{X}$ and $A_\nu \to A$ in $L^1_{loc}(J, \mathcal{X})$ as $\nu \to \infty$. Let $x_\nu : J \to \mathcal{X}$ denote a solution of the problem*

$$\begin{cases} \dot{x} &= A_\nu(t)x \\ x(t_0) &= l_\nu \end{cases} \tag{24}$$

for $\nu \in \mathbb{N}$, and let $x : J \to \mathcal{X}$ be a solution of

$$\begin{cases} \dot{x} &= A(t)x \\ x(t_0) &= l. \end{cases} \tag{25}$$

Then $x_\nu \to x$ almost uniformly in $\mathcal{X}$ as $\nu \to \infty$.

PROOF. $J_n \circ \pi_n \in \mathcal{L}(\mathcal{X}, B_n)$ so, according to Theorem 18, the functions $J_n \circ \pi_n \circ A_\nu : J \to B_n$ and $J_n \circ \pi_n \circ A : J \to B_n$ are locally summable $(\forall \nu, n \in \mathbb{N})$.

Let us denote by $y_{\nu,n} : J \to B_n$, $y_n : J \to B_n$ the solutions of the Cauchy problems

$$\begin{cases} \dot{y} &= (J_n \circ \pi_n \circ A_\nu(t))y \\ y(t_0) &= (J_n \circ \pi_n)(l_\nu), \end{cases} \qquad \begin{cases} \dot{y} &= (J_n \circ \pi_n \circ A(t))y \\ y(t_0) &= (J_n \circ \pi_n)(l), \end{cases}$$

in the Banach algebras B_n. By assumption $l_\nu \to l$ in $\mathcal{X}$ as $\nu \to \infty$, thus $(J_n \circ \pi_n)(l_\nu) \to (J_n \circ \pi_n)(l)$ in B_n as $\nu \to \infty$ $(\forall n \in \mathbb{N})$.

Let $\mathrm{x} \in \mathcal{X}$. Then

$$|(J_n \circ \pi_n)(\mathrm{x})|_{B_n} = |\pi_n(\mathrm{x})|_{X_n} = p_n(\mathrm{x}) \qquad \forall n \in \mathbb{N}. \tag{26}$$

Since $A_\nu \stackrel{\nu\to\infty}{\longrightarrow} A$ in $L^1_{loc}(J, \mathcal{X})$, (26) shows that $J_n \circ \pi_n \circ A_\nu \stackrel{\nu\to\infty}{\longrightarrow} J_n \circ \pi_n \circ A$ in $L^1_{loc}(J, B_n)$. In consequence, $y_{\nu,n} \to y_n$ almost uniformly in B_n as $\nu \to \infty$ $(\forall n \in \mathbb{N})$.

As in the proof of Theorem 20, $x_\nu := \iota^{-1} \circ y_\nu$ is the solution of (24), while $x := \iota^{-1} \circ y$ is the solution of (25), where $y_\nu := (y_{\nu,n})_{n\in\mathbb{N}}$, $y := (y_n)_{n\in\mathbb{N}}$. Now we prove that $x_\nu \to x$ almost uniformly as $\nu \to \infty$.

Let I be a compact subset of J. Let $t \in I$, $n \in \mathbb{N}$. There are $C > 0$ and $k \in \mathbb{N}$ such that

$$p_n(x_\nu(t) - x(t)) \leq C\, q_k(y_\nu(t) - y(t)) = C\, |y_{\nu,k}(t) - y_k(t)|_{B_k}. \tag{27}$$

Since, in particular, $y_{\nu,k} \to y_k$ uniformly in I as $\nu \to \infty$, (27) shows, (as $n \in \mathbb{N}$ is arbitrary) that also $x_\nu \to x$ uniformly in I as $\nu \to \infty$. □

We are now in a position to define the product integral in a Fréchet algebra.

Let $\mathcal{X}$ be a Fréchet algebra with unit $\mathbb{I}$. Consider a locally summable function $A : J \to \mathcal{X}$.

DEFINITION 28. Let $a, b \in J$. Assume that $x : J \to \mathcal{X}$ is a solution of the Cauchy problem

$$\begin{cases} \dot{x} &= A(t)x \\ x(a) &= \mathbb{I}. \end{cases} \tag{29}$$

The element $x(b) \in \mathcal{X}$ is called the product integral of A and is denoted by $\prod_a^b e^{A(s)ds}$.

