ON TWO-DIMENSIONAL ANALOGUES FOR SHELL-LIKE BODIE

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The paper deals with the question of reduction of the three-dimensional problem of the geometrical and physical nonlinear theory of elasticity to the two-dimensional one, for shell-like elastic bodies.

I. Let R and $\overset{*}{R}$ denote the radius-vectors before and after deformation of the bodies Ω and $\overset{*}{\Omega}$ respectively, moreover

$$\overset{*}{R}(x_1, x_2, x_3) = R(x_1, x_2, x_3) + U(x_1, x_2, x_3),$$

$$\partial_j \overset{*}{R} = \overset{*}{R}_j = \partial_j R + \partial_j U = R_j + \partial_j U$$
$$\left(\partial_j = \frac{\partial}{\partial x_j}\right),$$

where U is the displacement vector, (x_1, x_2, x_3) are the curvilinear coordinates in the space.

The equation of equilibrium has the form [1]

$$\frac{1}{\sqrt{\overset{*}{g}}} \frac{\partial \sqrt{\overset{*}{g}}\, \overset{*}{\boldsymbol{\sigma}}{}^i}{\partial x_i} + \overset{*}{\boldsymbol{\Phi}} = 0$$

where $\overset{*}{g}$ is the discriminant of the metric tensor of the domain $\overset{*}{\Omega}$, $\overset{*}{\boldsymbol{\sigma}}{}^i$ are ,,contravariant stress vectors“, $\overset{*}{\Phi}$ is an external force.

Under repeating indices summation is meant unless otherwise stated. The Latin letters take the values 1,2,3, while the Greek letters take the values 1,2.

The equilibrium equation can be written as

$$\frac{1}{\sqrt{g}} \frac{\partial \sqrt{g} \boldsymbol{\sigma}^i}{\partial x_i} + \Phi = 0, \quad \boldsymbol{\sigma}^i = \sigma^{ij}(R_j + \partial_j U),$$

where g is the discriminant of the metric tensor of the domain Ω, σ^{ij} are contravariant components of the stress tensor,

$$\boldsymbol{\sigma}^i = \sqrt{\frac{\overset{*}{g}}{g}}\, \overset{*}{\boldsymbol{\sigma}}{}^i, \quad \Phi = \sqrt{\frac{\overset{*}{g}}{g}}\, \overset{*}{\boldsymbol{\Phi}}\,.$$

The stress-strain relation has the form

$$\sigma^{ij} = (E^{ijmn} + E^{ijmnpq} \varepsilon_{pq}) \varepsilon_{mn},$$

where E^{ijmn} and E^{ijmnpq} are tensors of elasticity of the fourth and sixth rank, respectively, ε_{ij} are covariant components of the strain tensor, moreover

$$E^{ijmn} = \lambda g^{ij} g^{mn} + \mu(g^{im} g^{jn} + g^{in} g^{jm}),$$

$$E^{ijmnpq} = \eta_1 g^{ij} g^{mn} g^{pq} + \eta_2 g^{ij} (g^{mp} g^{nq} + g^{mq} g^{np}) +$$
$$+\eta_3 g^{mn} (g^{ip} g^{jq} + g^{iq} g^{jp}) + \eta_4 g^{pq} (g^{im} g^{jn} + g^{in} g^{jm}),$$
$$\varepsilon_{mn} = \frac{1}{2}(R_m \partial_n U + R_n \partial_m U + \partial_m U \partial_n U),$$

λ and μ are Lame's constants of elasticity, and $\eta_1, \eta_2, \eta_3, \eta_4$ are modules of elasticity of the second order for isotropic elastic bodies, $g^{ij} = R^i R^j$ are contravariant components of the metric tensor, R_i and R^i are co and contravariant base vectors.

II. 1. We consider the coordinate system of lines of curvature, which is connected normally to the midsurface S of the shell Ω, i.e.

$$R(x_1, x_2, x_3) = r(x_1, x_2) + x_3 n(x_1, x_2),$$

where r and n are radius-vectors and a normal of the S, x_3 is the thickness coordinate $-h \leq x_3 \leq h$, $h = const$ is the semi-thickness.

