

Stochastic Integral Representation of One Stochastically Non-smooth Wiener Functional

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(Received September 19, 2016; Revised November 29, 2016; Accepted December 9, 2016)

In this paper we obtain the Clark-Ocone's stochastic integral representation formula with explicit form of integrand in case, when path-dependent Wiener functional is not stochastically (in Malliavin sense) smooth. To achieve this aim, we check that the conditional mathematical expectation of the considered functional is stochastically smooth, and apply the generalization of the Clark-Ocone's formula, obtained by us earlier.

Keywords: Wiener functional, stochastic derivative, Clark's integral representation formula, Clark-Ocone's formula.

AMS Subject Classification: 60H07, 60H30, 62P05

1. Introduction

It is well-known from Ito's calculus, that the stochastic integral (as process) from a square integrable adapted process is a square integrable martingale. The answer to the inverse question: is it possible to represent the square integrable martingale adapted to the natural filtration of Wiener process, as the stochastic integral given by the well-known Clark formula ([1]). In particular, let W_t ($t \in [0, T]$) be a standard Wiener process and \mathfrak{F}_t^W is a natural filtration generated by this Wiener process. If F is a square integrable \mathfrak{F}_t^W -measurable random variable, then there exist a unique $\{\mathfrak{F}_t^W\}$ -adapted square integrable in $L_2([0, T])$ random process ψ_t such that

$$F = EF + \int_0^T \psi_t dW_t.$$

The representation of functionals of Wiener process by the stochastic integral, also known as the martingale representation, was studied by several authors. Martingale representation theorems (including Girsanov's measure transformation theorem) are widely known to play essentially important role in modern financial mathematics ([2]). Karatzas and Ocone ([3]) have shown how to use Ocone-Haussmann-Clark formula in financial mathematics, in particular for constructing hedging strategies in the complete financial markets driven by Wiener process.

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Since that time interest to Malliavin calculus has been significantly increasing. Therefore developing of the theory has intensively begun together with looking for the new sphere of its applications ([4]). Among them the applications in mathematical statistics are especially important (regularity of density, hypothesis testing).

At the same time, finding of explicit expression for ψ_t is a very difficult problem. In this direction, is known one general results, which is called Ocone-Clark formula (5), according to which $\psi_t = E(D_t F | \mathfrak{F}_t^W)$, where D_t is so called Malliavin stochastic derivative. But, on the one hand, here the stochastically smoothness of considered functional is required and on the other hand, even in case of smoothness, calculations of Malliavin derivative and conditional mathematical expectation are rather difficult.

Absolutely different method for finding of ψ_t was offered by Shyriaev, Yor and Graversen ([6], [7]). This method was based on using Ito's (generalized) formula and Levy's theorem for Levy's martingale $m_t = E(F | \mathfrak{F}_t^W)$ associated with F . Our approach (see, Jaoshvili, Purtukhia [8]) within the classical Ito's calculus allows to construct ψ_t explicitly by using both the standard L_2 theory and the theories of weighted Sobolev spaces, in case when the functional F has no stochastic derivative (in particular, the class of functionals considered by us includes, for example, the functional $F = I_{\{W_T > K\}}$ which is not stochastically differentiable).

Later, we (with prof. O. Glonti [9]) considered the case when the functional F is stochastically non-smooth, but from Levy's martingale associated with it one can select a stochastically smooth subsequence and in this case we have offered the method for finding the integrand. It is known, that if the random variable is stochastically differentiable (in Malliavin sense), then conditional mathematical expectation of this variable is stochastically differentiable as well ([10]). In particular, if $F \in D_{2,1}$, then $E(F | \mathfrak{F}_s^W) \in D_{2,1}$ and $D_t[E(F | \mathfrak{F}_s^W)] = E(D_t F | \mathfrak{F}_s^W) I_{[0,s]}(t)$, where $D_{2,1}$ denotes the Hilbert space which is the closure of the smooth Wiener functionals class with corresponding (Sobolev type) norm (see below). We generalized ([9]) Clark-Ocone formula for the case, when the functional is not stochastically smooth, but its conditional mathematical expectation is smooth (for example, $F = I_{\{W_T > K\}} \notin D_{2,1}$, but $E(F | \mathfrak{F}_t^W) = 1 - \Phi(\frac{K - W_t}{\sqrt{T-t}}) \in D_{2,1}$ for all $t \in [0, T)$, where $\Phi(\cdot)$ is the standard normal distribution function).

