Bulletin of TICMI
Vol. 23, No. 1, 2019, 9–17

The First Group of Necessary Conditions for the Boundary Value Problem of the Fourth-Order Partial Differential Equation

Sevinj Valiyeva*

Baku State University, Baku, Azerbaijan
(*Received November 12, 2018; Revised May 13, 2019; Accepted May 21, 2019*)

The paper is devoted to obtaining the first basic relation, which is obtained from the second Green's formula of the self-adjoint fourth-order partial differential equation and the fundamental solution of this equation. Then the necessary conditions are obtained from this basic relation.

Keywords: Fourth-order partial differential equations, Boundary value problem, Second Green's formula, Fundamental solution, First fundamental relation, Necessary conditions.

AMS Subject Classification: 35A08, 35A20, 35G15.

1. Introduction

In the course of the equation of mathematical physics and in the course of partial differential equations, three types of equations are mainly considered. Equations of parabolic, hyperbolic and elliptic types [1]-[2]. For the equation of parabolic and hyperbolic types, the Cauchy problem and the mixed problem are considered, and for the equation of elliptic type the boundary problem [3]-[4] is considered.

Later, boundary problems were considered for equations of parabolic and hyperbolic type [5]-[6], in spite of the fact that the boundary value problem for an equation of hyperbolic type is not correct.

Finally, we should note that the boundary value problem with nonlocal boundary conditions was considered for solving mixed and composite types, as well as for integro-differential and loaded equations [7] - [9].

Recently, boundary problems have appeared for the study of the elliptic first-order type of the so-called Cauchy-Riemann equations [10]-[12].

In this paper we consider the boundary value problem for a two-dimensional fourth-order equation, related to the process of motion of a compressible exponentially stratified fluid. Using the analog of the second Green's formula, all linearly independent necessary conditions are determined, the definition of which belongs to the authors of this article.

Using these necessary conditions, we obtain an analytical form of the solution of the boundary problem under consideration, which makes it possible to determine the approximate solution.

* Email: sevinj_veliyeva@mail.ru

ISSN: 1512-0082 print

2. Formulation of the problem

Consider the following boundary value problem:

$$\frac{1}{c^2}\frac{\partial^4 u(x)}{\partial x_1^4}-\frac{\partial^4 u(x)}{\partial x_1^2\partial x_2^2}+b^2\frac{\partial^2 u(x)}{\partial x_1^2}-\omega_0^2\frac{\partial^2 u(x)}{\partial x_2^2}=0,\ \ x=(x_1,x_2)\in D\subset R^2, \quad (1)$$

$$\sum_{0\le j_1+j_2\le 3}\left\{\alpha^{(1)}_{kj_1,j_2}(x_1)\left.\frac{\partial^{j_1+j_2}u(x)}{\partial x_1^{j_1}\partial x_2^{j_2}}\right|_{x_2=\gamma_1(x_1)}+\alpha^{(2)}_{kj_1,j_2}(x_1)\left.\frac{\partial^{j_1+j_2}u(x)}{\partial x_1^{j_1}\partial x_2^{j_2}}\right|_{x_2=\gamma_2(x_1)}\right\}$$

$$=\alpha_k(x_1), \qquad k=\overline{1,4},\quad x_1=[a_1,b_1], \quad (2)$$

where $c,\ b\ \omega_0$ is a constant number, D is bounded convex in the direction x_2 flat are, boundary $\Gamma=\partial D$ is the Lyapunov line, $\alpha^{(s)}_{kj_1,j_2}(x_1)$ and $\alpha_k(x_1)$, if $k=\overline{1,4}$, $0\le j_1+j_2\le 3$, $s=1,2$ and $x_1=[a_1,b_1]$, are continuous functions, and the boundary conditions (2) are linearly independent. When designing an area D on the direction x_1, parallel to x_2 boundary Γ is divided into two parts, Γ_1 and Γ_2, with equations $x_2=\gamma_k(x_1)$, $k=1,2$, $x_1=[a_1,b_1]$.

