Bulletin of TICMI
Vol. 21, No. 2, 2017, 65–79

Investigation of the Three-dimensional Boundary Value Problem for Thermoelastic Piezoelectric Solids

Gia Avalishvili[a*], Mariam Avalishvili[b], and Wofgang H. Müller[c]

[a]*Faculty of Exact and Natural Sciences, I. Javakhishvili Tbilisi State University 3 I. Tchavtchavadze Ave., 0179, Tbilisi, Georgia*;
[b]*School of Informatics, Engineering and Mathematics, University of Georgia 77a M. Kostava Str., 0175, Tbilisi, Georgia*;
[c]*Institute of Mechanics, LKM, Technische Universität Berlin, 5 Einsteinufer D-10587, Berlin, Germany*

(*Received September 4, 2017; Revised October 10, 2017; Accepted November 20, 2017*)

In this paper we consider static three-dimensional model of elastic body consisting of inhomogeneous anisotropic thermoelastic piezoelectric material with regard to magnetic field with continuous or piecewise continuous characteristics. General boundary value problem corresponding to the static model is studied, where on certain parts of the boundary displacement, electric and magnetic potentials, and temperature vanish, and on the corresponding remaining parts components of stress-vector, electric displacement and magnetic induction, and heat flux along the outward normal vector of the boundary are given. The variational formulation of the boundary value problem is obtained, which is equivalent to the original differential formulation of three-dimensional boundary value problem in the spaces of smooth enough functions. On the basis of the variational formulation existence, uniqueness and continuous dependence of solution on the given data is proved in suitable factor spaces of Sobolev spaces.

Keywords: Piezoelectric thermoelastic solids, Boundary value problem, Variational formulation, Existence and uniqueness of solution, Sobolev spaces.

AMS Subject Classification: 35J50, 35J57, 35Q74, 74F05, 74F15, 74G30.

1. Introduction

The modern approach for construction of sensors and actuators for control of various engineering structures is based on application of adaptive materials with specific properties, which enable to change their shape or material characteristics, and thereby avoiding the problems of mechanical actuators and sensors. Adaptive materials are integrated with the structure and replace complex mechanical linkages and joints, resulting in essential reduction of weight and structure complexity. Piezoelectric materials are currently widely used and intensively investigated for possible application as adaptive materials, because they can be easily embedded into existing structure and controlled by voltage, they have low weight, and low power requirements, low-field linearity and high bandwidth.

After discovery of the direct piezoelectric effect by Jacques and Pierre Curie, and theoretical prediction of the converse effect by G. Lippmann, which was con-

*Corresponding author. Email: gavalish@yahoo.com

ISSN: 1512-0082 print

firmed experimentally by the Curies, W. Voigt [12] developed the first rigorous theoretical model of piezoelectricity, which describes the interaction between elastic, electric and thermal properties of the elastic body. The first technically relevant application of the piezoelectric effect was developed by P. Langevin, who constructed a piezoelectric ultrasonic transducer assembling piezoelectric crystals. Subsequently, W. Cady [1] treated the physical properties of piezoelectric crystals as well as their practical applications. H. Tiersten [11] studied problems of vibration of piezoelectric plates. The widespread use of adaptive materials in diverse engineering construction, in particular, in aerospace industry, where sensors and actuators might undergo high thermal as well as mechanical stresses, has activated researches on thermal along with the mechanical and electro-magnetic properties of materials. A three-dimensional model of piezoelectric body taking into account thermal properties of the constituting material was derived by R. Mindlin [7] on the basis of variational principle. Further, W. Nowacki [10] developed some general theorems for thermoelastic piezoelectric materials. R. Dhaliwal and J. Wang [4] proved uniqueness theorem for linear three-dimensional model of the theory of thermo-piezoelectricity, which was generalized by J. Li in the paper [5], where a generalization of the reciprocity theorem of Nowacki [9] was also obtained. Applying the potential method and the theory of integral equations D. Natroshvili [8] studied problems of pseudo-oscillations with basic and crack type boundary conditions.

It should be pointed out that three-dimensional boundary value problems with general mixed boundary conditions for displacement, electric and magnetic potentials, and temperature corresponding to the linear static models for inhomogeneous anisotropic thermoelastic piezoelectric bodies with regard to the magnetic field have not been well investigated. The well-posedness results are mainly obtained for elastic bodies consisting of homogeneous materials. In the present paper, we investigate well-posedness of the linear three-dimensional boundary value problem with general mixed boundary conditions, provided that on certain parts of the boundary surface force and components of electric displacement, magnetic induction, and heat flux along the outward normal vector are prescribed, and on the remaining parts displacement, electric and magnetic potentials, and temperature vanish. We obtain new existence, uniqueness, and continuous dependence results in the corresponding factor spaces of Sobolev spaces.

In Section 2, we consider a differential and variational formulation of the boundary value problem corresponding to the linear static three-dimensional model for the inhomogeneous anisotropic thermoelastic piezoelectric body with regard to the magnetic field. More precisely, in Subsection 2.1 we give the differential formulation of the boundary value problem and in Subsection 2.2 we obtain integral equations, which are equivalent to the original problem in spaces of smooth enough functions, and on the basis of these integral equations we present a variational formulation of the three-dimensional problem in corresponding Sobolev spaces. In Section 3 we investigate the existence and uniqueness of solution of the boundary value problem. We study the structure of the set of solutions of the homogeneous boundary value problem and obtain the well-posedness results for the boundary value problem in suitable factor spaces of the corresponding Sobolev spaces.

2. Three-dimensional boundary value problem

2.1. *Differential formulation*

Let us consider a multilayer thermoelastic piezoelectric body with initial configuration $\overline{\Omega} = \bigcup_{k=1}^{K} \overline{\Omega_k}$, where each subdomain $\overline{\Omega_k}$, $k = 1, ..., K$, consists of a general inhomogeneous anisotropic material. The static linear three-dimensional model of the stress-strain state of thermoelastic piezoelectric body Ω with regard to the magnetic field is given by the following system [5], [8] of partial differential equaions:

$$-\sum_{j=1}^{3} \frac{\partial \sigma_{ij}^k}{\partial x_j} = f_i^k \qquad \text{in } \ \Omega_k, i = 1, 2, 3, \tag{1}$$

$$\sum_{i=1}^{3} \frac{\partial D_i^k}{\partial x_i} = f^{\varepsilon,k} \qquad \text{in } \ \Omega_k, \tag{2}$$

$$\sum_{i=1}^{3} \frac{\partial B_i^k}{\partial x_i} = 0 \qquad \text{in } \ \Omega_k, \tag{3}$$

$$-\sum_{i,j=1}^{3} \frac{\partial}{\partial x_i} \left(\eta_{ij}^k \frac{\partial \theta^k}{\partial x_j} \right) = f^{\theta,k} \qquad \text{in } \ \Omega_k, \tag{4}$$

where $k = 1, ..., K$, $\boldsymbol{f}^k = (f_i^k)_{i=1}^3 : \Omega_k \to \mathbb{R}^3$ is the density of the applied body force, $(\sigma_{ij}^k)_{i,j=1}^3$ is the mechanical stress tensor in the subdomain Ω_k, which is given by the following linear constitutive equations for thermoelastic piezoelectric solid:

$$\sigma_{ij}^k = \sum_{p,q=1}^{3} c_{ijpq}^k e_{pq}(\boldsymbol{u}^k) + \sum_{p=1}^{3} \varepsilon_{pij}^k \frac{\partial \varphi^k}{\partial x_p} + \sum_{p=1}^{3} b_{pij}^k \frac{\partial \psi^k}{\partial x_p} - \lambda_{ij}^k \theta^k, \qquad i, j = 1, 2, 3, \tag{5}$$

where $\boldsymbol{u}^k = (u_i^k)_{i=1}^3 : \overline{\Omega_k} \to \mathbb{R}^3$ is the displacement vector-function, $\varphi^k : \overline{\Omega_k} \to \mathbb{R}$ and $\psi^k : \overline{\Omega_k} \to \mathbb{R}$ stand for the electric and magnetic potentials such that electric and magnetic fields are $\boldsymbol{E}^k = -grad\,\varphi^k$ and $\boldsymbol{H}^k = -grad\,\psi^k$, $\theta^k : \overline{\Omega_k} \to \mathbb{R}$ is the temperature distribution, $e_{ij}(\boldsymbol{v}) = 1/2\,(\partial v_i/\partial x_j + \partial v_j/\partial x_i)$, $i, j = 1, 2, 3$, $\boldsymbol{v} = (v_i)_{i=1}^3$, is the strain tensor, $(c_{ijpq}^k)_{i,j,p,q=1}^3$ is the elasticity tensor, $(\varepsilon_{pij}^k)_{i,j,p=1}^3$ are piezoelectric and $(b_{pij}^k)_{i,j,p=1}^3$ are piezomagnetic coefficients, $(\lambda_{ij}^k)_{i,j=1}^3$ is the stress-temperature tensor. $\boldsymbol{D}^k = (D_j^k)_{j=1}^3$ is the electric displacement vector and $\boldsymbol{B} = (B_j)_{j=1}^3$ is the magnetic induction vector, which are given by the following constitutive equations:

$$D_i^k = \sum_{p,q=1}^{3} \varepsilon_{ipq}^k e_{pq}(\boldsymbol{u}^k) - \sum_{j=1}^{3} d_{ij}^k \frac{\partial \varphi^k}{\partial x_j} - \sum_{j=1}^{3} a_{ij}^k \frac{\partial \psi^k}{\partial x_j} + \mu_i^k \theta^k, \quad i = 1, 2, 3, \tag{6}$$

$$B_i^k = \sum_{p,q=1}^{3} b_{ipq}^k e_{pq}(\boldsymbol{u}^k) - \sum_{j=1}^{3} a_{ij}^k \frac{\partial \varphi}{\partial x_j} - \sum_{j=1}^{3} \zeta_{ij}^k \frac{\partial \psi}{\partial x_j} + m_i^k \theta^k, \quad i = 1, 2, 3, \tag{7}$$

where $(d_{ij}^k)_{i,j=1}^3$ and $(\zeta_{ij}^k)_{i,j=1}^3$ are the permittivity and permeability tensors, $(a_{ij}^k)_{i,j=1}^3$ are the coupling coefficients connecting electric and magnetic fields, $(\mu_i^k)_{i=1}^3$ and $(m_i^k)_{i=1}^3$ are coefficients characterizing the relation between thermal, electric and magnetic fields, $f^{\varepsilon,k}$ is the density of electric charges. $(\eta_{ij}^k)_{i,j=1}^3$ is the thermal conductivity tensor and $f^{\theta,k}$ is the density of heat sources.

