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On Some Solutions in the Fully Coupled Theory of Steady Vibrations for Solids with Double Porosity

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In this paper the 2D fully coupled theory of steady vibrations of poroelasticity for materials with double porosity is considered. The fundamental and singular matrices of solutions are obtained in terms of elementary functions. The single and double layer potentials are constructed and the basic properties of these potentials are established.

Keywords: Fully coupled theory of elasticity, Double porosity, Fundamental solution, Steady vibration.

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1. Introduction

In a material with two degrees of porosity there are two pore systems, the primary and the secondary. For example, in a fissured rock (i.s., a mass of porous blocks separated from each other by an interconnected and continuously distributed system of fissures) most of the porosity is provided by the pores of the blocks or primary porosity, while most of permeability is provided by the fissures or the secondary porosity. The physical and mathematical foundations of the theory of consolidation with double porosity was first proposed by Aifantis and co-authors in the papers [1-3](see [1-3] and the references cited therein.) This theory unifies a model proposed by Biot for the consolidation of deformable single porosity media with a model proposed by Barenblatt for seepage in undeformable media with two degrees of porosity. The basic results and the historical information on the theory of porous media may be found in book [4]. However, Aifantis' quasi-static theory ignores the cross-coupling effect between the volume change of the pores and fissures in the system. The cross-coupled terms were included in the equations of conservation of mass for the pore and fissure fluid and in Darcy's law for solids with double porosity by Khalili N. et al. in [5,6]. The phenomenological equations of the quasi-static theory for double porous media are established in [7,8], where a method to calculate the relevant coefficients is also presented. In [9] the fundamental solution in the theory of consolidation with double porosity is constructed. The fully coupled 3D linear theory of elasticity for solids with double porosity for the dynamical equations is considered in [10]. Uniqueness and existence theorems

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of solutions of the 2-dimensional boundary value problems of the theory of consolidation with double porosity are proved in [11]. Explicit solutions of the BVPs of the theory of consolidation with double porosity for Aifantis equations for the half-plane and half-space are considered in [12-13]. Explicit solutions of the BVPs of the fully coupled theory of elasticity for half-plane with double porosity are constructed in [14].

In the last years many authors have investigated the BVPs of the 2-dimensional and 3-dimensional theories of elasticity for materials with double porosity, publishing a large number of papers (for details see [15-23] and references therein).

In this paper the 2D fully coupled theory of steady vibrations of poroelasticity for materials with double porosity is considered. The fundamental and singular matrices of solutions are obtained in terms of elementary functions. The single and double layer potentials are constructed and the basic properties of these potentials are established.

2. Basic equations

Let $\mathbf{x} = (x_1, x_2)$ be a point of the Euclidean 2D space E^2. Let D^+ be a bounded 2D domain (with smooth boundary S) and let D^- be the complement of $D^+ \cup S$. We consider an isotropic material with double porosity that occupies a region $D^+(D^-)$ of space E^2.

The system of homogeneous equations of steady vibrations in the 2D fully coupled linear theory of elasticity for solids with double porosity can be written as follows [10]

$$\begin{aligned}
&\mu\Delta\mathbf{u} + (\lambda+\mu) graddiv\mathbf{u} - grad(\beta_1 p_1 + \beta_2 p_2) + \rho\omega^2\mathbf{u} = 0,\\
&i\omega\beta_1 div\mathbf{u} + (k_1\Delta + a_1)p_1 + (k_{12}\Delta + a_{12})p_2 = 0, \qquad (1)\\
&i\omega\beta_2 div\mathbf{u} + (k_{21}\Delta + a_{21})p_1 + (k_2\Delta + a_2)p_2 = 0,
\end{aligned}$$

where $\mathbf{u} = (u_1, u_2)^T$ is the displacement vector in a solid, p_1 and p_2 are the pore and fissure fluid pressures, respectively. $a_j = i\omega\alpha_j - \gamma$, $a_{ij} = i\omega\alpha_{ij} + \gamma$, $\omega > 0$ is the oscillation frequency, β_1 and β_2 are the effective stress parameters, $\gamma > 0$ is the internal transport coefficient and corresponds to a fluid transfer rate with respect to the intensity of flow between the pores and fissures, α_1 and α_2 measure the compressibilities of the pore and fissure system, respectively; α_{12} and α_{21} are the cross-coupling compressibility for fluid flow at the interface between the two-pore systems at a microscopic level [5,6]. However the coupled effect (α_{12} and α_{21}) is often neglected (see [1-3]). λ, μ, are constitutive coefficients, $k_j = \frac{\kappa_j}{\mu'}$, $k_{12} = \frac{\kappa_{12}}{\mu'}$, $k_{21} = \frac{\kappa_{21}}{\mu'}$; μ' is the fluid viscosity, κ_1 and κ_2 are the macroscopic intrinsic permeabilities associated with matrix and fissure porosity respectively, κ_{12} and κ_{21} are the cross-coupling permeabilities for fluid flow at the interface between the matrix and fissure phases. The cross-coupling terms with coefficients κ_{12} and κ_{21} are considered in [7,8]. We note that in the real porous media the fissure permeability κ_2 is much greater than the matrix permeability κ_1 while the fracture porosity is much smaller than the matrix porosity. Δ is the 2D Laplace operator. The superscript "T" denotes transposition.

