ON BIG DEFLECTIONS OF CUSPED PLATES

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Abstract: In the first part of the present paper we derive general equations for big deflections of cusped plates (i.e., symmetric prismatic shells), following an approach used for the case of plates of the constant thickness (see Timoshenko, Woinowsky-Krieger [1], §101). These equations represent a nonlinear system. The second part is devoted to the cylindrical bending of such plates.

Key words: cusped plates, cusped prismatic shells, big deflections, mathematical modelling.

MSC 2000: 74K20, 74K25, 74B20

1.The model of cusped plates with big deflections

In case of symmetric cusped shells (see Vekua [2]) the elastic body under consideration is bounded from the top and the bottom by the surfaces $z = h(x, y)$ and $z = -h(x, y)$, respectively, and from the lateral side by the cylindrical surface parallel to the axis Oz. The shell thickness $2h(x, y) \geq 0$ and case $2h(x, y) = 0$ can occur only on the projection boundary of the shell. The last part of the boundary is called a cusped edge of the shell. Let us consider bending of a cusped shell caused besides of transverse loadings by the forces acting in the middle plane of the cusped shell. Further, let us consider equilibrium of a small element cut out of the shell by two pairs of planes parallel to the coordinate planes Oxz and Oyz. On the Figures 1-3 are shown, acting on the above element (see Donnell [3], p.220), bending (M_x, M_y) and twisting (M_{xy}, M_{yx}) moments, shearing (Q_x, Q_y) forces, and forces $(N_x, N_y, N_{xy} = N_{yx})$, acting in the middle plane. Projecting all the forces to the axes Oz, Ox, Oy, we get, correspondingly,

$$\frac{\partial Q_x}{\partial x} dxdy + \frac{\partial Q_y}{\partial y} dydx + qdxdy = 0,$$

$$\frac{\partial N_x}{\partial x} dxdy + \frac{\partial N_{xy}}{\partial y} dydx + q_x dxdy = 0,$$

$$\frac{\partial N_{xy}}{\partial x} dxdy + \frac{\partial N_y}{\partial y} dydx + q_y dxdy = 0,$$

[1)] The research described in this publication was made possible in part by Award No. GTFPF-11 of the Georgian Research and Development Foundation (GRDF) and the U.S. Civilian Research & Development Foundation for the Independent States of the Former Soviet Union (CRDF)

where (q_x, q_y, q) is the intensity of loading[2]. Hence,

$$\frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y} + q = 0, \tag{1.1}$$

$$\frac{\partial N_x}{\partial x} + \frac{\partial N_{xy}}{\partial y} + q_x = 0, \tag{1.2}$$

$$\frac{\partial N_{xy}}{\partial x} + \frac{\partial N_y}{\partial y} + q_y = 0. \tag{1.3}$$

Taking moments with respect to the axes Ox and Oy of all the forces, acting on the element under consideration, and neglecting quantities of the third order smallness, we obtain, correspondingly

$$\frac{\partial M_{xy}}{\partial x} dxdy - \frac{\partial M_y}{\partial y} dydx + Q_y dxdy = 0,$$

$$\frac{\partial M_{yx}}{\partial y} dydx + \frac{\partial M_x}{\partial x} dxdy - Q_x dxdy = 0.$$

Therefore,

$$\frac{\partial M_{xy}}{\partial x} - \frac{\partial M_y}{\partial y} + Q_y = 0, \tag{1.4}$$

$$\frac{\partial M_x}{\partial x} + \frac{\partial M_{yx}}{\partial y} - Q_x = 0. \tag{1.5}$$

Considering projections on the axis Oz of the indicated on the Fig. 3 forces, we have to take into account, that the shell is subjected to the bending deformation and as a result there arise small angles between direction of the forces N_x, as well as between directions of the forces N_y, acting on the opposite sides of the element. In consequence of this bending, projecting on the axis Oz of the normal forces N_x implies

$$-N_x dy \frac{\partial w}{\partial x} + \left(N_x + \frac{\partial N_x}{\partial x} dx\right)\left(\frac{\partial w}{\partial x} + \frac{\partial^2 w}{\partial x^2} dx\right) dy,$$

where w is the defection of the points of the middle plane. Neglecting the quantities of the smallness higher than the second order, we get

$$N_x \frac{\partial^2 w}{\partial x^2} dxdy + \frac{\partial N_x}{\partial x}\frac{\partial w}{\partial x} dxdy. \tag{1.6}$$

Similarly, projecting on the axis Oz of the normal forces N_y implies

$$N_y \frac{\partial^2 w}{\partial y^2} dxdy + \frac{\partial N_y}{\partial y}\frac{\partial w}{\partial y} dxdy. \tag{1.7}$$

[2] If we take into account volume forces (X, Y, Z) as well, then q_x, q_y, q should be replaced by $q_x + X$, $q_y + Y$, $q + Z$. By bending deformation q_x and q_y are small quantities.