The dependence between covariant and contravariant base vectors of the shell Ω and the midsurface S, are expressed as follows

$$R_\alpha = (1 - k_\alpha x_3) r_\alpha, R^\alpha = \frac{r^\alpha}{1 - k_\alpha x_3}, \quad R_3 = R^3 = n, \ (\alpha = 1, 2)$$

(on α no summation!)

where k_1 and k_2 are main curvatures of the midsurface S, i.e.

$$R_i = (1 - k_i x_3) r_i, \quad R^i = \frac{r^i}{1 - k_i x_3}, \quad g^{ij} = \frac{a^{ij}}{(1 - k_i x_3)(1 - k_j x_3)},$$

$$\sqrt{g} = \sqrt{a}(1 - k_1 x_3)(1 - k_2 x_3), \quad a = a_{11} a_{22} - a_{12}^2,$$

$$a^{ij} = r^i r^j = \begin{cases} a^{\alpha\beta}, & i = \alpha, \ j = \beta; \\ 0, & i = 3, \ j = \beta, \quad or \ \ i = \alpha, \ j = 3; \\ 1, & i = j = 3, \end{cases}$$

$$a_{\alpha\beta} = r_\alpha r_\beta, \quad a_{\alpha 3} = a_{3\beta} = 0, \quad a_{33} = 0, \quad k_3 = 0$$

(on i, j, α, β no summation!).

For the tensor ε_{ij} we obtain

$$\varepsilon_{ij} = \frac{1}{2}(1 - k_i x_3)(1 - k_j x_3) \left(r_j \frac{\partial_i U}{1 - k_i x_3} + r_i \frac{\partial_j U}{1 - k_j x_3} + \frac{\partial_i U}{1 - k_i x_3} \frac{\partial_j U}{1 - k_j x_3} \right),$$

(on i, j no summation!) i.e.

$$\varepsilon_{\alpha\beta} = \frac{1}{2}(1-k_\alpha x_3)(1-k_\beta x_3)(r_\beta \frac{\partial_\alpha U}{1-k_\alpha x_3} + r_\alpha \frac{\partial_\beta U}{1-k_\beta x_3} + \frac{\partial_\alpha U}{1-k_\alpha x_3}\frac{\partial_\beta U}{1-k_\beta x_3}) =$$

$$= (1-k_\alpha x_3)(1-k_\beta x_3) e_{\alpha\beta},$$

$$\varepsilon_{\alpha 3} = \frac{1}{2}(1-k_\alpha x_3)(n\frac{\partial_\alpha U}{1-k_\alpha x_3} + r_\alpha \partial_3 U + \partial_3 U \frac{\partial_\alpha U}{1-k_\alpha x_3}) =$$

$$= (1-k_\alpha x_3) e_{\alpha 3},$$

$$\varepsilon_{33} = n\partial_3 U + \frac{1}{2}(\partial_3 U)^2 = e_{33}.$$

2. Now, we assume the validity of the representations:

$$\partial_1 U = (1-k_1 x_3)\partial_1 V(x_1, x_2),$$

$$\partial_2 U = (1-k_2 x_3)\partial_2 V(x_1, x_2),$$

$$\partial_3 U = \widehat{V}(x_1, x_2),$$

where V and $\widehat{V}$ are the two-dimensional vectors of x_1, x_2.

Taking into consideration the condition

$$\partial_1\partial_2 U = \partial_2\partial_1 U \ \ \Rightarrow\ \partial_2(k_1\partial_1 V) = \partial_1(k_2\partial_2 V),$$

$$\partial_3\partial_1 U = \partial_1\partial_3 U \ \ \Rightarrow\ \partial_1\widehat{V} = -k_1\partial_1 V,$$

$$\partial_3\partial_2 U = \partial_2\partial_3 U \ \ \Rightarrow\ \partial_2\widehat{V} = -k_2\partial_2 V,$$

for $V(x_1, x_2)$ we obtain the following equation

$$(k_1 - k_2)\frac{\partial^2 V}{\partial x_1 \partial x_2} + \frac{\partial k_1}{\partial x_2}\frac{\partial V}{\partial x_1} - \frac{\partial k_2}{\partial x_1}\frac{\partial V}{\partial x_2} = 0.$$

