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The Consistent Estimators and the Consistent Criteria for Hypotheses Testing for the Charlier's Statistical Structure

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In this paper, we consider the Charlier statistical Structures. Sufficient and necessary conditions for the existence of consistent estimators of parameters and sufficient and necessary conditions for the existence of consistent criteria for hypotheses testing are given.

Keywords: Consistent estimator, consistent criterion, Charlier statistical structure, strongly separable statistical structure.

AMS Subject Classification: 62H05, 62H12.

1. Introduction

Let (E, S) be a measurable space with a given family of probability measures: $\{\mu_i, i \in I\}$. We recall some definitions from [1] - [4].

The normal distribution is symmetrical, that is, the normal distribution density function

$$f(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-m)^2}{2\sigma^2}}$$

is symmetric with respect to the line $x = m$. However, in practice, asymmetric distributions are also often encountered. In the case when the asymmetry in absolute value is not very large, the density can be expressed using the so-called Charlier's law.

The density of Charlier's law is determined by the equality

$$f_{Ch}(x) = f(x) + \frac{1}{\sigma}\big[\frac{S_k(x)}{6} \cdot z_u \cdot (u^3 - 3u) + \frac{E_k(x)}{24} \cdot z_u \cdot (u^4 - 6u^2 + 3)\big], \qquad (1)$$

where $f(x)$ is the density of the normal distribution, $u = \frac{x-m}{\sigma}$, $z_u = \frac{1}{\sqrt{2\pi}} e^{-u^2/2}$, $S_k(x) = \mu_3/\sigma^3$ – asymmetry, and $E_k(x) = \mu_4/\sigma^4 - 3$ – kurtosis.

Let μ be the probability measure given on $(R, L(R))$ by the formula

$$\mu(A) = \int_A f_{Ch}(x)dx, \quad A \in L(R).$$

The probability measure determined in this way will be called the Charlier measure.

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Definition 1.1: A statistical structure $\{E, S, \mu_i, i \in I\}$ is called a statistical Charlier structure if μ_i, $\forall i \in I$ are Charlier measures.

Definition 1.2: A Charlier statistical structure $\{E, S, \mu_i, i \in I\}$ is called orthogonal (singular) if a family of Charlier probability measures $\{\mu_i, i \in I\}$ constists of pairwise singular measures (i.e. $\mu_i \perp \mu_j$, $\forall i \neq j$).

Definition 1.3: A Charlier statistical structure $\{E, S, \mu_i, i \in I\}$ is called weakly separable if there exists a family of S-measurable sets $\{X_i, i \in I\}$ such that the following relations are fulfilled:

$$\mu_i(X_j) = \begin{cases} 1, & if\ i = j; \\ 0, & if\ i \neq j \end{cases} \qquad (i, j \in I).$$

Definition 1.4: A Charlier statistical structure $\{E, S, \mu_i, i \in I\}$ is called separable (S) if there exists a family of S-measurable sets $\{X_i, i \in I\}$ such that the realations are fulfilled:

$$1)\ \forall i, j \in I: \quad \mu_i(X_j) = \begin{cases} 1, & if\ i = j; \\ 0, & if\ i \neq j. \end{cases}$$

$$2)\ \forall i, j \in I: \quad card(X_i \cap X_j) < c, \quad if\ i \neq j,$$

where c denotes the continuum power.

Definition 1.5: A Charlier statistical structure $\{E, S, \mu_i, i \in I\}$ is called strongly separable (SS) if there exists disjoint family of S-measurable sets $\{X_i, i \in I\}$ such that the realations are fulfilled:

$$\mu_i(X_i) = 1, \quad \forall\, i \in I.$$

Let I be the set of parameters and let $B(I)$ be σ-algebra of subsets of I which contains all finite subsets of I.

Definition 1.6: We will say that the Charlier statistical structure $\{E, S, \mu_i, i \in I\}$ admits a consistent estimators of parameters $i \in I$ if there exists at least one measurable mapping $\delta\ :\ (E, S)\ \longrightarrow\ (I, B(I))$, such that

$$\mu_i(\{x : \delta(x) = i\}) = 1, \ \ \forall\, i \in I.$$

2. The consistent estimators for the Charlier's statistical structure

Let H be set of hypotheses and let $B(H)$ be σ-algebra of subsets of H which contains all finite subsets of H.