Concerning projections on the axis Oz of the forces N_{xy} it should be mentioned that the slope of the bent middle surface in the direction y on the opposite sides of the element will be expressed by $\dfrac{\partial w}{\partial y}$ on the one side and by $\dfrac{\partial w}{\partial y}+\dfrac{\partial^2 w}{\partial x\partial y}dx$ on the opposite side, respectively. Hence, the projection on the axis Oz of the forces N_{xy} will be

$$N_{xy}\frac{\partial^2 w}{\partial x\partial y}dxdy+\frac{\partial N_{xy}}{\partial x}\frac{\partial w}{\partial y}dxdy. \tag{1.8}$$

Similarly, for the projection on the axis Oz of the forces $N_{yx}=N_{xy}$, we have

$$N_{yx}\frac{\partial^2 w}{\partial x\partial y}dxdy+\frac{\partial N_{yx}}{\partial y}\frac{\partial w}{\partial x}dxdy. \tag{1.9}$$

Thus, in order to get the sum q_z of all the projections on the axis Oz of the forces acting on the element under consideration we have to add together (1.6)-(1.9) and the prescribed loading $qdxdy$. Taking into account (1.2), (1.3), we get

$$q_z dxdy := \left(q+N_x\frac{\partial^2 w}{\partial x^2}+N_y\frac{\partial^2 w}{\partial y^2}+2N_{xy}\frac{\partial^2 w}{\partial x\partial y}-q_x\frac{\partial w}{\partial x}-q_y\frac{\partial w}{\partial y}\right)dxdy. \tag{1.10}$$

Now, substituting the expressions of Q_x, Q_y from (1.4), (1.5) and of q_z from (1.10) in (1.1), we arrive at

$$\begin{aligned}&\frac{\partial^2 M_x}{\partial x^2}-2\frac{\partial^2 M_{xy}}{\partial x\partial y}+\frac{\partial^2 M_y}{\partial y^2}\\ &=-\left(q+N_x\frac{\partial^2 w}{\partial x^2}+N_y\frac{\partial^2 w}{\partial y^2}+2N_{xy}\frac{\partial^2 w}{\partial x\partial y}-q_x\frac{\partial w}{\partial x}-q_y\frac{\partial w}{\partial y}\right).\end{aligned}$$

Therefore, by virtue of

$$\begin{aligned}&M_x=-D\left(\frac{\partial^2 w}{\partial x^2}+\nu\frac{\partial^2 w}{\partial y^2}\right),\quad M_y=-D\left(\frac{\partial^2 w}{\partial y^2}+\nu\frac{\partial^2 w}{\partial x^2}\right),\\ &M_{xy}=-M_{yx}=D(1-\nu)\frac{\partial^2 w}{\partial x\partial y},\\ &Q_x=\frac{\partial M_{yx}}{\partial y}+\frac{\partial M_x}{\partial x},\quad Q_y=\frac{\partial M_y}{\partial y}-\frac{\partial M_{xy}}{\partial x},\\ &Q_x^* := Q_x+\frac{\partial M_{yx}}{\partial y}=\frac{\partial M_x}{\partial x}+2\frac{\partial M_{yx}}{\partial y},\\ &Q_y^* := Q_y-\frac{\partial M_{xy}}{\partial x}=\frac{\partial M_y}{\partial y}-2\frac{\partial M_{xy}}{\partial x}\end{aligned} \tag{1.11}$$

