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Functionally Graded Couette Flow in a Duct

George Jaiani[a*]

[a] *I. Vekua Institute of Applied Mathematics of I. Javakhishvili Tbilisi State University & Faculty of Exact and Natural Sciences of I. Javakhishvili Tbilisi State University, 2 University St., 0186, Tbilisi, Georgia*

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Functionally Graded Couette flow when viscous coefficients vary from zero $\mu(x_2) \in C^1$, $\mu(0) = 0$, $\mu(x_2) > 0$ for $x_2 > 0$, in particular, as a power function of a width of a duct, where the fluid is contained at rest at the initial moment, is considered the peculiarities of non-classical setting BCs at the wall of the duct, where viscosity coefficients vanish, is investigated.

Keywords: Couette flow, functionally graded Couette flow, non-classical BCs

AMS Subject Classification: 76A05, 76D03, 35Q35

1. Introduction

To study of Couette flow [1] several works are devoted (see, e.g., [1]-[5] and the references given there). As far as we know, for the time being functionally graded Couette flows when viscosity coefficients vary from zero have not been studied. The aim of the present short paper is to investigate functionally Graded Couette flow when viscous coefficients vary from zero $\mu(x_2) \in C^1$, $\mu(0) = 0$, $\mu(x_2) > 0$ for $x_2 > 0$, in particular, as a power function

$$\mu = \mu_0 x_2^{\kappa}, \ \ \text{constants} \ \ \mu_0 > 0, \ \ \kappa \geq 0,$$

where the flow is contained at rest at the initial moment within the two planes. Namely, the setting BCs depends on convergence-divergence of a certain improper integral, in particular, on values of the exponent of the power function. The corresponding criteria are established.

The paper is organized as follows. Section 1 is intended to motivate our investigation and to this end some references are given. In Section 2 for readers convenience we review some of the standard facts on viscosity and the Newtonian viscous stress tensor. In Section 3 we proceed with the study of the title problem and our main results are stated and proved. Section 4 contains a brief summary of the obtained mathematical results and their physical (mechanical) interpretations.

*Corresponding author. Email: giorgi.jaiani@tsu.ge

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2. Some Auxiliary Materials

Here we follow [1] (see pp. 97-101): Modifications of Euler's equations needed to account real fluid at continuum level, introduce additional forces into the momentum balance equations. ... Because of the molecular structure of various Fluid materials, the nature of these forces can vary considerably, and there are many models that attempt to capture the observed properties of fluids under deformation. These models differ in what we shall call their rheology.

The simplest of these rheologies and one applicable to common fluids such as air or water, is the Newtonian viscous fluid. To understand the assumptions let us restrict attention to the determination of a viscous stress tensor at (x, t), $x := (x_1, x_2, x_3)$, which depends only upon the fluid properties within a fluid parcel at that point and time.

It is reasonable to assume that the forces associated with the rheology of the fluid are developed by the deformation of fluid parcels, and could be determined by the velocity field. If we allow only point properties, deformation of parcels must involve more than just the velocity itself; first and higher-order partial derivatives with respect to the spatial coordinates could be important. A moment's thought shows that viscous forces cannot depend on the velocity. The bulk translation of the fluid with constant velocity produces no force. Thus it is the deformation of a small fluid parcel that must be responsible for the viscous force, and the dominant measure of this deformation should come from first derivatives of the velocity field, i.e., from the components of the velocity derivative matrix

$$\frac{\partial u_i}{\partial x_j} := u_{i,j}.$$

Below, besides the above simplified notation for the partial derivatives, we use the Einstein summation convention on repeated indices.

The Newtonian viscous fluid is one where the stress tensor is linear in the components of the velocity derivative matrix. The specific form of this tensor will depend on other physical conditions.