The following properties are straightforward

REMARK 30. **i):** $\prod_a^a e^{A(s)ds} = \mathbb{I}$;

ii): $\frac{d}{dt} \prod_a^t e^{A(s)ds} = A(t) \prod_a^t e^{A(s)ds}$ for a.e. $t \in J$;

iii): if $A = \tilde{A}$ a.e. then $\prod_a^b e^{\tilde{A}(s)ds} = \prod_a^b e^{A(s)ds}$

REMARK 31. Fix $t_0 \in J$, $\mathrm{x}_0 \in \mathcal{X}$. The function $x : J \ni t \mapsto \prod_{t_0}^t e^{A(s)ds} \cdot \mathrm{x}_0 \in \mathcal{X}$ is the unique solution of the Cauchy problem

$$\begin{cases} \dot{x} &= A(t)x \\ x(t_0) &= \mathrm{x}_0. \end{cases}$$

REMARK 32. For all $a, b, c \in J$

$$\prod_a^c e^{A(s)ds} = \prod_b^c e^{A(s)ds} \cdot \prod_a^b e^{A(s)ds} \quad \text{and} \quad \left(\prod_a^b e^{A(s)ds} \right)^{-1} = \prod_b^a e^{A(s)ds}.$$

REMARK 33. For all $n \in \mathbb{N}$

$$p_n\left(\prod_a^b e^{A(s)ds}\right) \le \exp\left(\int_{[a,b]} p_n(A(s))ds\right) \qquad \text{and}$$

$$p_n\left(\prod_a^b e^{A(s)ds} - \mathbb{I}\right) \le \exp\left(\int_{[a,b]} p_n(A(s))ds\right) - 1.$$

PROOF. Denoting by $\varphi(t) := \prod_a^t e^{A(s)ds}$, we have $\varphi(t) = \mathbb{I} + \int_a^t A(\tau)\varphi(\tau)d\tau$. Thus

$$p_n(\varphi(t)) \le 1 + p_n\left(\int_a^t A(\tau)\varphi(\tau)d\tau\right) \le 1 + \int_{[a,t]} p_n(A(\tau))p_n(\varphi(\tau))d\tau$$

and from the Gronwall inequality

$$p_n(\varphi(b)) \le \exp\left(\int_{[a,b]} p_n(A(s))ds\right).$$

The proof of the second inequality is similar. □

Let $\mathcal{X}$ be a Fréchet algebra with unit $\mathbb{I}$. Setting $x^0 = \mathbb{I}$ for $x \in \mathcal{X}$, we define

$$\exp x := \sum_{n=0}^{\infty} \frac{x^n}{n!}.$$

Obviously, $\exp 0 = \mathbb{I}$ and $p_s(\exp x) \le \exp(p_s(x))\quad \forall x \in \mathcal{X}\ \forall s \in \mathbb{N}$.

Moreover, if $\mathcal{X}$, $\mathcal{Y}$ are Fréchet algebras and $L : \mathcal{X} \to \mathcal{Y}$ is a continuous homomorphism of algebras, then

$$\exp \circ L = L \circ \exp. \tag{34}$$

LEMMA 35. *If for a.e.* $t, s \in J \quad A(t)A(s) = A(s)A(t)$ *then*

$$\prod_a^b e^{A(s)ds} = \exp\left(\int_a^b A(s)ds\right).$$

PROOF. Let $\varphi(t) := \prod_a^t e^{A(\tau)d\tau} \in \mathcal{X}$, $y_n(t) := \prod_a^t e^{(J_n\circ\pi_n\circ A)(\tau)d\tau} \in B_n$ for $n \in \mathbb{N}$. According to the assumption,

$$(J_n \circ \pi_n \circ A)(t) \cdot (J_n \circ \pi_n \circ A)(s) = (J_n \circ \pi_n \circ A)(s) \cdot (J_n \circ \pi_n \circ A)(t)$$

for almost all $t, s \in J$ and for all $n \in \mathbb{N}$. But in the case of product integral in Banach algebra B_n (see [**3**], Cor. 3.5.1.3) we have

$$y_n(b) = \exp\left(\int_a^b (J_n \circ \pi_n \circ A)(\tau)d\tau\right)$$

and by virtue of Theorem 18 and (34) we obtain

$$y_n(b) = (J_n \circ \pi_n)\left(e^{\int_a^b A(\tau)d\tau}\right).$$

As in the proof of Theorem 20, we finally get

$$\varphi(b) = \iota^{-1}\left((y_n(b))_{n\in\mathbb{N}}\right) = \iota^{-1}\left(\left((J_n \circ \pi_n)\left(e^{\int_a^b A(\tau)d\tau}\right)\right)_{n\in\mathbb{N}}\right) = e^{\int_a^b A(\tau)d\tau}.$$