We introduce the matrix differential operator

$$\mathbf{A}(\partial_{\mathbf{x}},\omega)=\parallel A_{lj}(\partial_{\mathbf{x}})\parallel_{4x4},\;\; l,j=1,2,3,4,$$

where

$$A_{lj}(\partial_{\mathbf{x}}):=\delta_{lj}(\mu\Delta+\rho\omega^2)+(\lambda+\mu)\frac{\partial^2}{\partial x_l\partial x_j},\quad l,j=1,2,$$

$$A_{j3}(\partial_{\mathbf{x}}):=-\beta_1\frac{\partial}{\partial x_j},\quad A_{j4}:=-\beta_2\frac{\partial}{\partial x_j}\quad j=1,2$$

$$A_{3j}(\partial_{\mathbf{x}}):=i\omega\beta_1\frac{\partial}{\partial x_j},\quad A_{4j}(\partial_{\mathbf{x}}):=i\omega\beta_2\frac{\partial}{\partial x_j},j=1,2,$$

$$A_{33}(\partial_{\mathbf{x}}):=k_1\Delta+a_1,\quad A_{44}(\partial_{\mathbf{x}}):=k_2\Delta+a_2,$$

$$A_{34}(\partial_{\mathbf{x}}):=k_{12}\Delta+a_{12},\quad A_{43}(\partial_{\mathbf{x}}):=k_{21}\Delta+a_{21}$$

$\delta_{\alpha\gamma}$ is the Kronecker delta. Then the system (1) can be rewritten as

$$\mathbf{A}(\partial_{\mathbf{x}},\omega)\mathbf{U}=0, \tag{2}$$

where $\mathbf{U}=(u_1,u_2,p_1,p_2)^T$.

Let us consider the system of the equations

$$\begin{aligned}&\mu\Delta\mathbf{u}+\rho\omega^2\mathbf{u}+(\lambda+\mu)graddiv\mathbf{u}+i\omega grad(\beta_1p_1+\beta_2p_2)=0,\\&-\beta_1div\mathbf{u}+(k_1\Delta+a_1)p_1+(k_{21}\Delta+a_{21})p_2=0,\\&-\beta_2div\mathbf{u}+(k_{12}\Delta+a_{12})p_1+(k_2\Delta+a_2)p_2=0,\end{aligned} \tag{3}$$

As one may easily verify, the system (3) may be written in the form

$$\mathbf{A}^T(\partial_{\mathbf{x}},\omega)\mathbf{U}=0. \tag{4}$$

where $\mathbf{A}^T(\partial_{\mathbf{x}},\omega)$ is the transpose of matrix $\mathbf{A}(\partial_{\mathbf{x}},\omega)$.

We assume that $\mu\mu_0(k_1k_2-k_{12}k_{21})\neq 0$, where $\mu_0:=\lambda+2\mu$. Obviously, if the last condition is satisfied, then $\mathbf{A}(\partial_{\mathbf{x}},\omega)$ is the elliptic differential operator.

3. The basic fundamental matrix

In this section, we will construct the basic fundamental matrix of solutions for the system (2).

By the direct calculation, we get

$$det\mathbf{A}=\mu\mu_0\alpha_0(\Delta+\lambda_1^2)(\Delta+\lambda_2^2)(\Delta+\lambda_3^2)(\Delta+\lambda_4^2),$$

where λ_j^2, $j = 1, 2, 3$, are roots of cubic algebraic equation

$$\begin{aligned}&\mu_0\alpha_0\xi^3 - [\mu_0 k_0 + i\omega\alpha_{11} + \rho\omega^2\alpha_0]\xi^2\\&+[\mu_0(a_1a_2 - a_{12}a_{21}) + i\omega\alpha_{22} + \rho\omega^2 k_0]\xi - \rho\omega^2(a_1a_2 - a_{12}a_{21}) = 0,\end{aligned} \tag{5}$$

$$\alpha_{11} = k_2\beta_1^2 + k_1\beta_2^2 - \beta_1\beta_2(k_{12} + k_{21}), \quad \alpha_{22} = a_2\beta_1^2 + a_1\beta_2^2 - \beta_1\beta_2(a_{12} + a_{21}),$$

$$\alpha_0 = k_1k_2 - k_{12}k_{21}, \quad k_0 = a_1k_2 + a_2k_1 - k_{12}a_{21} - k_{21}a_{12}, \quad \lambda_4^2 = \frac{\rho\omega^2}{\mu}.$$