In this paper we consider a path-dependent Wiener functional

$$F = (W_T - K)^- I_{\{W_{T^*} \leq B\}}$$

which isn't stochastically smooth (here and bellow $W_{t^*} = \min_{0 \leq s \leq t} W_s$). For this functional the stochastic integral representation formula with the explicit form of integrand is obtained. With this aim in mind we find the conditional density function of joint distribution low of Wiener process and its minimum process under the given value of Wiener process, calculate the conditional mathematical expectation of the considered functional, check if it is stochastically smooth and apply above-mentioned generalization of the Clark-Ocone's formula. Note that this functional is a typical example of payoff function of so called European barrier¹ and lookback²

¹The barrier option is either nullified, activated or exercised when the underlying asset price breaches a barrier during the life of the option.

²The payoff of a lookback option depends on the minimum or maximum price of the underlying asset attained during certain period of the life of the option.

Options. Hence, the stochastic integral representation formula obtained here could be used to compute the explicit hedging portfolio of such barrier and lookback option.

2. Auxiliary results

On the probability space $(\Omega, \mathfrak{F}, P)$ the standard Wiener process $W = (W_t)$, $t \in [0, T]$ is given and (\mathfrak{F}_t^W) , $t \in [0, T]$, is the natural filtration generated by the Wiener process W . We consider functionals of the Wiener process, i.e. the random variables that are \mathfrak{F}_T^W -measurable.

The derivative (see [10]) of a smooth random variable F of the form

$$F = f(W(h_1), \dots, W(h_n)), \quad f \in C_p^\infty(R^n), \quad h_i \in L_2([0, T])$$

is the stochastic process $D_t F$ given by

$$D_t F = \sum_{i=1}^n \frac{\partial f}{\partial x_i}((W(h_1), \dots, W(h_n))) h_i(t)$$

(where $W(h_i) = \int_0^T h_i(t) dW_t$).

D is closable as an operator from $L_2(\Omega)$ to $L_2(\Omega; L_2([0, T]))$. We will denote its domain by $D_{2,1}$. That means, $D_{2,1}$ is equal to the adherence of the class of smooth random variables with respect to the norm

$$\|F\|_{2,1} = \|F\|_{L_2(\Omega)} + \|D F\|_{L_2(\Omega; L_2([0, T]))}.$$

Proposition 2.1: *Let $\psi : R^m \rightarrow R^1$ be a continuously differentiable function with bounded partial derivatives. Suppose that $F = (F^1, \dots, F^m)$ is a random vector whose components belong to the space $D_{2,1}$. Then $\psi(F) \in D_{2,1}$, and*

$$D_t \psi(F) = \sum_{i=1}^m \frac{\partial \psi}{\partial x^i}(F) D_t F^i.$$

(see, [11], Proposition 1.2.3.).

Let $p(u, t, W_u, A)$ be the transition probability of the Wiener process W , i.e. $P[W_t \in A | W_u] = p(u, t, W_u, A)$, where $0 \leq u \leq t$, A is a Borel subset of R and

$$p(u, t, x, A) = \frac{1}{\sqrt{2\pi(t-u)}} \int_A \exp\left\{-\frac{(y-x)^2}{2(t-u)}\right\} dy.$$

For the computation of conditional mathematical expectation below we use the well-known statement:

Proposition 2.2: *For any bounded or positive measurable function f we have the relation*

$$E[f(W_t) | W_u] = \int_R f(y) p(u, t, W_u, dy) \quad (P - a.s.).$$

Theorem 2.3: Suppose that $g_t = E[F|\mathfrak{S}_t^W]$ is Malliavin differentiable ($g_t(\cdot) \in D_{2,1}$) for almost all $t \in [0, T]$. Then we have the stochastic integral representation

$$g_T = F = EF + \int_0^T \nu_u dW_u \quad (P - a.s.),$$

where

$$\nu_u = \lim_{t \uparrow T} E[D_u g_t | \mathfrak{S}_u^W] \quad \text{in the } L_2([0, T] \times \Omega)$$

(see, [9], Theorem 1)).