$\left(D := \dfrac{2Eh^3}{3(1-\nu^2)}\right.$ is the flexural rigidity, ν is Poisson's ratio, E is Young's modulus), we obtain

$$\begin{aligned}
&\frac{\partial^2}{\partial x^2}\left[D\left(\frac{\partial^2 w}{\partial x^2}+\nu\frac{\partial^2 w}{\partial y^2}\right)\right]+2\frac{\partial^2}{\partial x\partial y}\left[D(1-\nu)\frac{\partial^2 w}{\partial x\partial y}\right]\\
&+\frac{\partial^2}{\partial y^2}\left[D\left(\frac{\partial^2 w}{\partial y^2}+\nu\frac{\partial^2 w}{\partial x^2}\right)\right]\\
&=q+N_x\frac{\partial^2 w}{\partial x^2}+N_y\frac{\partial^2 w}{\partial y^2}+2N_{xy}\frac{\partial^2 w}{\partial x\partial y}-q_x\frac{\partial w}{\partial x}-q_y\frac{\partial w}{\partial y}.
\end{aligned}\tag{1.12}$$

For the deformation components of the middle plane we have (see, Timoshenko, Woinowsky-Krieger [1], §92)

$$\begin{gathered}
\varepsilon_x=\frac{\partial u}{\partial x}+\frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2,\ \varepsilon_y=\frac{\partial v}{\partial y}+\frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2,\\
\varepsilon_{xy}=\frac{1}{2}\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}+\frac{\partial w}{\partial x}\frac{\partial w}{\partial y}\right),
\end{gathered}\tag{1.13}$$

where u and v are displacement vector components with respect to the axes Ox and Oy in the middle plane. Evidently, in view of (1.13),

$$\frac{\partial^2\varepsilon_x}{\partial y^2}+\frac{\partial^2\varepsilon_y}{\partial x^2}-2\frac{\partial^2\varepsilon_{xy}}{\partial x\partial y}=\left(\frac{\partial^2 w}{\partial x\partial y}\right)^2-\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2}.\tag{1.14}$$

After integration within the limits $-h(x,y)$ and $h(x,y)$ from Hooke's law we get the following expressions for the components of the deformation in the middle plane

$$\varepsilon_x=\frac{1}{2hE}\left(N_x-\nu N_y\right),\ \ \varepsilon_y=\frac{1}{2hE}\left(N_y-\nu N_x\right),\ \ \varepsilon_{xy}=\frac{1+\nu}{2hE}N_{xy},\tag{1.15}$$

since full deformations in the layer of the prismatic shell parallel to the middle surface on the distance z can be written as follows (see Donnell [3], p.214):

$$\begin{gathered}
e_x(x,y,z)=\varepsilon_x(x,y)-z\frac{\partial^2 w(x,y)}{\partial x^2},\\
e_y(x,y,z)=\varepsilon_y(x,y)-z\frac{\partial^2 w(x,y)}{\partial y^2},\\
e_{xy}(x,y,z)=\varepsilon_{xy}(x,y)-z\frac{\partial^2 w(x,y)}{\partial x\partial y},\\
e_{xz}(x,y,z)=0,\ \ e_{yz}(x,y,z)=0,\ \ e_z(x,y,z)=0,
\end{gathered}$$

$$N_x:=\int\limits_{-h(x,y)}^{h(x,y)}\sigma_x(x,y,z)dz, N_y:=\int\limits_{-h(x,y)}^{h(x,y)}\sigma_y(x,y,z)dz, N_{xy}:=\int\limits_{-h(x,y)}^{h(x,y)}\tau_{xy}(x,y,z)dz,$$

where σ_x, σ_y, τ_{xy} are the stress tensor components.

Let us assume $q_x = 0$, $q_y = 0$ and introduce the stress function $F(x,y)$[3]:

$$N_x = \frac{\partial^2 F}{\partial y^2}, \quad N_y = \frac{\partial^2 F}{\partial x^2}, \quad N_{xy} = -\frac{\partial^2 F}{\partial x \partial y}. \tag{1.16}$$

Substituting the expressions (1.16) in (1.15), we have

$$\begin{aligned} \varepsilon_x &= \frac{1}{2hE}\left(\frac{\partial^2 F}{\partial y^2} - \nu\frac{\partial^2 F}{\partial x^2}\right), \\ \varepsilon_y &= \frac{1}{2hE}\left(\frac{\partial^2 F}{\partial x^2} - \nu\frac{\partial^2 F}{\partial y^2}\right), \\ \varepsilon_{xy} &= -\frac{1+\nu}{2hE}\frac{\partial^2 F}{\partial x \partial y}. \end{aligned} \tag{1.17}$$

