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Some Properties of the Initial-Boundary Value Problem for One System of Nonlinear Partial Differential Equations

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The linear stability and Hopf bifurcation of a solution of the initial-boundary value problem for one system of nonlinear partial differential equations (NPDEs) is studied. A blow up result is given.

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Nonlinear evolution equations as mathematical models are widely used in almost all scientific disciplines. A lot of natural processes are described using the nonlinear systems of partial differential equations.

The main aim of the present paper is to study the linear stability and Hopf bifurcation of a solution of the initial-boundary value problem for one diffusion system of NPDEs. Such systems arise in mathematical modeling of the process of penetration of an electromagnetic field into a substance [11].

In this note, at first, we illustrate two reasonably simple problems, examples of blow up to obtain nonexistence results for classes of problems that arise in the studied NPDEs. The conditions which imply that the solution must blow up in finite time are given.

For most of NPDEs it is very difficult to find exact solutions and there is no general solution available in a closed form. It is known that, in some cases, it is possible to construct specific exact solutions of the initial-boundary value problem for NPDEs. The exact analytical solution is constructed in this note too.

Now, in the domain $Q = (0; 1) \times (0; \infty)$, let us consider the following initial-boundary value problem:

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$$\frac{\partial U}{\partial t}=\frac{\partial}{\partial x}\left(S^{\alpha}\frac{\partial U}{\partial x}\right),\quad \frac{\partial V}{\partial t}=\frac{\partial}{\partial x}\left(S^{\alpha}\frac{\partial V}{\partial x}\right),\quad \frac{\partial W}{\partial t}=\frac{\partial}{\partial x}\left(S^{\alpha}\frac{\partial W}{\partial x}\right),$$
$$\frac{\partial S}{\partial t}=-aS^{\beta}+bS^{\gamma}\left[\left(\frac{\partial U}{\partial x}\right)^2+\left(\frac{\partial V}{\partial x}\right)^2+\left(\frac{\partial W}{\partial x}\right)^2\right], \tag{1}$$

$$U(0,t)=V(0,t)=W(0,t)=0,$$
$$U(1,t)=\psi_1>0,\quad V(1,t)=\psi_2>0,\quad W(1,t)=\psi_3>0, \tag{2}$$

$$U(x,0)=U_0(x),\quad V(x,0)=V_0(x),$$
$$W(x,0)=W_0(x),\quad S(x,0)=S_0(x)>0. \tag{3}$$

Here $(x,t)\in Q$; $\alpha,\beta,\gamma\in R$; a,b,ψ_1,ψ_2,ψ_3 are positive constants, and $U_0(x)$, $V_0(x)$, $W_0(x)$, $S_0(x)$ are the given functions.

Systems of (1) type arise in mathematical modeling of many practical processes and in theoretical spears too (see, for example, [1] - [4], [6], [8], [13] - [15] and references therein). Some qualitative and structural properties of solutions of (1) type systems are established in many works. If $a=0$, $b=1$, when system (1), (2) may be considered as one-dimensional analogue of the model of process of penetration of an electromagnetic field into a substance [11].

It is easy to check that if $a=0$, $b=1$, $\gamma=\alpha$, $U_0(x)=\psi_1 x$, $V_0(x)=\psi_2 x$, $W_0(x)=\psi_3 x$ and $S_0(x)=S_0=const>0$, then when $\alpha\neq 1$ the solution of the problem (1) - (3) is:

$$U(x,t)=\psi_1 x,\quad V(x,t)=\psi_2 x,\quad W(x,t)=\psi_3 x,$$
$$S(x,t)=\left[S_0^{1-\alpha}+(1-\alpha)\left(\psi_1^2+\psi_2^2+\psi_3^2\right)t\right]^{\frac{1}{1-\alpha}} \tag{4}$$

As it can be seen from (4), for a finite value of time, namely, when

$$t_0=S_0^{1-\alpha}/\left[\left(\psi_1^2+\psi_2^2+\psi_3^2\right)(\alpha-1)\right]$$

and $\alpha>1$, the function $S(x,t)$ is not bounded.

The above example shows that (1) - (3) has no global solution at all. So, the solution of problem (1) - (3) with smooth initial and boundary conditions can be blown up at a finite time.

The questions of unique solvability of some cases of problems of this type are studied in above-mentioned literature and in the number of other works as well.