Definition 2.1: We will say the Charlier statistical structure $\{E, S, \mu_h, h \in H\}$ admit a consistent criterion for hypothesis testing if there exists at least one measurable mapping

$$\delta\ :\ (E, S)\ \longrightarrow\ (H, B(H)),$$

such that

$$\mu_h(\{x : \delta(x) = h\}) = 1, \ \ \forall\, h \in H.$$

Remark 1: In order for the Charlier Statistical Structure $\{E, S_1, \overline{\mu}_i, i \in I\}$ to admit a consistent criterion for hypothesis testing or admit a consistent estimators of parameters $i \in I$, then the Charlier Statistical Structure $\{E, S_1, \overline{\mu}_i, i \in I\}$ is strongly separable but not vice versa.

Theorem 2.2: *Let $\{E, S_1, \mu_i, i \in N\}$ be a Charlier Statistical Structure, then it admit a consistent estimators of parameters $i \in N$ and admit a consistent criterion for hypothesis testing if it is orthogonal.*

Proof: From the orthogonality of Charlier Statistical Structure $\{E, S_1, \mu_i, i \in N\}$ it follows that $\mu_i \perp \mu_j$, for any $i \neq j$, $i, j \in N$.

The singularity (orthogonality) of the probability measures implies the existence of the family of S-measurable sets $\{X_{ij}\}$ such that for any $i \neq j$, $i, j \in N$ we have $\mu_j(X_{ij}) = 0$ and $\mu_i(E \setminus X_{ij}) = 0$. Therefore, if we now consider the sets $X_i = \cup_{i \neq j}(E \setminus X_{ij})$, we will see that $\mu_i(X_i) = 0$ and $\mu_j(E \setminus X_i) = 0$, $\forall i \neq j$. It means that the Charlier statistical structure $\{E, S, \mu_i, i \in N\}$ is weakly separable and there exists the family of S-measurable sets $\{\widetilde{X}_i, i \in N\}$ such that

$$\mu_j(\widetilde{X}_i) = \begin{cases} 1, & if\ i = j; \\ 0, & if\ i \neq j. \end{cases}$$

Let us consider the sets

$$\overline{X}_i = \widetilde{X}_i \setminus (\widetilde{X}_i \cap (\cup_{j \neq i} \widetilde{X}_j)), \quad i \in N.$$

It is clear that $\overline{X}_i \cap \overline{X}_j = \emptyset$, $\forall i \neq j$ and $\mu_i(\overline{X}_i) = 1$, $\forall i \in N$. We define the mapping $\delta : (E, S) \longrightarrow (H, B(H))$ as follows $\delta(\overline{X}_i) = 1$, $\forall i \in N$. Then we have $\mu_i(\{x : \delta(x) = i\}) = 1$, $\forall i \in N$, i.e. the Charlier statistical structure $\{E, S, \mu_i, i \in N\}$ admits consistent estimators of parameters and admit a consistent criterion for hypothesis testing. □

Let $\{\mu_i, i \in I\}$ be a Charlier probability measures defined on the measurable space (E, S). For each $i \in I$ we denote by $\overline{\mu}_i$ the completion of the measure μ_i, and by $dom(\overline{\mu}_i)$ – the σ-algebra of all μ_i-measurable subsets of E.

We denote

$$S_1 = \cap_{i \in I} dom(\overline{\mu}_i).$$

Definition 2.3: A Charlier statistical structure $\{E, S_1, \overline{\mu}_i, i \in I\}$ is called strongly separable if there exists the family of S_1-measurable sets $\{Z_i, i \in I\}$ such that the following relations are fulfilled:

1) $\mu_i(Z_i) = 1 \ \forall i \in I$;
2) $Z_i \cap Z_j = \emptyset, \ \forall i \neq j, \ i, j \in I$;
3) $\cup_{i \in I} Z_i = E$.

Definition 2.4: We will say that the orthogonal Charlier statistical structure $\{E, S_1, \overline{\mu}_i, i \in I\}$ admits consistent estimators of parameters $i \in I$ if there exists at least one measurable mapping $\delta : (E, S_1) \longrightarrow (I, B(I))$, such that

$$\overline{\mu}_i(\{x : \delta(x) = i\}) = 1, \ \ \forall\, i \in I.$$

Theorem 2.5: *In order for the orthogonal Charlier statistical structure* $\{E, S_1, \overline{\mu}_i, i \in I\}$, *cardI* $= c$ *to admit consistent estimators of parameters* $i \in I$ *it is necessary and sufficient that this statistical structure was strongly separable.*

Proof: Necessity. The existence of consistent estimators of parameters $i \in I$ means that there exists at least one measurable mapping $\delta : (E, S_1) \longrightarrow (I, B(I))$ such that

$$\overline{\mu}_i(\{x : \delta(x) = i\}) = 1, \ \ \forall i \in I.$$

Denoting $Z_i = \{x : \delta(x) = i\}$ for $i \in I$, we get:
1) $\overline{\mu}_i(Z_i) = \overline{\mu}_i(\{x : \delta(x) = i\}) = 1, \ \ \forall i \in I$;
2) $Z_{i_1} \cap Z_{i_2} = \{x : \delta(x) = i_1\} \cap \{x : \delta(x) = i_2\} = \emptyset$, $\forall i_1 \neq i_2$, $i_1, i_2 \in I$;
3) $\cup_{i \in I} Z_i = E$.
Hence, the statistical structure $\{E, S_1, \overline{\mu}_i, i \in I\}$ is strongly separable.