Let $L_2([0, T]) = L_2([0, T], \mathcal{B}([0, T]), \lambda)$ (where λ is the Lebesgue measure). We denote by $L_{2,T}$ the set of measurable functions $u : R \rightarrow R$, such that $u(\cdot)\rho(\cdot, T) \in L_2 := L_2(R, \mathcal{B}(R), \lambda)$, where $\rho(x, T) = \exp\{-\frac{x^2}{2T}\}$.

Theorem 2.4: Let a function $f \in L_{2,T/\alpha}$, $0 < \alpha < 1$, and it has the first order generalized derivative $\partial f/\partial x$, such that $\partial f/\partial x \in L_{2,T/\beta}$, $0 < \beta < 1/2$. Then the following stochastic integral representation holds

$$f(W_T) = Ef(W_T) + \int_0^T E\left[\frac{\partial f}{\partial x}(W_T) | \mathfrak{S}_t^W\right] dW_t$$

(see, [8], Theorem 2)).

Proposition 2.5: The joint conditional distribution density ($t > s, y \leq 0, y \leq x$)

$$f_{W_t, W_{t^*} | W_s = z} = \frac{\partial^2 P\{W_t \leq x, W_{t^*} \leq y | W_s = z\}}{\partial x \partial y}$$

can be express as follows

$$f_{W_t, W_{t^*} | W_s = z} = \frac{2(x - 2y + z)}{\sqrt{2\pi(t-s)^3}} \exp\left\{-\frac{(x - 2y + z)^2}{2(t-s)}\right\}. \quad (1)$$

Proof: (Proof of Proposition 2.5.) It's clear that if $x < y$ or $y > 0$ then $\{W_t \leq x\} \subseteq \{W_{t^*} \leq y\}$ and the conditional joint distribution function of W_t and W_{t^*} under given $W_s = z$ is independent from y :

$$\begin{aligned} P\{W_t \leq x, W_{t^*} \leq y | W_s = z\} &= P\{W_t \leq x | W_s = z\} \\ &= P\{W_t - W_s \leq x - z | W_s = z\} = P\{W_t - W_s \leq x - z\}. \end{aligned}$$

Hence, in this case

$$f_{W_t, W_{t^*} | W_s = z} = \frac{\partial^2 P\{W_t - W_s \leq x - z\}}{\partial x \partial y} = 0.$$

Suppose now that $y \leq x$ and $y \leq 0$. According to the elementary relations

$$W_{t*} = W_{s*} \wedge \min_{s < l \leq t} W_l$$

and

$$(W_{s*} \wedge \min_{s < l \leq t} W_l) - W_s = (W_{s*} - W_s) \wedge (\min_{s < l \leq t} W_l - W_s),$$

we have

$$\begin{aligned} & P\{W_t \leq x, W_{t*} \leq y | W_s = z, W_{s*} = u\} \\ &= P\{W_t \leq x, W_{s*} \wedge \min_{s < l \leq t} W_l \leq y | W_s = z, W_{s*} = u\} \\ &= P\{W_t - W_s \leq x - z, (u \wedge \min_{s < l \leq t} W_l) - W_s \leq y - z | W_s = z, W_{s*} = u\} \\ &= P\{W_t - W_s \leq x - z, (u - W_s) \wedge \min_{s < l \leq t} (W_l - W_s) \leq y - z | W_s = z, W_{s*} = u\}. \end{aligned}$$