Note that if we add to (2) the following boundary conditions:

$$\left.\frac{\partial S}{\partial x}\right|_{x=0}=\left.\frac{\partial S}{\partial x}\right|_{x=1}=0, \tag{5}$$

then U, V, W and S defined by formulas (4) are also solutions of the following system:

$$\frac{\partial U}{\partial t} = \frac{\partial}{\partial x}\left(S^\alpha \frac{\partial U}{\partial x}\right), \quad \frac{\partial V}{\partial t} = \frac{\partial}{\partial x}\left(S^\alpha \frac{\partial V}{\partial x}\right), \quad \frac{\partial W}{\partial t} = \frac{\partial}{\partial x}\left(S^\alpha \frac{\partial W}{\partial x}\right),$$
$$\frac{\partial S}{\partial t} = S^\alpha \left[\left(\frac{\partial U}{\partial x}\right)^2 + \left(\frac{\partial V}{\partial x}\right)^2 + \left(\frac{\partial W}{\partial x}\right)^2\right] + \frac{\partial^2 S}{\partial x^2}, \tag{6}$$

with (2), (3), (5) boundary and initial conditions. We conclude that for $\alpha > 1$, neither (2), (3), (5), (6) the problem has no global solution.

In the remaining part, we give detailed formulations of the most important results mentioned in the abstract. Now, let us consider general system (1), (2). In some cases, linear and global problems of stability of stationary solutions are studied. There appears the possibility of Hopf bifurcation. The small perturbations may cause the transformation of a solution into periodic oscillations [12].

The study of similar problems in this area was firstly carried out in the article [5] for the two component case. The following works [6], [7], [9], [10] are devoted to similar studies for two and three component (U, V, S) cases of (1), (2) type systems.

It is not difficult to show that if $\beta \neq \gamma$ the stationary solution (U_s, V_s, W_s, S_s) of problem (1) - (3) has the form:

$$U_s = \psi_1 x, \quad V_s = \psi_2 x, \quad W_s = \psi_3 x, \quad S_s = \left[\frac{b}{a}\left(\psi_1^2 + \psi_2^2 + \psi_3^2\right)\right]^{\frac{1}{\beta-\gamma}}. \tag{7}$$

The following statement takes place.

Theorem 1: *Let $2\alpha + \beta - \gamma > 0$, $\beta \neq \gamma$, then stationary solution (7) of the problem (1) - (3) is linearly stable if and only if the following inequality is fulfilled*

$$a(\gamma - \beta)\left[\frac{b}{a}\left(\psi_1^2 + \psi_2^2 + \psi_3^2\right)\right]^{\frac{\beta-\alpha-1}{\beta-\gamma}} < \pi^2. \tag{8}$$

Proof: Assume, that the solution of problem (1) - (3) has the following form:

$$U(x,t) = U_s + u(x,t), \quad V(x,t) = V_s + v(x,t),$$
$$W(x,t) = W_s + w(x,t), \quad S(x,t) = S_s + s(x,t), \tag{9}$$

where $u(x,t)$, $v(x,t)$, $w(x,t)$, $s(x,t)$ are small perturbations.

As a result of system (1) linearization, we obtain:

$$\frac{\partial u}{\partial t} = \alpha_s \frac{\partial s}{\partial x} + \beta_s \frac{\partial^2 u}{\partial x^2},$$
$$\frac{\partial v}{\partial t} = \gamma s \frac{\partial s}{\partial x} + \beta_s \frac{\partial^2 v}{\partial x^2},$$
$$\frac{\partial w}{\partial t} = \rho_s \frac{\partial s}{\partial x} + \beta_s \frac{\partial^2 w}{\partial x^2},$$
$$\frac{\partial s}{\partial t} = \nu_s s + \eta_s \frac{\partial u}{\partial x} + \mu_s \frac{\partial v}{\partial x} + \tau_s \frac{\partial w}{\partial x}, \tag{10}$$

where the following notations are introduced:

$$\alpha_s = \alpha\psi_1 \left[\frac{b}{a}\left(\psi_1^2+\psi_2^2+\psi_3^2\right)\right]^{\frac{\alpha-1}{\beta-\gamma}}, \quad \beta_s = \left[\frac{b}{a}\left(\psi_1^2+\psi_2^2+\psi_3^2\right)\right]^{\frac{\alpha}{\beta-\gamma}},$$