Now, from the system of Gauss equations

$$\frac{\partial k_1}{\partial x_2} = (k_2 - k_1)\frac{\partial \ln\sqrt{a_{11}}}{\partial x_2},$$

$$\frac{\partial k_2}{\partial x_1} = (k_1 - k_2)\frac{\partial \ln\sqrt{a_{22}}}{\partial x_1},$$

we have

$$(k_1 - k_2)\left[\frac{\partial^2 V}{\partial x_1 \partial x_2} - \frac{\partial \ln\sqrt{a_{11}}}{\partial x_2}\frac{\partial V}{\partial x_1} - \frac{\partial \ln\sqrt{a_{22}}}{\partial x_1}\frac{\partial V}{\partial x_2}\right] = 0.$$

The general solution of this equation has the form [2]

$$V(x_1, x_2) = u(x_1) + v(x_2) - \int_{x_1^0}^{x_1} u(t)\frac{\partial R(t, x_2^0, x_1, x_2)}{\partial t} dt - \int_{x_2^0}^{x_2} v(\tau)\frac{\partial R(x_1^0, t, x_1, x_2)}{\partial \tau} d\tau.$$

where $R(t,\tau,x_1,x_2)$ is a Riemann function, $u(x_1)$ and $v(x_2)$ are arbitrary vectors.

For the vector $U(x_1,x_2,x_3)$ we obtain

$$\begin{aligned} U(x_1,x_2,x_3) &= \int\limits_{x_1^0}^{x_2}[1-x_3k_1(x_1,x_2)]\frac{\partial V(x_1,x_2)}{\partial x_1}dx_1 + \\ &+ \int\limits_{x_2^0}^{x_2}[1-x_3k_2(x_1^0,x_2)]\frac{\partial V(x_1^0,x_2)}{\partial x_2}dx_2 + \\ &+ (x_3-x_3^0)\widehat{V}(x_1^0,x_2^0)+U(x_1^0,x_2^0,x_3^0), \end{aligned}$$

and

$$\widehat{V} = -\int\limits_{x_1^0}^{x_1} k_1(x_1,x_2)\frac{\partial V}{\partial x_1}dx_2 - \int\limits_{x_2^0}^{x_2} k_2(x_1^0,x_2)\frac{\partial V(x_1^0,x_2)}{\partial x_2}dx_2 + \widehat{V}(x_1^0,x_2^0).$$

Now for e_{mn} we have the following two-dimensional expressions:

$$e_{\alpha\beta} = \frac{1}{2}(r_\alpha\partial_\beta V + r_\beta\partial_\alpha V + \partial_\alpha V\partial_\beta V),$$

$$e_{\alpha 3} = \frac{1}{2}(n\partial_\alpha V + r_\alpha V\widehat{V} + \widehat{V}\partial_\alpha V),$$

$$e_{33} = n\widehat{V} + \frac{1}{2}\widehat{V}^2.$$

The ,,contravariant stress vector“ $\boldsymbol{\sigma}^i$ has the form

$$\boldsymbol{\sigma}^i = (1-k_jx_3)\sigma^{ij}(r_j+\partial_j v) = \frac{T^i}{1-k_ix_3}$$

(on i no summation!), where

$$T^i = (M^{ijmn} + M^{ijmnpq}e_{pq})e_{mn}(r_j+\partial_j V),$$

$$\partial_j V = \begin{cases} \partial_\alpha V, & j=\alpha, \\ \widehat{V}, & j=3. \end{cases}$$

Here

$$\begin{gathered} M^{ijmn} = \lambda a^{ij}a^{mn} + \mu(a^{im}a^{jn} + a^{in}a^{jm}) \\ M^{ijmnpq} = \eta_1 a^{ij}a^{mn}a^{pq} + \eta_2 a^{ij}(a^{pm}a^{nq} + a^{mq}a^{np}) + \\ +\eta_3 a^{mn}(a^{ip}a^{jq} + a^{iq}a^{jp}) + \eta_4 a^{pq}(a^{im}a^{jn} + a^{in}a^{jm}). \end{gathered}$$