$\square$

LEMMA 36. *Suppose that $A_\nu \to A$ in $L^1_{loc}(J, \mathcal{X})$, $a_\nu \to a$, $b_\nu \to b$ in J as $\nu \to \infty$. Then*

$$\prod_{a_\nu}^{b_\nu} e^{A_\nu(s)ds} \longrightarrow \prod_{a}^{b} e^{A(s)ds} \qquad \text{in } \mathcal{X} \text{ as } \nu \to \infty.$$

PROOF. Let $x_\nu(t) = \prod_{a}^{t} e^{A_\nu(s)ds}$, $x(t) = \prod_{a}^{t} e^{A(s)ds}$. By virtue of Theorem 23, $x_\nu \to x$ almost uniformly in $\mathcal{X}$. The set $K := \{b, b_1, b_2, \ldots\}$ is compact in J. Hence $x_\nu \to x$ uniformly in K. Let $n \in \mathbb{N}$.

$$p_n(x_\nu(b_\nu) - x(b)) \leq p_n(x_\nu(b_\nu) - x(b_\nu)) + p_n(x(b_\nu) - x(b)) \to 0$$

as $\nu \to \infty$. According to Remark 33 also $p_n\left(\prod_{a_\nu}^{a} e^{A_\nu(s)ds} - \mathbb{I}\right)$ converges to 0, as $\nu \to \infty$. As $n \in \mathbb{N}$ is arbitrary,

$$\prod_{a}^{b_\nu} e^{A_\nu(s)ds} \to \prod_{a}^{b} e^{A(s)ds}, \qquad \prod_{a_\nu}^{a} e^{A_\nu(s)ds} \to \mathbb{I} \tag{37}$$

in $\mathcal{X}$ as $\nu \to \infty$. Finally, note that

$$\prod_{a_\nu}^{b_\nu} e^{A_\nu(s)ds} = \prod_{a}^{b_\nu} e^{A_\nu(s)ds} \cdot \prod_{a_\nu}^{a} e^{A_\nu(s)ds},$$

which by (37) and the continuity of multiplication in $\mathcal{X}$ completes the proof. $\square$

LEMMA 38. *Given are $a_1, \ldots, a_n \in \mathcal{X}$, real numbers $t_0 < t_1 < \ldots < t_n$ and the function $A : [t_0, t_n] \to \mathcal{X}$ such that $A := \sum_{i=1}^{n} \chi_{[t_{i-1}, t_i[} \cdot a_i$. Then*

$$\prod_{t_0}^{t_n} e^{A(s)ds} = e^{(t_n - t_{n-1})a_n} \cdot e^{(t_{n-1} - t_{n-2})a_{n-1}} \cdot \ldots \cdot e^{(t_1 - t_0)a_1}.$$

PROOF. Obviously,

$$\prod_{t_0}^{t_n} e^{A(s)ds} = \prod_{t_{n-1}}^{t_n} e^{A(s)ds} \cdot \prod_{t_{n-2}}^{t_{n-1}} e^{A(s)ds} \cdot \ldots \cdot \prod_{t_0}^{t_1} e^{A(s)ds}.$$

Since for all $s, t \in [t_{j-1}, t_j[\quad A(s) = A(t) = a_j$, Remark 35 leads to

$$\prod_{t_{j-1}}^{t_j} e^{A(s)ds} = e^{\int_{t_{j-1}}^{j} A(s)ds} = e^{a_j(t_j - t_{j-1})}.$$

□

LEMMA 39. *For a.e. $t \in J$:* $\displaystyle \frac{d}{dt}\prod_t^b e^{A(s)ds} = -\prod_t^b e^{A(s)ds} \cdot A(t).$

