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The Analysis of Vortex Equations Using Methods of Generalized Analytic Functions

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In this paper we continue our investigation of vortex equation and related topics in framework of generalized analytic functions. We show that solution space of systems of vortex equations does not depend on location of zeros of Higgs field and in this way we obtain another proof of the well known Taubes' theorem on description of solution space of vortex equation modulo gauge equivalence. It turns out that the first equation of this system is a particular case of Carleman-Bers-Vekua equation, and the second equation is a property of non dependence of the solution space of the first equation on complex structure of the noncompact Riemann surface, which is a Riemann sphere without zeros of the Higgs field.

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The main object of analysis will be

$$S[\psi] = \frac{1}{4\pi}\int_M \left(\partial_z\psi\partial_{\bar z}\psi + 4\pi\mu e^{\psi}\right) d^2z$$

type functional, related to uniformization of Riemann surfaces and the following statements

$ds^2 = e^{\psi}dzd\bar z$ have constant negative curvature and $S[\psi]$ is minimized iff ψ satisfies the Liouville equation of motion

$$\partial_z\psi\partial_{\bar z}\psi = \mu e^{\psi}. \tag{1}$$

The general solution of (1) can be written as

$$\psi(z,\bar z) = \log\left(\sqrt{2\pi\mu}\frac{|\partial_z a(z)|^2}{(1+|a(z)|^2)^2}\right).$$

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It is parameterized by a holomorphic function $a(z)$, which describes the uniformizing mapping.

By definition a conformal structure on a Riemann surface Σ is an equivalence class of metrics

$$[g] = \{e^{2u}g\colon u \in \mathcal{C}^{\infty}(\Sigma)\}.$$

Besides a complex structure on a Riemann surface Σ is an equivalence class of complex atlases, where two atlases are considered equivalent iff their union forms a new complex atlas.

From Riemann's uniformization Theorem follows, that for any given conformal structure there exists a unique metric with constant curvature of either 1,0 or -1. This gives a means of choosing a canonical representative for each conformal structure.

Specifying a complex structure completely specifies the conformal structure, and vice-versa. One might see this from the following Theorem:

Theorem 1: *Let R and S be Riemann surfaces induced by oriented 2-dimensional Riemannian manifolds (M, ds^2) and (N, ds_1^2) respectively. Then the map $f : (M, ds^2) \to (N, ds_1^2)$ is conformal if and only if $f : R \to S$ is biholomorphic.*

Consequence of this theorem is equivalence of conformal and complex structures.

Let $K(x)$ be a function defined on a sphere S^2. $K(x)$ is Gauss curvature of a metric g on S^2 conformally equivalent to g_0, if an only if there exists the function v on S^2, which satisfies the following equation

$$-\Delta_{g_0} v + 1 = K(x)e^{2v}, \tag{2}$$

where Δ_{g_0} denotes the Laplace-Beltrami operator associated with the metric g_0.

A necessary condition for solving equation (2) is that K has to be positive somewhere and, in general, statement is a corollary of the Gauss-Bonnet theorem.

The charged planar matter interacting with "photons" whose dynamics is governed not only by the Maxwell Lagrange density $-\frac{1}{4}F^{ij}F_{ij}$, but also by the Chern-Simons term $\frac{\kappa}{4}\epsilon^{ijl}F_{ij}A_l$ gives rise to topologically massive (2+ 1)-dimensional "electrodynamics." Such model called Abelian Chern-Simons theory with spontaneous symmetry breaking [1]. Hence, the static Abelian Chern-Simons-Higgs energy functional has the form [3]:

$$\mathcal{CSH}(A, \phi) = \int_{\mathbf{R}^2} |D_A\phi|^2 + \frac{\kappa^2}{4}\frac{|F_A|^2}{|\phi|^2} + \frac{\lambda}{\kappa^2}|\phi|^2(1 - |\phi|^2)^2 d^2z, \tag{3}$$

where $z = (x, y)$ are coordinates on $\mathbf{R}^2$, $\phi : \mathbf{R}^2 \to \mathbf{C}$ is a complex valued function and called Higgs field, $A : \mathbf{R}^2 \to \mathbf{R}^2$ is 1-form on $\mathbf{R}^2$ (mathematically connection form) called a gauge potential or gauge field, $D_A = d - iA$ is a covariant derivative, $F_A = dA + A \wedge A$ is the magnetic field (mathematically curvature) and in our consideration is equal to $\frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial y}$, κ and λ are constants.

This model describes vortices which are charged both electrically and magnetically. Such vortices are important in several areas of theoretical physics such as anyonic superconductivity and fractional quantum Hall effect (see [2], [1],[3]).

