A COUNTEREXAMPLE TO AN ASSERTION DUE TO BLUMENTHAL

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Abstract. Let the orthogonal polynomials $P_n(x)$ be defined by
$$P_n(x) = (x - c_n)P_{n-1}(x) - \lambda_n P_{n-2}(x),$$
$$P_{-1}(x) = 0, \quad P_0(x) = 1,$$
where $\lambda_{n+1} > 0$, c_n real, $\lim_{n \to \infty} c_n = c$ and $\lim_{n \to \infty} \lambda_n = \lambda$, $(n \geq 1)$.
Blumenthal has proved that the true interval of orthogonality $[\sigma, \tau]$ of the above polynomials is given by $\sigma = c - 2\sqrt{\lambda}$, $\tau = c + 2\sqrt{\lambda}$ and the zeros of $P_n(x)$ are dense in $[\sigma, \tau]$. Blumenthal also asserted that the spectrum of the distribution function ψ corresponding to the polynomials P_n has at most finite points in the complement of $[\sigma, \tau]$. In other words the limit points of the zeros of the polynomials P_n outside the intrval $[\sigma, \tau]$ are finite. The falseness of this assertion has been proved first in 1968 with the use of a series of results concerning chain sequences and a theorem due to Szegö. Now although one can find many other ways of proving this in the literature, a concrete counter example fails to exist. Here a counterexample is given which proves the invalidity of Blumenthal's assertion. This example is also of some interest because it exhibits a particular class of the Pollaczek polynomials where the support of the measure of orthogonality is extented beyond the interval $[-1, 1]$.

1. Introduction. We consider the sequence of monic polynomials $P_n(x)$, defined by
$$P_n(x) = (x - c_n)P_{n-1}(x) - \lambda_n P_{n-2}(x), \quad n = 1, 2, ...$$
$$P_{-1}(x) = 0, \quad P_0(x) = 1,$$
where c_n real, $\lambda_{n+1} > 0$, $(n \geq 1)$.

Let $x_{n1} < x_{n2} < ... < x_{nn}$ be the n real and distinct zeros of P_n which satisfy the separation theorem
$$x_{n+1,i} < x_{n,i} < x_{n+1,i+1}, \quad i = 1, 2, ..., n,$$

so that

$$\xi_i = \lim_{n \to \infty} x_{ni}, \quad n_j = \lim_{n \to \infty} x_{n,n-j+1}$$

are finite or take the values $-\infty, +\infty$.

Moreover, since

$$\xi_i \leq \xi_{i+1} \leq n_{j+1} \leq n_j$$

the following limits σ and τ exist:

$$\sigma = \lim_{i \to \infty} \xi_i, \quad \tau = \lim_{j \to \infty} n_j.$$

In the case $\lim_{n \to \infty} c_n = c$ and $\lim_{n \to \infty} \lambda_n = \lambda$, Blumenthal [1] has proved that the true interval of orthogonality $[\sigma, \tau]$ of the above polynomials is given by $\sigma = c - 2\sqrt{\lambda}$, $\tau = c + 2\sqrt{\lambda}$, and that the set $\{x_{ni} : 1 \leq i \leq n, \ n = 1, 2, ...\}$ is dence in $[\sigma, \tau]$. Blumenthal also asserted that the spectrum of the distribution function ψ corresponding to the polynomials P_n has at most finite points in the complement of $[\sigma, \tau]$. T.S. Chihara in [2] proved that Blumenthal's assertion is false. To do this Chihara used a series of theorems concerning, chain sequences and proved that it is possible to have $\xi_i < \sigma, \quad i = 1, 2, ...$ Consequently the use of a Szegö's theorem [6, th.3.41.2] which asserts that ψ is not constant in the interval (ξ_i, ξ_{i+1}), proves that the set of points in the spectrum of ψ smaller than σ, is denumerable. Now one can find numerous other proofs of this result in the literature . However, a concrete counterexample fails to exist.

The purpose here is to give a counterexample which proves the falsity of Blumenthal's assertion. Moreover, the support of the measure of orthogonality of a "singular" case of the Pollaczek polynomials is determined exactly.

