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On Some Dynamical Problems of the Antiplane Strain (Shear) of Isotropic Non-Homogeneous Prismatic Shell-Like Bodies

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Dynamical problem of the antiplane strain (shear) of an isotropic non-homogeneous prismatic shell-like body is considered when the shear modulus depending on the body projection (i.e., on a domain lying in the plane of interest) variables vanishes either on a part or on the entire boundary of the projection.

Keywords: Antiplane strain (shear) deformation, Degenerate partial differential equations.

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1. Introduction

We consider dynamical problems of the antiplane strain (shear) of an isotropic non-homogeneous prismatic shell-like body. The motion equation has the following form (see [1])

$$(\mu(x_1, x_2)u_{3,\alpha}(x_1, x_2, t))_{,\alpha} + \Phi_3(x_1, x_2, t) = \rho\ddot{u}_3(x_1, x_2, t), \tag{1}$$

where u_3 is a displacement vector component, μ is the shear modules, ρ is the density, Φ_3 is the component of the volume force. The static problems of the antiplane strain (shear) of isotropic non-homogeneous prismatic shell-like bodies are investigated in [1], where peculiarities of the correct setting of boundary conditions are clarified. In particular, some boundary value problems are solved in the explicit form.

The aim of the present paper is to investigate initial boundary value problems for the symmetric prismatic shell-like body (see [2,3]), when the shear modulus may vanish (i.e. degenerate) on the boundary of the projection on the plane of interest Ox_1x_2.

Admissible dynamical problems for cylindrical vibrations are investigated for a plate with a strip as a middle plane and a variable shear modules which may vanish at the edges of the strip. The setting of boundary conditions at the boundaries of the strip depends on the kind of degeneration of the shear modules, while setting

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of initial conditions is independent of it. In some cases the problem under consideration leads to the second type integral equations with symmetric kernels. To this end Hilbert-Schmidt theory is used.

2. Cylindrical vibration

Let us consider the shell-like body whose projection on $x_3 = 0$ occupies the domain ω

$$\omega := \{(x_1; x_2) : -\infty < x_1 < \infty; 0 \le x_2 \le l\}$$

and the shear modulus $\mu(x_2) \in C^1(\omega) \cap C(\overline{\omega})$ satisfies the following conditions

$$\mu(x_2) \begin{cases} > 0, \ x_2 \in (0, l); \\ \ge 0, \text{ at the boundaries } x_2 = 0 \text{ and } x_2 = l, \end{cases}$$

where $l = const > 0$ is the width of the body projection.

Since all the quantities depend only on one space variable x_2, equation (1) can be rewritten as follows

$$(\mu(x_2)u_{3,2}(x_2,t))_{,2} + \Phi_3(x_2,t) = \rho\ddot{u}_3(x_2,t), \ x_2 \in]0,l[, \ t > 0. \tag{2}$$

Equation (2) is a degenerate hyperbolic equation which we will solve with the following initial conditions (ICs)

$$u_3(x_2, 0) = \varphi_1(x_2), \ \ u_{3,t}(x_2, 0) = \varphi_2(x_2), \tag{3}$$

where $\varphi_i(x_2) \in C^2(]0,l[)$, $i = 1, 2$, are given functions.

Let us denote by I_0 and I_l the following integrals

$$I_0 := \int_0^{\varepsilon} \frac{d\xi}{\mu(\xi)}, \ \ \varepsilon = const > 0,$$

$$I_l := \int_{l-\varepsilon}^{l} \frac{d\xi}{\mu(\xi)}, \ \ \varepsilon = const > 0.$$