The Euler-Lagrange equations for Lagrangian (3) is the system of equations for ϕ and A:

$$-D_A^2\phi - \frac{\kappa^2}{4}\frac{|F_A|^2}{|\phi|^4}\phi + \frac{\lambda}{\kappa^2}\phi(|\phi|^2-1)(3|\phi|^2-1) = 0, \tag{4}$$

$$-\kappa^2\nabla\left(\frac{|F_A|^2}{|\phi|^2}\right) + 2i(\overline{\phi}D_A\phi - \phi\overline{D_A\phi}) = 0, \tag{5}$$

where ∇ in (5) is an operator acting on scalar functions by rule $(\frac{\partial}{\partial y}, -\frac{\partial}{\partial x})$ and $\overline{\phi}$ denotes the complex conjugate of ϕ.

The finite energy condition for functional (3) is given by boundary condition for ϕ :

$$|\phi| \to 1 \text{ as } |z| \to \infty \tag{6}$$

and called topological condition. The corresponding solutions called topological (vortex).

Remark 1 : The boundary condition (6) also gives finiteness condition for energy functional and is this case the solutions of (4),(5) are called non-topological (vortex).

The gauge field dynamics of Cern-Simons-Higgs model is governed by the Chern-Simons term. If this term is absent and the dynamics is given by the Maxwell term we obtain the Ginzburg-Landau energy functional for the superconductivity:

$$\mathcal{GL}(A,\phi) = \int_{\mathbf{R}^2} |D_A\phi|^2 + |F_A|^2 + \frac{\lambda}{4}(1-|\phi|^2)^2 d^2z, \tag{7}$$

The Euler-Lagrange equation for $\mathcal{GL}$ is the following couple of Ginzburg-Landau equations:

$$-D_A^2\phi + \frac{\lambda}{2}\phi(|\phi|^2-1) = 0, \tag{8}$$

$$\nabla^2 A + \frac{i}{2}(\overline{\phi}D_A\phi - \phi\overline{D_A\phi}) = 0, \tag{9}$$

with the topological boundary condition (6).

Let E denote the vector bundle $p : \mathbf{R}^2 \times \mathbf{C} \to \mathbf{R}^2$. Denote by $\mathcal{C}(E)$ the space of smooth $U(1)$ - connections on E and let $C^\infty(E)$ be the space of smooth cross sections of E. Then the Ginzburg-Landau action $\mathcal{GL}$ is a functional on the space $\mathcal{C}(E) \times C^\infty(E)$. Because E is trivial, the space of connections, $\mathcal{C}(E)$, can be identified with the space of C^∞ sections of the cotangent bundle which is 1-forms on $\mathbf{R}^2$, denoted by $\Lambda^1(\mathbf{R}^2)$ and the same reason, $C^\infty(E)$ can be identified with the space of C^∞ complex valued functions on $\mathbf{R}^2$. Let $A(x,y) = A_1dx + A_2dy \in \Lambda^1(\mathbf{R}^2)$, then a connection in $\mathcal{C}(E)$ is given by $-ip^*A$. The curvature form of the connection will be denoted $-ip^*F_A$, where $F_A = dA \in \Lambda^2(\mathbf{R}^2)$:

$$F_A = dA = \sum_{k,l=1}^{2} F_{kl} dx_k \wedge dx_l.$$

It is clear that

$$\frac{1}{2} F_A = F_{12}.$$

Indeed, nontrivial members in the above sum are F_{12} and F_{21}, with $F_{12} = -F_{21}$.

$$F_{12} = \frac{\partial A_2}{\partial x} dx \wedge dy - \frac{\partial A_1}{\partial y} dx \wedge dy = \left(\frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial y} \right) dx \wedge dy.$$

In a similar way we obtain

$$F_{21} = \frac{\partial A_1}{\partial y} dx \wedge dy - \frac{\partial A_2}{\partial x} dx \wedge de = \left(\frac{\partial A_1}{\partial y} - \frac{\partial A_2}{\partial x} \right) dx \wedge dy.$$

If $\phi = \phi_1 + i\phi_2 \in C^\infty(E)$ is any section of E, then $\phi^*(p^*F_A) = F_A$. The connection defines a map from $C^\infty(E)$ to $\mathcal{C}(E) \times C^\infty(E)$ via the covariant derivative, i.e. for $\phi \in C^\infty(E)$ the operator D_A acts on ϕ in the following way

$$D_A \phi = d\phi - iA\phi. \tag{10}$$

Let $* : \Lambda^k(\mathbf{R}^2) \to \Lambda^{2-k}(\mathbf{R}^2)$ be a Hodge operator (duality isomorphism), then in this notation the Ginzburg-Landau action (7) may be considered as a functional on $\mathcal{C}(E) \times C^\infty(E)$ by

$$\mathcal{GL} = \frac{1}{2} \int_{\mathbf{R}^2} [F_A \wedge *F_A + D_A\phi \wedge *\overline{D_A\phi} + \frac{\lambda}{4} * (\phi\overline{\phi} - 1)^2] d^2 z \tag{11}$$

and if $\lambda = 1$, the variational equations (4), (5) get

$$d * F_A - \frac{i}{2} * (\phi * \overline{D_A\phi} - \overline{\phi} D_A \phi) = 0, \tag{12}$$

$$-D * D_A \phi + * \frac{i}{2} (\phi * \overline{\phi} - 1)\phi = 0. \tag{13}$$

The boundary condition follows from the inequality $|\mathcal{GL}| < \infty$ and is expressed by the Chern number N of vector bundles E as

$$\int_{\mathbf{R}^2} F_A = 2\pi N \tag{14}$$

and is called the vortex number.