Hence, due to the equality

$$P\{AB|C\} = P\{A|C\} - P\{A\bar{B}|C\},$$

using properties of the Wiener process and conditional probability, one can easily see that ($u > y$)

$$\begin{aligned} & P\{W_t \leq x, W_{t*} \leq y | W_s = z, W_{s*} = u\} \\ &= P\{W_t - W_s \leq x - z | W_s = z, W_{s*} = u\} \\ &\quad - P\{W_t - W_s \leq x - z, (u - W_s) \wedge \min_{s < l \leq t} (W_l - W_s) > y - z | W_s = z, W_{s*} = u\} \\ &= P\{W_t - W_s \leq x - z\} \\ &\quad - P\{W_t - W_s \leq x - z, u - z > y - z, \min_{s < l \leq t} (W_l - W_s) > y - z | W_s = z, W_{s*} = u\} \\ &= P\{W_t - W_s \leq x - z\} \end{aligned}$$

$$\begin{aligned}
& -P\{W_t - W_s \leq x - z, \min_{s < l \leq t} (W_l - W_s) > y - z | W_s = z, W_{s^*} = u\} \\
& = P\{W_t - W_s \leq x - z\} - P\{W_t - W_s \leq x - z, \min_{s < l \leq t} (W_l - W_s) > y - z\} \\
& = P\{W_t - W_s \leq x - z, \min_{s < l \leq t} (W_l - W_s) \leq y - z\}. \tag{2}
\end{aligned}$$

Let us define a new Wiener process \overline{W}_θ , $\theta \in [0, t]$, as follows $\overline{W}_\theta = W_t - W_{t-\theta}$. It is evident that $W_t - W_l = \overline{W}_{t-l}$ and therefore

$$\min_{s < l \leq t} (W_l - W_s) = \min_{s \leq l \leq t} (W_l - W_s) = \min_{s \leq l \leq t} \overline{W}_{l-s} = \min_{0 \leq l-s \leq t-s} \overline{W}_{l-s} = \overline{W}_{(t-s)^*}.$$

On the other hand, basing on the expressions for distribution law of minimum process W_{t^*} and for joint distribution low of W_t and W_{t^*} (see, for example, [12]), we have:

$$P\{W_{t^*} \leq b\} = \frac{2}{\sqrt{2\pi t}} \int_{-\infty}^b \exp\{-\frac{v^2}{2t}\} dv, \quad b \leq 0,$$

and

$$P\{W_t > a, W_{t^*} \leq b\} = \frac{1}{\sqrt{2\pi t}} \int_{-\infty}^{2b-a} \exp\{-\frac{v^2}{2t}\} dv, \quad b \leq \min(a, 0),$$

we conclude:

$$\begin{aligned}
P\{W_t \leq a, W_{t^*} \leq b\} & = P\{W_{t^*} \leq b\} - P\{W_t > a, W_{t^*} \leq b\} \\
& = \frac{2}{\sqrt{2\pi t}} \int_{-\infty}^b \exp\{-\frac{v^2}{2t}\} dv - \frac{1}{\sqrt{2\pi t}} \int_{-\infty}^{2b-a} \exp\{-\frac{v^2}{2t}\} dv.
\end{aligned}$$

Taking into account the last relation, it is possible to rewrite the conditional probability in (2) in the following form:

$$\begin{aligned}
& P\{W_t \leq x, W_{t^*} \leq y | W_s = z, W_{s^*} = u\} \\
& = P\{\overline{W}_{t-s} \leq x - z, \min_{s < l \leq t} \overline{W}_{(t-s)^*} \leq y - z\} I_{\{u > y\}} \\
& = \frac{1}{\sqrt{2\pi(t-s)}} [2 \int_{-\infty}^{y-z} \exp\{-\frac{v^2}{2(t-s)}\} dv - \int_{-\infty}^{2y-x-z} \exp\{-\frac{v^2}{2(t-s)}\} dv].
\end{aligned}$$

