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Mixed Boundary Value Problems for an Anisotropic Helmholts Type Pseudo-Oscillation Equation

Tornike Tsertsvadze[a*]

[a]*Department of Mathematics, Georgian Technical University, 77 M. Kostava St., Tbilisi 0160, Georgia*

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In this paper, we consider a special approach to investigate a three-dimensional mixed boundary value problem (BVP) for an anisotropic Helmholtz type equation which is a second order elliptic partial differential equation containing a complex parameter τ. The boundary surface $S = \partial\Omega$ of a domain under consideration, $\Omega \subset \mathbb{R}^3$, is divided into two disjoint parts, S_D and S_N, where the Dirichlet and Neumann type boundary conditions are prescribed respectively. Our approach is based on the potential method. We look for a solution to the mixed boundary value problem in the form of linear combination of the corresponding single layer and double layer potentials with the densities supported respectively on the Dirichlet and Neumann parts of the boundary. This approach reduces the mixed BVP to a system of pseudodifferential equations. It is shown that the corresponding pseudodifferential matrix operator is bounded and coercive in the appropriate L_2-based Bessel potential spaces. Consequently, the operator is invertible, which implies the unconditional unique solvability of the mixed BVP in the Sobolev space $W_2^1(\Omega)$. Using a special structure of the obtained pseudodifferential matrix operator, it is also shown that it is invertible in the L_p-based Besov spaces, which under appropriate boundary data implies C^α-Hölder continuity of the solution to the mixed BVP in the closed domain $\overline{\Omega}$ with $\alpha = \frac{1}{2} - \varepsilon$, where $\varepsilon > 0$ is an arbitrarily small number.

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1. Introduction

Here we consider a special approach to investigate a mixed boundary value problem (BVP) for anisotropic Helmholtz type equation in the case of three-dimensional bounded domain $\Omega \subset \mathbb{R}^3$, when the smooth boundary surface $S = \partial\Omega$ is divided into two smooth disjoint parts, S_D and S_N, where the Dirichlet and Neumann type boundary conditions are prescribed respectively. Our approach is based on the classical potential method. We look for a solution to the mixed boundary value problem in the form of linear combination of the single layer and double layer potentials with the different densities supported respectively on the Dirichlet and Neumann parts of the boundary. This approach reduces the mixed BVP under consideration to a system of pseudodifferential equations, where the right hand side functions coincide with the Dirichlet and Neumann boundary data. The corresponding pseu-

*Corresponding author. Email: tsertsvadze.tornike@yahoo.com

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dodifferential matrix operator is bounded and coercive in the appropriate L_2-based Bessel potential spaces. Consequently, the operator is invertible, which implies the unconditional unique solvability of the mixed BVP in the Sobolev space $W_2^1(\Omega)$. Using a special structure of the obtained pseudodifferential matrix operator with the help of the bootstrap arguments we show that the operator is invertible in the L_p-based Besov spaces, which under appropriate boundary data implies C^α-Hölder continuity of the solution to the mixed BVP in the closed domain $\overline{\Omega}$ with $\alpha = \frac{1}{2} - \varepsilon$, where $\varepsilon > 0$ is an arbitrarily small number.

This type of mixed boundary value problems are studied in scientific literature by using the potential methods (see, e.g. [4], [16], [20]). In contrast to the existing approaches, our alternative method has two essential advantages, on the one hand, it does not require extension of the given boundary data to the whole surface and, on the other hand, the representation of a solution doesn't contain the Steklov-Poincaré type operator containing the inverse operator of the single layer boundary operator, which is not available explicitly in general for arbitrary surface. Therefore we hope that our alternative approach will play a crucial role in the process of construction of efficient algorithms for numerical solutions to the mixed BVPs.

2. Formulation of the mixed BVP and uniqueness theorem

Let $\Omega = \Omega^+ \subset \mathbb{R}^3$ be a three-dimensional bounded domain with the boundary $\partial\Omega = S$, which is divided into two simply connected disjoint parts, S_D and S_N, $\overline{S}_D \cup \overline{S}_N = S$, $S_D \cap S_N = \varnothing$. For simplicity we assume that $S \in C^\infty$ and $\ell = \partial S_D = \partial S_N \in C^\infty$. We denote the complement of the domain Ω to the whole space by $\Omega^- = \mathbb{R}^3 \setminus \overline{\Omega}$.

By L_p, $L_{p,loc}$, W_p^r, $W_{p,loc}^r$, H_p^s, $H_{p,loc}^s$, $B_{p,q}^s$, and $B_{p,q,loc}^s$ (with $r \geq 0$, $s \in \mathbb{R}$, $1 < p < \infty$, $1 \leq q \leq \infty$) we denote the well-known Lebesgue, Sobolev-Slobodetskii, Bessel potential, and Besov spaces of complex valued functions of real variables, respectively (see, e.g., [13], [21], [22]). Recall that $H_2^r = W_2^r = B_{2,2}^r$, $H_2^s = B_{2,2}^s$, $W_p^t = B_{p,p}^t$, and $H_p^k = W_p^k$, for any $r \geq 0$, for any $s \in \mathbb{R}$, for any positive and non-integer t, and for any non-negative integer k. In our analysis we employ also the spaces:

$$\widetilde{H}_p^s(\mathcal{M}) := \{f \, : \, f \in H_p^s(\mathcal{M}_0), \ \operatorname{supp} f \subset \overline{\mathcal{M}}\},$$

$$\widetilde{B}_{p,q}^s(\mathcal{M}) := \{f \, : \, f \in B_{p,q}^s(\mathcal{M}_0), \ \operatorname{supp} f \subset \overline{\mathcal{M}}\},$$

$$H_p^s(\mathcal{M}) := \{r_{\mathcal{M}} f \, : \, f \in H_p^s(\mathcal{M}_0)\},$$

$$B_{p,q}^s(\mathcal{M}) := \{r_{\mathcal{M}} f \, : \, f \in B_{p,q}^s(\mathcal{M}_0)\,\},$$

where $\mathcal{M}_0$ is a closed manifold without a boundary and $\mathcal{M}$ is an open proper sub-manifold of $\mathcal{M}_0$ with a nonempty smooth boundary $\partial\mathcal{M} \neq \varnothing$; $r_{\mathcal{M}}$ is the restriction operator onto $\mathcal{M}$. The norms in these spaces are determined by the standard way:

$$\|u\|_{\widetilde{H}_p^t(\mathcal{M})} = \|u\|_{H_p^t(\mathcal{M}_0)}, \qquad \|u\|_{\widetilde{B}_{p,q}^t(\mathcal{M})} = \|u\|_{B_{p,q}^t(\mathcal{M}_0)},$$

$$\|u\|_{H_p^t(\mathcal{M})} = \inf \|v\|_{H_p^t(\mathcal{M}_0)}, \quad v \in H_p^t(\mathcal{M}_0), \quad r_{\mathcal{M}} v = u,$$

$$\|u\|_{B^t_{p,q}(\mathcal{M})} = \inf \|v\|_{B^t_{p,q}(\mathcal{M}_0)}, \quad v \in B^t_{p,q}(\mathcal{M}_0), \quad r_{\mathcal{M}} v = u.$$