At last, we obtain the following two-dimensional equation of equilibrium:

$$\frac{1}{\sqrt{a}}\left(\frac{\partial\sqrt{a}(1-k_2x_3)T^1}{\partial x_1} + \frac{\partial\sqrt{a}(1-k_1x_3)T^2}{\partial x_2}\right) +$$

$$+\frac{\partial(1-k_1x_3)(1-k_2x_3)}{\partial x_3}T^3+(1-k_1x_3)(1-k_2x_3)\mathbf{\Phi}=0,$$

where

$$\begin{aligned}T^\alpha &= (M^{\alpha\beta mn}+M^{\alpha\beta mnpq}e_{pq})e_{mn}(r_\beta+\partial_\beta V)+\\ &+(M^{\alpha 3mn}+M^{\alpha 3mnpq}e_{pq})e_{mn}(n+\widehat{V}),\\ T^3 &= (M^{3\beta mn}+M^{3\beta mnpq}e_{pq})e_{mn}(r_\beta+\partial_\beta V)+\\ &+(M^{33mn}+M^{33mnpq}e_{pq})e_{mn}(n+\widehat{V}).\end{aligned}$$

3. Let us consider the boundary condition for the stresses.

The stress vector $\boldsymbol{\sigma}_{\overset{*}{l}}$ acting onto area with the mormal $(\overset{*}{l})$ has the form

$$\boldsymbol{\sigma}_{(\overset{*}{l})}=\overset{*}{\boldsymbol{\sigma}}{}^i\,\overset{*}{l}_i\quad(\overset{*}{l}_i=\overset{*}{l}\,\overset{*}{R}_i).$$

The normal $\overset{*}{l}$ after deformation can be defined as

$$\overset{*}{l}=\frac{\overset{*}{s}_1\times\overset{*}{s}_2}{|\overset{*}{s}_1\times\overset{*}{s}_2|},$$

where $\overset{*}{s}_1$ and $\overset{*}{s}_2$ are unit tangent vectors of the boundary surface $\partial\overset{*}{\Omega}$,with the surface element

$$d\overset{*}{S}=|\overset{*}{s}_1\times\overset{*}{s}_2|d\overset{*}{s}_1\overset{*}{s}_2.$$

Then we have

$$\overset{*}{l}=\frac{1}{|\overset{*}{s}_1\times\overset{*}{s}_2|}\left(\frac{d\overset{*}{R}}{d\overset{*}{s}_1}\times\frac{d\overset{*}{R}}{d\overset{*}{s}_2}\right)=\frac{1}{|\overset{*}{s}_1\times\overset{*}{s}_2|}\left(\frac{d\overset{*}{R}}{ds_1}\times\frac{d\overset{*}{R}}{ds_2}\right)\frac{ds_1}{d\overset{*}{s}_1}\frac{ds_2}{d\overset{*}{s}_2}=$$

$$=\overset{*}{R}_i\times\overset{*}{R}_j\frac{dx_i}{ds_1}\frac{dx_j}{ds_2}\frac{ds_1ds_2}{d\overset{*}{S}}=\sqrt{\overset{*}{g}}\in_{ijk}\overset{*}{R}{}^k\frac{dx_i}{ds_1}\frac{dx_j}{ds_2}\frac{ds_1ds_2}{d\overset{*}{S}}=$$

$$=\sqrt{\frac{\overset{*}{g}}{g}}\sqrt{g}\in_{ijk}\overset{*}{R}{}^k\frac{dx_i}{ds_1}\frac{dx_j}{ds_2}\frac{ds_1ds_2}{d\overset{*}{S}}=\sqrt{\frac{\overset{*}{g}}{g}}(R_i\times R_j)R_k\overset{*}{R}{}^k\frac{dx_i}{ds_1}\frac{dx_j}{ds_2}\frac{ds_1ds_2}{d\overset{*}{S}}=$$

$$=\sqrt{\frac{\overset{*}{g}}{g}}(s_1\times s_2)R_k\overset{*}{R}{}^k\frac{ds_1ds_2}{d\overset{*}{S}}=\sqrt{\frac{\overset{*}{g}}{g}}\frac{s_1\times s_2}{|s_1\times s_2|}R_k\overset{*}{R}{}^k\frac{|s_1\times s_2|ds_1ds_2}{d\overset{*}{S}}\Rightarrow$$

$$\Rightarrow\overset{*}{l}=\sqrt{\frac{\overset{*}{g}}{g}}(lR_k)\,\overset{*}{R}{}^k\frac{dS}{d\overset{*}{S}}\quad\Rightarrow$$

$$\Rightarrow\overset{*}{l}_i=\overset{*}{l}\,\overset{*}{R}_i=\sqrt{\frac{\overset{*}{g}}{g}}l_i\frac{dS}{d\overset{*}{S}}\quad(l_i=lR_i),$$

where $l = \dfrac{s_1 \times s_2}{|s_1 \times s_2|}$ is the normal of the boundary surface before deformation, dS is the element of this surface,

$$dS = |s_1 \times s_2| ds_1 ds_2,$$

$\in_{ijk}$ are the Levi-Civita symbols.