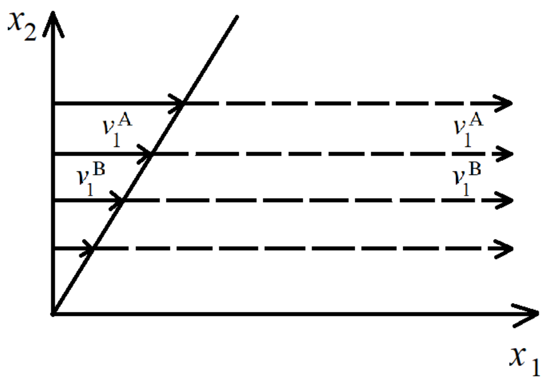

Figure 1. Momentum exchange by molecules between lamina in a shear flow.

To see why a linear relation of this kind might capture the dominant rheology of many fluids, consider a flow $(v_1, v_2) = (v_1(x_2), 0)$. Each different plane $x_2 = const$ or lamina of the fluid moves with a particular velocity. Now, we consider two lamina $x_2 = x_2^A$ and $x_2 = x_2^B$ as shown in Figure 1 moving with velocities $v_1^B < v_1^A$. If a molecule moves from B to A, then it is moving from an environment with velocity v_1^B to an environment with a larger velocity v_1^A. Consequently, it must accelerated to match the new velocity. According to Newton a force is therefore applied to the lamina $x_2 = x_2^A$ in the direction of negative x_1. Similarly, a molecule moving from x_2^A to x_2^B must slow down, exerting a force on lamina $x_2 = x_2^B$ in the direction of positive x_1. Thus these exchanges of molecules would tend to reduce the velocity difference between the two lamina. This tendency to reduce the difference in velocities can be thought of as a force applied to each lamina. Thus if we inset a virtual surface at some position x_2, a force should be exerted on the surface in the positive x_1-direction if

$$\frac{dv_1(x_2)}{dx_2} > 0.$$

Generally we expect the gradients of the velocity components to vary on a length scale L comparable to some macroscopic scale – the size of the container or the size of a body around which the fluid flows, for example. On the other hand, the scale of the molecular events envisaged above is very small compared to the macroscopic scale. Thus it is reasonable to assume that the force $\vec{F}(x_2)$ on the lamina is dominated by the first derivative

$$\vec{F}(x_2) := \sigma_{21}\vec{e}_1 = \mu \frac{dv_1(x_2)}{dx_2}\vec{e}_1, \quad \sigma_{21} = \sigma_{12}.$$

The constant of proportionality μ is called the viscosity, and a fluid obeying this law is our Newtonian viscous fluid.

In general, all of the components of the velocity derivative matrix need to be brought into the construction of the viscous stress tensor.

Let

$$\sigma_{ij} = -\delta_{ij}p + d_{ij}, \tag{1}$$

i.e., we have simply split off the pressure contribution and exhibited the deviadic stress tensor d_{ij}, which defines the viscous stress and determines the fluid rheology. For an isotropic fluid the linearity implies that the most general allowable deviatoric stress has the form

$$d_{ij} = \mu\Big(v_{i,j} + v_{j,i} - \frac{2}{3}u_{k,k}\delta_{ij}\Big) + \mu' v_{k,k}\delta_{ij}, \tag{2}$$

for certain scalars μ and μ' which are usually called the viscosity and the second visccosity, correspondingly. Often the last is taken as 0 for an approximation that is generally valid for liquids. The condition $\mu' = 0$ is equivalent to what is sometimes called the Stokes relation. In gases, in particular, μ' may be positive, in which case the thermodynamic pressure and the normal stresses are distinct.

3. The title Problem

No-slip condition. At rigid boundary the relative motion of fluid and boundary wall vanish. Thus at a nonmoving rigid wall the velocity of the fluid will be 0, while at any point on a moving boundary the fluid velocity must equal to the velocity of that point of the boundary. This condition is valid for gases and fluids in situations where the stress tensor is well approximated by (2). It can fall in small domains and in rarefied gases, where some slip may occur like the ideal fluid.

