ON THE SOME REMARKS ABOUT ONE CLASS OF GEOMETRICAL FIGURES

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Let $\mathbb{P}R_m \equiv A_1A_2\cdots A_mA'_1A'_2\cdots A'_m$ be an orthogonal prism, whose ends $A_1\cdots A_m$ and $A'_1\cdots A'_m$ are regular polygons $\mathbb{P}_m$ and m is a number of its angels (verteces). OO' is axis of symmetry of this prism.

Definition. *Generelized Möbius Listing's body* GML_n^m is obtained by identifying of the opposite ends of the prism $\mathbb{PR}_m$ in such a way that:

A) for any $n \in Z$ and $i = \overline{1,m}$, each vertex A_i coincides with $A'_{i+n} \equiv A'_{mod_m(i+n)}$, and each edge A_iA_{i+1} coincides with the edge

$$A'_{i+n}A'_{i+n+1} \equiv A'_{mod_m(i+n)}A'_{mod_m(i+n+1)}$$

correspondingly.[1]

B) $n \in Z$ is a number of rotations of the end of the prism with respect to the axis OO' before the identification.

if $n > 0$ rotations are counter-clockwise, and if $n < 0$ rotations are clockwise.

In particular, if m=2, then $\mathbb{PR}_2 \equiv A_1A_2A'_1A'_2$ is a rectangle, A_1A_2 is a segment of the straight line, and GML_1^2 becomes a classical Möbius band (see for example [1-3]); GML_0^2 is a cylinder or a ring.

I. In this parts of the article we give parametric represantation of the GML_n^m under the following restrictions:

i) middle line OO' transforms in the circle;

ii) the end rotation is evenly along the middle line.

Let

$$\begin{cases} x = p(\tau,\psi), \\ z = q(\tau,\psi) \end{cases} \tag{1}$$

parametric represantation of the regular polygon $\mathbb{P}_m$, where $(\tau,\psi) \in Q \subset \mathbb{R}^2$, such that $p(0,0) \equiv q(0,0) = 0$, the point $(0,0)$ be a center of symmetry of the $\mathbb{P}_m$.

Let $\Omega = \Omega_1 \cup \cdots \cup \Omega_m$ and $\Omega^* = \Omega_1^* \cup \cdots \cup \Omega_m^*$, where for any $i = \overline{1,m}$

$$\Omega_i = \{(x,z,\theta) \in \mathbb{R}^3;\ \ (x,z) \in \mathbb{P}_m,\ \ 2\pi(i-1)R \le \theta < 2\pi iR\}$$

$$\Omega_i^* = \{(\tau,\psi,\theta) \in \mathbb{R}^3;\ \ (\tau,\psi) \in Q,\ \ 2\pi(i-1)R \le \theta < 2\pi iR\}.$$

[1] If we have two numbers $m \in \mathbb{N}, n \in \mathbb{Z}$, then $n = km + i \equiv km + mod_m(n)$, where $k \in \mathbb{Z}$ and $i \equiv mod_m(n) \in \mathbb{N} \cup \{0\}$.

Theorem 1. *The transformation* $F : \Omega^* \to GML_n^m$ *with*

$$F = \begin{cases} x(\tau,\psi,\theta) = \left(R + p(\tau,\psi)\cos\dfrac{n\theta}{mR} - q(\tau,\psi)\ \sin\dfrac{n\theta}{mR}\right)\cos\dfrac{\theta}{R}, \\ y(\tau,\psi,\theta) = \left(R + p(\tau,\psi)\cos\dfrac{n\theta}{mR} - q(\tau,\psi)\ \sin\dfrac{n\theta}{mR}\right)\sin\dfrac{\theta}{R}, \\ z(\tau,\psi,\theta) = p(\tau,\psi)\sin\dfrac{n\theta}{mR} + q(\tau,\psi)\cos\dfrac{n\theta}{mR}, \end{cases} \tag{2}$$

where $(\tau,\psi,\theta) \in \Omega^*$ *is parametric represantation of* GML_n^m. R *is an arbitrary positive number, but* $R > \rho(0, A_i)$ *is distance between center of symmetry of polygon* $\mathbb{P}_m$ *and its vertex* A_i.

Examples:

a) if $m = 2,\ n = 0,\ q(\tau,\psi) \equiv 0,\ p(\tau,\psi) \equiv \tau,\ -\tau^* < \tau < \tau^*$, then GML_0^2 is a circular ring;

b) if $m = 2,\ n = 0,\ p(\tau,\psi) \equiv 0,\ q(\tau,\psi) \equiv \tau,\ -\tau^* < \tau < \tau^*$, then GML_0^2 is a cylinder;

c) if $m = 2$, $n = 1$, then (2) is a parametric represantation of Möbius band (see for example [2]);

d) if $m = 2$, n is even number, then $GML_n^2 \equiv M_n$ is Möbius-Listing's type surface (see [4]) which is one-sided surface and if n is an odd number, then $GML_n^2 = M_n$ is two-sided surface.

Remark 1. If k is the greatest common divisor of m and $mod_m(n)$ then GML_n^m is k - sided surface (i.e. it is possible to paint the surface of this figure in k different colours without taking away of the brush. It is prohibited to cross the edge of this figure).

In particular, GML_1^2 is one-sided surface, properly, the classical Möbius band, but GML_2^2 is two-sided surface. GML_0^m is m-sided surface, for any $m \in N$.

Remark 2. If $m = 2k$, for any $k \in N$, $mod_m(n) = k$, then any diagonal cross-section $A_iA_{i+k}A'_iA'_{i+k}$ of the prism $\mathbb{PR}_m$ after transformation (2) passes into one-sided surface, but if $n = k$, then the one-sided surface is the classical Möbius band.