Now the strees vector can be written as

$$\overset{*}{\boldsymbol{\sigma}}_{(\overset{*}{l})} = \overset{*}{\sigma} \overset{*}{l}_i = \sqrt{\frac{\overset{*}{g}}{g}} \overset{*}{\boldsymbol{\sigma}}{}^i l_i \frac{dS}{d\overset{*}{S}} = \boldsymbol{\sigma}^i l_i \frac{dS}{d\overset{*}{S}},$$

i.e.,

$$\overset{*}{\boldsymbol{\sigma}}_{(\overset{*}{l})} \frac{d\overset{*}{S}}{dS} = \boldsymbol{\sigma}^i l_i = \boldsymbol{\sigma}^\alpha l_\alpha + \boldsymbol{\sigma}^3 l_3,$$

where

$$l_\alpha = lR_\alpha, \quad l_3 = ln.$$

On the surfaces $x_3 = \pm h$ we have $l = n$ and so

$$\boldsymbol{\sigma}_{(n)}(x_1, x_2, \pm h) = \boldsymbol{\sigma}^3(x_1, x_2, \pm h).$$

The stress vector $\boldsymbol{\sigma}_{(\overset{*}{l})}$ acting on the lateral surface $d\widehat{S} = d\widehat{s}dx_3$ with the normal $\widehat{l}$ has the form

$$\boldsymbol{\sigma}_{(\widehat{l})} = \boldsymbol{\sigma}^\alpha(\widehat{l}R_\alpha).$$

The normal $\widehat{l}$ before deformation can be defined as:

$$\widehat{l} = \frac{dR}{d\widehat{s}} \times n,$$

where

$$\frac{dR}{d\widehat{s}} = \widehat{s} = \frac{dR}{ds}\frac{ds}{d\widehat{s}} = \frac{d(r + x_3 n)}{ds}\frac{ds}{d\widehat{s}} = \left(s + x_3 \frac{dn}{ds}\right)\frac{ds}{d\widehat{s}} \quad \Rightarrow$$

$$\Rightarrow \ \widehat{s} = [(1 - k_3 x_3)s + \tau_s x_3 l]\frac{ds}{d\widehat{s}}.$$

Therefore,

$$\widehat{l} = [(1 - k_3 k_s)l - x_3 \tau_s s]\frac{ds}{d\widehat{s}},$$

$$(\widehat{l} \times \widehat{s} = n, \quad l \times s = n).$$

where $\widehat{l}, \widehat{s}$ and l, s are the unit vectors of the tangential normal and tangent of the lateral curve of the surfaces $x_3 = \mathit{const}$ and $x_3 = 0$ (midsurface), respectively, k_s and τ_s are the normal curvature and geodesic torsion of the midsurface, $d\widehat{s}$ and ds are linear elements of the surfaces $x_3 = \mathit{const}$ and $x_3 = 0$, respectively, moreover

$$d\widehat{s} = \sqrt{1 - 2x_3 k_s + x_3^2(k_s^2 + \tau_s^2)}ds \Rightarrow$$

$$\Rightarrow\ d\widehat{s} = \sqrt{a_{11}(1-k_1x_3)^2\left(\frac{dx_1}{ds}\right)^2 + a_{22}(1-k_2x_3)^2\left(\frac{dx_2}{ds}\right)^2}\,ds.$$