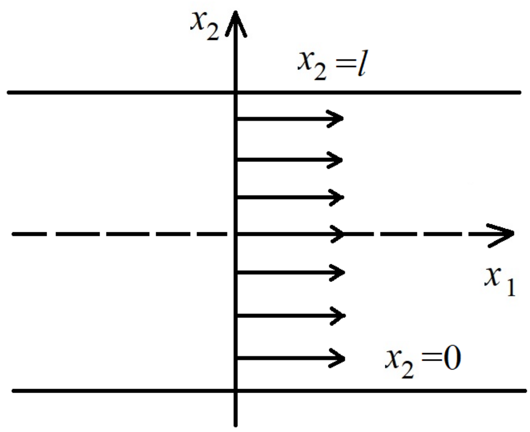

Figure 2.

Couette Flow. Let two rigid planes be $x_2 = 0, l$, where no-slip condition, in general, will be applied. The plane $x_2 = l$ moves in the x_1-direction with constant velocity V, while the plane $x_2 = 0$ is either stationary or moving in the same direction. An incompressible Newtonian viscous fluid with $\mu(x_2) \in C^1]0, l[\cap C^0[0, l]$[1], $\mu(0) = 0$, $\mu(x_2) > 0$ for $x_2 \in]0, l]$, in particular,

$$\mu = \mu_0 x_2^\kappa, \quad \text{constants } \mu_0 > 0, \quad \kappa \geq 0, \tag{3}$$

is contained at rest at the initial moment within the two planes. The flow is steady, so the velocity field must be a function of x_2 alone.

Assuming constant density, $v = (v_1(y), 0, 0)$, $\nabla p = 0$ from (1), (2) we have

$$\sigma_{21} = d_{21} = \mu v_{1,2}. \tag{4}$$

Neglecting body forces, on substituting (4) into the balance equation

$$\sigma_{21,2} = 0 \tag{5}$$

we obtain a momentum balance if

$$(\mu(x_2) v_{1,2})_{,2} = 0, \tag{6}$$

[1] C^0 and C^1 denote the sets of continuous and continuously differentiable functions, respectively.

in particular,

$$(x_2^\kappa v_{1,2})_{,2} = 0, \tag{7}$$

because of (3). Such a type and more general degenerate equations are analysed in [6]-[9] (see also references there).

The general solutions of (6) and (7) have the following forms

$$v_1(x_2) = c_1 \int\limits_l^{x_2} \frac{dx_2}{\mu(x_2)} + c_2 \tag{8}$$

and

$$v_1(x_2) = \begin{cases} \dfrac{c_1}{\mu_0(1-\kappa)}(x_2^{1-\kappa} - l^{1-\kappa}) + c_2, \ \kappa \neq 1; \\ \\ \dfrac{c_1}{\mu_0}(\ln x_2 - \ln l) + c_2, \qquad\qquad \kappa = 1, \end{cases} \tag{9}$$

respectively.

From (8) and (9) it is easily seen that we arrive at the following mathematical conclusions:

1. If $\int\limits_0^l \frac{dx_2}{\mu(x_2)} < +\infty$ $(0 \leq \kappa < 1)$, the BVPs (Diriclet type) with BCs

$$(i) \;\; v_1(0) = 0, \quad v_1(l) = V_l,$$
$$(ii) \;\; v_1(0) = V_0, \quad v_1(l) = 0$$

are well-posed and the unique solutions have the forms

$$(i) \;\; v_1(x_2) = V_l x_2^{1-\kappa} l^{\kappa-1},$$
$$(ii) \;\; v_1(x_2) = -V_0 l^{\kappa-1}(x_2^{1-\kappa} - l^{1-\kappa}),$$

respectively.