We introduce the matrix differential operator $\mathbf{B}(\partial\mathbf{x}, \omega)$ consisting of cofactors of the elements of the matrix $\mathbf{A}^T$ divided on $\mu\mu_0\alpha_0$:

$$\mathbf{B}(\partial_{\mathbf{x}}, \omega) = \frac{1}{\mu\mu_0\alpha_0} \parallel B_{lj}(\partial_{\mathbf{x}}) \parallel_{4x4}, \ \ l, j = 1, 2, 3, 4,$$

where

$$B_{lj}(\partial_{\mathbf{x}}) = \delta_{lj}\mu_0\alpha_0(\Delta + \lambda_1^2)(\Delta + \lambda_2^2)(\Delta + \lambda_3^2)$$

$$-\frac{\partial^2}{\partial x_j x_l}\{(\lambda + \mu)[\alpha_0\Delta\Delta + k_0\Delta + a_1a_2 - a_{12}a_{21}] + i\omega(\alpha_{11}\Delta + \alpha_{22})\},$$

$$B_{j3}(\partial_{\mathbf{x}}) = \mu(\Delta + \lambda_4^2)[(\beta_1k_2 - \beta_2k_{21})\Delta + (\beta_1a_2 - \beta_2a_{21})]\frac{\partial}{\partial x_j}, \quad l, j = 1, 2,$$

$$B_{j4}(\partial_{\mathbf{x}}) = -\mu(\Delta + \lambda_4^2)[(\beta_1k_{12} - \beta_2k_1)\Delta + (\beta_1a_{12} - \beta_2a_1)]\frac{\partial}{\partial x_j}, \quad j = 1, 2,$$

$$B_{3j}(\partial_{\mathbf{x}}) = -i\omega\mu(\Delta + \lambda_4^2)[(\beta_1k_2 - \beta_2k_{12})\Delta + (\beta_1a_2 - \beta_2a_{12})]\frac{\partial}{\partial x_j}, \quad j = 1, 2,$$

$$B_{4j}(\partial_{\mathbf{x}}) = i\omega\mu(\Delta + \lambda_4^2)[(\beta_1k_{21} - \beta_2k_1)\Delta + (\beta_1a_{21} - \beta_2a_1)]\frac{\partial}{\partial x_j}, \quad j = 1, 2,$$

$$B_{33}(\partial_{\mathbf{x}}) = \mu(\Delta + \lambda_4^2)[\mu_0k_2\Delta\Delta + (k_2\rho\omega^2 + a_2\mu_0 + i\omega\beta_2^2)\Delta + a_2\rho\omega^2],$$

$$B_{44}(\partial_{\mathbf{x}}) = \mu(\Delta + \lambda_4^2)[\mu_0k_1\Delta\Delta + (k_1\rho\omega^2 + a_1\mu_0 + i\omega\beta_1^2)\Delta + a_1\rho\omega^2],$$

$$B_{34}(\partial_{\mathbf{x}}) = -\mu(\Delta + \lambda_4^2)[\mu_0k_{12}\Delta\Delta + (k_{12}\rho\omega^2 + a_{12}\mu_0 + i\omega\beta_1\beta_2)\Delta + a_{12}\rho\omega^2],$$

$$B_{43}(\partial_{\mathbf{x}}) = -\mu(\Delta + \lambda_4^2)[\mu_0k_{21}\Delta\Delta + (k_{21}\rho\omega^2 + a_{21}\mu_0 + i\omega\beta_1\beta_2)\Delta + a_{21}\rho\omega^2].$$

By substituting the vector $\mathbf{U}(\mathbf{x}) = \mathbf{B}(\partial_{\mathbf{x}}, \omega)\mathbf{\Psi}$ into (2), where $\mathbf{\Psi}$ is four-component vector function, we get

$$(\Delta + \lambda_1^2)(\Delta + \lambda_2^2)(\Delta + \lambda_3^2)(\Delta + \lambda_4^2)\mathbf{\Psi} = 0.$$

From here, after some calculations, we obtain

$$\Psi = -\sum_{m=1}^{4} d_m \varphi_m, \tag{6}$$

where

$$\varphi_m = \frac{\pi}{2i} H_0^{(1)}(\lambda_m r),$$

$H_0^{(1)}(\lambda_m r)$ is Hankel's function of the first kind with the index 0

$$\begin{aligned} H_0^{(1)}(\lambda_m r) &= \frac{2i}{\pi} J_0(\lambda_m r) \ln r + \frac{2i}{\pi}\left(\ln\frac{\lambda_m}{2} + C - \frac{i\pi}{2}\right) J_0(\lambda_m r) \\ &- \frac{2i}{\pi}\sum_{k=1}^{\infty} \frac{(-1)^k}{(k!)^2}\left(\frac{\lambda_m r}{2}\right)^{2k}\left(\frac{1}{k} + \frac{1}{k-1} + \ldots + 1\right), \quad m = 1,2,3,4, \end{aligned} \tag{7}$$

$$J_0(\lambda_m r) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(k!)^2}\left(\frac{\lambda_m r}{2}\right)^{2k}, \quad r^2 = (x_1 - y_1)^2 + (x_2 - y_2)^2,$$

$$d_j = \prod_{\substack{m=1 \\ j\neq m}}^{4} \frac{1}{\lambda_j^2 - \lambda_m^2}, \quad \sum_{j=1}^{4} d_j = 0, \quad \sum_{j=1}^{4} d_j \lambda_j^2 = 0,$$

$$\sum_{j=1}^{4} d_j \lambda_j^4 = 0, \quad \sum_{j=1}^{4} d_j \lambda_j^6 = 1.$$