Therefore, in concordance with properties of the Wiener process and conditional mathematical expectation, we can write

$$\begin{aligned}
P\{W_t \leq x, W_{t^*} \leq y | W_s = z\} &= [E(I_{\{W_t \leq x, W_{t^*} \leq y\}} | W_s)] |_{W_s=z} \\
&= \{E[E(I_{\{W_t \leq x, W_{t^*} \leq y\}} | W_s, W_{s^*}) | W_s]\} |_{W_s=z} \\
&= \{E[E(I_{\{W_t \leq x, W_{t^*} \leq y\}} | W_s = z, W_{s^*} = u) |_{z=W_s, u=W_{s^*}} | W_s]\} |_{W_s=z} \\
&= \{E[P\{W_t \leq x, W_{t^*} \leq y | W_s = z, W_{s^*} = u\} |_{z=W_s, u=W_{s^*}} | W_s]\} |_{W_s=z} \\
&= \left\{ E \left[\frac{2}{\sqrt{2\pi(t-s)}} \int_{-\infty}^{y-z} \exp \left\{ -\frac{v^2}{2(t-s)} \right\} dv \Big|_{z=W_s, u=W_{s^*}} \Big| W_s \right] \right\} |_{W_s=z} \\
&\quad - \left\{ E \left[\frac{1}{\sqrt{2\pi(t-s)}} \int_{-\infty}^{2y-x-z} \exp \left\{ -\frac{v^2}{2(t-s)} \right\} dv \Big|_{z=W_s, u=W_{s^*}} \Big| W_s \right] \right\} |_{W_s=z} \\
&= \left\{ E \left[\frac{2}{\sqrt{2\pi(t-s)}} \int_{-\infty}^{y-W_s} \exp \left\{ -\frac{v^2}{2(t-s)} \right\} dv \Big| W_s \right] \right\} |_{W_s=z} \\
&\quad - \left\{ E \left[\frac{1}{\sqrt{2\pi(t-s)}} \int_{-\infty}^{2y-x-W_s} \exp \left\{ -\frac{v^2}{2(t-s)} \right\} dv \Big| W_s \right] \right\} |_{W_s=z} \\
&= \frac{1}{\sqrt{2\pi(t-s)}} \left[2 \int_{-\infty}^{y-z} \exp \left\{ -\frac{v^2}{2(t-s)} \right\} dv - \int_{-\infty}^{2y-x-z} \exp \left\{ -\frac{v^2}{2(t-s)} \right\} dv \right].
\end{aligned}$$

From this it follows, that

$$\begin{aligned}
&\frac{\partial^2}{\partial x \partial y} P\{W_t \leq x, W_{t^*} \leq y | W_s = z\} \\
&= \frac{\partial}{\partial y} \left[\frac{1}{\sqrt{2\pi(t-s)}} \exp \left\{ -\frac{(2y-x-z)^2}{2(t-s)} \right\} \right] \\
&= \frac{2(x-2y+z)}{\sqrt{2\pi(t-s)^3}} \exp \left\{ -\frac{(x-2y+z)^2}{2(t-s)} \right\},
\end{aligned}$$

that ends the proof of proposition. \square

3. Main result

Theorem 3.1: For the Wiener functional $F = (W_T - K)^- I_{\{W_{T^*} \leq B\}}$ ($T > t, B \leq 0, B \leq K$) the following stochastic integral representation holds

$$F = EF - \int_0^T \Phi\left(\frac{2B - K - W_t}{\sqrt{T-t}}\right) dW_t, \quad (3)$$

where $\Phi(\cdot)$ is a standard normal distribution function.

Proof: According to the Markov property of the Wiener process and the well-known properties of conditional mathematical expectation, in accordance with the Proposition 2.2, we have

$$\begin{aligned} g_t &= E[F | \mathfrak{S}_t^W] = E[(W_T - K)^- I_{\{W_{T^*} \leq B\}} | \mathfrak{S}_t^W] \\ &= \{E[(W_T - K)^- I_{\{W_{T^*} \leq B\}} | W_t = z]\}_{z=W_t} \\ &= \left[- \int_{-\infty}^B \int_{-\infty}^K (x - K) \frac{2(x - 2y + z)}{\sqrt{2\pi(T-t)^3}} \exp\left\{-\frac{(x - 2y + z)^2}{2(T-t)}\right\} dx dy \right]_{z=W_t}. \end{aligned}$$