PROOF. Set $\varphi(t,b) := \prod_t^b e^{A(s)ds}$, $y_n(t,b) := \prod_t^b e^{(J_n\circ\pi_n\circ A)(s)ds}$ for each $n \in \mathbb{N}$. As in the proof of Theorem 20,

$$(y_n(t,b))_{n\in\mathbb{N}} = \iota(\varphi(t,b)).$$

It is well known (see for example [**3**], Cor. 3.5.3.1) that

$$\frac{d}{dt}\prod_t^b e^{\mathbf{A}(s)ds} = -\prod_t^b e^{\mathbf{A}(s)ds} \cdot \mathbf{A}(t)$$

for almost all $t \in J$, when $\mathbf{A} : J \to \mathcal{A}$ is locally summable as a function with values in a Banach algebra $\mathcal{A}$. In particular,

$$\frac{d}{dt}y_n(t,b) = -y_n(t,b)\cdot (J_n\circ\pi_n\circ A)(t)$$

for all $n \in \mathbb{N}$ and a.e. $t \in J$. Therefore,

$$\iota^{-1}\left(\left(\frac{d}{dt}y_n(t,b)\right)_{n\in\mathbb{N}}\right) = -\iota^{-1}\left((y_n(t,b)\cdot(J_n\circ\pi_n\circ A)(t))_{n\in\mathbb{N}}\right). \tag{40}$$

Again, as in the proof of Theorem 20, we have

$$\frac{d}{dt}\left((y_n(t,b))_{n\in\mathbb{N}}\right) = \left(\frac{d}{dt}y_n(t,b)\right)_{n\in\mathbb{N}}.$$

But ι^{-1} is a continuous homomorphism of algebras, so

$$\iota^{-1}\left(\left(\frac{d}{dt}y_n(t,b)\right)_{n\in\mathbb{N}}\right) = \frac{d}{dt}(\iota^{-1}((y_n(t,b))_{n\in\mathbb{N}})) = \frac{d}{dt}\varphi(t,b).$$

On the other hand

$$\iota^{-1}\left((y_n(t,b)\cdot(J_n\circ\pi_n\circ A)(t))_{n\in\mathbb{N}}\right) = \varphi(t,b)\cdot A(t),$$

and finally (40) gives our claim. □

Let us consider a Fréchet space M being a module over a Fréchet algebra $\mathcal{X}$ and a continuous external multiplication $\mathcal{X}\times M \to M$.

Theorem 41. *Fix* $\mathrm{x}_0 \in M$, $t_0 \in J$ *and let* $A : J \to \mathcal{X}$, $B : J \to M$ *be locally summable. Set* $\varphi(t) := \prod_{t_0}^{t} e^{A(s)ds}$. *Then the function* $J \ni t \mapsto \varphi(t)\left(\mathrm{x}_0 + \int_{t_0}^{t} \varphi(\tau)^{-1} B(\tau)\, d\tau\right)$ *is the unique solution of the problem*

$$\left\{ \begin{array}{rcl} \dot{x} & = & A(t)x + B(t) \\ x(t_0) & = & \mathrm{x}_0. \end{array} \right. \tag{42}$$

Proof. Let $g(t) := \varphi(t)^{-1} f(t)$, f being a solution of (42). By Lemma 39,

$$g'(t) = -\varphi(t)^{-1} A(t) f(t) + \varphi(t)^{-1} f'(t) =$$

$$= -\varphi(t)^{-1} A(t) f(t) + \varphi(t)^{-1}(A(t)f(t) + B(t)) = \varphi(t)^{-1} B(t).$$

But $g : J \to M$ is absolutely continuous, thus

$$g(t) = g(t_0) + \int_{t_0}^{t} g'(\tau) d\tau = g(t_0) + \int_{t_0}^{t} \varphi(\tau)^{-1} B(\tau) d\tau.$$

Consequently,

$$f(t) = \varphi(t)\left(\mathrm{x}_0 + \int_{t_0}^{t} \varphi(\tau)^{-1} B(\tau)\, d\tau\right). \tag{43}$$

We have thus proved that every solution of (42) is of the form (43). Conversely, the function $f : J \to M$ given by (43) is a solution of (42). Indeed,

$$f'(t) = \varphi'(t)\mathrm{x}_0 + \varphi'(t) \int_{t_0}^{t} \varphi(\tau)^{-1} B(\tau)\, d\tau + \varphi(t)\varphi(t)^{-1} B(t) =$$

$$= A(t)\varphi(t)\mathrm{x}_0 + A(t)\varphi(t) \int_{t_0}^{t} \varphi(\tau)^{-1} B(\tau)\, d\tau + B(t) = A(t) f(t) + B(t)$$

and

$$f(t_0) = \varphi(t_0)\left(\mathrm{x}_0 + \int_{t_0}^{t_0} \varphi(\tau)^{-1} B(\tau)\, d\tau\right) = \mathrm{x}_0.$$