Sufficiency. Since the statistical structure $\{E, S_1, \overline{\mu}_i, i \in I\}$, *cardI* $= c$ is strongly separable, there exists a family $\{Z_i, \ i \in I\}$ of elements of the σ-aalgebra $S_1 = \cap_{i \in I} dom(\overline{\mu}_i)$ such that:
1) $\overline{\mu}_i(Z_i) = 1, \ \ \forall i \in I$;
2) $Z_{I_1} \cap Z_{i_2} = \emptyset, \ \ \forall i_1 \neq i_2$; $i_1, i_2 \in I$;
3) $\cup_{i \in I} Z_i = E$.
For $x \in E$, we put $\delta(x) = i$, where i is the unique parameter from the set I for which $x \in Z_i$. The existence and uniqueness of such parameter i can be proved using conditions 2) and 3).

Take now $Y \in B(I)$. Then $\{x : \delta(x) \in Y\} = \cup_{i \in Y} Z_i$. We have to show that $\{x : \delta(x) \in Y\} \subset dom(\overline{\mu}_{I_0})$ for each $i_0 \in I$.

If $i_0 \in Y$, then

$$\{x : \delta(x) \in Y\} = \cup_{i \in Y} Z_i = Z_{i_0} \cup (\cup_{i \in Y \setminus \{i_0\}} Z_i).$$

On the one hand, from the conditions 1), 2), 3) it follows that

$$Z_{i_0} \in S_1 = \cap_{i \in I} dom(\overline{\mu}_i) \subseteq dom(\overline{\mu}_{i_0}).$$

On the other hand, the inclusion

$$\cup_{i \in Y \setminus \{i_0\}} Z_i \subseteq (E \setminus Z_{i_0})$$

implies that $\overline{\mu}_{i_0}(\cup_{i \in Y \setminus \{i_0\}} Z_i) = 0$, and hence,

$$\cup_{i \in Y \setminus \{i_0\}} Z_i \subset dom(\overline{\mu}_{i_0}).$$

Since $dom(\overline{\mu}_{i_0})$ is a σ-algebra, we conclude that

$$\{x : \delta(x) \in Y\} = Z_{i_0} \cup (\cup_{i \in Y \setminus \{i_0\}} Z_i) \in dom(\overline{\mu}_{i_0}).$$

If $i_0 \notin Y$, then $\{x : \delta(x) \in Y\} = \cup_{i \in Y} Z_i \subseteq (E \setminus Z_{i_0})$ and we conclude that $\overline{\mu}_{i_0}\{x : \delta(x) \in Y\} = 0$. The last relation implies that

$$\{x : \delta(x) \in Y\} \subset dom(\overline{\mu}_{i_0}), \ \ \forall Y \in B(I).$$

Thus, we have shown the validity of the relation $\{x : \delta(x) \in Y\} \subset dom(\overline{\mu}_{i_0})$ for an arbitrary $i_0 \in I$. Hence, $\{x : \delta(x) \in Y\} \subset \cap_{i\in I} dom(\overline{\mu}_i) = S_1$. Therefore, the mapping

$$\delta \,:\, (E, S_1) \longrightarrow (I, B(I))$$

is a measurable mapping.

Since $B(I)$ contains all singletons of I, we ascertain that

$$\overline{\mu}_i(\{x : \delta(x) = i\}) = \overline{\mu}_i(Z_i) = 1, \;\; \forall i \in I.$$

□

3. The consistent criteria for hypotheses testing for the Charlier's statistical structure

Definition 3.1: We will say that the orthogonal Charlier statistical structure $\{E, S_1, \mu_h, h \in H\}$ admit a consistent criterion for hypothesis testing if there exists at least one measurable mapping

$$\delta \,:\, (E, S_1) \;\longrightarrow\; (H, B(H)),$$

such that

$$\overline{\mu}_h(\{x : \delta(x) = h\}) = 1, \;\; \forall\, h \in H.$$

Theorem 3.2: *In order for the orthogonal Charlier statistical structure $\{E, S_1, \overline{\mu}_h, h \in H\}$, $cardH = c$ to admit consistent criterion for hypothesis testing it is necessary and sufficient that this statistical structure was strongly separable.*