On the other hand, we have [3]

$$\widehat{l} = \frac{dR}{d\widehat{s}} \times n = \frac{dR}{ds} \times n\frac{ds}{d\widehat{s}} = R_\alpha \times n\frac{dx_\alpha}{ds}\frac{ds}{d\widehat{s}} = \sqrt{g}\in_{\alpha 3\beta} R^\beta \frac{dx_\alpha}{ds}\frac{ds}{d\widehat{s}} =$$

$$= \sqrt{\frac{g}{a}}\sqrt{a}\in_{\alpha 3\beta} R^\beta\frac{dx_\alpha}{ds}\frac{ds}{d\widehat{s}} = \sqrt{\frac{g}{a}}(r_\alpha \times n)r_\beta R^\beta \frac{dx_\alpha}{ds}\frac{ds}{d\widehat{s}} =$$

$$= \sqrt{\frac{g}{a}}(s\times n)r_\beta R^\beta\frac{ds}{d\widehat{s}} = \sqrt{\frac{g}{a}}(lr_\beta)R^\beta\frac{ds}{d\widehat{s}} \Rightarrow$$

$$\Rightarrow \widehat{l}R_\beta = \widehat{l}_\beta = \sqrt{\frac{g}{a}}l_\beta\frac{ds}{d\widehat{s}}.$$

Therefore,

$$\boldsymbol{\sigma}_{(\widehat{l})} = \boldsymbol{\sigma}^\alpha\widehat{l}_\alpha = \sqrt{\frac{g}{a}}\boldsymbol{\sigma}^\alpha l_\alpha\frac{ds}{d\widehat{s}} \Rightarrow$$

$$\Rightarrow \boldsymbol{\sigma}_{(\widehat{l})}\frac{ds}{d\widehat{s}} = (1-k_1x_3)(1-k_2x_3)(\boldsymbol{\sigma}^1 l_1 + \boldsymbol{\sigma}^2 l_2) \Rightarrow$$

$$\boldsymbol{\sigma}_{(\widehat{l})}\frac{ds}{d\widehat{s}} = (1-k_2x_3)T^1 l_1 + (1-k_1x_3)T^2 l_2.$$

Thus, we obtain the following system of two-dimensional equations of the geometrically and physically non-linear theory for shell-like elastic bodies:

a) Equilibrium equations

$$\frac{1}{\sqrt{a}}\left(\frac{\partial\sqrt{a}(1-k_2x_3)T^1}{\partial x_1} + \frac{\partial\sqrt{a}(1-k_1x_3)T^1}{\partial x_2}\right) - 2(H-Kx_3)T^3 + F = 0$$

$$\left(2H = k_1+k_2,\ K = k_1k_2,\ (1-k_1x_3)(1-k_2x_3)\Phi = F = \overset{(0)}{F} + x_3\overset{(1)}{F}\right);$$

b) Stress-strain relation

$$T^i(x_1,x_2) = (M^{i\beta mn} + M^{i\beta mnpq}e_{pq})e_{mn}(r_\beta + \partial_\beta V) +$$

$$+ (M^{i3mn} + M^{i3mnpq}e_{pq})e_{mn}(n + \widehat{V}),$$

where

$$e_{\alpha\beta} = \frac{1}{2}(r_\alpha\partial_\beta V + r_\beta\partial_\alpha V + \partial_\alpha V\partial_\beta V),$$

$$e_{\alpha 3} = \frac{1}{2}(n\partial_\alpha V + n\widehat{V} + \widehat{V}\partial_\alpha V),$$

$$e_{33} = n\widehat{V} + \frac{1}{2}\widehat{V}^2.$$

III. Special cases

1. Spherical shell ($k_1 = k_2 = -\dfrac{1}{R}$)

The vector of displacement U for the spherical shell has the form

$$U(x_1, x_2, x_3) = \left(1 + \frac{x_3}{R}\right) V(x_1, x_2) \Rightarrow$$

$$\Rightarrow \begin{cases} \partial_\alpha U = \left(1 + \dfrac{x_3}{R}\right) \partial_\alpha V \quad (\alpha = 1, 2), \\ \partial_3 U = \dfrac{1}{R} V. \end{cases}$$