Problem 1: Let

$$I_0 < +\infty, \quad I_l < +\infty.$$

Find

$$\begin{aligned} &u_3(\cdot, t) \in C^2(]0, l[) \cap C([0, l]), \\ &u_3(x_2, \cdot) \in C^2(t > 0) \cap C^1(t \ge 0), \ \ u_3(x_2, t) \in C(0 \le x_2 \le l, \ t \ge 0), \end{aligned} \tag{4}$$

satisfying equation (2), the boundary conditions (BCs)

$$u_3(0,t) = u_3(l,t) = 0 \tag{5}$$

and ICs (3), where

$$\varphi_i(x_2) \in C^2(]0,l[) \cap C([0,l]), \quad i = 1,2, \tag{6}$$

$$\varphi_i(0) = \varphi_i(l) = 0, \quad i = 1,2. \tag{7}$$

Problem 2: Let

$$I_0 < +\infty, \quad I_l \leq +\infty.$$

Find

$$u_3(\cdot,t) \in C^2(]0,l[) \cap C([0,l[), \quad (\mu u_{3,2})(\cdot,t) \in C([0,l]),$$

$$u_3(x_2,\cdot) \in C^2(t>0) \cap C^1(t \geq 0), \quad u_3(x_2,t) \in C(0 \leq x_2 < l, \ t \geq 0),$$

satisfying equation (2), BCs

$$u_3(0,t) = (\mu u_{3,2})(l,t) = 0, \tag{8}$$

and ICs (3), where

$$\varphi_i(x_2) \in\in C^2(]0,l[) \cap C([0,l[), \quad (\mu\varphi_{i,2})(x_2) \in C([0,l]), \quad i = 1,2,$$

$$\varphi_i(0) = (\mu\varphi_{i,2})(l) = 0, \quad i = 1,2.$$

Solution (of the Problem 1). We use the Fourier method and, consequently, we are looking for $u_3(x_2,t)$ in the following form

$$u_3(x_2,t) = X(x_2)T(t). \tag{9}$$

Let first $\Phi_3 \equiv 0$. Then from (2) we get

$$\frac{(\mu(x_2)X'(x_2))'}{\rho X(x_2)} = \frac{T''(t)}{T(t)} = -\lambda = \text{const},$$

Hence,

$$T''(t) + \lambda T(t) = 0 \tag{10}$$

and

$$(\mu(x_2)X'(x_2))' = -\lambda\rho X(x_2). \tag{11}$$

Now, in view of (4), (5) we have to solve the following BVP:
Find

$$X(x_2) \in C^2(]0,l[) \cap C([0,l]), \tag{12}$$

which satisfies equation (11) and BCs

$$X(0) = X(l) = 0. \tag{13}$$

After two times integration of (11) and using BCs (13) for $X(x_2)$, we obtain

$$X(x_2) = \lambda \int_0^l K(x_2,\xi)X(\xi)d\xi, \tag{14}$$

where

$$K(x_2,\xi) = \begin{cases} \dfrac{\rho}{\Delta}\displaystyle\int\limits_{x_2}^{l} \dfrac{d\eta}{\mu(\eta)} \int\limits_{0}^{\xi} \dfrac{d\eta}{\mu(\eta)}, \ 0 \le \xi \le x_2, \\ \\ \dfrac{\rho}{\Delta}\displaystyle\int\limits_{\xi}^{l} \dfrac{d\eta}{\mu(\eta)} \int\limits_{0}^{x_2} \dfrac{d\eta}{\mu(\eta)}, \ \xi \le x_2 \le l. \end{cases} \tag{15}$$

Evidently,

$$\Delta := \int_0^l \frac{d\eta}{\mu(\eta)} \neq 0.$$

Proposition 2.1: *$K(x_2,\xi)$ is a symmetric with respect to x_2 and ξ.*

Proposition 2.2: *Number of eigenvalues λ_n is not finite.*

Proof: Let it be finite, and $n = \overline{1,m}$. Then we can express $K(x_2,\xi)$ as follows (see, e.g., [4])

$$K(x_2,\xi) = \sum_{n=1}^{m} \frac{X_n(x_2)X_n(\xi)}{\lambda_n},$$

where $X_n(x_2) \in C^2(]0,l[)$, i.e.,

$$K(x_2,\xi) \in C^2(]0,l[\times]0,l[). \tag{16}$$

On the other hand, by virtue of (15),

$$K'_{x_2}(x_2,\xi)|_{\xi\to x_2-} - K'_{x_2}(x_2,\xi)|_{\xi\to x_2+} = -\frac{\rho}{\mu(x_2)},$$

then kernel

$$K(x_2,\xi) \notin C^2(]0,l[\times]0,l[). \tag{17}$$

But, (16) and (17) contradict to each other, thus the number of λ_n cannot be finite. □