$$\gamma_s = \alpha\psi_2 \left[\frac{b}{a}\left(\psi_1^2+\psi_2^2+\psi_3^2\right)\right]^{\frac{\alpha-1}{\beta-\gamma}}, \quad \rho_s = \alpha\psi_3 \left[\frac{b}{a}\left(\psi_1^2+\psi_2^2+\psi_3^2\right)\right]^{\frac{\alpha-1}{\beta-\gamma}},$$

$$\nu_s = (\gamma-\beta)a \left[\frac{b}{a}\left(\psi_1^2+\psi_2^2+\psi_3^2\right)\right]^{\frac{\beta-1}{\beta-\gamma}}, \quad \eta_s = 2\psi_1 b \left[\frac{b}{a}\left(\psi_1^2+\psi_2^2+\psi_3^2\right)\right]^{\frac{\gamma}{\beta-\gamma}},$$

$$\mu_s = 2\psi_2 b \left[\frac{b}{a}\left(\psi_1^2+\psi_3^2+\psi_3^2\right)\right]^{\frac{\gamma}{\beta-\gamma}}, \quad \tau_s = 2\psi_3 b \left[\frac{b}{a}\left(\psi_1^2+\psi_2^2+\psi_3^2\right)\right]^{\frac{\gamma}{\beta-\gamma}}.$$

Let us seek the solution of system (10) in the following form:

$$\begin{aligned} u(x,t) &= u(x)e^{\omega t}, \quad v(x,t) = v(x)e^{\omega t}, \\ w(x,t) &= w(x)e^{\omega t}, \quad s(x,t) = s(x)e^{\omega t}, \end{aligned} \tag{11}$$

then we get the problem on eigenvalues for the following system of ordinary differential equations:

$$\begin{aligned} \omega u &= \alpha_s \frac{ds}{dx} + \beta_s \frac{d^2u}{dx^2}, \\ \omega v &= \gamma s \frac{ds}{dx} + \beta_s \frac{d^2v}{dx^2}, \\ \omega w &= \rho_s \frac{ds}{dx} + \beta_s \frac{d^2w}{dx^2}, \\ \omega s &= \nu_s s + \eta_s \frac{du}{dx} + \mu_s \frac{dv}{dx} + \tau_s \frac{dw}{dx}. \end{aligned} \tag{12}$$

Now, assume that the solution of system (12) has the following form:

$$u(x) = u_0 e^{ikx}, \quad v(x) = v_0 e^{ikx}, \quad w(x) = w_0 e^{ikx}, \quad s(x) = s_0 e^{ikx}.$$

Substituting these functions in (12), after simple transformations, we get:

$$u_0(\omega + \beta_s k^2) - \alpha_s iks_0 = 0,$$

$$v_0(\omega + \beta_s k^2) - \gamma_s iks_0 = 0,$$

$$w_0(\omega + \beta_s k^2) - \rho_s iks_0 = 0,$$

$$\eta_s iku_0 + \mu_s ikv_0 + \tau_s ikw_0 + s_0(\nu_s - \omega) = 0.$$

It is clear that this system has a nontrivial solution if and only if the following condition is fulfilled

$$\Delta(\omega,k)=\begin{vmatrix}\omega+\beta_s k^2 & 0 & 0 & -\alpha_s ik\\ 0 & \omega+\beta_s k^2 & 0 & -\gamma_s ik\\ 0 & 0 & \omega+\beta_s k^2 & -\rho_s ik\\ \eta_s ik & \mu_s ik & \tau_s ik & \nu_s-\omega\end{vmatrix}$$

$$=(\omega+\beta_s k^2)^2[(\nu_s-\omega)(\omega+\beta_s k^2)-\alpha_s\eta_s k^2-\gamma_s\mu_s k^2-\tau_s\rho_s k^2]=0.$$

Since the case $\omega+\beta_s k^2=0$ is trivial, from this we get

$$k^2(\beta_s\nu_s-\beta_s\omega-\alpha_s\eta_s-\gamma_s\mu_s-\tau_s\rho_s)-\omega^2+\nu_s\omega=0. \tag{13}$$

The latest equality gives two values of the parameter k such as $k_1=-k_2$.