Remark 3. (Limiting case) If $m = \infty$, then $\mathbb{PR}_\infty$ is circular cylinder and its end $\mathbb{P}_\infty$ is a disk

$$\begin{array}{ll} p(\tau,\psi) = \tau\cos\psi, & \tau \in (0,\tau^*), \\ q(\tau,\psi) = \tau\sin\psi, & \psi \in (0,2\pi). \end{array} \tag{3}$$

In this case transformation (2) has the following form:

$$F = \begin{cases} x(\tau,\psi,\theta) = \left(R + \tau\cos\psi\cos\dfrac{n\theta}{R} - \tau\sin\psi\sin\dfrac{n\theta}{R}\right)\cos\dfrac{\theta}{R}, \\ y(\tau,\psi,\theta) = \left(R + \tau\cos\psi\cos\dfrac{n\theta}{R} - i\sin\psi\sin\dfrac{n\theta}{R}\right)\sin\dfrac{\theta}{R}, \\ z(\tau,\psi,\theta) = \tau\cos\psi\sin\dfrac{n\theta}{R} + \tau\sin\psi\cos\dfrac{n\theta}{R}, \end{cases} \tag{4}$$

where n is any real number.

Remark 4. If $n = 0$, formula (4) gives a parametric represantation of the classical torus (see, e.g., [3]).

Remark 5. If $n = \frac{1}{2}$, then every diametral cross-section of $\mathbb{PR}_\infty$ which contains OO' after transformation (4) passes into classical Möbius band $CLM_1^2 \equiv M_1$ (see [2] or [4]).

Remark 6. For any n figure GML_n^∞ is geometrically identical to the torus.

Remark 7. If (τ_0, ψ_0) is an arbitrary fixed point of $\partial\mathbb{P}_\infty$ (circle), then

$$l_n(\theta) = (x(\tau_0, \psi_0, \theta), y(\tau_0, \psi_0, \theta), z(\tau_0, \psi_0, \theta))$$

is a courve lying on the torus.

a) If $n \in Z$, the $l_n(\theta) = l_n(\theta + 2\pi)$ is a closed courve, and n is a number of coils around of little parts of the torus.

b) If $n = \frac{1}{k}, k \in Z$, then $l_n(\theta) = l_n(\theta + 2\pi k)$ is a closed courve, but after k rotations around of big parts of the torus we have only one coil around of little part of the torus.

c) If $n = \frac{p}{k},\ p, k \in Z$, then $l_n(\theta) = l_n(\theta + 2\pi k)$ is a closed courve, and after k rotations around of big parts of torus we have p coils around of little part of the torus.

d) If $n \in R \backslash Q$ is irrational number, then $l_n(\theta)$ is nonclosed courve. This courve makes infinite coils after infinite circuits arournd the torus, but this courve is not self-crossing.

II. In this part of the article we give parametric represantation of the GML_n^m under the following restrictions:

i) middle line OO' transforms in the some closed courve;

ii) the end rotation the end is semi-regular.

Let

$$\mathbb{L}_\rho = \begin{cases} x = f_1(\rho, \theta) \\ y = f_2(\rho, \theta) \end{cases} \tag{5}$$

be some one-parametric familly of closed courves, morever:

a) for every fixed $\rho \in [0, \rho^*]$, L_ρ is a closed courve and $f_i(\rho, \theta + 2\pi) = f_i(\rho, \theta),\ \ i = \overline{1,2}$

b) for any $\rho_1, \rho_2 \in [0, \rho^*],\ \rho_1 \neq \rho_2$, courves $\mathbb{L}_{\rho_1}$ and $\mathbb{L}_{\rho_2}$ have not common points.

Let $g(\theta) : [0, 2\pi] \to [0, 2\pi]$ be arbitrary functions and for every $\Phi \in [0, 2\pi]$ exist $\theta \in [0, 2\pi]$ such that $\Phi = g(\theta)$.

Theorem 2. *The transformation* $F:\Omega^*\to GML_n^m$ *with*

$$F=\begin{cases} x(\tau,\psi,\theta) &= f_1\left(\left(R+\rho(\tau,\psi)\cos\dfrac{ng(\theta)}{mR}-q(\tau,\psi)\sin\dfrac{ng(\theta)}{mR}\right),\dfrac{\theta}{R}\right),\\ \varphi(\tau,\psi,\theta) &= f_2\left(\left(R+\rho(\tau,\psi)\cos\dfrac{ng(\theta)}{mR}-q(\tau,\psi)\sin\dfrac{ng(\theta)}{mR}\right),\dfrac{\theta}{R}\right),\\ z(\tau,\psi,\theta) &= \rho(\tau,\psi)\sin\dfrac{ng(\theta)}{mR}+q(\tau,\psi)\cos\dfrac{ng(\theta)}{mR}, \end{cases} \tag{6}$$

where $(\tau,\psi,\theta)\in\Omega^*$, *is parametric represantation of* GML_n^m. R *is a arbitrary positive number, but* $R>\rho(0,A_i)$ *is a distance between center of symmetry of the polygon* $\mathbb{P}_m$ *and its vertex* A_i.

Remark 8. If (1) is a parametric represantation of an arbitrary plane figure, then in formula (6) $m\equiv 1$, for any $n\in Z$.

Remark 9. If $\mathbb{P}_\infty$ is a disk, then in formula (6) $m\equiv 1$ and n is an arbitrary real number.

References

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