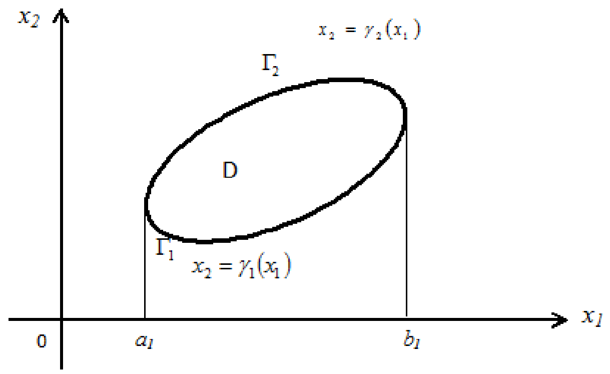

Figure 1.

As is known, the fundamental solution of equation (1) has the form:

$$\int_\Gamma U(x-\xi)=-\frac{e(x_1-\xi_1)\,e(x_2-\xi_2)}{cb^2}sinb(x_2-\xi_2)\ sincb(x_1-\xi_1)$$

$$-\frac{e(x_1-\xi_1)}{2cb^2}\ e(x_2-\xi_2+c(x_1-\xi_1))\,(cosb(x_2-\xi_2+c(x_1-\xi_1))\ -1)$$

$$+\frac{e\left(x_1-\xi_1\right)}{2cb^2}e\left(x_2-\xi_2-c\left(x_1-\xi_1\right)\right)\left(cosb\left(x_2-\xi_2-c\left(x_1-\xi_1\right)\right)\ -1\right). \qquad (3)$$

Let's create the second Green formula. To do this, we multiply equation (1) by the fundamental solution (3), integrate over the domain D. Applying the Ostragradskii-Gauss formula, we have:

$$-\frac{1}{c^2}\left(\int\limits_{\Gamma} U\left(x-\xi\right)\frac{\partial^3 u\left(x\right)}{\partial {x_1}^3}cos\left(\nu,x_1\right)dx-\int\limits_{\Gamma}\frac{\partial U\left(x-\xi\right)}{\partial x_1}\frac{\partial^2 u\left(x\right)}{\partial {x_1}^2}cos\left(\nu,x_1\right)d\ x\right.$$

$$\left.+\int\limits_{\Gamma}\frac{\partial^2 U\left(x-\xi\right)}{\partial {x_1}^2}\frac{\partial u\left(x\right)}{\partial x_1}cos\left(\nu,x_1\right)dx\ -\int\limits_{\Gamma}u\left(x\right)\frac{\partial^3 U\left(x-\xi\right)}{\partial {x_1}^3}cos\left(\nu,x_1\right)dx\right)$$

$$-\left(\int\limits_{\Gamma} U\left(x-\xi\right)\frac{\partial^3 u\left(x\right)}{\partial {x_2}^2\partial x_1}cos\left(\nu,x_1\right)dx-\int\limits_{\Gamma}\frac{\partial U\left(x-\xi\right)}{\partial x_1}\frac{\partial^2 u\left(x\right)}{\partial {x_2}^2}cos\left(\nu,x_1\right)dx\right.$$

$$\left.+\int\limits_{\Gamma}\frac{\partial^2 U\left(x-\xi\right)}{\partial {x_1}^2}\frac{\partial u\left(x\right)}{\partial x_2}cos\left(\nu,x_2\right)dx\ -\int\limits_{\Gamma}u\left(x\right)\frac{\partial^3 U\left(x-\xi\right)}{\partial {x_1}^2\partial x_2}cos\left(\nu,x_2\right)dx\right)$$

$$+b^2\left(\int\limits_{\Gamma} U\left(x-\xi\right)\frac{\partial u\left(x\right)}{\partial x_1}cos\left(\nu,x_1\right)dx-\int_{\Gamma}u\left(x\right)\frac{\partial U\left(x-\xi\right)}{\partial x_1}cos\left(\nu,x_1\right)dx\right)$$

$$-\omega_0^2\left(\int\limits_{\Gamma} U\left(x-\xi\right)\frac{\partial u\left(x\right)}{\partial x_2}cos\left(\nu,x_2\right)dx\ -\int\limits_{\Gamma}u\left(x\right)\frac{\partial U\left(x-\xi\right)}{\partial x_2}cos\left(\nu,x_2\right)dx\right)$$

$$=\begin{cases} u\left(\xi\right), & \xi\in D \\ \frac{1}{2}u\left(\xi\right), & \xi\in\Gamma \end{cases} \qquad (4)$$

where ν is an outer normal to the boundary Γ of the field of D, dx in the main, in the relation (4) means either $d\Gamma$.