Proof: Necessity. The existence of a consistent criterion for hypothesis testing means that there exists at least one measurable mapping $\delta \;:\; (E, S_1) \longrightarrow (H, B(H))$ such that

$$\overline{\mu}_h(\{x : \delta(x) = h\}) = 1, \;\; \forall h \in H.$$

Denoting $Z_h = \{x : \delta(x) = h\}$ for $h \in H$, we get:

1) $\overline{\mu}_h(Z_h) = \overline{\mu}_h(\{x : \delta(x) = h\}) = 1, \;\; \forall h \in H;$
2) $Z_{h_1} \cap Z_{h_2} = \{x : \delta(x) = h_1\} \cap \{x : \delta(x) = h_2\} = \emptyset$, $\forall h_1 \neq h_2$, $h_1, h_2 \in I$;
3) $\cup_{h\in H} Z_h = E$.

Hence, the Charlier statistical structure $\{E, S_1, \overline{\mu}_h, h \in H\}$ is strongly separable.

Sufficiency. Since the Charlier statistical structure $\{E, S_1, \overline{\mu}_h, h \in H\}$, $cardH = c$ is strongly separable, there exists a family $\{Z_h, \;\; h \in H\}$ of elements of the σ-aalgebra $S_1 = \cap_{h\in H} dom(\overline{\mu}_h)$ such that:

1) $\overline{\mu}_h(Z_h) = 1, \;\; \forall h \in H;$
2) $Z_{h_1} \cap Z_{h_2} = \emptyset, \;\; \forall h_1 \neq h_2$; $h_1, h_2 \in H$;
3) $\cup_{h\in H} Z_h = E$.

For $x \in E$, we put $\delta(x) = h$, where h is a hypothesis from the set H for which $x \in Z_h$. The existence and uniqueness of such hypothesis h can be proved using conditions 2) and 3).

Take now $Y \in B(H)$. Then $\{x : \delta(x) \in Y\} = \cup_{h\in Y} Z_h$. We have to show that $\{x : \delta(x) \in Y\} \subset dom(\overline{\mu}_{h_0})$ for each $h_0 \in H$.

If $h_0 \in Y$, then

$$\{x : \delta(x) \in Y\} = \cup_{h \in H} Z_h = Z_{h_0} \cup (\cup_{h \in Y \setminus \{h_0\}} Z_h).$$

On the one hand, from the conditions 1), 2), 3) it follows that

$$Z_{h_0} \in S_1 = \cap_{h \in H} dom(\overline{\mu}_h) \subseteq dom(\overline{\mu}_{h_0}).$$

On the other hand, the inclusion

$$\cup_{h \in Y \setminus \{h_0\}} Z_h \subseteq (E \setminus Z_{h_0})$$

implies that $\overline{\mu}_{h_0}(\cup_{h \in Y \setminus \{h_0\}} Z_h) = 0$, and hence,

$$\cup_{h \in Y \setminus \{h_0\}} Z_h \subset dom(\overline{\mu}_{h_0}).$$

Since $dom(\overline{\mu}_{h_0})$ is a σ-algebra, we conclude that

$$\{x : \delta(x) \in Y\} = Z_{h_0} \cup (\cup_{h \in Y \setminus \{h_0\}} Z_i) \in dom(\overline{\mu}_{h_0}).$$

If $h_0 \notin Y$, then $\{x : \delta(x) \in Y\} = \cup_{h \in Y} Z_h \subseteq (E \setminus Z_{h_0})$ and we conclude that $\overline{\mu}_{h_0}\{x : \delta(x) \in Y\} = 0$. The last ralation implies that

$$\{x : \delta(x) \in Y\} \subset dom(\overline{\mu}_{h_0}), \ \ \forall Y \in B(H).$$

Thus, we proved the validity of the relation $\{x : \delta(x) \in Y\} \subset dom(\overline{\mu}_{h_0})$ for any $h_0 \in H$. Hence, $\{x : \delta(x) \in Y\} \subset \cap_{h \in H} dom(\overline{\mu}_h) = S_1$. Therefore, the mapping

$$\delta \ : \ (E, S_1) \longrightarrow (H, B(H))$$

is a measurable mapping.

Since $B(H)$ contains all singletons of H, we ascertain that

$$\overline{\mu}_h(\{x : \delta(x) = h\}) = \overline{\mu}_h(Z_h) = 1, \ \ \forall h \in H.$$

□

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