Proposition 2.3: *All of λ_n are positive.*

Proof: Let X_n be orthonormalized eigenfunctions (it can be assumed without loss of generality) and let us multiply both sides of the following equation

$$(\mu(x_2)X_n'(x_2))' = -\lambda_n \rho X_n(x_2), \tag{18}$$

by X_n and integrate the obtained from 0 to l, then by virtue of BCs (13), we get

$$\begin{aligned} -\lambda_n\rho &= \int_0^l (\mu(x_2)X_n'(x_2))' X_n(x_2)dx_2 \\ &= \mu(x_2)X_n'(x_2)X_n(x_2)\big|_0^l - \int_0^l \mu(x_2)X_n'(x_2)X_n'(x_2)dx_2 \\ &= -\int_0^l \mu(x_2)\left[X_n'(x_2)\right]^2 dx_2 \le 0. \end{aligned}$$

Hence, $\lambda_n > 0$ for any n, provided $X_n \not\equiv 0$. □

The solution of (10) can be written as follows

$$T_n(t) = b_1^n \sin\left(\sqrt{\lambda_n}t\right) + b_2^n \cos\left(\sqrt{\lambda_n}t\right), \quad b_i^n = const, \quad i=1,2.$$

Now, we can formally represent the solution of the Problem 1 in the following form

$$u_3(x_2,t) = \sum_{n=1}^{\infty} X_n(x_2)\left(b_1^n \sin\left(\sqrt{\lambda_n}t\right) + b_2^n \cos\left(\sqrt{\lambda_n}t\right)\right). \tag{19}$$

In view of the initial conditions (3), we formally have

$$\sum_{n=1}^{\infty} X_n(x_2)b_2^n = \varphi_1, \quad \sum_{n=1}^{\infty} \sqrt{\lambda_n}X_n(x_2)b_1^n = \varphi_2. \tag{20}$$

If $\psi_i(x_2) := (\mu(x_2)\varphi_i'(x_2))'$ is an integrable function on $[0,l]$, taking into account symmetry of $K(x_2,\xi) \in C([0,l]\times[0,l])$, we get absolutely and uniformly convergence of the series (see, e.g. [4,5])

$$\varphi_i(x_2) = \sum_{n=1}^{\infty} \int_0^l \varphi_i(\xi)X_n(\xi)d\xi \cdot X_n(x_2), \quad i=1,2,$$

i.e., of the series (20) on $[0,l]$, and

$$b_1^n = \frac{1}{\sqrt{\lambda_n}} \int\limits_0^l X_n(x_2)\varphi_2(x_2)dx_2, \quad b_2^n = \int\limits_0^l X_n(x_2)\varphi_1(x_2)dx_2. \tag{21}$$

Therefore, the series (19) can be estimated as follows

$$|u_3(x_2,t)| \le \sum_{n=1}^{\infty} |X_n(x_2)b_1^n| + \sum_{n=1}^{\infty} |X_n(x_2)b_2^n|.$$

Since there exists positive minimum of the eigenvalues, from the convergence of (20) we obtain that (19) converges absolutely and uniformly on $[0,l]$.