Remark 1 : Let a function f be defined on an open proper submanifold $\mathcal{M}$ of a closed manifold $\mathcal{M}_0$ without a boundary. Let $f \in B^s_{p,q}(\mathcal{M})$ and let $\widetilde{f}$ be the extension of f by zero to $\mathcal{M}_0 \setminus \mathcal{M}$. If the extension preserves the space, i.e., if $\widetilde{f} \in \widetilde{B}^s_{p,q}(\mathcal{M})$, then we write $f \in \widetilde{B}^s_{p,q}(\mathcal{M})$ instead of $f \in r_{\mathcal{M}} \widetilde{B}^s_{p,q}(\mathcal{M})$, when it does not lead to misunderstanding.

Moreover, $\widetilde{B}^s_{p,q}(\mathcal{M})$ and $B^{-s}_{p',q'}(\mathcal{M})$ with $\frac{1}{p}+\frac{1}{p'}=1$ and $\frac{1}{q}+\frac{1}{q'}=1$ are dual spaces. Similarly, $\widetilde{H}^s_p(\mathcal{M})$ and $H^{-s}_{p'}(\mathcal{M})$ are dual spaces (for details see, e.g., [21], [22]).

Therefore, for functions $f \in B^{-s}_{p',q'}(\mathcal{M})$ and $g \in \widetilde{B}^s_{p,q}(\mathcal{M})$ (resp., $f \in H^{-s}_{p'}(\mathcal{M})$ and $g \in \widetilde{H}^s_p(\mathcal{M})$) the duality relation $\langle f,\, g\rangle_{\mathcal{M}}$ is well defined and it generalizes the classical L_2-inner product,

$$\langle f,\, g\rangle_{\mathcal{M}} = \overline{\langle g,\, f\rangle}_{\mathcal{M}} = \int\limits_{\mathcal{M}} f(x)\,\overline{g(x)}\,d\mathcal{M} \quad \text{for} \quad f, g \in L_2(\mathcal{M}),$$

where the overbar denotes complex conjugation operation.

By the symbols $\{\,\cdot\,\}^+$ and $\{\,\cdot\,\}^-$ we denote the one-sided traces on the surface $S = \partial\Omega^\pm$ from Ω^+ and Ω^- respectively.

It is well known that mixed boundary value problems for elliptic partial differential equations have not solutions in the class of regular functions in general (see, e.g., [10], [20] and the references therein). Therefore, the existence theorems of solutions are proved in the space of generalized functions and using the embedding theorems the Hölder regularity properties of solution are derived for smoother boundary data.

Now we formulate the interior mixed boundary value problem for the anisotropic Helmholtz type pseudo-oscillation equation in the domain $\Omega = \Omega^+$: Find a complex valued function $u \in H^1_2(\Omega)$, which satisfies the following conditions:

$$\sum_{k,j=1}^{3} a_{kj}\,\frac{\partial^2 u(x)}{\partial x_k \partial x_j} - \tau^2 u(x) = 0, \quad x \in \Omega, \tag{2.1}$$

$$\{\,u\,\}^+ = f \quad \text{on} \quad S_D, \tag{2.2}$$

$$\{\,T(\partial_x, n)u\,\}^+ = F \quad \text{on} \quad S_N, \tag{2.3}$$

where $\mathbf{a} = [a_{kj}]_{3\times 3}$ is a positive definite constant matrix with real entries, $\partial_x = (\partial_1, \partial_2, \partial_3)$, $\partial_j = \frac{\partial}{\partial x_j}$, $\tau = \sigma + i\,\omega$ is a complex parameter with $\sigma < 0$ and $\omega \in \mathbb{R}$, $n(n_1, n_2, n_3)$ stands for the outward unit normal vector to S, $T(\partial_x, n(x))$ denotes the conormal derivative

$$T(\partial_x, n(x))u(x) = \sum_{k,j=1}^{3} a_{kj} n_k(x) \partial_j u(x), \quad x \in S,$$

and the boundary data f and F meet the following natural inclusions

$$f \in H_2^{\frac{1}{2}}(S_D), \quad F \in H_2^{-\frac{1}{2}}(S_N). \tag{2.4}$$

Here, we understand equation (2.1) in the weak sense. The boundary condition (2.2) is understood in the trace sense, while for a weak solution u to equation (2.1) condition (2.3) is understood in the generalized functional sense defined with the help of Green's generalized formula (see, e.g., [15]),

$$\begin{aligned}\big\langle a_{kj}\partial_k\partial_j u(x) - \tau^2 u(x),\, v(x)\big\rangle_\Omega = -\big\langle a_{kj}\partial_j u(x)\,, \partial_k v(x)\big\rangle_\Omega - \tau^2\big\langle u(x),\, v(x)\big\rangle_\Omega \\ + \big\langle \{Tu(x)\}^+,\, \{v(x)\}^+\big\rangle_S \quad \text{for all } \ v \in H_2^1(\Omega),\end{aligned}$$

i.e.,

$$\begin{aligned}\big\langle \{Tu(x)\}^+,\, \{v(x)\}^+\big\rangle_S := \big\langle a_{kj}\partial_j u(x),\, \partial_k v(x)\big\rangle_\Omega + \tau^2\big\langle u(x),\, v(x)\big\rangle_\Omega \\ \text{for all } \ v \in H_2^1(\Omega).\end{aligned} \tag{2.5}$$

Here we assume summation over the repeated indices from 1 to 3 (Einstein summation convention). Since $v \in H_2^1(\Omega)$ is an arbitrary function and $\{v\}^+ \in H_2^{\frac{1}{2}}(S)$, it follows that relation (2.5) correctly defines the generalized trace $\{Tu\}^+ \in H_2^{-\frac{1}{2}}(S)$ for an arbitrary weak solution u of equation (2.1).