By substituting (6) into $\mathbf{U} = \mathbf{B}\mathbf{\Psi}$, we obtain the fundamental matrix of solutions for the equation (2),denoting it through $\mathbf{\Gamma}(\mathbf{x}\text{-}\mathbf{y}, \omega)$

$$\mathbf{\Gamma}(\mathbf{x}\text{-}\mathbf{y}, \omega) = \| \, \Gamma_{kj}(\mathbf{x}\text{-}\mathbf{y}, \omega) \, \|_{4\times 4} \tag{8}$$

where

$$\Gamma_{kj} = \frac{\delta_{kj}}{\mu}\varphi_4 + i\omega \sum_{l=1}^{3} m_l \frac{\partial^2(\varphi_l - \varphi_4)}{\partial x_k \partial x_j}, \quad k,j = 1,2,$$

$$m_l = \frac{\delta_l(\alpha_{22} - \alpha_{11}\lambda_l^2)}{\rho\omega^2 - \mu_0\lambda_l^2}, \quad \delta_l = \frac{d_l(\lambda_4^2 - \lambda_l^2)}{\mu_0 \alpha_0}, \quad \Gamma_{j3} = -\sum_{l=1}^{3} n_{l5}\frac{\partial \varphi_l}{\partial x_j},$$

$$\Gamma_{j4} = \sum_{l=1}^{3} n_{l4}\frac{\partial \varphi_l}{\partial x_j}, \quad \Gamma_{3j} = i\omega \sum_{l=1}^{3} n_{5\,l}\frac{\partial \varphi_l}{\partial x_j}, \quad \Gamma_{4j} = -i\omega \sum_{l=1}^{3} n_{4l}\frac{\partial \varphi_l}{\partial x_j},$$

$$j=1,2, \quad \Gamma_{33}=-\sum_{l=1}^{3}\eta_{3l}\varphi_l, \quad \Gamma_{44}=-\sum_{l=1}^{3}\eta_{4l}\varphi_l, \quad \Gamma_{34}=\sum_{l=1}^{3}\eta_{1l}\varphi_l,$$

$$\Gamma_{43}=\sum_{l=1}^{3}\eta_{2l}\varphi_l, \quad n_{l4}=\delta_l\left[(\beta_2 k_1-\beta_1 k_{12})\lambda_l^2+\beta_1 a_{12}-\beta_2 a_1\right],$$

$$n_{5l}=\delta_l\left[\beta_2 k_{12}-\beta_1 k_2)\lambda_l^2+\beta_1 a_2-\beta_2 a_{12}\right],$$

$$n_{4l}=\delta_l\left[(\beta_2 k_1-\beta_1 k_{21})\lambda_l^2+\beta_1 a_{21}-\beta_2 a_1\right],$$

$$n_{l\,5}=\delta_l\left[(\beta_2 k_{21}-\beta_1 k_2)\lambda_l^2+\beta_1 a_2-\beta_2 a_{21}\right],$$

$$\eta_{3l}=\delta_l[(\mu_0\lambda_l^2-\rho\omega^2)(k_2\lambda_l^2-a_2)-i\omega\beta_2^2\lambda_l^2],$$

$$\eta_{4l}=\delta_l[(\mu_0\lambda_l^2-\rho\omega^2)(k_1\lambda_l^2-a_1)-i\omega\beta_1^2\lambda_l^2],$$

$$\eta_{1l}=\delta_l[(\mu_0\lambda_l^2-\rho\omega^2)(k_{12}\lambda_l^2-a_{12})-i\omega\beta_1\beta_2\lambda_l^2],$$

$$\eta_{2l}=\delta_l[(\mu_0\lambda_l^2-\rho\omega^2)(k_{21}\lambda_l^2-a_{21})-i\omega\beta_1\beta_2\lambda_l^2].$$

Clearly,

$$\frac{\pi}{2i}H_0^{(1)}(\lambda r)=\ln|\mathbf{x}-\mathbf{y}|-\frac{\lambda^2}{4}|\mathbf{x}-\mathbf{y}|^2\ln|\mathbf{x}-\mathbf{y}|+const+O(|\mathbf{x}-\mathbf{y}|^2).$$

It is evident that all elements of $\mathbf{\Gamma}(\mathbf{x}\text{-}\mathbf{y},\omega)$ are single-valued functions on the whole plane and they have a logarithmic singularity at most. By applying the methods of classical theory of elasticity we can similarly prove the following:

Theorem 3.1 : *The elements of the matrix* $\mathbf{\Gamma}(\boldsymbol{x}\text{-}\boldsymbol{y},\omega)$ *have a logarithmic singularity as* $\boldsymbol{x}\to\boldsymbol{y}$ *and each column of the matrix* $\mathbf{\Gamma}(\boldsymbol{x}\text{-}\boldsymbol{y},\omega)$, *considered as a vector, is a solution of the system (2) at every point* $\boldsymbol{x}$, *if* $\boldsymbol{x}\neq\boldsymbol{y}$.