Obviously, the expressions (1.16) satisfy (1.2), (1.3) with $q_x = 0$, $q_y = 0$. Substituting (1.17) in (1.14), we arrive at

$$\begin{aligned} &\frac{\partial^2}{\partial y^2}\left[\frac{1}{2hE}\left(\frac{\partial^2 F}{\partial y^2} - \nu\frac{\partial^2 F}{\partial x^2}\right)\right] + \frac{\partial^2}{\partial x^2}\left[\frac{1}{2hE}\left(\frac{\partial^2 F}{\partial x^2} - \nu\frac{\partial^2 F}{\partial y^2}\right)\right] \\ &+\frac{\partial^2}{\partial x \partial y}\left(\frac{1+\nu}{hE}\frac{\partial^2 F}{\partial x \partial y}\right) = \left(\frac{\partial^2 w}{\partial x \partial y}\right)^2 - \frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2}. \end{aligned} \tag{1.18}$$

Along with (1.12) equation (1.18) constitutes the system for determination of two unknown functions F and w. In case $2h(x,y) > 0$, i.e., when the prismatic shell under consideration is non-cusped one, boundary conditions, e.g., for the edge normal to the axis Ox, look like (see Donnell [3], p. 289, and also Awrejcewicz et al. [4]):

- for bending boundary conditions

either

$$w = 0, \quad \frac{\partial w}{\partial x} = 0, \tag{1.19}$$

or

$$w = 0, \quad M_x = 0, \quad \text{i.e.,} \quad D\left(\frac{\partial^2 w}{\partial x^2} + \nu\frac{\partial^2 w}{\partial y^2}\right) = 0, \tag{1.20}$$

or

$$\begin{aligned} &M_x = 0, \quad Q_x^* = 0, \quad \text{i.e.,} \quad D\left(\frac{\partial^2 w}{\partial x^2} + \nu\frac{\partial^2 w}{\partial y^2}\right) = 0, \\ &2\frac{\partial}{\partial y}\left[D(1-\nu)\frac{\partial^2 w}{\partial x \partial y}\right] + \frac{\partial}{\partial x}\left[D\left(\frac{\partial^2 w}{\partial x^2} + \nu\frac{\partial^2 w}{\partial y^2}\right)\right] = 0, \end{aligned} \tag{1.21}$$

[3] Note that it differs from the following usual notion $N_x = 2h\frac{\partial^2 F}{\partial y^2}$, $N_y = 2h\frac{\partial^2 F}{\partial x^2}$, $N_{xy} = -2h\frac{\partial^2 F}{\partial x \partial y}$.

- for membrane (tension-compression) boundary conditions
either $u = 0$, $v = 0$ (when displacements in the middle plane are missing) or $N_x = 0$, $N_{xy} = 0$, i.e., $\dfrac{\partial^2 F}{\partial y^2} = 0$, $\dfrac{\partial^2 F}{\partial x \partial y} = 0$ (when resistance to the displacements in the middle plane is missing).

All the above homogeneous boundary conditions can be replaced by non-homogeneous ones.

In the case of cusped prismatic shells their arise some peculiarities, which will be discussed in the next section.

2. Cylindrical Bending

If we consider the particular case of bending along the cylindrical surface with the axis parallel to the axis Oy, w will be dependent only on x, and for $q_x = 0$, $q_y = 0$ the quantities $\dfrac{\partial^2 F}{\partial x^2}$ and $\dfrac{\partial^2 F}{\partial y^2}$ will be constants. Evidently, equation (1.18) will be satisfied identically, while equation (1.12) will be simplified as follows:

$$\frac{\partial^2}{\partial x^2}\left(D\frac{\partial^2 w}{\partial x^2}\right) = q + N_x\frac{\partial^2 w}{\partial x^2}. \tag{2.1}$$

Indeed, from (1.2), (1,3) with $q_x = 0$, $q_y = 0$ it follows

$$\frac{\partial N_x}{\partial x} = 0, \quad \frac{\partial N_{xy}}{\partial x} = 0.$$

Whence,

$$N_x = const, \quad N_{xy} = const. \tag{2.2}$$

On the other hand, from (1.13) we have

$$\varepsilon_y = 0.$$

Therefore, by virtue of (1.15),

$$N_y = \nu N_x = const. \tag{2.3}$$

Multiplying by D both the sides of (2.1) and taking into account (1.11), we get

$$D\frac{\partial^2 M_x}{\partial x^2} - N_x M_x = -qD. \tag{2.4}$$

After differentiation of (2.4), in view of (1.11), we obtain

$$\frac{\partial}{\partial x}\left(D\frac{\partial Q_x}{\partial x}\right) - N_x Q_x = -\frac{\partial qD}{\partial x}. \tag{2.5}$$