We assume that the thermoelastic piezoelectric body Ω is clamped along a part $\Gamma_0 \subset \Gamma = \partial\Omega$ of the Lipschitz boundary $\Gamma = \partial\Omega$, and on the remaining part $\Gamma_1 = \Gamma\backslash\overline{\Gamma_0}$ applied surface force with density $\boldsymbol{g} = (g_i) : \Gamma_1 \to \mathbb{R}^3$ is given, where $\partial\Omega = \Gamma_0 \cup \Gamma_{01} \cup \Gamma_1$, $\Gamma_0 \cap \Gamma_1 = \emptyset$, is a Lipschitz dissection [6] of $\partial\Omega$:

$$\boldsymbol{u}^k = \mathbf{0} \ \text{ on } \Gamma_{0,k} = \Gamma_0 \cap \partial\Omega_k, \quad \sum_{j=1}^3 \sigma_{ij}^k n_j^k = g_i \ \text{ on } \Gamma_{1,k} = \partial\Omega_k\backslash\overline{\Gamma_{0,k}}, \ i = 1,2,3, \tag{8}$$

where $\boldsymbol{n}^k = (n_i^k)_{i=1}^3$ is the unit outward normal vector to $\Gamma_{1,k}$. Along a part $\Gamma_0^\varphi \subset \Gamma = \partial\Omega$ of the boundary the electric potential vanishes and on the remaining part $\Gamma_1^\varphi = \Gamma\backslash\overline{\Gamma_0^\varphi}$ the normal component of the electric displacement with density $g^\varphi : \Gamma_1^\varphi \to \mathbb{R}$ is given, where $\partial\Omega = \Gamma_0^\varphi \cup \Gamma_{01}^\varphi \cup \Gamma_1^\varphi$, $\Gamma_0^\varphi \cap \Gamma_1^\varphi = \emptyset$, is a Lipschitz dissection of $\partial\Omega$:

$$\varphi^k = 0 \ \text{ on } \ \Gamma_{0,k}^\varphi = \Gamma_0^\varphi \cap \partial\Omega_k, \quad \sum_{i=1}^3 D_i^k n_i^k = g^\varphi \ \text{ on } \ \Gamma_{1,k}^\varphi = \partial\Omega_k\backslash\overline{\Gamma_{0,k}^\varphi}, \tag{9}$$

where $\boldsymbol{n}^k = (n_i^k)_{i=1}^3$ is the unit outward normal vector to $\Gamma_{1,k}^\varphi$. Along a part $\Gamma_0^\psi \subset \Gamma = \partial\Omega$ magnetic potential vanishes and on the remaining part $\Gamma_1^\psi = \Gamma\backslash\overline{\Gamma_0^\psi}$ the normal component of the magnetic induction with density $g^\psi : \Gamma_1^\psi \to \mathbb{R}$ is given, where $\partial\Omega = \Gamma_0^\psi \cup \Gamma_{01}^\psi \cup \Gamma_1^\psi$, $\Gamma_0^\psi \cap \Gamma_1^\psi = \emptyset$, is a Lipschitz dissection of $\partial\Omega$:

$$\psi^k = 0 \quad \text{on } \Gamma_{0,k}^\psi = \Gamma_0^\psi \cap \partial\Omega_k, \quad \sum_{i=1}^3 B_i^k n_i^k = g^\psi \quad \text{on } \ \Gamma_{1,k}^\psi = \partial\Omega_k\backslash\overline{\Gamma_{0,k}^\psi}, \tag{10}$$

where $\boldsymbol{n}^k = (n_i^k)_{i=1}^3$ is the unit outward normal vector to $\Gamma_{1,k}^\psi$. The temperature vanishes along a part $\Gamma_0^\theta \subset \Gamma = \partial\Omega$ of the boundary and heat flux along the outward normal of Γ with density $g^\theta : \Gamma_1^\theta \to \mathbb{R}$ is given on $\Gamma_1^\theta = \Gamma\backslash\overline{\Gamma_0^\theta}$, where $\partial\Omega = \Gamma_0^\theta \cup \Gamma_{01}^\theta \cup \Gamma_1^\theta$, $\Gamma_0^\theta \cap \Gamma_1^\theta = \emptyset$, is a Lipschitz dissection of $\partial\Omega$:

$$\theta^k = 0 \quad \text{on } \Gamma_{0,k}^\theta = \Gamma_0^\theta \cap \partial\Omega_k, \quad -\sum_{i,j=1}^3 \eta_{ij}^k \frac{\partial\theta^k}{\partial x_j} n_i^k = g^\theta \quad \text{on } \Gamma_{1,k}^\theta = \partial\Omega_k\backslash\overline{\Gamma_{0,k}^\theta}, \tag{11}$$

where $\boldsymbol{n}^k = (n_i^k)_{i=1}^3$ is the unit outward normal vector to $\Gamma_{1,k}^\theta$. Since Ω consists of several subdomains, on the common interfaces $\partial\Omega_k \cap \partial\Omega_{\overline{k}}$, $k, \overline{k} = 1, ..., K$, of the subdomains Ω_k and $\Omega_{\overline{k}}$ special transmission conditions should be satisfied. We consider the so-called rigid contact conditions, where the displacement and stress vectors, temperature, electric and magnetic potentials, and normal components of the heat flux, electric displacement and magnetic induction are continuous:

$$\boldsymbol{u}^k = \boldsymbol{u}^{\overline{k}}, \quad \sum_{j=1}^{3} \sigma_{ij}^k n_j = \sum_{j=1}^{3} \sigma_{ij}^{\overline{k}} n_j \quad \text{on} \quad \partial\Omega_k \cap \partial\Omega_{\overline{k}}, \ \ i = 1, 2, 3, \tag{12}$$

$$\varphi^k = \varphi^{\overline{k}}, \quad \sum_{i=1}^{3} D_i^k n_i = \sum_{i=1}^{3} D_i^{\overline{k}} n_i \quad \text{on} \quad \partial\Omega_k \cap \partial\Omega_{\overline{k}}, \tag{13}$$

$$\psi^k = \psi^{\overline{k}}, \quad \sum_{i=1}^{3} B_i^k n_i = \sum_{i=1}^{3} B_i^{\overline{k}} n_i \quad \text{on} \quad \partial\Omega_k \cap \partial\Omega_{\overline{k}}, \tag{14}$$

$$\theta^k = \theta^{\overline{k}}, \quad \sum_{i,j=1}^{3} \eta_{ij}^k \frac{\partial \theta^k}{\partial x_j} n_i = \sum_{i,j=1}^{3} \eta_{ij}^{\overline{k}} \frac{\partial \theta^{\overline{k}}}{\partial x_j} n_i \quad \text{on} \quad \partial\Omega_k \cap \partial\Omega_{\overline{k}}, \tag{15}$$

where $\boldsymbol{n} = (n_i)_{i=1}^3$ is the unit normal vector of $\partial\Omega_k \cap \partial\Omega_{\overline{k}}$, $k, \overline{k} = 1, ..., K$.

2.2. *Variational formulation*

In order to investigate the three-dimensional boundary value problem (1)-(15) let us obtain variational formulation in corresponding Sobolev spaces. Throughout this article for each real $s \geq 0$ we denote by $H^s(\Omega)$ and $H^s(\widetilde{\Gamma})$ the Sobolev spaces of functions based on $H^0(\Omega) = L^2(\Omega)$ and $H^0(\widetilde{\Gamma}) = L^2(\widetilde{\Gamma})$, respectively, and $tr_{\widetilde{\Gamma}} : H^1(\Omega) \to H^{1/2}(\widetilde{\Gamma})$ are the trace operators, where $\Omega \subset \mathbb{R}^n$, $n \in \mathbb{N}$, is a bounded Lipschitz domain and $\widetilde{\Gamma}$ is an element of a Lipschitz dissection of the boundary $\Gamma = \partial\Omega$ [6]. $H_0^s(\Omega)$ denotes the closure of the set $\mathfrak{D}(\Omega)$ of infinitely differentiable functions with compact support in Ω in the space $H^s(\Omega)$. We denote the corresponding spaces of vector-valued functions by $\mathbf{H}^s(\Omega) = [H^s(\Omega)]^3$, $\mathbf{H}_0^s(\Omega) = [H_0^s(\Omega)]^3$, $\mathbf{H}^s(\widetilde{\Gamma}) = [H^s(\widetilde{\Gamma})]^3$, $s \geq 0$, $\mathbf{L}^{s_1}(\widetilde{\Gamma}) = [L^{s_1}(\widetilde{\Gamma})]^3$, $s_1 \geq 1$, and by $\boldsymbol{tr}_{\widetilde{\Gamma}} : \mathbf{H}^1(\Omega) \to \mathbf{H}^{1/2}(\widetilde{\Gamma})$. Hereafter, we use c_1, c_2 to denote generic constants that are independent of the main parameters involved, but whose values may differ from line to line and may change even within a single chain of estimates.

We assume that the elasticity tensors $(c_{ijpq}^k)_{i,j,p,q=1}^3$, are symmetric

$$c_{ijpq}^k = c_{ijqp}^k = c_{jipq}^k, \quad i, j, p, q = 1, 2, 3, \quad k = 1, ..., K; \tag{16}$$

tensors $(\varepsilon_{pij}^k)_{i,j,p=1}^3$ and $(b_{pij}^k)_{i,j,p=1}^3$, consisting of piezoelectric and piezomagnetic coefficients are symmetric with respect to the second and third indices

$$\varepsilon_{pij}^k = \varepsilon_{pji}^k, \quad b_{pij}^k = b_{pji}^k, \quad i, j, p = 1, 2, 3, \quad k = 1, ..., K; \tag{17}$$

the stress-temperature tensors $(\lambda_{ij}^k)_{i,j=1}^3$, are symmetric

$$\lambda_{ij}^k = \lambda_{ji}^k, \quad i, j = 1, 2, 3, \quad k = 1, ..., K. \tag{18}$$