Remark 1 : The operator $\mathbf{A}(\partial_{\mathbf{x}},\omega)\mathbf{U}$ is not self adjoint. Obviously, it is possible to construct the fundamental solution of adjoint operator in quite similar manner.

Let's consider the matrices $\widetilde{\mathbf{\Gamma}}(\mathbf{x},\omega):=\mathbf{\Gamma}^T(-\mathbf{x},\omega)$ and $\widetilde{\boldsymbol{A}}(\partial_{\mathbf{x}},\omega):=\boldsymbol{A}^T(-\partial_{\mathbf{x}},\omega)$. The following basic properties of matrix $\widetilde{\mathbf{\Gamma}}(\mathbf{x},\omega)$ may be easily verified.

Theorem 3.2 : *Each column of the matrix* $\widetilde{\mathbf{\Gamma}}(\boldsymbol{x}\text{-}\boldsymbol{y},\omega)$, *considered as a vector, satisfies the associated system* $\widetilde{\boldsymbol{A}}(\partial_{\boldsymbol{x}},\omega)\widetilde{\mathbf{\Gamma}}(\boldsymbol{x}\text{-}\boldsymbol{y},\omega)=0$, *at every point* $\boldsymbol{x}$, *if* $\boldsymbol{x}\neq\boldsymbol{y}$ *and the elements of the matrix* $\widetilde{\mathbf{\Gamma}}(\boldsymbol{x}\text{-}\boldsymbol{y},\omega)$ *have a logarithmic singularity as* $\boldsymbol{x}\to\boldsymbol{y}$.

4. Singular matrix of solutions

By using the basic fundamental matrix we can construct new singular solutions of the same equations playing an important role in the theory of boundary value problems.

Let $\boldsymbol{P}(\partial\mathbf{x},\mathbf{n})$ be the stress operator in the linear theory of elasticity for solids with double porosity and $\boldsymbol{P}(\partial\mathbf{x},\mathbf{n})\mathbf{U}$ is the stress vector which acts on an element of the arc with the normal $\mathbf{n}=(n_1,n_2)$

$$\boldsymbol{P}(\partial\mathbf{x},\mathbf{n})\mathbf{U}=\boldsymbol{T}(\partial\mathbf{x},\mathbf{n})\mathbf{u}-\mathbf{n}(\beta_1p_1+\beta_2p_2), \tag{9}$$

where $\boldsymbol{T}(\partial\mathbf{x},\mathbf{n})$ is the stress operator of the classical theory of elasticity

$$\boldsymbol{T}(\partial_{\mathbf{x}},\mathbf{n})=\begin{pmatrix}\mu\frac{\partial}{\partial\mathbf{n}}+(\lambda+\mu)n_1\frac{\partial}{\partial x_1} & (\lambda+\mu)n_1\frac{\partial}{\partial x_2}+\mu\frac{\partial}{\partial s}\\ (\lambda+\mu)n_2\frac{\partial}{\partial x_1}-\mu\frac{\partial}{\partial s} & \mu\frac{\partial}{\partial\mathbf{n}}+(\lambda+\mu)n_2\frac{\partial}{\partial x_2}\end{pmatrix},$$

$$\frac{\partial}{\partial\mathbf{n}}=n_1\frac{\partial}{\partial x_1}+n_2\frac{\partial}{\partial x_2},\quad \frac{\partial}{\partial s}=n_2\frac{\partial}{\partial x_1}-n_1\frac{\partial}{\partial x_2}.$$

Let us introduce the following matrix differential operators of dimension 4x4

$$\boldsymbol{R}(\partial_{\mathbf{x}},\mathbf{n})=\begin{pmatrix}T_{11}(\partial\mathbf{x},\mathbf{n}) & T_{12}(\partial_{\mathbf{x}},\mathbf{n}) & -\beta_1n_1 & -\beta_2n_1\\ T_{21}(\partial_{\mathbf{x}},\mathbf{n}) & T_{22}(\partial_{\mathbf{x}},\mathbf{n}) & -\beta_1n_2 & -\beta_2n_2\\ 0 & 0 & k_1\frac{\partial}{\partial\mathbf{n}} & k_{12}\frac{\partial}{\partial\mathbf{n}}\\ 0 & 0 & k_{21}\frac{\partial}{\partial\mathbf{n}} & k_2\frac{\partial}{\partial\mathbf{n}}\end{pmatrix},$$

$$\widetilde{\boldsymbol{R}}(\partial_{\mathbf{x}},\mathbf{n})=\begin{pmatrix}T_{11}(\partial_{\mathbf{x}},\mathbf{n}) & T_{12}(\partial_{\mathbf{x}},\mathbf{n}) & -i\omega n_1\beta_1 & -i\omega n_1\beta_2\\ T_{21}(\partial_{\mathbf{x}},\mathbf{n}) & T_{22}(\partial_{\mathbf{x}},\mathbf{n}) & -i\omega n_2\beta_1 & -i\omega n_2\beta_2\\ 0 & 0 & k_1\frac{\partial}{\partial\mathbf{n}} & k_{21}\frac{\partial}{\partial\mathbf{n}}\\ 0 & 0 & k_{12}\frac{\partial}{\partial\mathbf{n}} & k_2\frac{\partial}{\partial\mathbf{n}}\end{pmatrix}.$$