In the case of cusped prismatic shells, evidently, equations (1.12), (1.18), (2.4), (2.5) are singular differential equations and, therefore, setting of boundary conditions is characterized with some peculiarities (in some cases Dirichlet problem should be replaced by Keldysh problem (see Keldysh [5]) or by weighted Dirichlet problem; Neumann problem should be replaced by weighted Neumann problem (see Bitsadze [6]). Evidently, boundary conditions (1.20), (1.21) will become weighted boundary conditions since

$$D(0, y) = 0 \text{ and } D(x, y) > 0 \text{ for } x > 0.$$

The classical bending problem within the framework of geometrically linear theory is investigated in detail in Jaiani [7]. His main results states that the boundary condition (1.19) can be set if and only if

$$\int_P^Q D^{-1}(x, y)dy < +\infty;$$

(1.20) can be set if and only if

$$\int_P^Q xD^{-1}(x, y)dy < +\infty,$$

where P runs all the points of the cusped prismatic shell edge, e.g.,with the normal parallel to the axis Ox; Q belongs to the shell middle plane, while (1.21) can be set without any restrictions. As we see from (2.4) and (2.5), in the case of geometrically nonlinear theory the analogous restrictions arise for the case of boundary conditions (1.21) as well. This topic will be discussed in detail in the forthcoming paper.

Acknowledgment. We are grateful to Eng. M.Bitsadze for her help in the preparation of the paper for publication.

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Received May, 6, 2005; accepted July, 30, 2005.

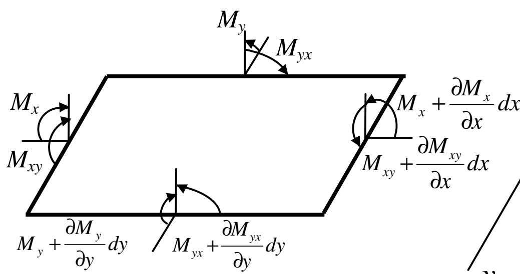

Fig. 1

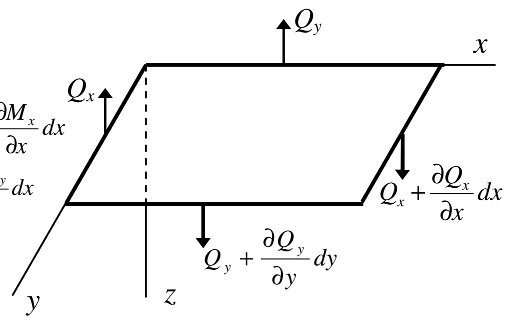

Fig. 2

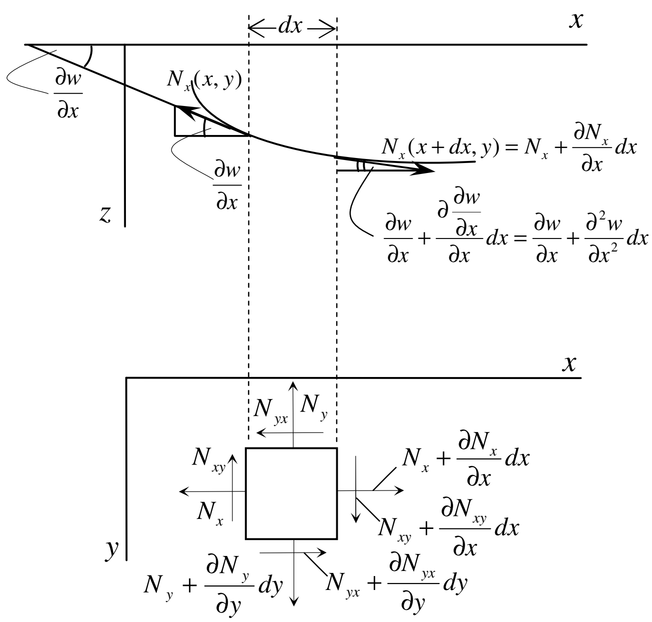

Fig. 3