The equation of equilibrium can be written as:

$$\frac{1}{\sqrt{a}} \frac{\partial \sqrt{a} T^\alpha}{\partial x_\alpha} + \frac{2}{R} T^3 + F = 0,$$

where

$$T^i(x_1, x_2) = (M^{ijmn} + M^{ijmnpq} e_{pq}) e_{mn} (r_j + \partial_j V) \quad (\partial_3 V = \frac{1}{R} V),$$

$$e_{mn}(x_1, x_2) = \frac{1}{2}(r_m \partial_n V + r_n \partial_m V + \partial_m V \partial_n V),$$

$$F(x_1, x_2) = \left(1 + \frac{x_3}{R}\right) \mathbf{\Phi}.$$

The stress vector has the form

$$\boldsymbol{\sigma}_{(l)} = \left(1 + \frac{x_3}{R}\right) T^\alpha l_\alpha \frac{ds}{d\hat{s}} = T^\alpha l_\alpha = T_{(l)},$$

$$\left(d\hat{s} = \left(1 + \frac{x_3}{R}\right) ds\right).$$

2. Cylindrical shell ($k_1 = -\frac{1}{R}, k_2 = 0$)

The vector of displacement for the cylindrical shell has the form

$$U(x_1, x_2, x_3) = \left(1 + \frac{x_3}{R}\right) u(x_1) + v(x_2).$$

Then

$$\begin{cases} \partial_1 v = \left(1 + \dfrac{x_3}{R}\right) \dfrac{du(x_1)}{dx_1} \\ \partial_2 v = \dfrac{dv(x_2)}{dx_2} \\ \partial_3 v = \dfrac{1}{R} u(x_1) \end{cases}$$

The equilibrium equation looks like:

$$\frac{1}{\sqrt{a}} \frac{\partial \sqrt{a} T^1}{\partial x_1} + \frac{1}{R} T^3 = 0,$$

$$\frac{1}{\sqrt{a}}\frac{\partial\sqrt{a}T^2}{\partial x_2}+\boldsymbol{\Phi}=0 \quad (\boldsymbol{\Phi}=\boldsymbol{\Phi}(x_1,x_2)),$$

where

$$T^i=(M^{i1mn}+M^{i1mnpq}e_{pq})e_{mn}\left(r_1+\frac{du(x_1)}{dx_1}\right)$$
$$+(M^{i2mn}+M^{i2mnpq}e_{pq})e_{mn}\left(r_2+\frac{dv(x_2)}{dx_2}\right)$$
$$+(M^{i3mn}+M^{i3mnpq}e_{pq})e_{mn}\left(n+\frac{1}{R}u(x_1)\right).$$

Here

$$M^{ijmn}=\lambda\delta^{ij}\delta^{mn}+\mu(\delta^{im}\delta^{jn}+\delta^{in}\delta^{jm}),$$
$$M^{ijmnpq}=\eta_1\delta^{ij}\delta^{mn}\delta^{pq}+\eta_2\delta^{ij}(\delta^{mn}\delta^{nq}+\delta^{mn}\delta^{np})+$$
$$+\eta_3\delta^{mn}(\delta^{ip}\delta^{jq}+\delta^{iq}\delta^{jp})+\eta_4\delta^{pq}(\delta^{im}\delta^{jn}+\delta^{in}\delta^{jm}),$$
$$\left(\delta^{ij}=\begin{cases}1, & i=j\\ 0, & i\neq j\end{cases}\right).$$

For the e_{ij} we obtain

$$e_{12}=\frac{1}{2}\left(r_1\frac{du(x_1)}{dx_1}+r_2\frac{dv(x_2)}{dx_2}+\frac{du}{dx_1}\frac{dv}{dx_2}\right)=e_{21},$$

$$e_{13}=\frac{1}{2}\left(n\frac{du}{dx_1}+\frac{r_1u}{R}+\frac{u}{R}\frac{du}{dx_1}\right),$$

$$e_{23}=\frac{1}{2}\left(n\frac{dv}{dx_2}+\frac{r_2u}{R}+\frac{u}{R}\frac{dv}{dx_2}\right),$$

$$e_{11}=r_1\frac{du}{dx_1}+\frac{1}{2}\left(\frac{du}{dx_1}\right)^2,\quad e_{22}=r_2\frac{dv}{dx_2}+\frac{1}{2}\left(\frac{dv}{dx_2}\right)^2,\quad e_{33}=\frac{nu}{R}+\frac{u^2}{2R^2}.$$

The stress vector $\boldsymbol{\sigma}_{(\widehat{l})}$ has the form $(\widehat{l}=r_2)$:

$$\boldsymbol{\sigma}_{(\widehat{l})}=\boldsymbol{\sigma}_{(r_2)}=(1+\frac{x_3}{R})T^2l_2\frac{ds}{d\widehat{s}}=T^2.$$

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