If $\mathbf{U}=(\mathbf{u},\mathbf{p})$ then obviously

$$\boldsymbol{R}(\partial_{\mathbf{x}},\mathbf{n})\mathbf{U}=(\boldsymbol{P}\mathbf{U},\boldsymbol{P}^{(1)}\mathbf{p}),$$

where

$$\mathbf{P}^{(1)}(\partial_{\mathbf{x}},\mathbf{n})\mathbf{p}=\begin{pmatrix}k_1 & k_{12}\\ k_{21} & k_2\end{pmatrix}\frac{\partial\mathbf{p}}{\partial\mathbf{n}},$$

$$\mathbf{p} = (p_1, p_2).$$

By applying the operator $\boldsymbol{R}(\partial\mathbf{x}, \mathbf{n})$ to the matrix $\boldsymbol{\Gamma}(\mathbf{x}\text{-}\mathbf{y})$ and the operator $\widetilde{\boldsymbol{R}}(\partial\mathbf{x}, \mathbf{n})$ to the $\widetilde{\boldsymbol{\Gamma}}(\mathbf{x}\text{-}\mathbf{y}) = \boldsymbol{\Gamma}^T(\mathbf{y} - \mathbf{x})$, we obtain, respectively

$$\boldsymbol{R}(\partial_{\mathbf{x}}, \mathbf{n})\boldsymbol{\Gamma}(\mathbf{x} - \mathbf{y}) = \|R_{pq}\|_{4\times4}, \quad \widetilde{\boldsymbol{R}}(\partial_{\mathbf{x}}, \mathbf{n})\widetilde{\boldsymbol{\Gamma}}(\mathbf{y}\text{-}\mathbf{x}) = \|\widetilde{R}_{pq}\|_{4\times4},$$

The elements R_{pq} are the following:

$$R_{11} = \frac{\partial\varphi_4}{\partial n} + 2\mu i\omega\frac{\partial}{\partial s}\frac{\partial^2}{\partial x_1 x_2}\sum_{l=1}^{3} m_l(\varphi_l - \varphi_4) - i\omega\mu\lambda_4^2 n_1\xi_1\sum_{l=1}^{3} m_l(\varphi_l - \varphi_4),$$

$$R_{12} = \frac{\partial\varphi_4}{\partial s} + 2\mu i\omega\frac{\partial}{\partial s}\frac{\partial^2}{\partial x_2^2}\sum_{l=1}^{3} m_l(\varphi_l - \varphi_4) - i\omega\mu\lambda_4^2 n_1\xi_2\sum_{l=1}^{3} m_l(\varphi_l - \varphi_4),$$

$$R_{21} = -\frac{\partial\varphi_4}{\partial s} - 2\mu i\omega\frac{\partial}{\partial s}\frac{\partial^2}{\partial x_1^2}\sum_{l=1}^{3} m_l(\varphi_l - \varphi_4) - i\omega\mu\lambda_4^2 n_2\xi_1\sum_{l=1}^{3} m_l(\varphi_l - \varphi_4),$$

$$R_{22} = \frac{\partial\varphi_4}{\partial n} - 2\mu i\omega\frac{\partial}{\partial s}\frac{\partial^2}{\partial x_1 x_2}\sum_{l=1}^{3} m_l(\varphi_l - \varphi_4) - i\omega\mu\lambda_4^2 n_2\xi_2\sum_{l=1}^{3} m_l(\varphi_l - \varphi_4),$$

$$R_{13} = \left(n_1\rho\omega^2 - 2\mu\frac{\partial}{\partial s}\frac{\partial}{\partial x_2}\right)\sum_{l=1}^{3} n_{l5}\varphi_l,$$

$$R_{23} = \left(n_2\rho\omega^2 + 2\mu\frac{\partial}{\partial s}\frac{\partial}{\partial x_1}\right)\sum_{l=1}^{3} n_{l5}\varphi_l,$$

$$R_{14} = \left(-n_1\rho\omega^2 + 2\mu\frac{\partial}{\partial s}\frac{\partial}{\partial x_2}\right)\sum_{l=1}^{3} n_{l4}\varphi_l,$$

$$R_{24} = -\left(n_2\rho\omega^2 + 2\mu\frac{\partial}{\partial s}\frac{\partial}{\partial x_1}\right)\sum_{l=1}^{3} n_{l4}\varphi_l,$$

$$R_{3j} = i\omega\frac{\partial}{\partial n}\frac{\partial}{\partial x_j}\sum_{l=1}^{3}(k_1 n_{5l} - k_{12}n_{4l})\varphi_l,$$

$$R_{4j} = i\omega\frac{\partial}{\partial n}\frac{\partial}{\partial x_j}\sum_{l=1}^{3}(k_{21}n_{5l} - k_2 n_{4l})\varphi_l, \quad j = 1, 2,$$