Equation (2.5) is called Green's generalized identity implying the well known uniqueness theorem.

Theorem 2.1 : *The mixed boundary value problem* (2.1)-(2.3) *possesses at most one solution.*

Proof : It suffices to show that the homogeneous mixed BVP possesses only the trivial solution which immediately follows from Green's formula (2.5) by separating the real and imaginary parts.

□

Remark 2 : The exterior mixed boundary value problem for the anisotropic Helmholtz equation in the domain Ω^- is formulated analogously: Find a function $u \in H^1_{2,loc}(\Omega^-)$, which satisfies the following conditions:

$$a_{kj}\partial_k\partial_j u(x) - \tau^2 u(x) = 0 \quad \text{in} \quad \Omega^-, \tag{2.6}$$

$$\{\, u\,\}^- = f \quad \text{on} \quad S_D, \tag{2.7}$$

$$\{\, T(\partial_x, n)u\,\}^- = F \quad \text{on} \quad S_N, \tag{2.8}$$

$$\lim_{|x|\,\to\infty} u(x) = 0, \tag{2.9}$$

where f and F satisfy inclusions (2.4).

It is well known that due to the decaying asymptotic behaviour (2.9), this exterior mixed BVP possesses at most one solution as well.

It can be shown that a solution $u \in H^1_{2,loc}(\Omega^-)$ of equation (2.6) satisfying condition (2.9) actually decays exponentially at infinity and consequently $u \in H_2^1(\Omega^-)$.

3. Properties of potentials

To investigate the existence of a solution to the formulated mixed boundary value problem, we introduce the single and double layers potentials associated with the anisotropic Helmholtz type pseudo-oscillation operator (cf. [14])

$$V_\tau(\varphi)(x) = \int\limits_S \Gamma(x-y,\tau)\,\varphi(y)\,dS_y, \quad x \in \Omega^\pm,$$

$$W_\tau(\psi)(x) = \int\limits_S T(\partial_y, n(y))\Gamma(x-y,\tau)\,\psi(y)\,dS_y, \quad x \in \Omega^\pm,$$

where φ and ψ are densities and Γ is the fundamental solution of the anisotropic Helmholtz equation (2.1),

$$\Gamma(x-y,\tau) = -\frac{\exp\left\{\tau\big(\mathbf{a}^{-1}(x-y)\cdot(x-y)\big)\right\}}{4\pi\,\mathbf{a}^{-1}(x-y)\cdot(x-y)}\,.$$

Here $\mathbf{a}^{-1}$ stands for the inverse to the matrix $\mathbf{a}$ and the central dot denotes the scalar product. One can easily check the following relations for sufficiently small $|x-y|$:

$$\Gamma(x-y,\tau) - \Gamma(x-y) = -\frac{\tau}{4\pi} + \mathcal{O}(|x-y|), \tag{3.1}$$

$$\partial_k\Gamma(x-y,\tau) - \partial_k\Gamma(x-y) = \mathcal{O}(1), \tag{3.2}$$

$$\partial_k\partial_j\Gamma(x-y,\tau) - \partial_k\partial_j\Gamma(x-y) = \mathcal{O}(|x-y|^{-1}), \tag{3.3}$$

where

$$\Gamma(x-y) = \Gamma(x-y,0) = -\frac{1}{4\pi\,\mathbf{a}^{-1}(x-y)\cdot(x-y)}\,. \tag{3.4}$$

For these potentials, the following jump relations for smooth densities are well known (see, e.g., [14, Ch. 2]):

$$\{V_\tau(\varphi)(x)\}^+ = \{V_\tau(\varphi)(x)\}^- = \int\limits_S \Gamma(x-y,\tau)\,\varphi(y)\,dS_y, \quad x \in S,$$

$$\{T(\partial_x, n(x))V_\tau(\varphi)(x)\}^\pm = \mp\frac{1}{2}\,\varphi(x) + \int\limits_S T(\partial_x, n(x))\Gamma(x-y,\tau)\,\varphi(y)\,dS_y, \quad x \in S,$$

$$\{W_\tau(\psi)(x)\}^\pm = \pm\frac{1}{2}\,\psi(x) + \int\limits_S T(\partial_y, n(y))\Gamma(x-y,\tau)\,\psi(y)\,dS_y, \quad x \in S,$$

$$\{T(\partial_x, n(x))W_\tau(\psi)(x)\}^+ = \{T(\partial_x, n(x))W_\tau(\psi)(x)\}^-$$

$$= \lim_{\Omega^+ \ni z \to x \in S} \int\limits_S T(\partial_z, n(x)) T(\partial_y, n(y)) \Gamma(z-y,\tau)\, \psi(y)\, dS_y$$

$$= \lim_{\Omega^- \ni z \to x \in S} \int\limits_S T(\partial_z, n(x)) T(\partial_y, n(y)) \Gamma(z-y,\tau)\, \psi(y)\, dS_y, \quad x \in S.$$

Let us introduce the boundary integral operators generated by the above single layer and double layer potentials:

$$\mathcal{H}_\tau \varphi(x) := \int\limits_S \Gamma(x-y,\tau)\, \varphi(y)\, dS_y, \quad x \in S,$$

$$\mathcal{K}_\tau^\top \varphi(x) := \int\limits_S T(\partial_x, n(x)) \Gamma(x-y,\tau)\, \varphi(y)\, dS_y, \quad x \in S,$$

$$\mathcal{K}_\tau \psi(x) := \int\limits_S T(\partial_y, n(y)) \Gamma(x-y,\tau)\, \psi(y)\, dS_y, \quad x \in S,$$

$$\mathcal{L}_\tau \psi(x) := \lim_{\Omega^\pm \ni z \to x \in S} \int\limits_S T(\partial_z, n(x)) T(\partial_y, n(y)) \Gamma(z-y,\tau)\, \psi(y)\, dS_y, \quad x \in S.$$

Throughout the paper, the potentials and boundary operators constructed by the fundamental solution $\Gamma(x-y)$ are equipped with the subscript 0 and they correspond to the case $\tau = 0$.

The following mapping properties for the single and double layer potentials and the corresponding boundary operators are also well known (see, e.g., [3], [4], [15], [17] for general elliptic systems).