The first obtained basic relation (4) consists of two parts, the first part of which is any solution of equation (1) defined in the field D, and the second one connecting with the boundary Γ are the first necessary conditions.

Separating these conditions, we have:

$$\frac{1}{2}u\left(\xi_1,\gamma_1(\xi_1)\right)=\frac{1}{c^2}\int\limits_{a_1}^{b_1}u\left(x_1,\gamma_1(x_1)\right)\gamma_1{}'(x_1)\left[bc^2\ e\left(x_1-\xi_1\right)e\left(\gamma_1\left(x_1\right)-\gamma_1\left(\xi_1\right)\right)\right.$$

$$\times sinb\left(\gamma_1\left(x_1\right)-\gamma_1\left(\xi_1\right)\right)coscb\left(x_1-\xi_1\right)$$

$$+\frac{bc^2e\left(x_1-\xi_1\right)e\left(\gamma_1\left(x_1\right)-\gamma_1\left(\xi_1\right)+c\left(x_1-\xi_1\right)\right)}{2}$$

$$\times sinb\left(\gamma_1\left(x_1\right)-\gamma_1\left(\xi_1\right)+c\left(x_1-\xi_1\right)\right)$$

$$-\frac{bc^2e\left(x_1-\xi_1\right)e\left(\gamma_1\left(x_1\right)-\gamma_1\left(\xi_1\right)-c\left(x_1-\xi_1\right)\right)}{2}$$

$$\times sinb\left(\gamma_1\left(x_1\right)-\gamma_1\left(\xi_1\right)-c\left(x_1-\xi_1\right)\right)\ ]\,dx_1+\int\limits_{a_1}^{b_1}u\left(x_1,\gamma_1\left(x_1\right)\right)[bc\ e\left(x_1-\xi_1\right)$$

$$\times e\left(\gamma_1\left(x_1\right)-\gamma_1\left(\xi_1\right)\right)cosb\left(\gamma_1\left(x_1\right)-\gamma_1\left(\xi_1\right)\right)sincb\left(x_1-\xi_1\right)$$

$$-\frac{bce\left(x_1-\xi_1\right)e\left(\gamma_1\left(x_1\right)-\gamma_1\left(\xi_1\right)+c\left(x_1-\xi_1\right)\right)}{2}$$

$$\times sinb\left(\gamma_1\left(x_1\right)-\gamma_1\left(\xi_1\right)+c\left(x_1-\xi_1\right)\right)$$

$$+\frac{bce\left(x_1-\xi_1\right)e\left(\gamma_1\left(x_1\right)-\gamma_1\left(\xi_1\right)-c\left(x_1-\xi_1\right)\right)}{2}$$

$$\times sinb\left(\gamma_1\left(x_1\right)-\gamma_1\left(\xi_1\right)-c\left(x_1-\xi_1\right)\right)\ ]\,dx_1$$

$$+\frac{1}{c^2}\int\limits_{a_1}^{b_1} u\left(x_1,\gamma_2(x_1)\right)\gamma_2{}'\left(x_1\right)\left[bc^2\ e\left(x_1-\xi_1\right)\right.$$

$$\times e\left(\gamma_2\left(x_1\right)-\gamma_1\left(\xi_1\right)\right) sinb\left(\gamma_2\left(x_1\right)-\gamma_1\left(\xi_1\right)\right)\ coscb\left(x_1-\xi_1\right)\ +\frac{c^2 e\left(x_1-\xi_1\right)}{2}$$

$$\times\frac{\delta\left(\gamma_2\left(x_1\right)-\gamma_1\left(\xi_1\right)+c\left(x_1-\xi_1\right)\right)}{2}+\frac{bc^2 e\left(x_1-\xi_1\right) e\left(\gamma_2\left(x_1\right)-\gamma_1\left(\xi_1\right)+c\left(x_1-\xi_1\right)\right)}{2}$$

$$\times sinb\left(\gamma_2\left(x_1\right)-\gamma_1\left(\xi_1\right)+c\left(x_1-\xi_1\right)\right)$$