$$R_{33} = \frac{\partial}{\partial n}\sum_{l=1}^{3}(-k_1\eta_{3l} + k_{12}\eta_{2l})\varphi_l,$$

$$R_{34} = \frac{\partial}{\partial n} \sum_{l=1}^{3} (k_1 \eta_{1l} - k_{12} \eta_{4l}) \varphi_l,$$

$$R_{43} = \frac{\partial}{\partial n} \sum_{l=1}^{3} (-k_{21} \eta_{3l} + k_2 \eta_{2l}) \varphi_l,$$

$$R_{44} = \frac{\partial}{\partial n} \sum_{l=1}^{3} (k_{21} \eta_{1l} - k_2 \eta_{4l}) \varphi_l.$$

The elements $\tilde{R}_{pq}$ are the following:

$$\tilde{R}_{kl} = R_{kl}, \quad k, l = 1, 2,$$

$$\tilde{R}_{13} = i\omega \left[\rho\omega^2 n_1 - 2\mu \frac{\partial}{\partial s} \frac{\partial}{\partial x_2} \right] \sum_{l=1}^{3} n_{5l} \varphi_l,$$

$$\tilde{R}_{23} = i\omega \left[\rho\omega^2 n_2 + 2\mu \frac{\partial}{\partial s} \frac{\partial}{\partial x_1} \right] \sum_{l=1}^{3} n_{5l} \varphi_l,$$

$$\tilde{R}_{14} = i\omega \left[-\rho\omega^2 n_1 + 2\mu \frac{\partial}{\partial s} \frac{\partial}{\partial x_2} \right] \sum_{l=1}^{3} n_{4l} \varphi_l,$$

$$\tilde{R}_{24} = -i\omega \left[\rho\omega^2 n_2 + 2\mu \frac{\partial}{\partial s} \frac{\partial}{\partial x_1} \right] \sum_{l=1}^{3} n_{4l} \varphi_l,$$

$$\tilde{R}_{3j} = \frac{\partial}{\partial n} \frac{\partial}{\partial x_j} \sum_{l=1}^{3} [k_1 n_{l5} - k_{21} n_{l4}] \varphi_l,$$

$$\tilde{R}_{4j} = \frac{\partial}{\partial n} \frac{\partial}{\partial x_j} \sum_{l=1}^{3} [k_{12} n_{l5} - k_2 n_{l4}] \varphi_l, \quad j = 1, 2,$$

$$\tilde{R}_{33} = \frac{\partial}{\partial n} \sum_{l=1}^{3} [-k_1 \eta_{3l} + k_{21} \eta_{1l}] \varphi_l,$$

$$\tilde{R}_{34} = \frac{\partial}{\partial n} \sum_{l=1}^{3} [k_1 \eta_{2l} - k_{21} \eta_{4l}] \varphi_l,$$

$$\tilde{R}_{43} = \frac{\partial}{\partial n} \sum_{l=1}^{3} [-k_{12} \eta_{3l} + k_2 \eta_{1l}] \varphi_l,$$

$$\tilde{R}_{44} = \frac{\partial}{\partial n} \sum_{l=1}^{3} [k_{12} \eta_{2l} - k_2 \eta_{4l}] \varphi_l.$$

It is well-known that in the case of a Lyapunov curve $S \in C^{1,\alpha}$ the function $\dfrac{\partial \ln r}{\partial n}$ for $\mathbf{x}, \mathbf{y} \in S$ has a week singularity and $\dfrac{\partial \ln r}{\partial n}$ is integrable in the sense of the principal Cauchy value. Consequently $\dfrac{\partial \ln r}{\partial n}$ is singular kernel on S. It is obvious that $\tilde{\boldsymbol{R}}(\partial_{\mathbf{y}}, \mathbf{n})\boldsymbol{\Gamma}^T(\mathbf{x}\text{-}\mathbf{y}, \omega)$ and $\boldsymbol{R}(\partial_{\mathbf{x}}, \mathbf{n})\boldsymbol{\Gamma}(\mathbf{x}-\mathbf{y}, \omega)$ are singular kernels (in the sense of Cauchy).

Theorem 4.1 : *Every column of the matrix* $[\boldsymbol{R}(\partial_{\boldsymbol{y}}, \boldsymbol{n})\boldsymbol{\Gamma}(\boldsymbol{y}\text{-}\boldsymbol{x}, \omega)]^T$, *considered as a vector, is a solution of the system* $\widetilde{\boldsymbol{A}}(\partial_{\boldsymbol{x}}, \omega)\boldsymbol{U} = 0$ *at any point* $\boldsymbol{x}$ *if* $\boldsymbol{x} \neq \boldsymbol{y}$ *and the elements of the matrix* $[\boldsymbol{R}(\partial_{\boldsymbol{y}}, \boldsymbol{n})\boldsymbol{\Gamma}(\boldsymbol{y}\text{-}\boldsymbol{x}, \omega)]^T$ *contain a singular part which is integrable in the sense of the Cauchy principal value.*