Using an integration formula in parts in the integral with respect to dx , it is not difficult to see that

$$\begin{aligned} & - \int_{-\infty}^B \int_{-\infty}^K (x - K) \frac{2(x - 2y + z)}{\sqrt{2\pi(T-t)^3}} \exp\left\{-\frac{(x - 2y + z)^2}{2(T-t)}\right\} dx dy \\ &= \int_{-\infty}^B \frac{1}{\sqrt{2\pi(T-t)}} \int_{-\infty}^K 2(x - K) d\left(\exp\left\{-\frac{(x - 2y + z)^2}{2(T-t)}\right\}\right) dy \\ &= \int_{-\infty}^B \frac{1}{\sqrt{2\pi(T-t)}} 2(x - K) \exp\left\{-\frac{(x - 2y + z)^2}{2(T-t)}\right\} \Big|_{-\infty}^K dy \\ & \quad - \int_{-\infty}^B \frac{2}{\sqrt{2\pi(T-t)}} \int_{-\infty}^K \exp\left\{-\frac{(x - 2y + z)^2}{2(T-t)}\right\} dx dy \\ &= - \int_{-\infty}^B \frac{2}{\sqrt{2\pi(T-t)}} \int_{-\infty}^K \exp\left\{-\frac{(x - 2y + z)^2}{2(T-t)}\right\} dx dy. \end{aligned}$$

Note, that at the end of calculations we have used the relation: $\lim_{x \rightarrow \infty} x \exp\{-x^2\} = 0$.

Therefore, we conclude that

$$g_t = - \int_{-\infty}^B \frac{2}{\sqrt{2\pi(T-t)}} \int_{-\infty}^K \exp \left\{ - \frac{(x-2y+W_t)^2}{2(T-t)} \right\} dx dy.$$

According to Proposition 2.1, it is not difficult to see that the obtained expression for g_t is stochastically differentiable ($g_t \in D_{2,1}$ for all $t \in [0, T)$). Therefore, basing on the rule of stochastic differentiation of the ordinary integral as well as composite function, we can write

$$D_s g_t = \int_{-\infty}^B \frac{2}{\sqrt{2\pi(T-t)}} \int_{-\infty}^K \frac{x-2y+W_t}{T-t} \exp \left\{ - \frac{(x-2y+W_t)^2}{2(T-t)} \right\} I_{[0,t]}(s) dx dy.$$

Further, using again the standard technique of integration, we easily obtain that

$$\begin{aligned} D_s g_t &= - \int_{-\infty}^B \frac{2}{\sqrt{2\pi(T-t)}} \left[\int_{-\infty}^K d \left(\exp \left\{ - \frac{(x-2y+W_t)^2}{2(T-t)} \right\} \right) \right] dy I_{[0,t]}(s) \\ &= - \int_{-\infty}^B \frac{2}{\sqrt{2\pi(T-t)}} \exp \left\{ - \frac{(K-2y+W_t)^2}{2(T-t)} \right\} dy I_{[0,t]}(s) \\ &= - \int_{-\infty}^B \frac{1}{\sqrt{2\pi(T-t)/4}} \exp \left\{ - \frac{(y-K/2-W_t/2)^2}{2(T-t)/4} \right\} dy I_{[0,t]}(s) \\ &= - \Phi_{0,(T-t)/4}(B-K/2-W_t/2) I_{[0,t]}(s), \end{aligned}$$

where $\Phi_{0,\sigma^2}(\cdot)$ is the distribution function of normal distributed random variable $N(0, \sigma^2)$ with mean 0 and variance σ^2 respectively ($\Phi(\cdot) = \Phi_{0,1}(\cdot)$).

As a consequence the elementary relation $cN(a, \sigma^2) \cong N(ca, c^2\sigma^2)$, we can rewrite the last equality in the following form

$$D_s g_t = - \Phi_{0,T-t}(2B-K-W_t) I_{[0,t]}(s).$$

Now let us pass to calculation of conditional mathematical expectation of $D_s g_t$ with respect to σ -algebra \mathfrak{S}_s^W .