After formal differentiation of (19) with respect to t we get

$$u_{3,t}(x_2,t) = \sum_{n=1}^{\infty} X_n(x_2)\sqrt{\lambda_n}\left(b_1^n \cos(\sqrt{\lambda_n}t) - b_2^n \sin(\sqrt{\lambda_n}t)\right), \tag{22}$$

$$u_{3,tt}(x_2,t) = -\sum_{n=1}^{\infty} X_n(x_2)\lambda_n\left(b_1^n \sin(\sqrt{\lambda_n}t) + b_2^n \cos(\sqrt{\lambda_n}t)\right). \tag{23}$$

Theorem 2.4: *The series (22) – (23) are convergent absolutely and uniformly on any* $[a,b] \in]0,l[$ *if*

$$\psi_i(0) = \psi_i(l) \ \text{ for } \ i = 1,2, \tag{24}$$

and

$$\chi_i(x_2) := \left(\mu(x_2)\psi_i'(x_2)\right)', \ i=1,2, \ \textit{ are integrable ones on }]0,l[. \tag{25}$$

Proof: Substituting into (21) the function $X_n(x_2)$ found from (18), we get

$$b_1^n = \frac{1}{\rho\lambda_n\sqrt{\lambda_n}} \int\limits_0^l (\mu(x_2)X_n'(x_2))'\varphi_2(x_2)dx_2$$

$$= \frac{1}{\rho\lambda_n\sqrt{\lambda_n}} \left\{ (\mu(x_2)X_n'(x_2))\varphi_2(x_2)\big|_0^l - \int\limits_0^l \mu(x_2)X_n'(x_2)\varphi_2'(x_2)dx_2 \right\}$$

$$= \frac{1}{\rho\lambda_n\sqrt{\lambda_n}} \left\{ -\mu(x_2)X_n(x_2)\varphi_2'(x_2)\big|_0^l + \int\limits_0^l X_n(x_2)(\mu(x_2)\varphi(x_2)_2')'dx_2 \right\}$$

$$= \frac{1}{\rho \lambda_n \sqrt{\lambda_n}} \int_0^l X_n(x_2) \psi_2(x_2) dx_2. \tag{26}$$

Analogously,

$$b_2^n = \frac{1}{\rho \lambda_n} \int_0^l X_n(x_2) \psi_1(x_2) dx_2. \tag{27}$$

From (25), in view of (24), $\psi_i(x_2)$ can be expressed as follows

$$\psi_i(x_2) = \int_0^l K(x_2, \xi) \chi_i(\xi) d\xi, \quad i = 1, 2.$$

The following series (see, e.g., [4,5])

$$\sum_{n=1}^{\infty} \beta_i^n X_n(x_2),$$

where

$$\beta_i^n = \int_0^l X_n(x_2) \psi_i(x_2) dx_2, \quad i = 1, 2,$$

is convergent absolutely and uniformly on $]0, l[$, i.e.,

$$\sum_{n=1}^{\infty} |\beta_i^n| |X_n(x_2)| < +\infty, \quad i = 1, 2. \tag{28}$$

Further, from (22)

$$\begin{aligned}
|u_{3,t}(x_2, t)| &= \left| \sum_{n=1}^{\infty} X_n(x_2) \sqrt{\lambda_n} \left(b_1^n \cos(\sqrt{\lambda_n} t) - b_2^n \sin(\sqrt{\lambda_n} t) \right) \right| \\
&\leq \left| \sum_{n=1}^{\infty} X_n(x_2) \sqrt{\lambda_n} b_1^n \cos(\sqrt{\lambda_n} t) \right| \\
&+ \left| \sum_{n=1}^{\infty} X_n(x_2) \sqrt{\lambda_n} b_2^n \sin(\sqrt{\lambda_n} t) \right| \\
&\leq \sum_{n=1}^{\infty} \left| X_n(x_2) \sqrt{\lambda_n} b_1^n \right| + \sum_{n=1}^{\infty} \left| X_n(x_2) \sqrt{\lambda_n} b_2^n \right|.
\end{aligned} \tag{29}$$