If $\boldsymbol{u}^k = (u_i^k)_{i=1}^3 : \overline{\Omega_k} \to \mathbb{R}^3$, $\varphi^k : \overline{\Omega_k} \to \mathbb{R}$, $\psi^k : \overline{\Omega_k} \to \mathbb{R}$, and $\theta^k : \overline{\Omega_k} \to \mathbb{R}$, $k = 1, ..., K$, are smooth enough, then by multiplying the equations (1) by arbitrary continuously differentiable functions $v_i^k : \overline{\Omega_k} \to \mathbb{R}$ $(i = 1, 2, 3)$, which vanish on $\Gamma_{0,k}$ and $v_i^k = v_i^{\overline{k}}$ on $\partial\Omega_k \cap \partial\Omega_{\overline{k}}$, equation (2) by a continuously differentiable function $\overline{\varphi}^k : \overline{\Omega_k} \to \mathbb{R}$, such that $\overline{\varphi}^k = 0$ on $\Gamma^{\varphi}_{0,k}$ and $\overline{\varphi}^k = \overline{\varphi}^{\overline{k}}$ on $\partial\Omega_k \cap \partial\Omega_{\overline{k}}$, the equation (3) by a continuously differentiable function $\overline{\psi}^k : \overline{\Omega_k} \to \mathbb{R}$, which vanishes on $\Gamma^{\psi}_{0,k}$ and $\overline{\psi}^k = \overline{\psi}^{\overline{k}}$ on $\partial\Omega_k \cap \partial\Omega_{\overline{k}}$, and equation (4) by a continuously differentiable function $\overline{\theta}^k : \overline{\Omega_k} \to \mathbb{R}$, such that $\overline{\theta}^k = 0$ on $\Gamma^{\theta}_{0,k}$ and $\overline{\theta}^k = \overline{\theta}^{\overline{k}}$ on $\partial\Omega_k \cap \partial\Omega_{\overline{k}}$, $k, \overline{k} = 1, ..., K$, by integrating on Ω_k, using Green's formula, and taking into account constitutive equations (5)-(7) and symmetry conditions (16)-(18) we obtain the following integral equations:

$$-\int\limits_{\partial\Omega_k} \sum_{i,j=1}^{3} \sigma^k_{ij} n^k_j v^k_i d\Gamma + \int\limits_{\Omega_k} \left(\sum_{i,j,p,q=1}^{3} c^k_{ijpq} e_{pq}(u^k) e_{ij}(\boldsymbol{v}^k) + \sum_{i,j,p=1}^{3} \varepsilon^k_{pij} \frac{\partial \varphi^k}{\partial x_p} e_{ij}(\boldsymbol{v}^k) \right) dx$$

$$+\int\limits_{\Omega_k} \sum_{i,j,p=1}^{3} b^k_{pij} \frac{\partial \psi^k}{\partial x_p} e_{ij}(\boldsymbol{v}^k) dx - \int\limits_{\Omega_k} \sum_{i,j=1}^{3} \lambda^k_{ij} \theta^k e_{ij}(\boldsymbol{v}^k) dx = \int\limits_{\Omega_k} \sum_{i=1}^{3} f^k_i v^k_i dx, \tag{19}$$

$$\int\limits_{\partial\Omega_k} \sum_{i=1}^{3} D^k_i n^k_i \overline{\varphi}^k d\Gamma - \int\limits_{\Omega_k} \sum_{i,j,p=1}^{3} \varepsilon^k_{ipq} e_{pq}(\boldsymbol{u}^k) \frac{\partial \overline{\varphi}^k}{\partial x_i} dx + \int\limits_{\Omega_k} \sum_{i,j=1}^{3} d^k_{ij} \frac{\partial \varphi^k}{\partial x_j} \frac{\partial \overline{\varphi}^k}{\partial x_i} dx$$

$$+\int\limits_{\Omega_k} \sum_{i,j=1}^{3} a^k_{ij} \frac{\partial \psi^k}{\partial x_j} \frac{\partial \overline{\varphi}^k}{\partial x_i} dx - \int\limits_{\Omega_k} \sum_{i=1}^{3} \mu^k_i \theta^k \frac{\partial \overline{\varphi}^k}{\partial x_i} dx = \int\limits_{\Omega_k} f^{\varepsilon,k} \overline{\varphi}^k dx, \tag{20}$$

$$\int\limits_{\partial\Omega_k} \sum_{i=1}^{3} B^k_i n^k_i \overline{\psi}^k d\Gamma - \int\limits_{\Omega_k} \sum_{i,j,p=1}^{3} b^k_{ipq} e_{pq}(\boldsymbol{u}^k) \frac{\partial \overline{\psi}^k}{\partial x_i} dx + \int\limits_{\Omega_k} \sum_{i,j=1}^{3} a^k_{ij} \frac{\partial \varphi^k}{\partial x_j} \frac{\partial \overline{\psi}^k}{\partial x_i} dx$$

$$+\int\limits_{\Omega_k} \sum_{i,j=1}^{3} \zeta^k_{ij} \frac{\partial \psi^k}{\partial x_j} \frac{\partial \overline{\psi}^k}{\partial x_i} dx - \int\limits_{\Omega_k} \sum_{i=1}^{3} m^k_i \theta^k \frac{\partial \overline{\psi}^k}{\partial x_i} dx = 0, \tag{21}$$

$$-\int\limits_{\partial\Omega_k} \sum_{i,j=1}^{3} \eta^k_{ij} \frac{\partial \theta^k}{\partial x_j} n^k_i \overline{\theta}^k d\Gamma + \int\limits_{\Omega_k} \sum_{i,j=1}^{3} \eta^k_{ij} \frac{\partial \theta^k}{\partial x_j} \frac{\partial \overline{\theta}^k}{\partial x_i} dx = \int\limits_{\Omega_k} f^{\theta,k} \overline{\theta} dx, \tag{22}$$

where $\boldsymbol{n}^k = (n^k_i)^3_{i=1}$ is the unit outward normal vector to $\partial\Omega_k$. On the common interfaces $\partial\Omega_k \cap \partial\Omega_{\overline{k}}$, we have $v^k_i = v^{\overline{k}}_i$, $\overline{\varphi}^k = \overline{\varphi}^{\overline{k}}$, $\overline{\psi}^k = \overline{\psi}^{\overline{k}}$, $\overline{\theta}^k = \overline{\theta}^{\overline{k}}$ and $\boldsymbol{n}^k = -\boldsymbol{n}^{\overline{k}}$, $k, \overline{k} = 1, ..., K$, and therefore from the rigid contact conditions (12)-(15) after summation of equations (19)-(22) with respect to $k = 1, ..., K$, and taking into account boundary conditions (8)-(11) we obtain:

$$\sum_{k=1}^{K} \int\limits_{\Omega_k} \sum_{i,j,p,q=1}^{3} c^k_{ijpq} e_{pq}(\boldsymbol{u}^k) e_{ij}(\boldsymbol{v}^k) dx + \sum_{k=1}^{K} \int\limits_{\Omega_k} \sum_{i,j,p=1}^{3} \varepsilon^k_{pij} \frac{\partial \varphi^k}{\partial x_p} e_{ij}(\boldsymbol{v}^k) dx$$

$$+\sum_{k=1}^{K} \int\limits_{\Omega_k} \sum_{i,j,p=1}^{3} b^k_{pij} \frac{\partial \psi^k}{\partial x_p} e_{ij}(\boldsymbol{v}^k) dx - \sum_{k=1}^{K} \int\limits_{\Omega_k} \sum_{i,j=1}^{3} \lambda^k_{ij} \theta^k e_{ij}(\boldsymbol{v}^k) dx$$

$$= \int\limits_{\Omega} \sum_{i=1}^{3} f_i v_i dx + \int\limits_{\Gamma_1} \sum_{i=1}^{3} g_i v_i d\Gamma, \tag{23}$$

$$\sum_{k=1}^{K} \left(-\int\limits_{\Omega_k} \sum_{i,j,p=1}^{3} \varepsilon_{ipq}^k e_{pq}(\boldsymbol{u}^k) \frac{\partial \overline{\varphi}^k}{\partial x_i} dx + \int\limits_{\Omega_k} \sum_{i,j=1}^{3} \left(d_{ij}^k \frac{\partial \varphi^k}{\partial x_j} \frac{\partial \overline{\varphi}^k}{\partial x_i} + a_{ij}^k \frac{\partial \psi^k}{\partial x_j} \frac{\partial \overline{\varphi}^k}{\partial x_i} \right) dx \right)$$

$$- \sum_{k=1}^{K} \int\limits_{\Omega_k} \sum_{i=1}^{3} \mu_i^k \theta^k \frac{\partial \overline{\varphi}^k}{\partial x_i} dx = \int\limits_{\Omega} f^{\varepsilon} \overline{\varphi} dx - \int\limits_{\Gamma_1^{\varphi}} g^{\varphi} \overline{\varphi} d\Gamma, \tag{24}$$

$$- \sum_{k=1}^{K} \int\limits_{\Omega_k} \sum_{i,j,p=1}^{3} b_{ipq}^k e_{pq}(\boldsymbol{u}^k) \frac{\partial \overline{\psi}^k}{\partial x_i} dx + \sum_{k=1}^{K} \int\limits_{\Omega_k} \sum_{i,j=1}^{3} a_{ij}^k \frac{\partial \varphi^k}{\partial x_j} \frac{\partial \overline{\psi}^k}{\partial x_i} dx$$

$$+ \sum_{k=1}^{K} \int\limits_{\Omega_k} \sum_{i,j=1}^{3} \zeta_{ij}^k \frac{\partial \psi^k}{\partial x_j} \frac{\partial \overline{\psi}^k}{\partial x_i} dx - \sum_{k=1}^{K} \int\limits_{\Omega_k} \sum_{i=1}^{3} m_i^k \theta^k \frac{\partial \overline{\psi}^k}{\partial x_i} dx = - \int\limits_{\Gamma_1^{\psi}} g^{\psi} \overline{\psi} d\Gamma, \tag{25}$$

$$\sum_{k=1}^{K} \int\limits_{\Omega_k} \sum_{i,j=1}^{3} \eta_{ij}^k \frac{\partial \theta^k}{\partial x_j} \frac{\partial \overline{\theta}^k}{\partial x_i} dx = \int\limits_{\Omega} f^{\theta} \overline{\theta} dx - \int\limits_{\Gamma_1^{\theta}} g^{\theta} \overline{\theta} d\Gamma, \tag{26}$$

where $f_i = f_i^k$, $f^{\varepsilon} = f^{\varepsilon,k}$, $f^{\theta} = f^{\theta,k}$ in Ω_k, $\boldsymbol{v} = \boldsymbol{v}^k$, $\overline{\varphi} = \overline{\varphi}^k$, $\overline{\psi} = \overline{\psi}^k$, $\overline{\theta} = \overline{\theta}^k$ on $\overline{\Omega_k}$, $k = 1, ..., K$.