□

Lemma 44. *Let* $\mathcal{X}, \mathcal{Y}$ *be Fréchet algebras and* $h : \mathcal{X} \to \mathcal{Y}$ *a continuous homomorphism. Assume that* $A : J \to \mathcal{X}$ *is locally summable. Then for all* $t_1, t_2 \in J$*:*

$$h\left(\prod_{t_1}^{t_2} e^{A(s)ds}\right) = \prod_{t_1}^{t_2} e^{(h \circ A)(s)ds}.$$

PROOF. Let $t_1, t_2 \in J$, $x(t) := \prod_{t_1}^{t} e^{A(s)ds}$ and $h \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$. By Lemmas 8 and 9, the function $h \circ x : J \to \mathcal{Y}$ is absolutely continuous and

$$(h \circ x)'(t) = h(\dot{x}(t)) = h(A(t)x(t)) = (h \circ A)(t) \cdot (h \circ x)(t)$$

a.e. in J. Due to Lemma 14 $h \circ A : J \to \mathcal{Y}$ is locally summable. Furthermore,

$$(h \circ x)(t_1) = h(x(t_1)) = h(\mathbb{I}_{\mathcal{X}}) = \mathbb{I}_{\mathcal{Y}},$$

where $\mathbb{I}_{\mathcal{X}}$, $\mathbb{I}_{\mathcal{Y}}$ are units in $\mathcal{X}$, $\mathcal{Y}$, respectively. Thus $h \circ x$ is a solution of

$$\begin{cases} \dot{y} &= (h \circ A)(t)y \\ y(t_1) &= \mathbb{I}_{\mathcal{Y}} \end{cases}$$

in $\mathcal{Y}$. But the only solution of the above problem is $\prod_{t_1}^{t} e^{(h \circ A)(s)ds}$. In particular, for $t = t_2$

$$h\left(\prod_{t_1}^{t_2} e^{A(s)ds}\right) = \prod_{t_1}^{t_2} e^{(h \circ A)(s)ds}.$$

□

Here follow two theorems on the differentation of the product integral with respect to a parameter. The obtained formulas are similar to that of Duhamel type (see **[3]**) and, in the case of a Banach algebra, were proved by K. Holly **[4]**.

THEOREM 45. *Let $G \subset Y$ be an open subset of a normed space Y and $\mathcal{X}$ a Fréchet algebra. Fix $a, b \in \mathbb{R}$. Consider $A : [a, b] \times G \to \mathcal{X}$ such that for all $x \in G$ the function $A(\cdot, x)$ is summable and for all $t \in [a, b]$ the mapping $A(t, \cdot)$ is differentiable in direction $u \in Y$. Assume that for every $p \in \operatorname{sn} \mathcal{X}$ there is a function $\varphi : [a, b] \to [0, \infty]$ summable and such that $p(\partial^u A(t, x)) \le \varphi(t)$, $\forall (t, x) \in [a, b] \times G$. Then the function $G \ni x \mapsto \prod_{a}^{b} e^{A(s,x)ds}$ is differentiable in direction u and*

$$\partial^u \prod_{a}^{b} e^{A(s,x)ds} = \int_a^b \prod_{\tau}^{b} e^{A(s,x)ds} \cdot \partial^u A(\tau, x) \cdot \prod_{a}^{\tau} e^{A(s,x)ds}\, d\tau. \tag{46}$$

Proof. The right-hand side of (46) makes sense since functions $\tau \mapsto \prod_{\tau}^{b} e^{A(s,x)ds}$, $\tau \mapsto \prod_{a}^{\tau} e^{A(s,x)ds}$ are continuous and $\tau \mapsto \partial^u A(\tau, x)$ is summable. Set $f(t,x) := \prod_{a}^{t} e^{A(s,x)ds}$. An easy computation shows that the function $t \mapsto \frac{1}{r}(f(t,x+ru) - f(t,x))$ is a solution of the problem

$$\begin{cases} \dot{y}(t) &= A(t,x)y(t) + \dfrac{A(t,x+ru) - A(t,x)}{r} f(t,x+ru) \\ y(a) &= 0. \end{cases}$$