2. The counterexample. Consider the polynomials $P_n(x)$ defined by

(2.1)
$$P_{n+1}(x) + P_{n-1}(x) + \frac{2b}{n}P_n(x) = 2xP_n(x)$$

$$P_0(x) = 0, \quad P_1(x) = 1.$$

It is known that the support of the measure of orthogonality of these polynomials is the spectrum of the self-adjoint tridiagonal operator

$$Te_n = \frac{1}{2}(e_{n+1} + e_{n-1}) + \frac{2b}{n}e_n$$

$$Te_1 = \frac{1}{2}e_2 + be_1, \quad b \neq 0, b \in R.$$

This operator is defined on finite linear combinations of the orthonormal basis e_n, $n = 1, 2, ...$, of a Hilbert space H and can be easily extented on all the elements of H. It can be written as follows

(2.2)
$$T = \frac{1}{2}(V + V^*) + 2bB,$$

where V is the unilateral shift with respect to the basis e_n, V^* is its adjoint and B is the compact tridiagonal operator $B : Be_n = \frac{1}{n}e_n$. Since B is compact and selfadjoint, Weyl's theorem implies that the operators T and $T_0 = \frac{1}{2}(V + V^*)$ have the same continuous spectrum. Since the spectrum of T_0 is purely continuous (in fact absolutely continuous) and covers the interval $[-1, 1]$, it follows that the interval $[\sigma, \tau]$ here is the interval $[-1, 1]$ and the zeros of the polynomials (2.1) are dense in $[-1, 1]$, [**5**]. On the other hand it is known [**3, 4**, Appendix B] that for every $b \neq 0$ the operator T has an infinite set of eigenvalues outside the interval $[-1, 1]$. These eigenvalues are

$$\epsilon_n = \sqrt{1 + (b/n)^2}, \quad n = 1, 2, \dots \text{ for } b > 0$$

and

$$(2.3) \qquad \epsilon_n = -\sqrt{1 + (b/n)^2}, \quad n = 1, 2, \dots \text{ for } b < 0.$$

Since the normalized distribution function of the polynomials (2.2) is the function $\psi(t) = (E_t e_1, e_1)$, where E_t is the spectral family of T, this example proves the falsity of Blumenthal assertion. The eigenvalues (2.3) have been obtained by several methods. For the sake of completeness, we recall the simplest one.

The eigenvalue equation

$$(2.4) \qquad (V + V^* + 2bB)f = 2\epsilon f$$

is equivalent to

$$(2.5) \qquad (C_0 V + C_0 V^* + 2b)f = 2\epsilon \, C_0 f,$$

where C_0, the inverse of B, is the diagonal operator $C_0 e_n = n e_n$, $n = 1, 2, \dots$, defined on the range of the bounded operator B. Using the set of elements $f_z = e_1 + z e_2 + z^2 e_3 + \dots$, which is complete in H in the sense that $(f_z, f) = 0$ for every z such that $|z| < 1$ implies that $f = 0$, and setting

$$(2.6) \qquad \phi(z) = (f_z, f)$$

we have an isomorphism from H onto the Hilbert space $H_2(\Delta)$ consisting of analytic functions $\phi(z) = \sum_{n=1}^{\infty} \alpha_n z^{n-1}$ in the open unit disk $\Delta = \{z : |z| < 1\}$, which satisfy the condition $\sum_{n=1}^{\infty} |\alpha_n|^2 < \infty$. It is known [**4**] that if (2.6) holds, then

$$(f_z, C_0 V^* f) = \phi'(z) = \frac{d\phi}{dz}$$
$$(f_z, C_0 f) = (f_z, C_0 V^* V f) = (z\phi)' = z\phi' + \phi$$
$$(f_z, C_0 V f) = z^2 \phi' + 2z\phi.$$

So (2.5) in H is equivalent to the following differential equation in $H_2(\Delta)$

$$(f_z, C_0 V f) + (f_z, C_0 V^* f) + 2b(f_z, f) = 2\epsilon(f_z, C_0 f),$$

or

$$z^2\phi' + 2z\phi + \phi' + 2b\phi = 2\epsilon(z\phi' + \phi)$$

and

(2.7) $$\frac{\phi'(z)}{\phi(z)} = -\frac{(z^2 - 2\epsilon z + 1)'}{z^2 - 2\epsilon z + 1} - \frac{b}{\sqrt{\epsilon^2 - 1}}\Big(\frac{1}{z - z_1} - \frac{1}{z - z_2}\Big),$$

where

(2.8) $$z_1 = \epsilon + \sqrt{\epsilon^2 - 1}, \quad z_2 = \epsilon - \sqrt{\epsilon^2 - 1}.$$

For $b > 0$ the solutions of (2.7) are

(2.9) $$\phi(z) = c\frac{(z - z_2)^{\frac{b}{\sqrt{\epsilon^2-1}}-1}}{(z - z_1)^{\frac{b}{\sqrt{\epsilon^2-1}}+1}}, \quad c = constant.$$