Therefore, if $\boldsymbol{u}^k = (u_i^k)_{i=1}^3 : \overline{\Omega_k} \to \mathbb{R}^3$, $\varphi^k : \overline{\Omega_k} \to \mathbb{R}$, $\psi^k : \overline{\Omega_k} \to \mathbb{R}$, and $\theta^k : \overline{\Omega_k} \to \mathbb{R}$, $k = 1, ..., K$, are solutions of equations (1)-(4) and satisfy boundary conditions (8)-(11), and rigid contact conditions (12)-(15), then $\boldsymbol{u}^k$, φ^k, ψ^k and θ^k are solutions of equations (23)-(26). Conversely, if $\boldsymbol{u}^k$, φ^k, ψ^k and θ^k are twice continuously differentiable solutions of integral equations (23)-(26), then by using Green's formula we have:

$$\sum_{k=1}^{K} \int\limits_{\partial\Omega_k} \sum_{i,j=1}^{3} \sigma_{ij}^k n_j^k v_i^k d\Gamma - \sum_{k=1}^{K} \int\limits_{\Omega_k} \sum_{j=1}^{3} \frac{\partial}{\partial x_j} \left(\sum_{p,q=1}^{3} c_{ijpq}^k e_{pq}(\boldsymbol{u}^k) + \sum_{p=1}^{3} \varepsilon_{pij}^k \frac{\partial \varphi^k}{\partial x_p} \right.$$

$$\left. + \sum_{p=1}^{3} b_{pij}^k \frac{\partial \psi^k}{\partial x_p} - \lambda_{ij}^k \theta^k \right) v_i^k dx = \int\limits_{\Omega} \sum_{i=1}^{3} f_i v_i dx + \int\limits_{\Gamma_1} \sum_{i=1}^{3} g_i v_i d\Gamma, \tag{27}$$

$$- \sum_{k=1}^{K} \int\limits_{\partial\Omega_k} \sum_{i=1}^{3} D_i^k n_i^k \overline{\varphi}^k d\Gamma + \sum_{k=1}^{K} \int\limits_{\Omega_k} \sum_{i=1}^{3} \frac{\partial}{\partial x_i} \left(\sum_{p,q=1}^{3} \varepsilon_{ipq}^k e_{pq}(\boldsymbol{u}^k) - \sum_{j=1}^{3} d_{ij}^k \frac{\partial \varphi^k}{\partial x_j} \right.$$

$$\left. - \sum_{j=1}^{3} a_{ij}^k \frac{\partial \psi^k}{\partial x_j} + \mu_i^k \theta^k \right) \overline{\varphi}^k dx = \int\limits_{\Omega} f^{\varepsilon} \overline{\varphi} dx - \int\limits_{\Gamma_1^{\varphi}} g^{\varphi} \overline{\varphi} d\Gamma, \tag{28}$$

$$- \sum_{k=1}^{K} \int\limits_{\partial\Omega_k} \sum_{i=1}^{3} B_i^k n_i^k \overline{\psi}^k d\Gamma + \sum_{k=1}^{K} \int\limits_{\Omega_k} \sum_{i=1}^{3} \frac{\partial}{\partial x_i} \left(\sum_{p,q=1}^{3} b_{ipq}^k e_{pq}(\boldsymbol{u}^k) - \sum_{j=1}^{3} a_{ij}^k \frac{\partial \varphi}{\partial x_j} \right.$$

$$
\left. -\sum_{j=1}^{3} \zeta_{ij}^{k} \frac{\partial \psi}{\partial x_j} + m_i^k \theta^k \right) \overline{\psi}^k dx = -\int\limits_{\Gamma_1^{\psi}} g^{\psi} \overline{\psi} d\Gamma, \tag{29}
$$

$$
\sum_{k=1}^{K} \int\limits_{\partial\Omega_k} \sum_{i,j=1}^{3} \eta_{ij}^{k} \frac{\partial \theta^k}{\partial x_j} n_i^k \overline{\theta}^k d\Gamma - \sum_{k=1}^{K} \int\limits_{\Omega_k} \sum_{i,j=1}^{3} \frac{\partial}{\partial x_i} \left(\eta_{ij}^{k} \frac{\partial \theta^k}{\partial x_j} \right) \overline{\theta}^k dx
$$

$$
= \int\limits_{\Omega} f^{\theta} \overline{\theta} dx - \int\limits_{\Gamma_1^{\theta}} g^{\theta} \overline{\theta} d\Gamma, \tag{30}
$$

where $\boldsymbol{v}^k = (v_i^k)_{i=1}^3$, $\overline{\varphi}^k$, $\overline{\psi}^k$, $\overline{\theta}^k$ are continuously differentiable functions on $\overline{\Omega_k}$, such that $v_i^k = 0$ on $\Gamma_{0,k}$, $\overline{\varphi}^k = 0$ on $\Gamma_{0,k}^{\varphi}$, $\overline{\psi}^k = 0$ on $\Gamma_{0,k}^{\psi}$, $\overline{\theta}^k = 0$ on $\Gamma_{0,k}^{\theta}$, and $v_i^k = v_i^{\overline{k}}$, $\overline{\varphi}^k = \overline{\varphi}^{\overline{k}}$, $\overline{\psi}^k = \overline{\psi}^{\overline{k}}$, $\overline{\theta}^k = \overline{\theta}^{\overline{k}}$ on $\partial\Omega_k \cap \partial\Omega_{\overline{k}}$, $i = 1, 2, 3,\ k, \overline{k} = 1, ..., K$. By letting $\boldsymbol{v}^k \in (C_0^1(\overline{\Omega_k}))^3$, $C_0^1(\overline{\Omega_k}) = \{v \in C^1(\overline{\Omega_k})\, | v = 0 \text{ on } \partial\Omega_k\}$, $\overline{\varphi}^k \in C_0^1(\overline{\Omega_k})$, $\overline{\psi}^k \in C_0^1(\overline{\Omega_k})$, $\overline{\theta}^k \in C_0^1(\overline{\Omega_k})$, from equations (27)-(30) we obtain:

$$
-\sum_{k=1}^{K} \int\limits_{\Omega_k} \sum_{j=1}^{3} \frac{\partial \sigma_{ij}^k}{\partial x_j} v_i^k dx = \sum_{k=1}^{K} \int\limits_{\Omega_k} \sum_{i=1}^{3} f_i^k v_i^k dx, \tag{31}
$$

$$
\sum_{k=1}^{K} \int\limits_{\Omega_k} \sum_{i=1}^{3} \frac{\partial D_i^k}{\partial x_i} \overline{\varphi}^k dx = \sum_{k=1}^{K} \int\limits_{\Omega_k} f^{\varepsilon,k} \overline{\varphi}^k dx\,, \tag{32}
$$

$$
\sum_{k=1}^{K} \int\limits_{\Omega_k} \sum_{i=1}^{3} \frac{\partial B_i^k}{\partial x_i} \overline{\psi}^k dx = 0, \tag{33}
$$

$$
-\sum_{k=1}^{K} \int\limits_{\Omega_k} \sum_{i,j=1}^{3} \frac{\partial}{\partial x_i} \left(\eta_{ij}^{k} \frac{\partial \theta^k}{\partial x_j} \right) \overline{\theta}^k dx = \sum_{k=1}^{K} \int\limits_{\Omega_k} f^{\theta,k} \overline{\theta}^k dx. \tag{34}
$$

By taking account of density of $C_0^1(\overline{\Omega})$ in $L^2(\Omega)$ from (31)-(34) we obtain that $\boldsymbol{u}^k$, φ^k, ψ^k and θ^k, $k = 1, ..., K$, satisfy equations (1)-(4). Now, if we assume that functions $\boldsymbol{v}^k$, $\overline{\varphi}^k$, $\overline{\psi}^k$ and $\overline{\theta}^k$ are arbitrary continuously differentiable functions on surfaces $\Gamma_{1,k}$, $\Gamma_{1,k}^{\varphi}$, $\Gamma_{1,k}^{\psi}$ and $\Gamma_{1,k}^{\theta}$, which vanish on the remaining parts of the boundary $\partial\Omega_k$, from equations (27)-(30) taking into account equations (1)-(4) we have:

$$
\sum_{k=1}^{K} \int\limits_{\Gamma_{1,k}} \sum_{i,j=1}^{3} \sigma_{ij}^k n_j^k v_i^k d\Gamma = \sum_{k=1}^{K} \int\limits_{\Gamma_{1,k}} \sum_{i=1}^{3} g_i v_i^k d\Gamma,
$$

$$
-\sum_{k=1}^{K} \int\limits_{\Gamma_{1,k}^{\varphi}} \sum_{i=1}^{3} D_i^k n_i^k \overline{\varphi}^k d\Gamma = -\sum_{k=1}^{K} \int\limits_{\Gamma_{1,k}^{\varphi}} g^{\varphi} \overline{\varphi}^k d\Gamma,
$$

$$-\sum_{k=1}^{K}\int_{\Gamma_{1,k}^{\psi}}\sum_{i=1}^{3}B_i^k n_i^k\overline{\psi}^k d\Gamma=-\sum_{k=1}^{K}\int_{\Gamma_{1,k}^{\psi}}g^{\psi}\overline{\psi}^k d\Gamma,$$

$$\sum_{k=1}^{K}\int_{\Gamma_{1,k}^{\theta}}\sum_{i,j=1}^{3}\eta_{ij}^k\frac{\partial\theta^k}{\partial x_j}n_i^k\overline{\theta}^k d\Gamma=-\sum_{k=1}^{K}\int_{\Gamma_{1,k}^{\theta}}g^{\theta}\overline{\theta}^k d\Gamma.$$

From the latter equations and density of the sets of continuously differentiable functions vanishing on the boundary of $\Gamma_{1,k}$, $\Gamma_{1,k}^{\varphi}$, $\Gamma_{1,k}^{\psi}$ and $\Gamma_{1,k}^{\theta}$ in spaces $L^2(\Gamma_{1,k})$, $L^2(\Gamma_{1,k}^{\varphi})$, $L^2(\Gamma_{1,k}^{\psi})$ and $L^2(\Gamma_{1,k}^{\theta})$ we infer that $\boldsymbol{u}^k$, φ^k, ψ^k and θ^k, $k=1,...,K$, satisfy boundary conditions (8)-(11). In order to obtain contact conditions we take functions $\boldsymbol{v}^k$, $\overline{\varphi}^k$, $\overline{\psi}^k$ and $\overline{\theta}^k$, which are arbitrary continuously differentiable functions on interface $\partial\Omega_k\cap\partial\Omega_{\overline{k}}$, $k,\overline{k}=1,...,K$, and vanish on the remaining part of the boundary $\partial\Omega_k$. From equations (27)-(30) taking into account equations (1)-(4) we obtain:

$$\int_{\partial\Omega_k\cap\partial\Omega_{\overline{k}}}\sum_{i,j=1}^{3}\sigma_{ij}^k n_j^k v_i^k d\Gamma+\int_{\partial\Omega_k\cap\partial\Omega_{\overline{k}}}\sum_{i,j=1}^{3}\sigma_{ij}^{\overline{k}} n_j^{\overline{k}} v_i^{\overline{k}} d\Gamma=0, \tag{35}$$

$$-\int_{\partial\Omega_k\cap\partial\Omega_{\overline{k}}}\sum_{i=1}^{3}D_i^k n_i^k\overline{\varphi}^k d\Gamma-\int_{\partial\Omega_k\cap\partial\Omega_{\overline{k}}}\sum_{i=1}^{3}D_i^{\overline{k}} n_i^{\overline{k}}\overline{\varphi}^{\overline{k}} d\Gamma=0\,, \tag{36}$$

$$-\int_{\partial\Omega_k\cap\partial\Omega_{\overline{k}}}\sum_{i=1}^{3}B_i^k n_i^k\overline{\psi}^k d\Gamma-\int_{\partial\Omega_k\cap\partial\Omega_{\overline{k}}}\sum_{i=1}^{3}B_i^{\overline{k}} n_i^{\overline{k}}\overline{\psi}^{\overline{k}} d\Gamma=0, \tag{37}$$

$$\int_{\partial\Omega_k\cap\partial\Omega_{\overline{k}}}\sum_{i,j=1}^{3}\eta_{ij}^k\frac{\partial\theta^k}{\partial x_j}n_i^k\overline{\theta}^k d\Gamma+\int_{\partial\Omega_k\cap\partial\Omega_{\overline{k}}}\sum_{i,j=1}^{3}\eta_{ij}^{\overline{k}}\frac{\partial\theta^{\overline{k}}}{\partial x_j}n_i^{\overline{k}}\overline{\theta}^{\overline{k}} d\Gamma=0. \tag{38}$$

The equations (35)-(38) and density of the set of continuously differentiable functions vanishing on the boundary of $\partial\Omega_k\cap\partial\Omega_{\overline{k}}$ in the space $L^2(\partial\Omega_k\cap\partial\Omega_{\overline{k}})$ imply that $\boldsymbol{u}^k$, φ^k, ψ^k and θ^k, $k=1,...,K$, satisfy rigid contact conditions (12)-(15).