$$+\frac{c^2 e\left(x_1-\xi_1\right)\delta\left(\gamma_2\left(x_1\right)-\gamma_1\left(\xi_1\right)-c\left(x_1-\xi_1\right)\right)}{2}$$

$$-\frac{bc^2 e\left(x_1-\xi_1\right) e\left(\gamma_2\left(x_1\right)-\gamma_1\left(\xi_1\right)-c\left(x_1-\xi_1\right)\right)}{2}$$

$$\times\ sinb\left(\gamma_2\left(x_1\right)-\gamma_1\left(\xi_1\right)-c\left(x_1-\xi_1\right)\right)\ \left.\right]dx_1$$

$$-\int\limits_{a_1}^{b_1} u\left(x_1,\gamma_2\left(x_1\right)\right)\left[bc\ e\left(x_1-\xi_1\right)\ e\left(\gamma_2\left(t\right)-\gamma_1\left(\xi_1\right)\right)\right.$$

$$\times cosb\left(\gamma_2\left(x_1\right)-\gamma_1\left(\xi_1\right)\right)\ sincb\left(x_1-\xi_1\right)$$

$$+\frac{ce\left(x_1-\xi_1\right)\delta\left(\gamma_2\left(x_1\right)-\gamma_1\left(\xi_1\right)+c\left(x_1-\xi_1\right)\right)}{2}$$

$$-\frac{bce\left(x_1-\xi_1\right) e\left(\gamma_2\left(x_1\right)-\gamma_1\left(\xi_1\right)+c\left(x_1-\xi_1\right)\right)}{2}$$

$$\times sinb\left(\gamma_2\left(x_1\right)-\gamma_1\left(\xi_1\right)+\ c\left(x_1-\xi_1\right)\right)$$

$$-\frac{ce\left(x_1-\xi_1\right)\delta\left(\gamma_2\left(x_1\right)-\gamma_1\left(\xi_1\right)-c\left(x_1-\xi_1\right)\right)}{2}$$

$$+\frac{bce\left(x_1-\xi_1\right)e\left(\gamma_2\left(x_1\right)-\gamma_1\left(\xi_1\right)-c\left(x_1-\xi_1\right)\right)}{2}$$
$$sinb\left(\gamma_2\left(x_1\right)-\gamma_1\left(\xi_1\right)-c\left(x_1-\xi_1\right)\right)\ \Big]\,dx_1. \qquad (5)$$

Where through $e(t)$ denotes the Heaviside symmetric unit function, and through $\delta(t)$ denotes the Dirac delta function. Similarly to (5), from (4) we obtain the second necessary condition for $u\left(\xi_1,\gamma_2(\xi_1)\right)$.

$$\frac{1}{2}u\left(\xi_1,\gamma_2(\xi_1)\right)=\frac{1}{c^2}\int\limits_{a_1}^{b_1}u\left(x_1,\gamma_1\ \left(x_1\right)\right)\gamma_1{}'(x_1)\left[bc^2\ e\left(x_1-\xi_1\right)e\left(\gamma_1\left(x_1\right)-\gamma_2\left(\xi_1\right)\right)\right.$$

$$\times sinb\left(\gamma_1\left(x_1\right)-\gamma_2\left(\xi_1\right)\right)coscb\left(x_1-\xi_1\right)$$

$$+\frac{c\delta\left(x_1-\xi_1\right)e\left(\gamma_1\left(x_1\right)-\gamma_2\left(\xi_1\right)\right)}{2}cosb\left(\gamma_1\left(x_1\right)-\gamma_2\left(\xi_1\right)\right)$$

$$+\frac{c^2e\left(x_1-\xi_1\right)\delta\left(\gamma_1\left(x_1\right)-\gamma_2\left(\xi_1\right)+c\left(x_1-\xi_1\right)\right)}{2}$$

$$-\frac{bc^2e\left(x_1-\xi_1\right)e\left(\gamma_1\left(x_1\right)-\gamma_2\left(\xi_1\right)+c\left(x_1-\xi_1\right)\right)}{2}$$

$$\times sinb\left(\gamma_1\left(x_1\right)-\gamma_2\left(\xi_1\right)+c\left(x_1-\xi_1\right)\right)$$