Theorem 3.1 : *Let $s \in \mathbb{R}$, $1 < p < \infty$, and $1 \leqslant q \leqslant \infty$.*
(i) *The following potential operators are continuous*

$$V_\tau : H_2^{-\frac{1}{2}}(S) \to H_2^1(\Omega^+), \qquad W_\tau : H_2^{\frac{1}{2}}(S) \to H_2^1(\Omega^+),$$

$$V_\tau : B_{p,p}^s(S) \to H_p^{s+1+\frac{1}{p}}(\Omega^+), \quad W_\tau : B_{p,p}^s(S) \to H_p^{s+\frac{1}{p}}(\Omega^+),$$

$$V_\tau : B_{p,q}^s(S) \to B_{p,q}^{s+1+\frac{1}{p}}(\Omega^+), \quad W_\tau : B_{p,q}^s(S) \to B_{p,q}^{s+\frac{1}{p}}(\Omega^+).$$

$$V_\tau : H_2^{-\frac{1}{2}}(S) \to H_{2,loc}^1(\Omega^-), \qquad W_\tau : H_2^{\frac{1}{2}}(S) \to H_{2,loc}^1(\Omega^-),$$

$$V_\tau : B_{p,p}^s(S) \to H_{p,loc}^{s+1+\frac{1}{p}}(\Omega^-), \qquad W_\tau : B_{p,p}^s(S) \to H_{p,loc}^{s+\frac{1}{p}}(\Omega^-),$$

$$V_\tau : B_{p,q}^s(S) \to B_{p,q,loc}^{s+1+\frac{1}{p}}(\Omega^-), \qquad W_\tau : B_{p,q}^s(S) \to B_{p,q,loc}^{s+\frac{1}{p}}(\Omega^-).$$

(ii) *The following boundary operators are continuous*

$$\begin{aligned} \mathcal{H}_\tau &: H_2^s(S) \to H_2^{s+1}(S), & \mathcal{H}_\tau &: B_{p,q}^s(S) \to B_{p,q}^{s+1}(S), \\ \mathcal{K}_\tau, \mathcal{K}_\tau^\top &: H_2^s(S) \to H_2^s(S), & \mathcal{K}_\tau, \mathcal{K}_\tau^\top &: B_{p,q}^s(S) \to B_{p,q}^s(S), \\ \mathcal{L}_\tau &: H_2^s(S) \to H_2^{s-1}(S), & \mathcal{L}_\tau &: B_{p,q}^s(S) \to B_{p,q}^{s-1}(S), \end{aligned} \tag{3.5}$$

(iii) *Moreover, the following jump relations*

$$\begin{aligned} &\{V_\tau(\varphi)(x)\}^+ = \{V_\tau(\varphi)(x)\}^- = \mathcal{H}_\tau\varphi(x), \ \ x \in S, \\ &\{T(\partial_x, n(x))V_\tau(\varphi)(x)\}^\pm = \mp\frac{1}{2}\varphi(x) + \mathcal{K}_\tau^\top\varphi(x), \ \ x \in S, \\ &\{W_\tau(\psi)(x)\}^\pm = \pm\frac{1}{2}\psi(x) + \mathcal{K}_\tau\psi(x), \ \ x \in S, \\ &\{T(\partial_x, n(x))W_\tau(\psi)(x)\}^+ = \{T(\partial_x, n(x))W_\tau(\psi)(x)\}^- = \mathcal{L}_\tau\psi(x), \ \ x \in S, \end{aligned}$$

hold in the sense of appropriate function spaces for arbitrary $\varphi \in B_{p,q}^{-\frac{1}{p}}(S)$ *and* $\psi \in B_{p,q}^{1-\frac{1}{p}}(S)$.

Remark 1: (Cf. [5], [20]) It is evident that in the case of a smooth surface $S \in C^2$, the operators $\mathcal{K}_\tau$ and $\mathcal{K}_\tau^* = \overline{\mathcal{K}_\tau^\top} = \mathcal{K}_{\overline{\tau}}^\top$ are mutually adjoint weakly singular integral operators and for functions $\psi \in H_2^{\frac{1}{2}}(S)$ and $\varphi \in H_2^{-\frac{1}{2}}(S)$ the following relation holds

$$\langle \mathcal{K}_\tau^*\varphi,\, \psi\rangle_S = \langle \varphi,\, \mathcal{K}_\tau\psi\rangle_S. \tag{3.6}$$

Actually, due to the weak singularity of the kernel functions of the integral operators $\mathcal{K}_\tau$ and $\mathcal{K}_\tau^\top$ we have the following smoothing mapping properties:

$$\begin{aligned} &\mathcal{K}_\tau,\ \mathcal{K}_\tau^\top \ : \ H_p^s(S) \to H_p^{s+1}(S) \subset H_p^s(S), \quad p > 1, \\ &\mathcal{K}_\tau,\ \mathcal{K}_\tau^\top \ : \ B_{p,q}^s(S) \to B_{p,q}^{s+1}(S) \subset B_{p,q}^s(S), \quad p > 1,\ q \geqslant 1, \end{aligned}$$

which imply that operators (3.5) are compact thanks to the compact embedding theorems for the Bessel potential and Besov spaces.

Remark 2: (Cf. [7], [8], [9], [11], [12], [20]) The operators $\mathcal{H}_0$ and $\mathcal{L}_0$ are strongly elliptic self-adjoint pseudodifferential operators of order -1 and $+1$ respectively, and for complex valued functions $\varphi_1, \varphi_2 \in H_2^{-\frac{1}{2}}(S)$ and $\psi_1, \psi_2 \in H_2^{\frac{1}{2}}(S)$ the following relations hold

$$\langle \mathcal{H}_0\varphi_1,\, \varphi_2\rangle_S = \langle \varphi_1,\, \mathcal{H}_0\varphi_2\rangle_S, \quad \langle \mathcal{L}_0\psi_1,\, \psi_2\rangle_S = \langle \psi_1,\, \mathcal{L}_0\psi_2\rangle_S.$$

Moreover, there are positive constants δ_1 and δ_2 such that the following inequalities hold

$$\big\langle -\mathcal{H}_0\varphi,\, \varphi\big\rangle_S \geqslant \delta_1\, \|\varphi\|^2_{H_2^{-\frac{1}{2}}(S)} \quad \text{for arbitrary} \quad \varphi\in H_2^{-\frac{1}{2}}(S), \tag{3.7}$$

$$\big\langle \mathcal{L}_0\psi,\, \psi\big\rangle_{S_0} \geqslant \delta_2\, \|\psi\|^2_{H_2^{\frac{1}{2}}(S_0)} \quad \text{for arbitrary} \quad \psi\in \widetilde{H}_2^{\frac{1}{2}}(S_0), \tag{3.8}$$

where S_0 is a non-empty open proper part of S, $\overline{S}_0\neq S$.