2. If $\int\limits_0^l \frac{dx_2}{\mu(x_2)} = +\infty$ $(\kappa \geq 1)$ the BVP (Keldysh type) has a unique bounded solution

$$v_1(x_2) = V_l$$

under BC

$$v_1(l) = V_l$$

and the condition

$$v_1(x_2) = O(1) \;\text{ as }\; x_2 \to 0+.$$

Moreover,

1) If $\int\limits_0^l \frac{dx_2}{\mu(x_2)} < +\infty$ $(0 \leq \kappa < 1)$, the BVP (Dirichlet type) with BCs

$$v_1(0) = V_0, \quad v_1(l) = V_l, \quad V_0 \neq V_l \tag{10}$$

has a unique solution of the form

$$v_1(x_2) = (V_0 - V_l)\Big[\int\limits_l^0 \frac{dx_2}{\mu(x_2)}\Big]^{-1}\Big[\int\limits_l^{x_2} \frac{dx_2}{\mu(x_2)}\Big] + V_l, \tag{11}$$

the constant viscous stress

$$\sigma_{21} = \mu(x_2)\frac{dv_1(x_2)}{dx_2} = (V_0 - V_l)\Big[\int\limits_l^0 \frac{dx_2}{\mu(x_2)}\Big]^{-1}$$

$$\Big(v_1(x_2) = (V_l - V_0)l^{\kappa-1}(x_2^{1-\kappa} - l^{1-\kappa}) + V_l, \tag{12}$$

$$\sigma_{21} = \mu(x_2)\frac{dv_1(x_2)}{dx_2} = (V_l - V_0)\mu_0(1-\kappa)l^{\kappa-1}\Big).$$

2) If $\int\limits_0^l \frac{dx_2}{\mu(x_2)} = +\infty$ $(\kappa \geq 1)$ the BVP (Keldysh type) with BC

$$v_1(l) = V_l$$

and the condition

$$v_1(x_2) = O(1) \text{ as } x_2 \to 0+$$

is well-posed and has a unique bounded solution

$$v_1(x_2) = V_l, \quad \sigma_{21} = 0.$$

When $V_0 = V_l$ we have bulk translation of the fluid together with both the walls $x_2 = 0$ and $x_2 = l$.

Indeed, after integrating twice (6) and (7) we get

$$v_1(x_2) = c_1\int\limits_l^{x_2} \frac{dx_2}{\mu(x_2)} + c_2, \quad x_2 \in]0, l] \tag{13}$$

and

$$v_1(x_2) = \frac{c_1}{\mu_0(1-\kappa)}(x_2^{1-\kappa} - l^{1-\kappa}) + c_2, \quad \kappa < 1 \tag{14}$$

respectively.

Now satisfying BCs (10) we obtain

$$c_2 = V_l,$$

$$c_1 = (V_0 - V_l)\Big[\int\limits_l^0 \frac{dx_2}{\mu(x_2)}\Big]^{-1} \text{ if } \int\limits_0^l \frac{dx_2}{\mu(x_2)} < +\infty;$$

$$c_1 = (V_l - V_0)\mu_0(1-\kappa)l^{\kappa-1} \text{ if } \kappa < 1.$$

If

$$\int\limits_0^l \frac{dx_2}{\mu(x_2)} = +\infty \ (\kappa \geq 1),$$

we must take

$$c_1 = 0$$

otherwise (13) will be unbounded as $x_2 \to 0+$.

3) If the wall $x_2 = l$ moves with constant velocity $V_l \geq 0$ and at the wall $x_2 = 0$ we apply the constant tangent force

$$F_0 = \sigma_{21}(0) \tag{15}$$

then taking into account (8), at the wall $x_2 = 0$ we arrive at the weighted Nuemann BC

$$F_0 = \sigma_{21}(0) = \lim_{x_2 \to 0+} \mu(x_2)\frac{dv_1(x_2)}{dx_2}, \tag{16}$$

i.e.,

$$F_0 = \lim_{x_2 \to 0+} \mu(x_2)\frac{dv_1(x_2)}{dx_2} = \lim_{x_2 \to 0+} \mu(x_2)c_1\frac{1}{\mu(x_2)} = c_1.$$

Satisfying BC

$$v(l) = V_l \tag{17}$$

we obtain

$$c_2 = V_l.$$

Thus the mixed BVP (6), (16), (17) is always (in particular, independent on values of κ) well-posed and the unique explicit solution has the following form

$$v_1(x_2) = F_0 \int\limits_l^{x_2} \frac{dx_2}{\mu(x_2)} + V_l. \tag{18}$$

Note that in this case velocity according to expectations is unbounded, in general, for $x_2 \to 0+$.