According to Proposition 2.3, all of λ_n are positive. Therefore, we can find λ_0 such that $\lambda_0 \leq \min\limits_{1\leq i\leq\infty}\{\lambda_i\}$, and, by virtue of (26)-(28), we obtain

$$\sum_{n=1}^{\infty}\left|X_n(x_2)\sqrt{\lambda_n}b_2^n\right| = \sum_{n=1}^{\infty}\left|X_n\sqrt{\lambda_n}\frac{1}{\lambda_n}\beta_1^n\right| \leq \frac{1}{\sqrt{\lambda_0}}\sum_{n=1}^{\infty}|X_n||\beta_1^n| < \infty,$$

$$\sum_{n=1}^{\infty}\left|X_n(x_2)\sqrt{\lambda_n}b_1^n\right| = \sum_{n=1}^{\infty}\left|X_n\sqrt{\lambda_n}\frac{1}{\lambda_n\sqrt{\lambda_n}}\beta_2^n\right| \leq \frac{1}{\lambda_0}\sum_{n=1}^{\infty}|X_n||\beta_2^n| < \infty, \quad x_2 \in]0,l[.$$

Hence, the series in (29) are convergent. Thus, (22) is convergent absolutely and uniformly on $]0,l[$. Similarly, we get the absolute and uniform convergence of (23) on $]0,l[$. □

Let us now differentiate (19) formally i-times with respect to x_2 and consider the following expressions

$$\frac{\partial^i}{\partial x_2^i}u_3(x_2,t) = \sum_{n=1}^{\infty}\frac{d^i}{dx_2^i}X_n(x_2)\left(b_1^n\sin(\sqrt{\lambda_n}t) + b_2^n\cos(\sqrt{\lambda_n}t)\right), i = 1,2, \qquad (31_{\mathrm{i}})$$

Theorem 2.5: *The series* (31_i) $(i = 1, ..., 4)$ *are convergent absolutely and uniformly on any* $[a,b] \in]0,l[$.

Proof: Obviously, in view of (11) and (15), we get

$$X_n'(x_2) = \lambda_n\int_0^l K_1(x_2,\xi)X_n(\xi)d\xi, \qquad (32)$$

where

$$K_1(x_2,\xi) = \begin{cases} -\dfrac{\rho}{\Delta}\dfrac{\int_0^{\xi}\frac{d\eta}{\mu(\eta)}}{\mu(x_2)}, & 0 < \xi \leq x_2, \\[2ex] \dfrac{\rho}{\Delta}\dfrac{\int_{\xi}^{l}\frac{d\eta}{\mu(\eta)}}{\mu(x_2)}, & x_2 \leq \xi < l, \end{cases}$$

and

$$K_1(x_2,\xi) \in C(]0,l[\times]0,l[), \qquad (33)$$

because of $I_0,\ I_l < +\infty$.

Substituting (32) into (31_1), we obtain

$$\frac{\partial}{\partial x_2}u_3(x_2,t) = \sum_{n=1}^{\infty}\lambda_n\int_0^l K_1(x_2,\xi)X_n(\xi)d\xi\left(b_1^n\sin(\sqrt{\lambda_n}t) + b_2^n\cos(\sqrt{\lambda_n}t)\right) =$$

$$= \int_0^l K_1(x_2,\xi)\left[\sum_{n=1}^{\infty} X_n(\xi)\lambda_n\left(b_1^n \sin(\sqrt{\lambda_n}t) + b_2^n \cos(\sqrt{\lambda_n}t)\right)\right] d\xi, \qquad (34)$$

since (23) is absolutely and uniformly convergent on $]0,l[$, in view of (33) and $X_n(x_2) \in C([0,l])$, we conclude that the corresponding integral in (34) is absolutely convergent on $]0,l[$. Similarly, we can prove the convergence of the series (31_2), on $]0,l[$. □

Thus, (19) is the solution of the Problem 1 for $\Phi_3(x_2,t) \equiv 0$.