In our analysis we essentially use the following assertion (for details see, e.g., [1], [2], [6], [10], [16], [19]).

Lemma 3.2: *Let $S_1\in C^\infty$ be a bounded, 2-dimensional, non–self–intersecting, two–sided surface with the boundary $\partial S_1\in C^\infty$, and $s\in\mathbb{R}$, $1<p<\infty$, $1\le q\le\infty$. Further, let $\mathcal{A}$ be a strongly elliptic pseudodifferential operator of order $\alpha\in\mathbb{R}$ on S_1 having a uniformly positive principal homogeneous symbol, $\mathfrak{S}(\mathcal{A};x,\xi)\ge c_0=const>0$ for $x\in\overline{S}_1$, $\xi\in\mathbb{R}^2$ with $|\xi|=1$.*

Then the operators

$$\mathcal{A}:\widetilde{H}^s_p(S_1)\to H^{s-\alpha}_p(S_1) \tag{3.9}$$

$$\mathcal{A}:\widetilde{B}^s_{p,q}(S_1)\to B^{s-\alpha}_{p,q}(S_1) \tag{3.10}$$

are Fredholm operators of index zero if

$$\frac{1}{p}-1<s-\frac{\alpha}{2}<\frac{1}{p}. \tag{3.11}$$

Moreover, the null-spaces of operators (3.9) *and* (3.10) *are the same for all values of the parameter s, p, and $1\leqslant q\leqslant\infty$, provided s and p satisfy inequality* (3.11).

This lemma and Remark **2** lead to the following theorem for the boundary operators $\mathcal{H}_\tau$ and $\mathcal{L}_\tau$ generated by the single and double layer potentials.

Theorem 3.3: *Let $S_1\in\{S_D,\, S_N\}$, $s\in\mathbb{R}$, $1<p<+\infty$, $1\le q\le+\infty$ and $-\frac{1}{2}<s-\frac{1}{p}<\frac{1}{2}$.*
Then the pseudodifferential operators

$$r_{S_1}\mathcal{H}_\tau:\widetilde{H}^{s-1}_p(S_1)\to H^s_p(S_1),\qquad r_{S_1}\mathcal{H}_\tau:\widetilde{B}^{s-1}_{p,q}(S_1)\to B^s_{p,q}(S_1),$$

$$r_{S_1}\mathcal{L}_\tau:\widetilde{H}^{s}_p(S_1)\to H^{s-1}_p(S_1),\qquad r_{S_1}\mathcal{L}_\tau:\widetilde{B}^{s}_{p,q}(S_1)\to B^{s-1}_{p,q}(S_1),$$

are invertible.

4. Existence results

We look for a solution to the mixed boundary value problem in the form of the linear combination of single and double layer potentials

$$\begin{aligned} u(x) &= -\,V_\tau(\varphi)(x) + W_\tau(\psi)(x) \qquad (4.1)\\ &= -\int\limits_S \Gamma(x-y,\tau)\,\varphi(y)\,dS_y + \int\limits_S T(\partial_y, n(y))\Gamma(x-y,\tau)\,\psi(y)\,dS_y, \quad x\in\Omega, \end{aligned}$$

with unknown densities

$$\varphi \in \widetilde{H}_2^{-\frac{1}{2}}(S_D), \qquad \psi \in \widetilde{H}_2^{\frac{1}{2}}(S_N). \tag{4.2}$$

Evidently, by Theorem 3.1 we have $u \in C^\infty(\Omega) \cap H_2^1(\Omega)$ and equation (2.1) is automatically satisfied in the classical sense in Ω.

The mixed boundary conditions (2.2) and (2.3) lead to the following integral equations:

$$\begin{aligned} -\mathcal{H}_\tau\varphi + \frac{1}{2}\,\psi + \mathcal{K}_\tau\psi &= f \quad \text{on} \quad S_D,\\ \frac{1}{2}\,\varphi - \mathcal{K}_\tau^\top\varphi + \mathcal{L}_\tau\psi &= F \quad \text{on} \quad S_N. \end{aligned}$$

In view of inclusions (4.2), these equation can be rewritten as follows

$$-\mathcal{H}_\tau\varphi + \mathcal{K}_\tau\psi = f \quad \text{on} \quad S_D, \tag{4.3}$$

$$-\mathcal{K}_\tau^\top\varphi + \mathcal{L}_\tau\psi = F \quad \text{on} \quad S_N. \tag{4.4}$$

Let us introduce the notation:

$$\mathbf{A}_\tau := \begin{bmatrix} r_{S_D}(-\mathcal{H}_\tau) & r_{S_D}(\mathcal{K}_\tau)\\ r_{S_N}(-\mathcal{K}_\tau^\top) & r_{S_N}(\mathcal{L}_\tau)\end{bmatrix}_{2\times 2}, \quad X := \begin{bmatrix}\varphi\\ \psi\end{bmatrix}, \quad G := \begin{bmatrix} f\\ F\end{bmatrix}, \tag{4.5}$$

where r_{S_D} and r_{S_N} are the restriction operators onto S_D and S_N respectively.