So, the boundary value problem (1)-(15) corresponding to the static three-dimensional model of multilayer anisotropic inhomogeneous thermoelastic piezoelectric solid with regard to magnetic field is equivalent to integral equations (23)-(26) in spaces of twice continuously differentiable functions. Note that if functions v^k belong to $H^1(\Omega_k)$, $k=1,...,K$, and on the common interfaces $\partial\Omega_k\cap\partial\Omega_{\overline{k}}$ we have $tr_{\partial\Omega_k\cap\partial\Omega_{\overline{k}}}(v^k)=tr_{\partial\Omega_k\cap\partial\Omega_{\overline{k}}}(v^{\overline{k}})$, then there exists the function $v\in H^1(\Omega)$ such that $v=v^k$ in Ω_k, $k=1,...,K$. Therefore, on the basis of integral equations (23)-(26) we obtain the following variational formulation of the boundary value problem (1)-(15): Find $\boldsymbol{u}\in\boldsymbol{V}(\Omega)=\{\boldsymbol{v}\in\boldsymbol{H}^1(\Omega);\mathbf{tr}_{\Gamma}(\boldsymbol{v})=\mathbf{0}\text{ on }\Gamma_0\}$, $\varphi\in V^{\varphi}(\Omega)=\{\overline{\varphi}\in H^1(\Omega);tr_{\Gamma}(\overline{\varphi})=0\text{ on }\Gamma_0^{\varphi}\}$, $\psi\in V^{\psi}(\Omega)=\{\overline{\psi}\in H^1(\Omega);tr_{\Gamma}(\overline{\psi})=0\text{ on }\Gamma_0^{\psi}\}$, $\theta\in V^{\theta}(\Omega)=\{\overline{\theta}\in H^1(\Omega);tr_{\Gamma}(\overline{\theta})=0\text{ on }\Gamma_0^{\theta}\}$ such that

$$c(\boldsymbol{u},\boldsymbol{v})+\varepsilon(\varphi,\boldsymbol{v})+b(\psi,\boldsymbol{v})-\lambda(\theta,\boldsymbol{v})=L^{\boldsymbol{u}}(\boldsymbol{v}),\quad\forall\boldsymbol{v}\in\boldsymbol{V}(\Omega), \tag{39}$$

$$-\varepsilon(\overline{\varphi}, \boldsymbol{u}) + d(\varphi, \overline{\varphi}) + a(\psi, \overline{\varphi}) - \mu(\theta, \overline{\varphi}) = L^{\varphi}(\overline{\varphi}), \quad \forall \overline{\varphi} \in V^{\varphi}(\Omega), \tag{40}$$

$$-b(\overline{\psi}, \boldsymbol{u}) + a(\varphi, \overline{\psi}) + \zeta(\psi, \overline{\psi}) - m(\theta, \overline{\psi}) = L^{\psi}(\overline{\psi}), \quad \forall \overline{\psi} \in V^{\psi}(\Omega), \tag{41}$$

$$\eta(\theta, \overline{\theta}) = L^{\theta}(\overline{\theta}), \quad \forall \overline{\theta} \in V^{\theta}(\Omega), \tag{42}$$

where

$$c(\boldsymbol{u}, \boldsymbol{v}) = \int\limits_{\Omega} \sum_{i,j,p,q=1}^{3} c_{ijpq} e_{pq}(\boldsymbol{u}) e_{ij}(\boldsymbol{v}) dx, \quad \varepsilon(\varphi, \boldsymbol{v}) = \int\limits_{\Omega} \sum_{i,j,p=1}^{3} \varepsilon_{pij} \frac{\partial \varphi}{\partial x_p} e_{ij}(\boldsymbol{v}) dx,$$

$$b(\psi, \boldsymbol{v}) = \int\limits_{\Omega} \sum_{i,j,p=1}^{3} b_{pij} \frac{\partial \psi}{\partial x_p} e_{ij}(\boldsymbol{v}) dx, \quad \lambda(\theta, \boldsymbol{v}) = \int\limits_{\Omega} \sum_{i,j=1}^{3} \lambda_{ij} \theta e_{ij}(\boldsymbol{v}) dx,$$

$$d(\varphi, \overline{\varphi}) = \int\limits_{\Omega} \sum_{i,j=1}^{3} d_{ij} \frac{\partial \varphi}{\partial x_j} \frac{\partial \overline{\varphi}}{\partial x_i} dx, \quad a(\psi, \overline{\varphi}) = \int\limits_{\Omega} \sum_{i,j=1}^{3} a_{ij} \frac{\partial \psi}{\partial x_j} \frac{\partial \overline{\varphi}}{\partial x_i} dx,$$

$$\mu(\theta, \overline{\varphi}) = \int\limits_{\Omega} \sum_{i=1}^{3} \mu_i \theta \frac{\partial \overline{\varphi}}{\partial x_i} dx, \quad \zeta(\psi, \overline{\psi}) = \int\limits_{\Omega} \sum_{i,j=1}^{3} \zeta_{ij} \frac{\partial \psi}{\partial x_j} \frac{\partial \overline{\psi}}{\partial x_i} dx,$$

$$m(\theta, \overline{\psi}) = \int\limits_{\Omega} \sum_{i=1}^{3} m_i \theta \frac{\partial \overline{\psi}}{\partial x_i} dx, \quad \eta(\theta, \overline{\theta}) = \int\limits_{\Omega} \sum_{i,j=1}^{3} \eta_{ij} \frac{\partial \theta}{\partial x_j} \frac{\partial \overline{\theta}}{\partial x_i} dx,$$

$$L^{\boldsymbol{u}}(\boldsymbol{v}) = \int\limits_{\Omega} \sum_{i=1}^{3} f_i v_i dx + \int\limits_{\Gamma_1} \sum_{i,j=1}^{3} g_i v_i d\Gamma, \quad L^{\varphi}(\overline{\varphi}) = \int\limits_{\Omega} f^{\varepsilon} \overline{\varphi} dx - \int\limits_{\Gamma_1^{\varphi}} g^{\varphi} \overline{\varphi} d\Gamma,$$

$$L^{\psi}(\overline{\psi}) = -\int\limits_{\Gamma_1^{\psi}} g^{\psi} \overline{\psi} d\Gamma, \quad L^{\theta}(\overline{\theta}) = \int\limits_{\Omega} f^{\theta} \overline{\theta} dx - \int\limits_{\Gamma_1^{\theta}} g^{\theta} \overline{\theta} d\Gamma,$$

and $\boldsymbol{u} = \boldsymbol{u}^k$, $\varphi = \varphi^k$, $\psi = \psi^k$, $\theta = \theta^k$, $c_{ijpq} = c_{ijpq}^k$, $\varepsilon_{pij} = \varepsilon_{pij}^k$, $b_{pij} = b_{pij}^k$, $\lambda_{ij} = \lambda_{ij}^k$, $d_{ij} = d_{ij}^k$, $a_{ij} = a_{ij}^k$, $\mu_i = \mu_i^k$, $\zeta_{ij} = \zeta_{ij}^k$, $m_i = m_i^k$, $\eta_{ij} = \eta_{ij}^k$ in Ω_k, $k = 1, ..., K$.

Since functions $\boldsymbol{v} \in \boldsymbol{V}(\Omega)$, $\overline{\varphi} \in V^{\varphi}(\Omega)$ and $\overline{\psi} \in V^{\psi}(\Omega)$ are independent of each other, problem (39)-(42) is equivalent to the following problem: Find $(\boldsymbol{u}, \varphi, \psi) \in \mathcal{V}(\Omega) = \boldsymbol{V}(\Omega) \times V^{\varphi}(\Omega) \times V^{\psi}(\Omega)$, $\theta \in V^{\theta}(\Omega)$ such that

$$A((\boldsymbol{u}, \varphi, \psi), (\boldsymbol{v}, \overline{\varphi}, \overline{\psi})) = L(\theta, (\boldsymbol{v}, \overline{\varphi}, \overline{\psi})), \quad \forall (\boldsymbol{v}, \overline{\varphi}, \overline{\psi}) \in \mathcal{V}(\Omega), \tag{43}$$

$$\eta(\theta, \overline{\theta}) = L^{\theta}(\overline{\theta}), \quad \forall \overline{\theta} \in V^{\theta}(\Omega), \tag{44}$$

where $L(\theta, (\boldsymbol{v}, \overline{\varphi}, \overline{\psi})) = L^{\boldsymbol{u}}(\boldsymbol{v}) + L^{\varphi}(\overline{\varphi}) + L^{\psi}(\overline{\psi}) + \Lambda(\theta, (\boldsymbol{v}, \overline{\varphi}, \overline{\psi}))$,

$$\Lambda(\theta, (\boldsymbol{v}, \overline{\varphi}, \overline{\psi})) = \lambda(\theta, \boldsymbol{v}) + \mu(\theta, \overline{\varphi}) + m(\theta, \overline{\psi}),$$

$$\begin{aligned} A((\boldsymbol{u}, \varphi, \psi), (\boldsymbol{v}, \overline{\varphi}, \overline{\psi})) &= c(\boldsymbol{u}, \boldsymbol{v}) + d(\varphi, \overline{\varphi}) + a(\psi, \overline{\varphi}) + a(\varphi, \overline{\psi}) \\ &\quad + \zeta(\psi, \overline{\psi}) + \varepsilon(\varphi, \boldsymbol{v}) - \varepsilon(\overline{\varphi}, \boldsymbol{u}) + b(\psi, \boldsymbol{v}) - b(\overline{\psi}, \boldsymbol{u}). \end{aligned}$$

3. Existence and uniqueness of solution

Note that if the parts Γ_0, Γ_0^φ, Γ_0^ψ and Γ_0^θ of the body Ω, where displacement vector-function, electric and magnetic potentials, and temperature vanish, are empty sets, then the homogeneous problem (43), (44) has non-trivial solutions. Hence, the solution of problem (43), (44) is not unique in the first order Sobolev spaces mentioned in the variational formulation and it is necessary to introduce suitable factor spaces, where solution of problem (43), (44) will be unique.