Further, using the Markov property and the transition probabilities of the Wiener process, we have

$$\begin{aligned} E \left[\Phi_{0,T-t}(C-W_t) | \mathfrak{S}_s^W \right] &= E \left[\frac{1}{\sqrt{2\pi(T-t)}} \int_{-\infty}^{C-W_t} \exp \left\{ - \frac{u^2}{2(T-t)} \right\} du | \mathfrak{S}_s^W \right] \\ &= E \left[\frac{1}{\sqrt{2\pi(T-t)}} \int_{-\infty}^C \exp \left\{ - \frac{(u-W_t)^2}{2(T-t)} \right\} du | \mathfrak{S}_s^W \right] \end{aligned}$$

$$\begin{aligned}
 &= E\left[\frac{1}{\sqrt{2\pi(T-t)}} \int_{-\infty}^C \exp\left\{-\frac{(u-W_t)^2}{2(T-t)}\right\} du \mid W_s\right] \\
 &= \frac{1}{\sqrt{2\pi(T-t)}} \frac{1}{\sqrt{2\pi(t-s)}} \int_{-\infty}^{\infty} \left[\int_{-\infty}^C \exp\left\{-\frac{(u-x)^2}{2(T-t)}\right\} du \right] \exp\left\{-\frac{(x-W_s)^2}{2(t-s)}\right\} dx \\
 &= \frac{1}{\sqrt{2\pi(T-t)}} \frac{1}{\sqrt{2\pi(t-s)}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} I_{(-\infty, C)}(u) \exp\left\{-\frac{(u-x)^2}{2(T-t)} - \frac{(x-W_s)^2}{2(t-s)}\right\} dudx.
 \end{aligned}$$

According to the Fubini's theorem, highlighting the full square in the argument of the exponential function and using the properties of the distribution density function, it is not difficult to see that

$$\begin{aligned}
 &E[\Phi_{0, T-t}(C - W_t) | \mathfrak{S}_s^W] \\
 &= \frac{1}{\sqrt{2\pi(T-t)}} \frac{1}{\sqrt{2\pi(t-s)}} \\
 &\times \int_{-\infty}^{\infty} I_{(-\infty, C)}(u) \left[\int_{-\infty}^{\infty} \exp\left\{-\frac{(u-x)^2}{2(T-t)} - \frac{(x-W_s)^2}{2(t-s)}\right\} dx \right] du \\
 &= \frac{1}{\sqrt{2\pi(T-t)}} \frac{1}{\sqrt{2\pi(t-s)}} \\
 &\times \int_{-\infty}^C \left[\int_{-\infty}^{\infty} \exp\left\{-\frac{(u-x)^2}{2(T-t)} - \frac{(x-W_s)^2}{2(t-s)}\right\} dx \right] du \\
 &= \frac{1}{\sqrt{2\pi(T-t)}} \frac{1}{\sqrt{2\pi(t-s)}} \\
 &\times \int_{-\infty}^C \exp\left\{-\frac{(u-W_s)^2}{2(T-s)}\right\} \left\{ \int_{-\infty}^{\infty} \exp\left\{-\frac{\left[x - \frac{u(t-s)+W_s(T-t)}{T-s}\right]^2}{\frac{2(T-t)(t-s)}{T-s}}\right\} dx \right\} du \\
 &= \frac{1}{\sqrt{2\pi(T-t)}} \frac{1}{\sqrt{2\pi(t-s)}} \sqrt{2\pi \frac{(T-t)(t-s)}{T-s}} \int_{-\infty}^C \exp\left\{-\frac{(u-W_s)^2}{2(T-s)}\right\} du
 \end{aligned}$$

$$= \frac{1}{\sqrt{2\pi(T-s)}} \int_{-\infty}^C \exp\left\{-\frac{(u-W_s)^2}{2(T-s)}\right\} du = \Phi_{0,T-s}(C-W_s).$$