Then the system of equations (4.3)-(4.4) can be rewritten in the vector-matrix form,

$$\mathbf{A}_\tau\, X = G. \tag{4.6}$$

Further, let

$$\widetilde{\mathbb{H}}_2 := \widetilde{H}_2^{-\frac{1}{2}}(S_D) \times \widetilde{H}_2^{\frac{1}{2}}(S_N), \quad \mathbb{H}_2 := H_2^{\frac{1}{2}}(S_D) \times H_2^{-\frac{1}{2}}(S_N).$$

Obviously, $\widetilde{\mathbb{H}}_2$ and $\mathbb{H}_2$ are mutually adjoint spaces. The norms in these spaces are determined as follows:

$$\|X\|^2_{\widetilde{\mathbb{H}}_2} := \|\varphi\|^2_{\widetilde{H}_2^{-\frac{1}{2}}(S_D)} + \|\psi\|^2_{\widetilde{H}_2^{\frac{1}{2}}(S_N)} \quad \text{for} \quad X = \begin{bmatrix} \varphi \\ \psi \end{bmatrix} \in \widetilde{\mathbb{H}}_2,$$

$$\|G\|^2_{\mathbb{H}_2} := \|f\|^2_{H_2^{\frac{1}{2}}(S_D)} + \|F\|^2_{H_2^{-\frac{1}{2}}(S_N)} \quad \text{for} \quad G = \begin{bmatrix} f \\ F \end{bmatrix} \in \mathbb{H}_2.$$

We will analyze solvability of equation (4.6) in the space $\widetilde{\mathbb{H}}_2$ for arbitrary right hand side vector function $G \in \mathbb{H}_2$. By Theorem 3.1(ii), the operator $\mathbf{A}_\tau$ has the following mapping property:

$$\mathbf{A}_\tau \,:\, \widetilde{\mathbb{H}}_2 \to \mathbb{H}_2. \tag{4.7}$$

Let us show that (4.7) is isomorphism. To this end, first of all, let us note that in view of relations (3.1)-(3.3) and the Relich-Kondrashov compact embedding theorems, the operator

$$\mathbf{A}_\tau - \mathbf{A}_0 \,:\, \widetilde{\mathbb{H}}_2 \to \mathbb{H}_2$$

is compact. Here, the operator $\mathbf{A}_0$ is defined by (4.5) with $\tau = 0$.
Further, we show that the operator

$$\mathbf{A}_0 \,:\, \widetilde{\mathbb{H}}_2 \to \mathbb{H}_2 \tag{4.8}$$

generates a bounded and coercive bilinear form. Indeed, let $X' := \begin{bmatrix} \varphi' \\ \psi' \end{bmatrix}$ and $X'' := \begin{bmatrix} \varphi'' \\ \psi'' \end{bmatrix}$ be arbitrary elements of the space $\widetilde{\mathbb{H}}_2$. Then $\mathbf{A}_0 X' \in \mathbb{H}_2$ and the following duality is well-defined

$$\begin{aligned} \big\langle \mathbf{A}_0 X',\, X'' \big\rangle_{(\mathbb{H}_2,\, \widetilde{\mathbb{H}}_2)} := \big\langle -\mathcal{H}_0 \varphi',\, \varphi'' \big\rangle_{S_D} + \big\langle \mathcal{K}_0 \psi',\, \varphi'' \big\rangle_{S_D} \\ - \big\langle \mathcal{K}_0^\top \varphi',\, \psi'' \big\rangle_{S_N} + \big\langle \mathcal{L}_0 \psi',\, \psi'' \big\rangle_{S_N}. \end{aligned} \tag{4.9}$$

Here the symbol $\langle \cdot\,,\, \cdot \rangle_{(\mathbb{H}_2,\, \widetilde{\mathbb{H}}_2)}$ denotes the duality between the mutually adjoint spaces $\mathbb{H}_2$ and $\widetilde{\mathbb{H}}_2$.

Note that due to inclusions (4.2), the duality brackets in equality (4.9) over S_D and over S_N can be replaced by the duality brackets over S. In view of relation (3.6) and since the kernels of the integral operators for $\tau = 0$ are real valued functions, we have

$$\big\langle \mathcal{K}_0 \psi',\, \varphi'' \big\rangle_S = \big\langle \psi',\, \mathcal{K}_0^* \varphi'' \big\rangle_S = \big\langle \psi',\, \mathcal{K}_0^\top \varphi'' \big\rangle_S,$$

which due to inclusions (4.2) can be rewritten as

$$\big\langle \mathcal{K}_0 \psi',\, \varphi'' \big\rangle_{S_D} = \big\langle \psi',\, \mathcal{K}_0^\top \varphi'' \big\rangle_{S_N} = \overline{\big\langle \mathcal{K}_0^\top \varphi'',\, \psi' \big\rangle}_{S_N}. \tag{4.10}$$

By Theorem 3.1(ii) the operators $\mathcal{H}_0$, $\mathcal{K}_0$, $\mathcal{K}_0^\top$, and $\mathcal{L}_0$ are bounded. Therefore we deduce

$$\begin{aligned}\Big|\big\langle \mathbf{A}_0 X',\, X''\big\rangle_{(\mathbb{H}_2,\widetilde{\mathbb{H}}_2)}\Big| &\leqslant C_1\Big(\|\varphi''\|_{\widetilde{H}_2^{-\frac12}(S_D)}\,\|\varphi'\|_{\widetilde{H}_2^{-\frac12}(S_D)} + \|\varphi''\|_{\widetilde{H}_2^{-\frac12}(S_D)}\,\|\psi'\|_{\widetilde{H}_2^{\frac12}(S_N)}\\ &\qquad + \|\varphi'\|_{\widetilde{H}^{-\frac12}(S_D)}\,\|\psi''\|_{\widetilde{H}_2^{\frac12}(S_N)} + \|\psi'\|_{\widetilde{H}^{\frac12}(S_N)}\,\|\psi''\|_{\widetilde{H}_2^{\frac12}(S_N)}\Big)\\ &\leqslant C_2\,\|X'\|_{\widetilde{\mathbb{H}}_2}\,\|X''\|_{\widetilde{\mathbb{H}}_2},\end{aligned}$$

implying the boundedness of the bilinear form $\big\langle \mathbf{A}_0 X',\, X''\big\rangle_{(\mathbb{H}_2,\widetilde{\mathbb{H}}_2)}$ on $\widetilde{\mathbb{H}}_2\times\widetilde{\mathbb{H}}_2$.

Now, let us consider the quadratic form $\big\langle \mathbf{A}_0 X,\, X\big\rangle_{(\mathbb{H}_2,\widetilde{\mathbb{H}}_2)}$, where $X := \begin{bmatrix}\varphi\\ \psi\end{bmatrix} \in \widetilde{\mathbb{H}}_2$, and show that it is coercive on $\widetilde{\mathbb{H}}_2\times\widetilde{\mathbb{H}}_2$.