Let us determine the structure of the set $\mathfrak{R}$ of solutions of the homogeneous problem (43), (44), where $L^{\boldsymbol{u}}(\boldsymbol{v}) = 0$, $L^\varphi(\overline{\varphi}) = 0$, $L^\psi(\overline{\psi}) = 0$, $L^\theta(\overline{\theta}) = 0$, for all $(\boldsymbol{v}, \overline{\varphi}, \overline{\psi}) \in \mathcal{V}(\Omega)$ and $\overline{\theta} \in V^\theta(\Omega)$. We assume that c_{ijpq}, ε_{pij}, b_{pij}, d_{ij}, ζ_{ij}, a_{ij}, λ_{ij}, μ_i, m_i, $\eta_{ij} \in L^\infty(\Omega)$, $i, j, p, q = 1, 2, 3$, satisfy the following positive definiteness conditions

$$\sum_{i,j,p,q=1}^{3} c_{ijpq}\xi_{ij}\xi_{pq} \geq \alpha_c \sum_{i,j=1}^{3} (\xi_{ij})^2, \quad \sum_{i,j=1}^{3} \eta_{ij}\xi_j\xi_j \geq \alpha_\eta \sum_{i=1}^{3} (\xi_i)^2, \tag{45}$$

$$\sum_{i,j=1}^{3} d_{ij}\xi_j\xi_i + \sum_{i,j=1}^{3} a_{ij}\widetilde{\xi}_j\xi_i + \sum_{i,j=1}^{3} a_{ij}\xi_j\widetilde{\xi}_i + \sum_{i,j=1}^{3} \zeta_{ij}\widetilde{\xi}_j\widetilde{\xi}_i \geq \alpha \sum_{i=1}^{3}((\xi_i)^2 + (\widetilde{\xi}_i)^2), \tag{46}$$

for all $\xi_{ij} \in \mathbb{R}$, $\xi_{ij} = \xi_{ji}$, ξ_i, $\widetilde{\xi}_i \in \mathbb{R}$, where α_c, α_η, α are positive constants.

We denote by $(\boldsymbol{u}^{r\theta^r}, \varphi^{r\theta^r}, \psi^{r\theta^r}, \theta^r) \in \mathcal{V}(\Omega) \times V^\theta(\Omega)$ solution of the homogeneous problem (43), (44). From positive definiteness condition (45) for the tensor $(\eta_{ij})_{i,j=1}^3$ we obtain that $\theta^r \in \mathfrak{R}_\theta = \{v \in V^\theta(\Omega); v = \alpha_\theta,\ \alpha_\theta = const\}$. Hence from the equation (43) we have:

$$A((\boldsymbol{u}^{r\theta^r}, \varphi^{r\theta^r}, \psi^{r\theta^r}), (\boldsymbol{v}, \overline{\varphi}, \overline{\psi})) = \Lambda(\theta^r, (\boldsymbol{v}, \overline{\varphi}, \overline{\psi})), \quad \forall (\boldsymbol{v}, \overline{\varphi}, \overline{\psi}) \in \mathcal{V}(\Omega). \tag{47}$$

From conditions (45), (46) it follows that the solution $(\boldsymbol{u}^{r\theta^r}, \varphi^{r\theta^r}, \psi^{r\theta^r})$ of the latter equaion for $\theta^r = 0$ is a rigid displacement for $\boldsymbol{u}^{\theta^r}$ and constants for φ^{θ^r} and ψ^{θ^r}, i.e. $(\boldsymbol{u}^{r0}, \varphi^{r0}, \psi^{r0}) \in \mathfrak{R}_V = \{(\boldsymbol{v}^r, \overline{\varphi}^r, \overline{\psi}^r) \in \mathcal{V}(\Omega); \boldsymbol{v}^r = \vec{\alpha} + \vec{\beta} \times \overrightarrow{Ox}, \vec{\alpha}, \vec{\beta} \in \mathbb{R}^3, \overline{\varphi}^r = \alpha_\varphi,\ \alpha_\varphi = const,\ \overline{\psi}^r = \alpha_\psi, \alpha_\psi = const\}$. Hence, in order to study the equation (47) we introduce the factor space $\mathcal{V}(\Omega)/\mathfrak{R}_V$, consisting of the following equivalence classes $(\boldsymbol{v}, \overline{\varphi}, \overline{\psi})^{\mathfrak{R}_V} = \{(\boldsymbol{v}, \overline{\varphi}, \overline{\psi}) + (\boldsymbol{v}^r, \overline{\varphi}^r, \overline{\psi}^r); (\boldsymbol{v}^r, \overline{\varphi}^r, \overline{\psi}^r) \in \mathfrak{R}_V\}$ for each $(\boldsymbol{v}, \overline{\varphi}, \overline{\psi}) \in \mathcal{V}(\Omega)$, which is a Hilbert space with respect to the norm

$$\left\| (\boldsymbol{v}, \overline{\varphi}, \overline{\psi})^{\mathfrak{R}_V} \right\|_{\mathcal{V}(\Omega)/\mathfrak{R}_V} = \inf\{||(\boldsymbol{v}, \overline{\varphi}, \overline{\psi}) + (\boldsymbol{v}^r, \overline{\varphi}^r, \overline{\psi}^r)||_{(H^1(\Omega))^5}; (\boldsymbol{v}^r, \overline{\varphi}^r, \overline{\psi}^r) \in \mathfrak{R}_V\}.$$

In the factor space $\mathcal{V}(\Omega)/\mathfrak{R}_V$ the nonhomogeneous equation (47) is equivalent to the following equaion: Find $(\boldsymbol{u}^{r\theta^r}, \varphi^{r\theta^r}, \psi^{r\theta^r})^{\mathfrak{R}_V} \in \mathcal{V}(\Omega)/\mathfrak{R}_V$,

$$A^{\mathfrak{R}_V}((\boldsymbol{u}^{r\theta^r}, \varphi^{r\theta^r}, \psi^{r\theta^r})^{\mathfrak{R}_V}, (\boldsymbol{v}, \overline{\varphi}, \overline{\psi})^{\mathfrak{R}_V}) = \Lambda^{\mathfrak{R}_V}(\theta^r, (\boldsymbol{v}, \overline{\varphi}, \overline{\psi})^{\mathfrak{R}_V}), \tag{48}$$

for all $(\boldsymbol{v}, \overline{\varphi}, \overline{\psi})^{\mathfrak{R}_V} \in \mathcal{V}(\Omega)/\mathfrak{R}_V$, where $A^{\mathfrak{R}_V}((\boldsymbol{u}^{r\theta^r}, \varphi^{r\theta^r}, \psi^{r\theta^r})^{\mathfrak{R}_V}, (\boldsymbol{v}, \overline{\varphi}, \overline{\psi})^{\mathfrak{R}_V}) = A((\boldsymbol{u}^{r\theta^r}, \varphi^{r\theta^r}, \psi^{r\theta^r}), (\boldsymbol{v}, \overline{\varphi}, \overline{\psi}))$, $\Lambda^{\mathfrak{R}_V}(\theta^r, (\boldsymbol{v}, \overline{\varphi}, \overline{\psi})^{\mathfrak{R}_V}) = \Lambda(\theta^r, (\boldsymbol{v}, \overline{\varphi}, \overline{\psi}))$. The equation (48) has a unique solution. The uniqueness of solution of (48) directly follows from the construction of the factor space, so it is sufficient to show the existence.

Sience c_{ijpq}, d_{ij}, a_{ij}, ζ_{ij}, ε_{pij}, b_{pij}, λ_{ij}, μ_i, $m_i \in L^\infty(\Omega)$, $i,j,p,q = 1,2,3$, we infer that the bilinear forms $c(.,.)$, $d(.,.)$, $a(.,.)$, $\zeta(.,.)$, $\varepsilon(.,.)$ and $b(.,.)$, and linear forms $\lambda(\overline{\theta},.), \mu(\overline{\theta},.), m(\overline{\theta},.)$, $\overline{\theta} \in \overline{H}^1(\Omega)$, are continuous in the corresponding first order Sobolev spaces, and thus we have:

$$\begin{aligned} & A^{\mathfrak{R}_V}((\boldsymbol{w},\widetilde{\varphi},\widetilde{\psi})^{\mathfrak{R}_V},(\boldsymbol{v},\overline{\varphi},\overline{\psi})^{\mathfrak{R}_V}) \\ & \leq c_A \left\|(\boldsymbol{w},\widetilde{\varphi},\widetilde{\psi})^{\mathfrak{R}_V}\right\|_{\mathcal{V}(\Omega)/\mathfrak{R}_V} \left\|(\boldsymbol{v},\overline{\varphi},\overline{\psi})^{\mathfrak{R}_V}\right\|_{\mathcal{V}(\Omega)/\mathfrak{R}_V}, \end{aligned} \tag{49}$$

$$\Lambda^{\mathfrak{R}_V}(\overline{\theta},(\boldsymbol{v},\overline{\varphi},\overline{\psi})^{\mathfrak{R}_V}) \leq c_\Lambda \left\|(\boldsymbol{v},\overline{\varphi},\overline{\psi})^{\mathfrak{R}_V}\right\|_{\mathcal{V}(\Omega)/\mathfrak{R}_V}, \tag{50}$$

for all $(\boldsymbol{w},\widetilde{\varphi},\widetilde{\psi})^{\mathfrak{R}_V},(\boldsymbol{v},\overline{\varphi},\overline{\psi})^{\mathfrak{R}_V} \in \mathcal{V}(\Omega)/\mathfrak{R}_V$. From positive definiteness conditions (45), (46) we obtain:

$$\begin{aligned} & A((\boldsymbol{v},\overline{\varphi},\overline{\psi}),(\boldsymbol{v},\overline{\varphi},\overline{\psi})) = c(\boldsymbol{v},\boldsymbol{v}) + d(\overline{\varphi},\overline{\varphi}) + a(\overline{\psi},\overline{\varphi}) + a(\overline{\varphi},\overline{\psi}) + \zeta(\overline{\psi},\overline{\psi}) \\ & \geq \alpha_c \int\limits_\Omega \sum_{i,j=1}^3 (e_{ij}(\boldsymbol{v}))^2 dx + \alpha \int\limits_\Omega \sum_{i=1}^3 \left(\left(\frac{\partial \overline{\varphi}}{\partial x_i}\right)^2 + \left(\frac{\partial \overline{\psi}}{\partial x_i}\right)^2 \right) dx. \end{aligned}$$