$$-\frac{c\delta\left(t-\xi_1\right)e\left(\gamma_1\left(x_1\right)-\gamma_2\left(\xi_1\right)\right)}{2}cosb\left(\gamma_1\left(x_1\right)-\gamma_2\left(\xi_1\right)\right)$$

$$+\frac{c^2e\left(x_1-\xi_1\right)\delta\left(\gamma_1\left(x_1\right)-\gamma_2\left(\xi_1\right)-c\left(x_1-\xi_1\right)\right)}{2}$$

$$-\frac{bc^2e\left(x_1-\xi_1\right)}{2}\frac{e\left(\gamma_1\left(x_1\right)-\gamma_2\left(\xi_1\right)-c\left(x_1-\xi_1\right)\right)}{2}$$

$$\times sinb\left(\gamma_1\left(x_1\right)-\gamma_2\left(\xi_1\right)-c\left(x_1-\xi_1\right)\right)\ ]\,dx_1$$

$$+\int_{a_1}^{b_1}u\left(x_1,\gamma(x_1)\right)\left[bc\ e\left(x_1-\xi_1\right)e\left(\gamma_1\left(x_1\right)-\gamma_1\left(\xi_1\right)\right)\right.$$

$$\times cosb\left(\gamma_1\left(x_1\right)-\gamma_2\left(\xi_1\right)\right)\ sincb\left(x_1-\xi_1\right)$$

$$+\frac{ce\left(x_1-\xi_1\right)\delta\left(\gamma_1\left(x_1\right)-\gamma_2\left(\xi_1\right)+c\left(x_1-\xi_1\right)\right)}{2}$$

$$-\frac{bce\left(x_1-\xi_1\right)e\left(\gamma_1\left(x_1\right)-\gamma_2\left(\xi_1\right)+c\left(x_1-\xi_1\right)\right)}{2}$$

$$\times sinb\left(\gamma_1\left(x_1\right)-\gamma_2\left(\xi_1\right)+c\left(x_1-\xi_1\right)\right)$$

$$-\frac{ce\left(x_1-\xi_1\right)\delta\left(\gamma_1\left(x_1\right)-\gamma_2\left(\xi_1\right)-c\left(x_1-\xi_1\right)\right)}{2}$$

$$+\frac{bce\left(x_1-\xi_1\right)}{2}\frac{e\left(\gamma_1\left(x_1\right)-\gamma_2\left(\xi_1\right)-c\left(x_1-\xi_1\right)\right)}{2}$$

$$\times sinb\left(\gamma_1\left(x_1\right)-\gamma_2\left(\xi_1\right)-c\left(x_1-\xi_1\right)\right)\ ]\,dx_1$$

$$+\frac{1}{c^2}\int_{a_1}^{b_1} u\left(x_1,\gamma(x_1)\right)\gamma_2{}'(x_1)\,\big[bc^2\ e\left(x_1-\xi_1\right)$$

$$\times e\left(\gamma_2\left(x_1\right)-\gamma_2\left(\xi_1\right)\right) sinb\left(\gamma_2\left(x_1\right)-\gamma_2\left(\xi_1\right)\right)\ coscb\left(x_1-\xi_1\right)\ +\frac{bc^2 e\left(x_1-\xi_1\right)}{2}$$

$$\times\frac{e\left(\gamma_2\left(x_1\right)-\gamma_2\left(\xi_1\right)+c\left(x_1-\xi_1\right)\right)}{2}$$

$$\times sinb\left(\gamma_2\left(x_1\right)-\gamma_2\left(\xi_1\right)+c\left(x_1-\xi_1\right)\right)-\frac{bc^2 e\left(x_1-\xi_1\right)}{2}$$

$$\times\frac{e\left(\gamma_2\left(x_1\right)-\gamma_2\left(\xi_1\right)-c\left(x_1-\xi_1\right)\right)}{2}\ sinb\left(\gamma_2\left(x_1\right)-\gamma_2\left(\xi_1\right)-c\left(x_1-\xi_1\right)\right)\ \big]\,dx_1$$

$$-\int_{a_1}^{b_1} u\left(x_1,\gamma\left(x_1\right)\right)\left[bc\ e\left(x_1-\xi_1\right)e\left(\gamma_2\left(x_1\right)-\gamma_2\left(\xi_1\right)\right)\right.$$