Using (4.10) we get

$$\big\langle \mathcal{K}_0\psi,\, \varphi\big\rangle_{S_D} = \overline{\big\langle \mathcal{K}_0^\top\varphi,\, \psi\big\rangle}_{S_N}$$

and equation (4.9) takes the form

$$\big\langle \mathbf{A}_0 X,\, X\big\rangle_{(\mathbb{H}_2,\widetilde{\mathbb{H}}_2)} = \big\langle -\mathcal{H}_0\varphi,\, \varphi\big\rangle_{S_D} + \big\langle \mathcal{K}_0\psi,\, \varphi\big\rangle_{S_D} - \big\langle \mathcal{K}_0^\top\varphi, \psi\big\rangle_{S_N} + \big\langle \mathcal{L}_0\psi, \psi\big\rangle_{S_N},$$

i.e.

$$\mathrm{Re}\,\big\langle \mathbf{A}_0 X,\, X\big\rangle_{(\mathbb{H}_2,\widetilde{\mathbb{H}}_2)} = \big\langle -\mathcal{H}_0\varphi,\, \varphi\big\rangle_{S_D} + \big\langle \mathcal{L}_0\psi, \psi\big\rangle_{S_N}.$$

In view of inequalities (3.7) and (3.8) in Remark **2**, we deduce the following coercivity property for the operator $\mathbf{A}_0$

$$\big\langle \mathbf{A}_0 X,\, X\big\rangle_{(\mathbb{H}_2,\widetilde{\mathbb{H}}_2)} \geqslant \delta_3\|X\|^2_{\widetilde{\mathbb{H}}_2} \quad \text{for all} \quad X\in\widetilde{\mathbb{H}}_2,$$

where $\delta_3 = \min\{\delta_1,\, \delta_2\} > 0$. Consequently, the operator $\mathbf{A}_0$ is coercive on $\widetilde{\mathbb{H}}_2\times\widetilde{\mathbb{H}}_2$.

Therefore, by the well-known Lax-Milgram theorem it follows that for an arbitrary $G_0\in\mathbb{H}_2$ the equation

$$\big\langle \mathbf{A}_0 X,\, Y\big\rangle_{(\mathbb{H}_2,\widetilde{\mathbb{H}}_2)} = \big\langle G_0,\, Y\big\rangle_{(\mathbb{H}_2,\widetilde{\mathbb{H}}_2)} \quad \text{for all} \quad Y\in\widetilde{\mathbb{H}}_2$$

is uniquely solvable in the space $\widetilde{\mathbb{H}}_2$, which yields that the operator (4.8) is invertible.

Therefore, operator (4.7) is Fredholm with zero index. Let us show that the null space of the operator (4.7) is trivial. Indeed, let the pair $(\varphi,\psi)\in\widetilde{\mathbb{H}}_2$ be a solution to the homogeneous system (4.3)-(4.4), which implies that the function u represented by formula (4.1) solves the homogeneous mixed BVP (2.1)-(2.3). Due to the uniqueness Theorem 2.1, we have $u(x) = 0$ for $x\in\Omega$. Applying the properties of the layer potentials, it is easy to show that

$$\{u\}^+ - \{u\}^- = \psi, \quad \{Tu\}^+ - \{Tu\}^- = -\varphi \quad \text{on} \;\; S, \tag{4.11}$$

implying

$$\{u\}^- = 0 \quad \text{on} \;\; S_D, \quad \{Tu\}^- = 0 \quad \text{on} \;\; S_N.$$

Consequently, the same function u solves the homogeneous exterior mixed BVP (2.6)-(2.9) and $u(x) = 0$ for $x \in \Omega^-$ (see Remark **2**). Therefore by (4.11) we find $\psi = 0$ and $\varphi = 0$, which proves that the null space of operator (4.7) is trivial.

Thus, we have proved the following assertions.

Lemma 4.1: *Operator* (4.7) *is invertible.*

Lemma 4.2: *Let f and F be arbitrary functions satisfying conditions* (2.4)*. The system of pseudodifferential equations* (4.3)-(4.4) *possesses a unique solution*

$$(\varphi, \psi) \in \widetilde{H}_2^{-\frac{1}{2}}(S_D) \times \widetilde{H}_2^{\frac{1}{2}}(S_N).$$

This lemma and Theorem 2.1 imply the following existence result.

Theorem 4.3: *Let f and F be arbitrary functions satisfying conditions* (2.4)*. Then the mixed boundary value problem* (2.1)-(2.3) *is uniquely solvable in the space $H_2^1(\Omega)$ and the solution can be represented as a linear combination of single and double layer potentials by formula* (4.1)*, where the densities $\varphi \in \widetilde{H}_2^{-\frac{1}{2}}(S_D)$ and $\psi \in \widetilde{H}_2^{\frac{1}{2}}(S_N)$ are unique solutions of the system of pseudodifferential equations* (4.3)-(4.4)*.*

Remark 1: It is evident that using the representation (4.1), the exterior mixed BVP (2.6)-(2.9) is reduced to the same system of pseudodifferential equations (4.3)-(4.4). Therefore, in view of Lemma 4.2, we conclude that the function u defined by formula (4.1), where the densities $\varphi \in \widetilde{H}_2^{-\frac{1}{2}}(S_D)$ and $\psi \in \widetilde{H}_2^{\frac{1}{2}}(S_N)$ are unique solutions of the system of pseudodifferential equations (4.3)-(4.4), solves the exterior mixed BVP (2.6)-(2.9).

With the help of Lemma 3.2 and embedding properties of the Besov spaces, using the word for word arguments employed in [18, Section 5] one can prove the following regularity results for solutions to the mixed BVP under consideration.

Theorem 4.4: *Let the data of the mixed boundary value problem* (2.1)-(2.3) *satisfy the conditions*

$$f \in B^s_{p,2}(S_D) \subset B^{\frac{1}{2}}_{2,2}(S_D) = H_2^{\frac{1}{2}}(S_D), \quad F \in B^{s-1}_{p,2}(S_N) \subset B^{-\frac{1}{2}}_{2,2}(S_N) = H_2^{-\frac{1}{2}}(S_N)$$
$$\text{with} \;\; \frac{1}{2} \leqslant s < \frac{1}{2} + \frac{1}{p} \quad \text{and} \;\; p > 4.$$

(i) *The system of pseudodifferential equations* (4.3)-(4.4) *is uniquely solvable and for the solution pair (φ, ψ) we have the inclusions*

$$\begin{aligned} &\varphi \in \widetilde{B}^{s-1}_{p,2}(S_D) \subset \widetilde{B}^{-\frac{1}{2}}_{2,2}(S_D) = \widetilde{H}_2^{-\frac{1}{2}}(S_D), \\ &\psi \in \widetilde{B}^{s}_{p,2}(S_N) \subset \widetilde{B}^{\frac{1}{2}}_{2,2}(S_D) = \widetilde{H}_2^{\frac{1}{2}}(S_N). \end{aligned} \tag{4.12}$$