Thus, we have

$$E[\Phi_{0,T-t}(2B-K-W_t)|\mathfrak{S}_s^W] = \Phi_{0,T-s}(2B-K-W_s).$$

Now, combining all the relations obtained above, we easily conclude that

$$E[D_s g_t | \mathfrak{S}_s^W] = -\Phi_{0,T-s}(2B-K-W_s)I_{[0,t]}(s).$$

Passing now to the limit in the latter expression as $t \rightarrow T$ we obtain

$$\begin{aligned} \nu_s &= \lim_{t \rightarrow T} E[D_s g_t | \mathfrak{S}_s^W] = -\Phi_{0,T-s}(2B-K-W_s)I_{[0,T]}(s) \\ &= -\Phi_{0,1}\left(\frac{2B-K-W_s}{\sqrt{T-s}}\right)I_{[0,T]}(s). \end{aligned}$$

From here, using Theorem 2.3, we complete the proof of the theorem. \square

Corollary 3.2: *Taking in Theorem 3.1 $B = K$, we will see that the Wiener functional $F = (W_T - K)^-$ permits the following stochastic integral representation*

$$F = EF - \int_0^T \Phi\left(\frac{K-W_t}{\sqrt{T-t}}\right) dW_t,$$

where

$$EF = K\Phi\left(\frac{K}{\sqrt{T}}\right) + T\varphi\left(\frac{K}{\sqrt{T}}\right).$$

Really, in the case $B = K$ it is evident that $(W_T - K)^- I_{\{W_{T^*} \leq K\}} = (W_T - K)^-$. On the other hand, we have

$$\begin{aligned} E(W_T - K)^- &= \frac{1}{\sqrt{2\pi T}} \int_{-\infty}^K (K-x) \exp\left\{-\frac{x^2}{2T}\right\} dx \\ &= \frac{K}{\sqrt{2\pi T}} \int_{-\infty}^K \exp\left\{-\frac{x^2}{2T}\right\} dx + \frac{T}{\sqrt{2\pi T}} \int_{-\infty}^K d\left(\exp\left\{-\frac{x^2}{2T}\right\}\right) \\ &= K\Phi\left(\frac{K}{\sqrt{T}}\right) + T\varphi\left(\frac{K}{\sqrt{T}}\right). \end{aligned}$$

Remark 1: It is not difficult to see that the same result can be obtained from

result [8]. Indeed, according to Theorem 2.4 we have

$$\begin{aligned}
(W_T - K)^- &= E(W_T - K)^- + \int_0^T E \left[\left\{ \frac{\partial}{\partial x} (x - K)^- \right\} \Big|_{x=W_T} \Big| \mathfrak{S}_t^W \right] dW_t \\
&= E(W_T - K)^- - \int_0^T E [I_{\{W_T \leq K\}} | \mathfrak{S}_t^W] \\
&= K\Phi\left(\frac{K}{\sqrt{T}}\right) + T\varphi\left(\frac{K}{\sqrt{T}}\right) - \int_0^T E [I_{\{W_T \leq K\}} | W_t] dW_t \\
&= K\Phi\left(\frac{K}{\sqrt{T}}\right) + T\varphi\left(\frac{K}{\sqrt{T}}\right) - \int_0^T E [I_{\{W_T - W_t \leq K - x\}} | W_t = x] \Big|_{x=W_t} dW_t \\
&= K\Phi\left(\frac{K}{\sqrt{T}}\right) + T\varphi\left(\frac{K}{\sqrt{T}}\right) - \int_0^T P\{W_T - W_t \leq K - x\} \Big|_{x=W_t} dW_t \\
&= K\Phi\left(\frac{K}{\sqrt{T}}\right) + T\varphi\left(\frac{K}{\sqrt{T}}\right) - \int_0^T [\Phi_{0, T-t}(K - W_t)] dW_t \\
&= K\Phi\left(\frac{K}{\sqrt{T}}\right) + T\varphi\left(\frac{K}{\sqrt{T}}\right) - \int_0^T \Phi\left(\frac{K - W_t}{\sqrt{T - t}}\right) dW_t.
\end{aligned}$$

Corollary 3.3: *In case $B = K = 0$, we obtain the known result (see, for example, [8])*

$$W_T^- = \sqrt{\frac{T}{2\pi}} + \int_0^T \left[\Phi\left(\frac{W_t}{\sqrt{T - t}}\right) - 1 \right] dW_t.$$

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