$$\times cosb\left(\gamma_2\left(x_1\right)-\gamma_2\left(\xi_1\right)\right) sincb\left(x_1-\xi_1\right)$$

$$-\frac{bce\left(x_1-\xi_1\right)e\left(\gamma_2\left(x_1\right)-\gamma_2\left(\xi_1\right)+c\left(x_1-\xi_1\right)\right)}{2}$$

$$\times sinb\left(\gamma_2\left(x_1\right)-\gamma_2\left(\xi_1\right)+c\left(x_1-\xi_1\right)\right)$$

$$+\frac{bce\left(x_1-\xi_1\right)e\left(\gamma_2\left(x_1\right)-\gamma_2\left(\xi_1\right)-c\left(x_1-\xi_1\right)\right)}{2}$$

$$\times\ sinb\left(\gamma_2\left(x_1\right)-\gamma_2\left(\xi_1\right)-c\left(x_1-\xi_1\right)\right)\ \left.\right]dx_1.$$

As seen from (5), some terms mutually cancel out or turn into zero.

If $\gamma_2(x_1) - \gamma_1(\xi_1) \neq 0$ if $x_1,\ \xi_1 \in (a_1, b_1)$ then in (5) there is no delta Dirac function.

Thus, the following theorem is established.

Theorem 2.1 : *Let D be bounded, convex toward to x_2 a plane flat with the boundary of the Lyapunov Γ line, c, b and ω_0 are positive constant numbers, then each solution of equation* (1) *defined in the field D satisfies the necessary conditions obtained from the basic relation* (4)*.*

References

[1] S.L. Sobolev, *Equations of Mathematical Physics*, GITTL, Moscow, 1954

[2] L. Berg, F. John, M. Schechter, *Partial Differential Equations*, Mir, Moscow, 1966

[3] S.G. Mikhlin, *Course of Mathematical Physics*, Nauka, Moscow, 1968

[4] S.G.Mikhlin, *Linear Partial Differential Equations, Higher Mathematics*, Moscow, 1977

[5] N.A. Aliyev, S.M. Hosseini, *An analysis of a parabolic problem with a general (non-local and global) supplementary linear conditions 2*, Italian Journal of Pure and Applied Mathematics, **13** (2003), 15-127

[6] N.A. Aliev, R.M. Aliguliyev, *Boundary Value Problem for Hyperbolic Type Equations*, Spectral Theory of Differential Operators (thematic collection of scientific papers) of ASU named after S.M. Kirov, Baku (1984), 3-9

[7] N.A. Aliev, F.O. Mamedova, *A mixed type problem for a first order equation*, Izv. AN Azerbaijan SSR., Series FT and MN, **6** (1983), 110-113

[8] N.A. Aliyev, A.M. Aliyev, *Investigation of solutions of Boundary Value Problems for a composite type equation with non-local boundary conditions*, USA, arxiv:0807.1697v1 [math.AP] 10 Jul (2008), 1-6

[9] M.R. Fatemi, N.A. Aliyev, *General Linear Boundary Value Problem for the Second-Order Integro-Differential Loaded Equation with Boundary Conditions Containing Both Nonlocal and Global Terms*, (English) Hindawi Publishing Corporation, Abstract and Applied Analysis, 2010

[10] N.A. Aliyev, A.M. Aliyev, *Data of One Nonlinear Differential Equation with the Partial Derivatives to the Cauchy-Riemann Equation, Book of Abstracts*, The Fourth Congress of The Turkic World Mathematical Society, Azerbaijan, Baku, 1-3 July, 2011

[11] N.A. Aliev, Y.Y. Mustafayeva, S.M. Murtuzayeva, *The Influence of the Carleman Condition on the Fredholm Property of the Boundary Value Problem for Cauchy-Riemann Equation*, Proceedings of the Institute of Applied Mathematics, Baku, Azerbaijan, **1**, 2 (2012), 153-162

[12] M. Sajjadmanesh, M. Jahanshahi, N.A. Aliyev, *Tikhonov-Lavrentev type inverse problem including Cauchy-Riemann equation*, Azerbaijan Journal of Mathematics, Baku, **3**, 1 (2013), 104-110

[13] G.W. BegehrH, et al. (eds) *Partial Differential and Integral Equations*, Kliwer Academic Publishers, Printed in the Netherlands (1999), 155-175