Moreover,

$$\psi \in C^t(S) \quad \text{with} \quad \frac{1}{2} - \frac{1}{p} > t = s - \frac{2}{p} \geqslant \frac{1}{2} - \frac{2}{p} > 0.$$

(ii) *The unique solution u to the mixed BVP belongs to the class $H_2^1(\Omega) \cap B_{p,2}^{s+\frac{1}{p}}(\Omega)$ and it can be represented as the linear combination of the single and double layer potentials*

$$u(x) = -V(\varphi)(x) + W(\psi)(x), \quad x \in \Omega,$$

with densities φ and ψ being solutions to the system of pseudodifferential equations (4.3)-(4.4) *belonging to the spaces* (4.12).
Moreover, the solution u possesses the following smoothness property

$$u \in C^t(\overline{\Omega}) \quad \text{with} \quad \frac{1}{2} - \frac{1}{p} > t = s - \frac{2}{p} \geqslant \frac{1}{2} - \frac{2}{p} > 0.$$

Corollary 4.5: *Let the data of the mixed boundary value problem satisfy the relations*

$$f \in B_{\infty,2}^{\frac{1}{2}}(S_D), \qquad F \in B_{\infty,2}^{-\frac{1}{2}}(S_N).$$

Then the items (i) and (ii) of Theorem 4.4 hold for all $p > 4$ and the density function ψ and solution u to the mixed BVP have the following Hölder continuity properties

$$\psi \in \bigcap_{\beta < \frac{1}{2}} C^{\beta}(S), \quad u \in \bigcap_{\beta < \frac{1}{2}} C^{\beta}(\overline{\Omega}).$$

References

[1] M.S. Agranovich. *Elliptic singular integro–differential operators* (Russian), Uspekhi Mat. Nauk, **20**, 5 (1965), 3-120

[2] A. Brener, E. Shargorodsky. *Boundary value problems for elliptic pseudodifferential operators*, In: Partial Differential Equations IX: Elliptic Boundary Value Problems. Encyclopaedia of Mathematical Sciences, (Agranovich, M.S., Egorov, Yu.V., Shubin, M.A. Eds.), Springer, Heidelberg, **79** (1997), 145-215

[3] T. Buchukuri, O. Chkadua, R. Duduchava, D. Natroshvili. *Interface crack problems for metallic-piezoelectric composite structures*, Mem. Differential Equations Math. Phys., **55** (2012), 1150

[4] T. Buchukuri, O. Chkadua, D. Natroshvili. *Mathematical problems of generalized thermo-electro-magneto-elasticity theory*, Mem. Differential Equations Math. Phys., **68** (2016), 1-166

[5] O. Chkadua, S.E. Mikhailov, D. Natroshvili. *Analysis of direct boundary-domain integral equations for a mixed BVP with variable coefficient*, I: Equivalence and invertibility, J.Integral Equations Appl., **21**, 4 (2009), 499-543

[6] O. Chkadua, S.E. Mikhailov, D. Natroshvili. *Analysis of direct boundary-domain integral equations for a mixed BVP with variable coefficient*, II: Solution regularity and asymptotics, J. Integral Equations Appl., **21**, 4 (2009), 499-543

[7] M. Costabel, W.L. Wendland. *Strong ellipticity of boundary integral operators*, J. Reine Angew. Mathematik, **372** (1986), 34-63

[8] R. Duduchava, D. Natroshvili, E. Shargorodsky. *Basic boundary value problems of thermoelasticity for anisotropic bodies with cuts*, I, Georgian Math. J., **2**, 2 (1995), 123-140

[9] R. Duduchava, D. Natroshvili, E. Shargorodsky. *Basic boundary value problems of thermoelasticity for anisotropic bodies with cuts*, II, Georgian Math. J., **2**, 3 (1995), 259-276

[10] G. Eskin. *Boundary value problems for elliptic pseudodifferential equations*, Transl. of Mathem. Monographs, Amer. Math. Soc., Providence, Rhode Island, **52** (1981)
[11] L. Jentsch, D. Natroshvili. *Three-dimensional mathematical problems of thermoelasticity of anisotropic bodies*, Part I, Mem. Differential Equations Math. Phys., **17** (1999), 7-126
[12] L. Jentsch, D. Natroshvili. *Three-dimensional mathematical problems of thermoelasticity of anisotropic bodies*, Part II, Mem. Differential Equations Math. Phys., **18** (1999), 1-50
[13] J.-L. Lions, E. Magenes. *Non-homogeneous Boundary Value Problems and Applications.*, Springer-Verlag, New York–Heidelberg, **I** (1972)
[14] C. Miranda. *Partial Differential Equations of Elliptic Type,* 2-nd ed. Berlin – Heidelberg – New York: Springer, 1970
[15] W. McLean. *Strongly Elliptic Systems and Boundary Integral Equations,* Cambridge University Press, Cambridge, UK, 2000
[16] D. Natroshvili, O. Chkadua, E. Shargorodsky. *Mixed boundary value problems of the anisotropic elasticity* (Russian), Proc. I. Vekua Inst. Appl. Math. Tbilisi State University, **39** (1990), 133-181
[17] D. Natroshvili. *Mathematical Problems of Thermo-Electro-Magneto-Elasticity*, Lecture Notes of TICMI, Tbilisi University Press, Tbilisi, **12** (2011)
[18] D. Natroshvili, T. Tsertsvadze. *On an alternative approach for mixed boundary value problems for the Laplace equation*, To appear: Georgian Marth. J.
[19] E. Shargorodsky. *An L_p-analogue of the Vishik-Eskin theory*, Mem. Differential Equations Math. Phys., **2** (1994) 41-148
[20] E.P. Stephan. *Boundary integral equations for mixed boundary value problems in* $\mathbb{R}^3$, Math. Nachr., **184** (1987), 21-53
[21] H. Triebel. *Interpolation Theory, Function Spaces, differential operators*, North–Holland Publication, Amsterdam, 1978
[22] H. Triebel. *Theory of Function Spaces*, Birkhäuser Verlag, Basel-Boston-Stuttgart, 1983