These belong to $H_2(\Delta)$ if and only if

$$\frac{b}{\sqrt{\epsilon^2 - 1}} = n, \quad n = 1, 2, ..., \quad b > 0, \quad \epsilon > 1.$$

Thus the eigenvalues ϵ_n are the following

$$\epsilon_n = \sqrt{1 + \Big(\frac{b}{n}\Big)^2}, \quad n = 1, 2, ...$$

For $b < 0$ we find that

$$\epsilon_n = -\sqrt{1 + \Big(\frac{b}{n}\Big)^2}, \quad n = 1, 2, ...$$

REMARK 1. Another way to see that for $b < 0$ the eigenvalues are negative is to observe that if $2\epsilon > 0$ is an eigenvalue of the operator $V + V^* + 2bB$ corresponding to the eigenvector f, then -2ϵ is an eigenvalue of the operator $V + V^* - 2bB$ corresponding to the eigenvector Uf, where U is the unitary operator $Ue_n = (-1)^n e_n$.

REMARK 2. Unfortunately, for the polynomials (2.1) we know only the support of the measure of orthogonality. This support consists of the continuous part, which is the entire interval $[-1, 1]$, and the discrete part, consisting of the points $\sqrt{1 + (\frac{b}{n})^2}$, $n = 1, 2, ...$ in the case $b > 0$ or the points $-\sqrt{1 + (\frac{b}{n})^2}$, $n = 1, 2, ...$ in the case $b < 0$. Note that one way to find the discrete part of the measure of orthogonality is to normalize the eigenvectors x_k of the operator (2.2). More precisely we have to compute the values $\sigma_k = |(e_1, x_k)|^2$, where $\|x_k\| = 1$. Then the discrete part of the distribution function $\mu(x)$, which corresponds to the measure of orthogonality μ, is given by $\mu(x) = \sum_{k:\lambda_k \leq x} \sigma_k$.

Note that since $\mu(\{\epsilon_n\}) = \frac{1}{\sum_{k=1}^{\infty} P_k^2(\epsilon_n)}$, we need a comprehensive expression for the sum $\sum_{k=1}^{\infty} P_k^2(\epsilon_n)$.

3. Connection with the Pollaczek polynomials. The Pollaczek polynomials $P_n^\lambda(x; a, b)$ are defined as follows

$$(n+1)P_{n+1}^\lambda(x; a, b) + (n + 2\lambda - 1)P_{n-1}^\lambda(x; a, b) + 2bP_n^\lambda(x; a, b) =$$

(3.1)
$$2x(n + \lambda + a)P_n^\lambda(x; a, b)$$

$$P_{-1}^\lambda(x; a, b) = 0, \quad P_0^\lambda(x; a, b) = 1.$$

It is well known [**6**] from Favard's theorem that these polynomials are orthogonal with respect to some positive measure on the real line whenever

(3.2)
$$\lambda > 0 \quad \text{and} \quad a + \lambda > 0$$

or

(3.3)
$$-\frac{1}{2} < \lambda < 0 \quad \text{and} \quad -1 < \lambda + a < 0.$$

Also, it is well-known [**7**] that if (3.2) or (3.3) holds and if moreover $a \geq |b|$, then the support of the measure of orthogonality is the entire interval $[-1, 1]$.

One can see that the polynomials (2.1) are a particular case of the Pollaczek polynomials. In fact the polynomials defined by the recurrence relation (2.1) are obtained from the recurrence relation (3.1) for $\lambda = 1$, $b \neq 0$ ans $a = 0$. (Note that the initial conditions are not the same.) Thus the polynomials (2.1) illustrate the irregular case of the Pollaczek polynomials in the sense that the support of the measure of orthogonality is extented outside the interval $[-1, 1]$ by an infinite denumerable set of points. In fact, the interval of orthogonality is the interval $[-1, \sqrt{1 + b^2}]$ in case $b > 0$ and the interval $[-\sqrt{1 + b^2}, 1]$ in case $b < 0$.

REMARK 3. We note that a necessary and sufficient condition for the existence of a mass point ϵ of the measure of orthogonality of the polynomials (3.1) is that the functions

(3.4)
$$\phi(z) = c \frac{(z - z_2)^{\frac{b - a\epsilon}{\sqrt{\epsilon^2 - 1}} - \lambda}}{(z - z_1)^{\frac{b - a\epsilon}{\sqrt{\epsilon^2 - 1}} + \lambda}},$$

where z_1 and z_2 are given by (2.8), belong to the space $H_2(\Delta)$. Relation (3.4) can be obtained in the same way as relation (2.9).

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References

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