Applying corollary from Korn's inequality in factor spaces [3] and generalized Poincare's inequality [2] we have:

$$\int\limits_\Omega \sum_{i,j=1}^3 (e_{ij}(\boldsymbol{v}))^2 dx \geq c_1 \inf\{\|\boldsymbol{v}+\boldsymbol{u}^r\|^2_{\boldsymbol{H}^1(\Omega)} \,|(\boldsymbol{u}^r,\varphi^r,\psi^r) \in \mathfrak{R}_V\},$$

$$\int\limits_\Omega \sum_{i=1}^3 \left(\frac{\partial v}{\partial x_i}\right)^2 dx \geq c_1 \left(\int\limits_\Omega v^2 dx - \left| \int\limits_\Omega v dx \right|^2 \right) = c_1 \inf\{\|v+c\|^2_{L^2(\Omega)} \,|c \in \mathbb{R}\}, \tag{51}$$

for all $v \in H^1(\Omega)$. Consequently, the bilinear form $A^{\mathfrak{R}_V} : \mathcal{V}(\Omega)/\mathfrak{R}_V \times \mathcal{V}(\Omega)/\mathfrak{R}_V \to \mathbb{R}$ satisfies the following inequality

$$A^{\mathfrak{R}_V}((\boldsymbol{v},\overline{\varphi},\overline{\psi})^{\mathfrak{R}_V},(\boldsymbol{v},\overline{\varphi},\overline{\psi})^{\mathfrak{R}_V}) \geq c_1 \left\|(\boldsymbol{v},\overline{\varphi},\overline{\psi})^{\mathfrak{R}_V}\right\|_{\mathcal{V}(\Omega)/\mathfrak{R}_V}, \tag{52}$$

for all $(\boldsymbol{v},\overline{\varphi},\overline{\psi})^{\mathfrak{R}_V} \in \mathcal{V}(\Omega)/\mathfrak{R}_V$. Hence $A^{\mathfrak{R}_V}$ is continuous and bounded below, and from Lax-Milgram theorem [6] we have that equation (48) possesses a unique solution in $\mathcal{V}(\Omega)/\mathfrak{R}_V$. Therefore, solution $(\boldsymbol{u}^{r\theta^r},\varphi^{r\theta^r},\psi^{r\theta^r},\theta^r) \in \mathcal{V}(\Omega) \times V^\theta(\Omega)$ of the equation (47) corresponding to $\theta^r \in \mathfrak{R}_\theta$ exists.

Thus, the set $\mathfrak{R}$ of solutions of the homogeneous problem (43), (44) is of the following form:

$$\begin{gathered} \mathfrak{R} = \{(\boldsymbol{v}^{r\theta^r},\overline{\varphi}^{r\theta^r},\overline{\psi}^{r\theta^r},\theta^r) \in \mathcal{V}(\Omega) \times V^\theta(\Omega);\ \boldsymbol{v}^{r\theta^r} = \vec{\alpha} + \vec{\beta} \times \overrightarrow{Ox} + \boldsymbol{u}^{r\theta^r}, \vec{\alpha},\vec{\beta} \in \mathbb{R}^3, \\ \overline{\varphi}^{r\theta^r} = \alpha_\varphi + \varphi^{r\theta^r},\ \alpha_\varphi = const,\ \overline{\psi}^{r\theta^r} = \alpha_\psi + \psi^{r\theta^r}, \alpha_\psi = const, \theta^r \in \mathfrak{R}_\theta\}. \end{gathered}$$

Applying the set $\mathfrak{R}$ we can define the factor space $(\mathcal{V}(\Omega) \times V^\theta(\Omega))/\mathfrak{R}$, which consists of the following elements $(\boldsymbol{v},\overline{\varphi},\overline{\psi},\overline{\theta})^{\mathfrak{R}} = \{(\boldsymbol{v},\overline{\varphi},\overline{\psi},\overline{\theta}) + (\boldsymbol{v}^{r\theta^r},\overline{\varphi}^{r\theta^r}, \overline{\psi}^{r\theta^r},\theta^r); (\boldsymbol{v}^{r\theta^r},\overline{\varphi}^{r\theta^r},\overline{\psi}^{r\theta^r},\theta^r) \in \mathfrak{R}\}$, which is the Hilbert space equipped with the following norm:

$$\left\|(\boldsymbol{v},\overline{\varphi},\overline{\psi},\overline{\theta})^{\mathfrak{R}}\right\|_* = \inf\{||(\boldsymbol{v},\overline{\varphi},\overline{\psi},\overline{\theta}) + (\boldsymbol{v}^{r\theta^r},\overline{\varphi}^{r\theta^r},\overline{\psi}^{r\theta^r},\theta^r)||_{(H^1(\Omega))^6};$$
$$(\boldsymbol{v}^{r\theta^r},\overline{\varphi}^{r\theta^r},\overline{\psi}^{r\theta^r},\theta^r) \in \mathfrak{R}\}.$$

Remark 1. If $\Gamma_1^\theta \neq \partial\Omega$, then the area of the surface Γ_0^θ is positive and, hence, the homogeneous equation (44) has only trivial solution, $\mathfrak{R}_\theta = \{0\}$and $\mathfrak{R} = \mathfrak{R}_V \times \{0\}$.

Remark 2. If the areas of the surfaces Γ_0, Γ_0^φ, Γ_0^ψ, Γ_0^θ are positive, then the homogeneous equations (43), (44) have only trivial solution, $\mathfrak{R}_V = \{(\mathbf{0},0,0)\}, \mathfrak{R}_\theta = \{0\}$ and $\mathfrak{R} = \{(\mathbf{0},0,0,0)\}$.

Note that if $(\boldsymbol{u},\varphi,\psi,\theta)$ is a solution of the problem (43), (44), then any function $(\boldsymbol{u},\varphi,\psi,\theta) + (\boldsymbol{v}^{r\theta^r},\overline{\varphi}^{r\theta^r},\overline{\psi}^{r\theta^r},\theta^r)$, where $(\boldsymbol{v}^{r\theta^r},\overline{\varphi}^{r\theta^r},\overline{\psi}^{r\theta^r},\theta^r) \in \mathfrak{R}$ is a solution of (43), (44). Therefore, we say that $(\boldsymbol{u},\varphi,\psi,\theta)^{\mathfrak{R}}$ is a solution of the problem (43), (44), if any function from the equivalence class $(\boldsymbol{u},\varphi,\psi,\theta)^{\mathfrak{R}}$ is a solution of the problem (43), (44).

For the problem (43), (44), which is equivalent to the boundary value problem (1)-(15) in the spaces of classical smooth enough function, we have the following existence, uniqueness and continuous dependence theorem.

Theorem 3.1: *Suppose that the parameters characterizing thermo-mechanical and electro-magnetic properties of the elastic body Ω satisfy the symmetry and positive definiteness conditions (16)-(18) and (45), (46), and c_{ijpq}, ε_{pij}, b_{pij},d_{ij}, ζ_{ij}, a_{ij}, λ_{ij}, μ_i, m_i, $\eta_{ij} \in L^\infty(\Omega)$, $i,j,p,q = 1,2,3$. If $\boldsymbol{f} \in \boldsymbol{L}^{6/5}(\Omega)$, $\boldsymbol{g} \in \boldsymbol{L}^{4/3}(\Gamma_1)$, $f^\varepsilon \in L^{6/5}(\Omega)$, $g^\varphi \in L^{4/3}(\Gamma_1^\varphi)$, $g^\psi \in L^{4/3}(\Gamma_1^\psi)$, $f^\theta \in L^{6/5}(\Omega)$, $g^\theta \in L^{4/3}(\Gamma_1^\theta)$, and $L^\theta(\theta^r) = 0$ and $L(\boldsymbol{v}^r) + L^\varphi(\overline{\varphi}^r) + L^\psi(\overline{\psi}^r) = 0$, for all $(\boldsymbol{v}^r,\overline{\varphi}^r,\overline{\psi}^r) \in \mathfrak{R}_V$, $\theta^r \in \mathfrak{R}_\theta$, then the problem (43), (44) possesses a unique solution $(\boldsymbol{u},\varphi,\psi,\theta)^{\mathfrak{R}} \in (\mathcal{V}(\Omega) \times V^\theta(\Omega))/\mathfrak{R}$, which continuously depends on the given data, i.e., the mapping $(\boldsymbol{f},\boldsymbol{g},f^\varepsilon, g^\varphi, g^\psi, f^\theta, g^\theta) \to (\boldsymbol{u},\varphi,\psi,\theta)^{\mathfrak{R}}$ is linear and continuous from the space $\boldsymbol{L}^{6/5}(\Omega) \times \boldsymbol{L}^{4/3}(\Gamma_1) \times L^{6/5}(\Omega) \times L^{4/3}(\Gamma_1^\varphi) \times L^{4/3}(\Gamma_1^\psi) \times L^{6/5}(\Omega) \times L^{4/3}(\Gamma_1^\theta)$ to the space $(\mathcal{V}(\Omega) \times V^\theta(\Omega))/\mathfrak{R}$.*

Proof: Note that from the linearity of the problem (43), (44) it follows that the solution is unique in the factor space $(\mathcal{V}(\Omega) \times V^\theta(\Omega))/\mathfrak{R}$. Hence, let us prove the existence and continuous dependence on the given data.

From the conditions of the theorem c_{ijpq}, d_{ij}, a_{ij}, ζ_{ij}, ε_{pij}, b_{pij}, $\eta_{ij} \in L^\infty(\Omega)$, $i,j,p,q = 1,2,3$, we infer that the bilinear forms $c(.,.)$, $d(.,.)$, $a(.,.)$, $\zeta(.,.)$, $\varepsilon(.,.)$, $b(.,.)$ and $\eta(.,.)$ are continuous in corresponding spaces, and hence the bilinear forms $A^{\mathfrak{R}_V}((\boldsymbol{u},\varphi,\psi)^{\mathfrak{R}_V},(\boldsymbol{v},\overline{\varphi},\overline{\psi})^{\mathfrak{R}_V}) = A((\boldsymbol{u},\varphi,\psi),(\boldsymbol{v},\overline{\varphi},\overline{\psi}))$: $\mathcal{V}(\Omega)/\mathfrak{R}_V \times \mathcal{V}(\Omega)/\mathfrak{R}_V \to \mathbb{R}$ and $\eta^{\mathfrak{R}_\theta}(\theta^r,\overline{\theta}^r) = \eta(\theta,\overline{\theta}) : V^\theta(\Omega)/\,\mathfrak{R}_\theta \times V^\theta(\Omega)/\,\mathfrak{R}_\theta \to \mathbb{R}$ are continuous, where $V^\theta(\Omega)/\,\mathfrak{R}_\theta$ is the factor space consisting of the equivalence classes $\overline{\theta}^{\mathfrak{R}_\theta} = \{\overline{\theta} + \theta^r; \theta^r \in \mathfrak{R}_\theta\}$ for each $\overline{\theta} \in V^\theta(\Omega)$, which is a Hilbert space with respect to the norm $||\overline{\theta}^{\mathfrak{R}_\theta}||_{V^\theta(\Omega)/\mathfrak{R}_\theta} = \inf\{\left\|\overline{\theta} + \theta^r\right\|_{H^1(\Omega)}; \theta^r \in \mathfrak{R}_\theta\}$.