After the integration by parts of (11) we obtain [4]:

$$\mathcal{GL} = \frac{1}{2}\int_{\mathbf{R}^2}[\frac{1}{2}(D_A\phi \pm i * D_A\phi) \wedge *(\overline{D_A\phi} \mp i * \overline{D_A\phi})$$

$$+(*F \mp \frac{1}{2}(\phi\overline{\phi}-1)) \wedge *(*F \mp \frac{1}{2}(\phi\overline{\phi}-1))] \mp \int_{\mathbf{R}^2} F.$$

From this it follows that

$$|\mathcal{GL}| \geq |N|\pi,$$

wherein the lower bound is realized if and only if (A, ϕ) satisfy the system of equations

$$D_A\phi - i * D_A\phi = 0, \tag{15}$$

$$*F + \frac{1}{2}(\phi\overline{\phi}-1) = 0 \tag{16}$$

when $N \geq 0$ and

$$D_A\phi - i * D_A\phi = 0, \tag{17}$$

$$*F + \frac{1}{2}(\phi\overline{\phi}-1) = 0 \tag{18}$$

when $N \leq 0$.

From the equivalence theorem of Taubes [5] it follows, that any solution of system of equations (12),(13) must by either the solutions of (15),(16) with boundary condition (14) if $N \geq 0$ or the solution (17),(18) if $N \leq 0$.

Taubes [6] also proved that the solutions to the Ginzburg-Landau equations (with coupling constant $\lambda = 1$) are uniquely determined by a set of N not necessarily distinct points in the plane corresponding to the zeros of the Higgs field. Every set of N points determines one such solution. The vortex number of this solution is N. The solution manifold of the Ginzburg-Landau equations with a vortex number N is isomorphic to $\mathbf{C}^{2N}$.

Remark 2 : The explicit solutions of equation (15), (16) do nor exist expect some particular cases (see [9], [10]). Thus the problem is description of solutions space of this equation modulo gauge equivalence. We say that (A_1, ϕ_1) and (A_2, ϕ_2) are gauge equivalent, if $A_2 = A_1 + d\theta$, $\phi_2 = e^{i\theta}\phi_1$, for some real function θ on $\mathbf{C}$.

We prove

Theorem 2 : *The solutions of (15),(16) are first kind pseudo analytic functions with zeros at given N points. The solutions do not depend on location of zeros of solutions.*

Proof: The analysis of the first part of theorem is given in [8] using the following notation and identifications. If $A = \alpha dz + \overline{\alpha dz}$, then

$$D_A\phi = \left(\left(\frac{\partial}{\partial z} - i\alpha\right)\phi\right)dz + \left(\left(\frac{\partial}{\partial \overline{z}} - i\overline{\alpha}\right)\phi\right)d\overline{z} = 0$$

and

$$D_A\phi - i * D_A\phi = 2\left(\left(\frac{\partial}{\partial \overline{z}} - i\overline{\alpha}\right)\phi\right)d\overline{z} = 0.$$

It is easy to show, that the real part of the last expression is equation (15). Therefore we obtain the generalized Caushy-Riemann equation and using similarity principle from the theory of generalized analytic functions we conclude that the analytic multiplier of solution is a polynomial respect to z with zeros at given points (see the details [8]). The second part of the theorem follows from the procedure of reducing equation (16) to the Liuville equation of type (1). For this introduce the complex valued function $w = u + iv$ and suppose $\phi = e^w$. Then

$$A_1 = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}, \quad A_2 = -\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}, \quad F_{12} = \Delta f_1.$$

The boundary condition (6) (or (14)) gives asymptotic condition for u: $\lim_{|z|\to\infty} u(z) = 0$. The equation (16) may be rewritten as

$$\Delta u - \frac{1}{2}(e^{2u} - 1) = 0.$$

Let g be a metric on $X_N = S^2 - \{z_1, ..., z_N\}$ induced from his conformal structure. If u is solution of the last equation, then as mentioned above, $e^u g$ have the same Gauss curvature as g. On the other hand, conformal structure on X_N is defined by Beltrami equation, corresponding to equation (15) (as equation which produce second kind pseudo analytic functions, (see [8]). The Beltrami coefficient in this case is equal to 0 and therefore turn out Cauchy-Riemann equation. It means, that the solutions of (15) are such pseudo analytic functions of first kind with polynomial multiplier which do not depend on the complex structure of X_N. □

Remark 3: The pseudo-analytic functions of the first and second kind are well defined on Riemann surfaces [8], thus results above may be extended to Riemann surfaces of genus $g \geq 0$ [11], [12].

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