Now, let us consider Problem 1 when $\Phi_3(x_2,t) \not\equiv 0$, $\varphi_i = 0$, $i = 1,2$, and let $\Phi_3(x_2,t)$ be represented as a convergent series in $C(0,l)$ (see, e.g. [4]):

$$\Phi_3(x_2,t) = \sum_{n=1}^{\infty} \left(\Phi_3(x_2,t), X_n\right) X_n = \sum_{n=1}^{\infty} (\Phi_3, X_n) X_n, \qquad (35)$$

then,

$$\Phi_3(x_2,t) = \sum_{n=1}^{\infty} X_n(x_2)\Phi_{3n}(t), \quad \Phi_{3n}(t) := \int_0^l \Phi_3(x_2,t) X_n(x_2) dx_2.$$

Further, we are looking for a solution in the form

$$u_3(x_2,t) = \sum_{n=1}^{\infty} u_{3n}(x_2,t),$$

where $u_{3n}(x_2,t)$ is a solution of the Problem 1 with $\Phi_3(x_2,t)$ replaced by $X_n(x_2)\Phi_{3n}(t)$. Using the method of separation of variables, we can write

$$u_{3n}(x_2,t) = X_n(x_2) T_{1n}(t),$$

where

$$T_{1n}''(t) + \lambda_n T_{1n}(t) = \Phi_{3n}(t)$$

and $X_n(x_2)$ satisfies (18).

Therefore, $u_3(x_2,t)$ can be expressed as follows

$$u_3(x_2,t) = \sum_{n=1}^{\infty} \frac{1}{\sqrt{\lambda_n}} X_n \int_0^t \sin(\sqrt{\lambda_n}(t-\tau))\Phi_{3n}(\tau) d\tau. \qquad (36)$$

Now, similarly to the proofs of Theorems 2.4 and 2.5, because of (35) we get the absolute and uniform convergence of the series (36) on $[0,l]$, and the absolute and uniform convergence of the series

$$\frac{\partial^i}{\partial x_2^i} u_3(x_2,t) = \sum_{n=1}^{\infty} \frac{d^i}{dx_2^i} X_n(x_2) T_{1n}(t), \quad i = 1, 2,$$

$$\frac{\partial^i}{\partial t^i} u_3(x_2,t) = \sum_{n=1}^{\infty} X_n(x_2) \frac{d^i}{dt^i} T_{1n}(t), \quad i = 1, 2,$$

on any $[a, b] \in]0, l[$.

Remark 1: If $\Phi_3(x_2,t)$, $\varphi_i(x_2) \not\equiv 0$, then the solution of the Problem 1 can be expressed as follows

$$u_3(x_2,t) = \sum_{n=1}^{\infty} u_{3n}(x_2,t),$$

where

$$u_{3n}(x_2,t) = X_n(x_2)(T_{1n}(t) + T_n(t)),$$

$X_n(x_2)T_{1n}(t)$ is given by the right hand side of (36) and $X_n(x_2)T_n(t)$ is given by the right hand side of the formula (19).

Remark 2: Similarly, we can solve Problem 2 for $\Phi_3(x_2,t) \not\equiv 0$.

Remark 3: In case $\mu(x_2)$ has we following form

$$\mu(x_2) = \mu_0 x_2^{\alpha} (l - x_2)^{\beta}, \quad \mu_0, l = const > 0,$$

$$\alpha, \beta = const \geq 0, \quad x_2 \in [0, l],$$

the conditions

$$I_0 := \int_0^{\varepsilon} \frac{d\xi}{\mu(\xi)} < +\infty, \quad I_l := \int_{l-\varepsilon}^{l} \frac{d\xi}{\mu(\xi)} < +\infty$$

are equivalent to

$$\alpha < 1, \text{ and } \beta < 1,$$

respectively.

The sufficient conditions (24)-(25) will be fulfilled if

$$\frac{d^j}{dx_2^j} \varphi_i(x_2) = O(x_2^{\gamma_{ij}}), \quad \gamma_{ij} = const \geq 3 - j - 2\alpha, \quad x_2 \to 0_+ \; i = 1, 2, \; j = \overline{1, 4},$$

$$\frac{d^j}{dx_2^j}\varphi_i(x_2) = O((l - x_2)^{\delta_{ij}}), \quad \delta_{ij} = const \geq 3 - j - 2\beta, \quad x_2 \to l_-, \ i = 1, 2, \ j = \overline{1, 4}.$$

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