It is easy to show that the solution of system (12) has the following form:

$$\begin{aligned} u(x)&=\frac{ik_1\alpha_s}{\omega+\beta_s k_1^2}\left(S_1e^{ik_1x}-S_2e^{-ik_1x}\right),\\ v(x)&=\frac{ik_1\gamma_s}{\omega+\beta_s k_1^2}\left(S_1e^{ik_1x}-S_2e^{-ik_1x}\right),\\ w(x)&=\frac{ik_1\rho_s}{\omega+\beta_s k_1^2}\left(S_1e^{ik_1x}-S_2e^{-ik_1x}\right),\\ s(x)&=S_1e^{ik_1x}+S_2e^{-ik_1x}, \end{aligned} \tag{14}$$

where S_1 and S_2 are constants.

Taking into account boundary conditions (2), from (9) and (11) we get

$$u(0)=u(1)=0.$$

From this, taking into account (14) we get the following system:

$$S_1-S_2=0,$$

$$S_1e^{ik_1}-S_2e^{-ik_1}=0.$$

The above system has a nontrivial solution when

$$\Delta=\begin{vmatrix}1 & -1\\ e^{ik_1} & -e^{ik_1}\end{vmatrix}=2isink_1=0,$$

or

$$k_{1n}=\pi n,\quad n\in Z.$$

Let us rewrite equation (13) in the following form

$$\omega_n^2+P_n(\beta_s,k_n,\nu_s)\omega_n+L_n(\beta_s,k_n,\nu_s,\eta_s,\mu_s,\gamma_s,\tau_s,\rho_s)=0,$$

where:

$$P_n(\beta_s, k_n, \nu_s) = \beta_s k_n^2 - \nu_s,$$

$$L_n(\beta_s, k_n, \nu_s, \eta_s, \mu_s, \gamma_s, \tau_s, \rho_s) = -\beta_s \nu_s k_n^2 + \alpha_s \eta_s k_n^2 + \gamma_s \mu_s k_n^2 + \tau_s \rho_s k_n^2.$$

One must note that the solution of problem (1) - (3) is linearly stable if and only if for all n the following inequality holds $Re(\omega_n) < 0$. It is easy to show that if $2\alpha + \beta - \gamma > 0$, then $L_n(\beta_s, k_n, \nu_s, \eta_s, \mu_s, \gamma_s, \tau_s, \rho_s) > 0$.

Therefore, for the linear stability of solution it is necessary and sufficient that the following inequality

$$P_n(\beta_s, k_n, \nu_s) = \beta_s k_n^2 - \nu_s$$

$$= \left[\frac{b}{a}\left(\psi_1^2 + \psi_2^2 + \psi_3^2\right)\right]^{\frac{\alpha}{\beta-\gamma}} \pi^2 n^2 - (\gamma - \beta) a \left[\frac{b}{a}\left(\psi_1^2 + \psi_2^2 + \psi_3^2\right)\right]^{\frac{\beta-1}{\beta-\gamma}} > 0,$$

holds or

$$(\gamma - \beta) a \left[\frac{b}{a}\left(\psi_1^2 + \psi_2^2 + \psi_3^2\right)\right]^{\frac{\beta-\alpha-1}{\beta-\gamma}} < \pi^2, \quad (n = 1).$$

□

Remark 1: As we see from the inequality i.e. from (8), when $\gamma < \beta$, then the solution of problem (1) - (3) is always linearly stable.

Assume, $\gamma > \beta$, $\beta - \alpha - 1 \neq 0$ and consider the value

$$\psi_s = \left[\frac{\pi^2}{\gamma - \beta} b^{\frac{\alpha-\beta+1}{\beta-\gamma}} a^{\frac{\gamma-\alpha-1}{\beta-\gamma}}\right]^{\frac{\beta-\gamma}{\beta-\alpha-1}},$$

for which we have

$$P_1(\psi_s, \alpha, \beta, \gamma) = 0, \quad P_n(\psi_s, \alpha, \beta, \gamma) > 0, \quad n = 2, 3, ...$$

In addition, if we assume that $\beta - \alpha - 1 < 0$, then for $\psi \in (0, \psi_s)$, $\psi = \psi_1^2 + \psi_2^2 + \psi_3^2$, we have $P_n(\psi, \alpha, \beta, \gamma) > 0$, $n \in Z_0$.

Therefore, when $\psi \in (0, \psi_s)$, then the solution of problem (1) - (3) is linearly stable, and when $\psi > \psi_s$ it is unstable. When $\psi = \psi_s$, we have $Re(\omega_1) = 0$ and $Im(\omega_1) \neq 0$, i.e. there appears possibility of Hoph bifurcation. The small perturbations may cause transformation of solution in periodic oscillations [12].

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