Because $\boldsymbol{f} \in \boldsymbol{L}^{6/5}(\Omega)$ and $\boldsymbol{g} \in \boldsymbol{L}^{4/3}(\Gamma_1)$ by applying Holder's inequality, and continuity of the embedding $H^1(\Omega) \to L^6(\Omega)$ and the trace operator $tr : H^1(\Omega) \to L^4(\Gamma)$ [2] we have:

$$|L(\boldsymbol{v})| \leq \left|(\boldsymbol{f},\boldsymbol{v})_{\boldsymbol{L}^2(\Omega)}\right| + \left|(\boldsymbol{g}, \boldsymbol{tr}_{\Gamma_1}(\boldsymbol{v}))_{\boldsymbol{L}^2(\Gamma_1)}\right| \leq \|\boldsymbol{f}\|_{\boldsymbol{L}^{6/5}(\Omega)} \|\boldsymbol{v}\|_{\boldsymbol{L}^6(\Omega)}$$
$$+ \|\boldsymbol{g}\|_{\boldsymbol{L}^{4/3}(\Gamma_1)} \|\boldsymbol{tr}_{\Gamma_1}(\boldsymbol{v})\|_{\boldsymbol{L}^4(\Gamma_1)} \leq c_1 \|\boldsymbol{f}\|_{\boldsymbol{L}^{6/5}(\Omega)} \|\boldsymbol{v}\|_{\boldsymbol{H}^1(\Omega)} + c_2 \|\boldsymbol{g}\|_{\boldsymbol{L}^{4/3}(\Gamma_1)} \|\boldsymbol{v}\|_{\boldsymbol{H}^1(\Omega)},$$

for all $\boldsymbol{v} \in \boldsymbol{H}^1(\Omega)$. Therefore, the linear form $L : \boldsymbol{H}^1(\Omega) \to \mathbb{R}$ is continuous. From $f^\varepsilon \in L^{6/5}(\Omega)$, $g^\varphi \in L^{4/3}(\Gamma_1^\varphi)$, $g^\psi \in L^{4/3}(\Gamma_1^\psi)$, $f^\theta \in L^{6/5}(\Omega)$, $g^\theta \in L^{4/3}(\Gamma_1^\theta)$ we similarly obtain that the linear forms $L^\varphi : H^1(\Omega) \to \mathbb{R}$, $L^\psi : H^1(\Omega) \to \mathbb{R}$, and $L^\theta : H^1(\Omega) \to \mathbb{R}$ are continuous.

Taking into account that $L^\theta(\theta^r) = 0$, for all $\theta^r \in \mathfrak{R}_\theta$, we have:

$$|L^\theta(\overline{\theta}^{\mathfrak{R}})| = |L^\theta(\overline{\theta}+\theta^r)| = |L^\theta(\overline{\theta})| \leq c_1 \inf\{||\overline{\theta}+\theta^r||_{H^1(\Omega)} | \theta^r \in \mathfrak{R}_\theta\} = c_1||\overline{\theta}^{\mathfrak{R}}||_{V^\theta(\Omega)/\mathfrak{R}_\theta},$$

for all $\overline{\theta}^{\mathfrak{R}_\theta} \in V^\theta(\Omega)\mathfrak{R}_\theta \to \mathbb{R}$. From positive definiteness conditions (45) for $(\eta_{ij})_{i,j=1}^3$ applying inequality (51) we obtain:

$$\eta(\overline{\theta},\overline{\theta}) \geq \alpha_\eta \int_\Omega \sum_{i=1}^3 \left(\frac{\partial \overline{\theta}}{\partial x_i}\right)^2 dx \geq c_1||\overline{\theta}^{\mathfrak{R}}||^2_{V^\theta(\Omega)/\mathfrak{R}_\theta}, \quad \forall \overline{\theta} \in V^\theta(\Omega).$$

Thus, $\eta^{\mathfrak{R}_\theta}(\overline{\theta}^{\mathfrak{R}}, \widetilde{\theta}^{\mathfrak{R}}) = \eta(\overline{\theta}, \widetilde{\theta})$ is continuous and bounded from below on $V^\theta(\Omega)/\mathfrak{R}_\theta$, and from Lax-Milgram theorem [6] we have that the equation (44) possesses a unique solution $\theta^{\mathfrak{R}_\theta} \in V^\theta(\Omega)/\mathfrak{R}_\theta$ and

$$\left\|\theta^{\mathfrak{R}_\theta}\right\|_{V^\theta(\Omega)/\mathfrak{R}_\theta} \leq c_1 \left(||f^\theta||_{L^{6/5}(\Omega)} + ||g^\theta||_{L^{4/3}(\Gamma_1^\theta)}\right). \tag{53}$$

As in the case of the linear form L^θ from the continuity of the linear forms $L : \boldsymbol{H}^1(\Omega) \to \mathbb{R}$, $L^\varphi : H^1(\Omega) \to \mathbb{R}$, $L^\psi : H^1(\Omega) \to \mathbb{R}$ taking into account that $L(\boldsymbol{u}^r) + L^\varphi(\varphi^r) + L^\psi(\psi^r) = 0$, for all $(\boldsymbol{u}^r, \varphi^r, \psi^r) \in \mathfrak{R}_V$, and using (50) we obtain:

$$|L(\overline{\theta}, (\boldsymbol{v}, \overline{\varphi}, \overline{\psi})^{\mathfrak{R}_V})| = |L(\boldsymbol{v}) + L^\varphi(\overline{\varphi}) + L^\psi(\overline{\psi}) + \Lambda(\overline{\theta}, (\boldsymbol{v}, \overline{\varphi}, \overline{\psi}))|$$
$$\leq c_1 \left\|(\boldsymbol{v}, \overline{\varphi}, \overline{\psi})^{\mathfrak{R}_V}\right\|_{\mathcal{V}(\Omega)/\mathfrak{R}_V},$$

for all $(\boldsymbol{v}, \overline{\varphi}, \overline{\psi})^{\mathfrak{R}_V} \in \mathcal{V}(\Omega)/\mathfrak{R}_V$ and $\overline{\theta} \in V^\theta(\Omega)$. Therefore, because $A^{\mathfrak{R}_V}$ is continuous and bounded from below, from Lax-Milgram theorem we have that the equation (43) for each $\theta \in \theta^{\mathfrak{R}_\theta}$ possesses a unique solution $(\boldsymbol{u}, \varphi, \psi)^{\mathfrak{R}_V} \in \mathcal{V}(\Omega)/\mathfrak{R}_V$, and

$$\left\|(\boldsymbol{u}, \varphi, \psi)^{\mathfrak{R}_V}\right\|_{\mathcal{V}(\Omega)/\mathfrak{R}_V} \leq c_{L^{\boldsymbol{u}}} + c_1 \|\theta + \theta^r\|_{H^1(\Omega)}, \quad \forall \theta^r \in \mathfrak{R}_\theta,$$

where $c_{L^{\boldsymbol{u}}} = \|\boldsymbol{f}\|_{\boldsymbol{L}^{6/5}(\Omega)} + \|\boldsymbol{g}\|_{\boldsymbol{L}^{4/3}(\Gamma_1)} + \|f^\varepsilon\|_{L^{6/5}(\Omega)} + \|g^\varphi\|_{L^{4/3}(\Gamma_1^\varphi)} + \|g^\varphi\|_{L^{4/3}(\Gamma_1^\psi)}$, and hence

$$\inf\{||(\boldsymbol{u}, \varphi, \psi)^{\mathfrak{R}_V}||_{\mathcal{V}(\Omega)/\mathfrak{R}_V}; \theta^r \in \mathfrak{R}_\theta\} \leq c_{L^{\boldsymbol{u}}}$$
$$+ c_1 \inf\{\|\theta + \theta^r\|_{H^1(\Omega)}; \theta^r \in \mathfrak{R}_\theta\}. \tag{54}$$

So, solution of the problem (43), (44) exists and is unique in the space $(\mathcal{V}(\Omega) \times V^\theta(\Omega))/\mathfrak{R}$, and combining estimates (53) and (54) we obtain the continuity of the mapping $(\boldsymbol{f}, \boldsymbol{g}, f^\varepsilon, g^\varphi, g^\psi, f^\theta, g^\theta) \to (\boldsymbol{u}, \varphi, \psi, \theta)^{\mathfrak{R}}$, because

$$\left\|(\boldsymbol{u}, \varphi, \psi, \theta)^{\mathfrak{R}}\right\|_* \leq c_1 \inf\{||(\boldsymbol{u}, \varphi, \psi)^{\mathfrak{R}_V}||_{\mathcal{V}(\Omega)/\mathfrak{R}_V}; \theta^r \in \mathfrak{R}_\theta\} + c_1||\theta^{\mathfrak{R}_\theta}||_{V^\theta(\Omega)/\mathfrak{R}_\theta}.$$

□

From Remark 2 it follows, that if the areas of the surfaces Γ_0, Γ_0^φ, Γ_0^ψ, Γ_0^θ are positive, then the factor space $(\mathcal{V}(\Omega) \times V^\theta(\Omega))/\mathfrak{R}$ coincides with $\mathcal{V}(\Omega) \times V^\theta(\Omega)$ and from Theorem 3.1 we have the following theorem.

Theorem 3.2: *Suppose that the parameters characterizing thermo-mechanical and electro-magnetic properties of the elastic body Ω satisfy the symmetry and positive definiteness conditions (16)-(18) and (45), (46), and c_{ijpq}, ε_{pij}, $b_{pij}, d_{ij}, \zeta_{ij}, a_{ij}, \lambda_{ij}, \mu_i, m_i, \eta_{ij} \in L^\infty(\Omega)$, $i, j, p, q = 1, 2, 3$. If $\boldsymbol{f} \in \boldsymbol{L}^{6/5}(\Omega)$, $\boldsymbol{g} \in \boldsymbol{L}^{4/3}(\Gamma_1)$, $f^\varepsilon \in L^{6/5}(\Omega)$, $g^\varphi \in L^{4/3}(\Gamma_1^\varphi)$, $g^\psi \in L^{4/3}(\Gamma_1^\psi)$, $f^\theta \in L^{6/5}(\Omega)$, $g^\theta \in L^{4/3}(\Gamma_1^\theta)$, then the problem (43), (44) possesses a unique solution $(\boldsymbol{u}, \varphi, \psi, \theta) \in \mathcal{V}(\Omega) \times V^\theta(\Omega)$, which continuously depends on the given data, i.e., the mapping $(\boldsymbol{f}, \boldsymbol{g}, f^\varepsilon, g^\varphi, g^\psi, f^\theta, g^\theta) \to (\boldsymbol{u}, \varphi, \psi, \theta)$ is linear and continuous from the space $\boldsymbol{L}^{6/5}(\Omega) \times \boldsymbol{L}^{4/3}(\Gamma_1) \times L^{6/5}(\Omega) \times L^{4/3}(\Gamma_1^\varphi) \times L^{4/3}(\Gamma_1^\psi) \times L^{6/5}(\Omega) \times L^{4/3}(\Gamma_1^\theta)$ to the space $\mathcal{V}(\Omega) \times V^\theta(\Omega)$.*

Acknowledgements

This work was supported by Shota Rustaveli National Science Foundation (SRNSF) [Grant number 217596, Construction and investigation of hierarchical models for thermoelastic piezoelectric structures].

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