According to Theorem 41 we obtain

$$\begin{aligned} &\frac{f(b,x+ru) - f(b,x)}{r} = \\ &= \int_a^b \prod_{\tau}^{b} e^{A(s,x)ds} \frac{A(\tau,x+ru) - A(\tau,x)}{r} \prod_{a}^{\tau} e^{A(s,x+ru)ds} d\tau. \end{aligned} \tag{47}$$

The function $\kappa(r) := A(\tau, x+ru)$ is differentiable for small r and $\dot{\kappa}(r) = \partial^u A(\tau, x+ru)$.

Fix $p \in \operatorname{sn} \mathcal{X}$. There is a summable $\varphi : [a,b] \to [0,\infty]$ such that $p(\dot{\kappa}(r)) \leq \varphi(\tau)$. Following the notations of the proof of Lemma 18, by the mean value theorem we have

(48)

$$\begin{aligned} p(A(\tau,x+ru) - A(\tau,x)) = p(\kappa(r) - \kappa(0)) = |(L \circ \kappa)(r) - (L \circ \kappa)(0)|_Y \leq \\ \leq |r| \cdot \sup_{s \in]0,r[} |L(\dot{\kappa}(s))|_Y = |r| \cdot \sup_{s \in]0,r[} p(\dot{\kappa}(s)) \leq |r|\varphi(\tau). \end{aligned}$$

In particular for $|r| \leq 1$

$$p(A(\tau,x+ru)) \leq p(A(\tau,x)) + \varphi(\tau)$$

and by Remark 33 we have

$$p\left(\prod_{a}^{\tau} e^{A(s,x+ru)ds}\right) \leq \exp\left(\int_{[a,b]} (p(A(s,x)) + \varphi(s))ds\right) < \infty.$$

In this way each seminorm $p \in \operatorname{sn} \mathcal{X}$ of the integrand of the right-hand side of (47) is dominated by a summable function. On the other hand, $\dfrac{A(\tau,x+ru) - A(\tau,x)}{r} \longrightarrow \partial^u A(\tau,x)$ as $r \to 0$. Simultaneously, by (48)

$A(\cdot, x+ru) \longrightarrow A(\cdot, x)$ in $L^1(a,b;\mathcal{X})$ as $r \to 0$ and by Lemma 36

$$\prod_a^\tau e^{A(s,x+ru)ds} \longrightarrow \prod_a^\tau e^{A(s,x)ds}$$

as $r \to 0$. Letting $r \to 0$ in (47) by virtue of the dominated convergence theorem we obtain our assertion. $\square$

THEOREM 49. *Let $G \in \operatorname{top}\mathbb{R}^{\mathrm{n}}$. Let $\mathcal{X}$ be a Fréchet algebra. Fix $a, b \in \mathbb{R}$. Consider $A : [a,b] \times G \to \mathcal{X}$ such that for all $x \in G$ the function $A(\cdot, x)$ is summable and for all $t \in [a,b]$ the mapping $A(t,\cdot)$ is of class $\mathcal{C}^k$ $(k \geq 1)$. Assume that for every $p \in \operatorname{sn}\mathcal{X}$ and $\alpha \in \mathbb{N}^n$ $(1 \leq |\alpha| \leq k)$ there is a function $\varphi_\alpha : [a,b] \to [0,\infty]$ summable and such that $p(D_x^\alpha A(t,x)) \leq \varphi_\alpha(t)$, $\forall (t,x) \in [a,b] \times G$. Then the function $G \ni x \mapsto \prod_a^b e^{A(s,x)ds}$ is of class $\mathcal{C}^k$, for every $|\beta| \leq k$ the mapping $(t,x) \longmapsto D_x^\beta \prod_a^t e^{A(s,x)ds}$ is continuous and $\forall 1 \leq |\alpha| \leq k$*

$$D_x^\alpha \prod_a^b e^{A(s,x)ds} = \sum_{\alpha \neq \beta \leq \alpha} \binom{\alpha}{\beta} \int_a^b \prod_\tau^b e^{A(s,x)ds} \cdot D_x^{\alpha-\beta} A(\tau,x) \cdot D_x^\beta \prod_a^\tau e^{A(s,x)ds}\, d\tau$$

PROOF. To prove the theorem, one may use induction with respect to k. $\square$

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