On the other hand we have arrived at the following physical (mechanical) conclusions:

If $\int\limits_0^l \frac{dx_2}{\mu(x_2)} < +\infty$ $(0 \leq \kappa < 1)$ the plane $x_2 = 0$ may be stationary or moving; while for $\int\limits_0^l \frac{dx_2}{\mu(x_2)} = +\infty$ $(\kappa \geq 1)$ it should be always stationary and therefore, will be slippery;

If $0 \leq \kappa < 1$ the plane $x_2 = 0$ may move with the velocity V_0 provided plane $x_2 = l$ is either stationary or moving with the velocity $V_l \neq V_0$.

If the wall $x_2 = l$ is either moving with the constant velocity or motionless and at the wall $x_2 = 0$ the constant tangent force is applied, the BVP for the Cuette flow is always uniquely solvable in the explicit form (see (18)).

4. Conclusions

A Functionally graded Couette flow between two rigid planes when at least one of them is moving with a constant velocity is investigated. Analysing the general solution of the governing equation we have obtained the following 1. mathematical and 2. Mechanical results:

1.1. If $\int\limits_0^l \frac{dx_2}{\mu(x_2)} < +\infty$ $(\kappa \in [0, 1[)$, the following BVP with Bcs (Dirichlet type)

$$v_1(0) = V_0, \;\; v_1(l) = V_l, \;\; V_0 \neq V_l$$

is well-posed and the unique explicit solution has been constructed (see (11) and (12). The constant viscous stress

$$\sigma_{21} = \mu(x_2)\frac{dv_1(x_2)}{dx_2} = (V_0 - V_l)\Big[\int\limits_l^0 \frac{dx_2}{\mu(x_2)}\Big]^{-1};$$

1.2. If $\int\limits_0^l \frac{dx_2}{\mu(x_2)} = +\infty$ $(\kappa \geq 1)$ the BVP (Keldysh type) with BC

$$v_1(l) = V_l$$

and the condition

$$v_1(x_2) = O(1), \;\; \text{as} \;\; x_2 \to 0+$$

is well-posed and has a unique solution

$$v_1(x_2) = V_l, \quad \sigma_{21} = 0;$$

1.3. If the wall $x_2 = l$ moves with the constant velocity $V_l \geq 0$ and at the wall $x_2 = 0$ we apply the tangent force F_0, then at the wall $x_2 = 0$ we have the weighted Neumann BC

$$\lim_{x_2 \to 0+} \mu(x_2) \frac{v_1(x_2)}{dx_2} = F_0.$$

The mixed BVP is well-posed and we have constructed the unique explicit solution (see (18)).

2.1. If $\int\limits_0^l \frac{dx_2}{\mu(x_2)} = +\infty$ ($\kappa \in [1, +\infty[$) the wall $x_2 = 0$ is slippery as it was expected since a boundary layer adjacent to well $x_2 = 0$ is actually ideal lamina for $\mu(0) = 0$ and we have the bulk translation of the fluid together with the moving wall $x_2 = l$ which produces no force, while if $\int\limits_0^l \frac{dx_2}{\mu(x_2)} < +\infty$ ($\kappa \in [0, 1[$) it is not the case since the adjacent boundary layer is practically viscous in spite of the fact that $\mu(0) = 0$;

2.2. When $V_0 = V_l$ we have bulk translation of the fluid together with both the walls $x_2 = 0$ and $x_2 = l$ which produces no force;

2.3. If the wall $x_2 = l$ is either moving with the constant velocity or motionless and at the wall $x_2 = 0$ the constant tangent force is applied, the BVP for the Cuette flow is always uniquely solvable in the explicit form (see (18)).

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