Theorem 4.2 : *Every column of the matrix* $\left[\tilde{\boldsymbol{R}}(\partial_{\boldsymbol{y}}, \boldsymbol{n})\tilde{\boldsymbol{\Gamma}}(\boldsymbol{y}\text{-}\boldsymbol{x}, \omega)\right]^T$, *considered as a vector, is a solution of the system* $\boldsymbol{A}(\partial_{\boldsymbol{x}}, \omega)\boldsymbol{U} = 0$ *at any point* $\boldsymbol{x}$ if $\boldsymbol{x} \neq \boldsymbol{y}$ *and the elements of the matrix* $\left[\tilde{\boldsymbol{R}}(\partial_{\boldsymbol{y}}, \boldsymbol{n})\tilde{\boldsymbol{\Gamma}}(\boldsymbol{y}\text{-}\boldsymbol{x}, \omega)\right]^T$ *contain a singular part which is integrable in the sense of the Cauchy principal value.*

We introduce the single-layer potential

$$\mathbf{Z}^{(1)}(\mathbf{x}, \mathbf{g}, \omega) = \frac{1}{4i}\int\limits_S \boldsymbol{\Gamma}(\mathbf{x}-\mathbf{y}, \omega)\mathbf{g}(\mathbf{y})dS$$

and the double-layer potential

$$\mathbf{Z}^{(2)}(\mathbf{x}, \mathbf{h}, \omega) = \frac{1}{4i}\int\limits_S [\tilde{\boldsymbol{R}}(\partial_{\mathbf{y}}, \mathbf{n})\boldsymbol{\Gamma}^T(\mathbf{x}\text{-}\mathbf{y}, \omega)]^T\mathbf{h}(\mathbf{y})dS, \tag{10}$$

where $\boldsymbol{\Gamma}(\mathbf{x}-\mathbf{y}, \omega)$ is given by (8), $\mathbf{g}$ and $\boldsymbol{h}$ are four-component continuous (or Holder continuous) vectors. S is a closed Lyapunov curve.

Let us consider the operation $\boldsymbol{R}(\partial_{\mathbf{x}}, \mathbf{n})$ acting on a single-layer potential. We obtain

$$\boldsymbol{R}(\partial_{\mathbf{x}}, \mathbf{n})\mathbf{Z}^{(1)}(\mathbf{x}, \boldsymbol{g}, \omega) = \frac{1}{4i}\int\limits_S \boldsymbol{R}(\partial_{\mathbf{x}}, \mathbf{n})\boldsymbol{\Gamma}(\mathbf{x}-\mathbf{y}, \omega)\boldsymbol{g}(\mathbf{y})d\mathbf{y}. \tag{11}$$

The following theorem is valid:

Theorem 4.3 : *The vectors* $\boldsymbol{Z}^{(j)}$, $j = 1, 2$, *are solutions of the system (2) in both the domains* D^+ *and* D^-. *When* $\boldsymbol{x} \longrightarrow \boldsymbol{z} \in S$, *from (10) and (11) we obtain*

$$[\boldsymbol{Z}^{(2)}(\boldsymbol{z}, \boldsymbol{h}), \omega]^{\pm} = \pm\boldsymbol{h}(\boldsymbol{z}) + \boldsymbol{Z}^{(2)}(\boldsymbol{z}, \boldsymbol{h}, \omega),$$

$$[\boldsymbol{R}(\partial_{\boldsymbol{z}}, \boldsymbol{n})\boldsymbol{Z}^{(1)}(\boldsymbol{z}, \boldsymbol{g}), \omega]^{\pm} = \mp\boldsymbol{g}(\boldsymbol{z}) + \boldsymbol{R}(\partial_{\boldsymbol{z}}, \boldsymbol{n})\boldsymbol{Z}^{(1)}(\boldsymbol{z}, \boldsymbol{g}, \omega),$$

where $[\cdot]^{\pm}$ *denotes the limiting value from* $D^{\pm}$, $\boldsymbol{Z}^{(2)}(\boldsymbol{z}, \boldsymbol{h}, \omega)$ *and* $\boldsymbol{R}(\partial_{\boldsymbol{z}}, \boldsymbol{n})\boldsymbol{Z}^{(1)}(\boldsymbol{z}, \boldsymbol{g}, \omega)$ *are given by formulas (10) and (11), respectively, at the point* $\boldsymbol{z} \in S$.

5. Concluding remarks

1. The fundamental and singular matrices of solutions, in terms of elementary functions, in the 2D fully coupled theory of steady vibrations of the linear theory of elasticity for materials with double porosity are constructed.

2. On the basis of fundamental solution of the system (1) it is possible; to construct the single-layer and double layer potentials and to establish their basic properties; to investigate 2D BVPs of the theory of the fully coupled linear theory of elasticity for solids with double porosity by means of boundary integral method and the theory of singular integral equations.

3. By using the above mentioned method it is possible to construct the fundamental solutions of the systems of equations in the modern linear theories of elasticity and thermoelasticity for homogeneous isotropic elastic materials with microstructure.

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