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Properties of Layer Potentials of Thermoelastostatics

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We construct explicitly the fundamental matrix for the equations of statics of the thermoelasticity theory and study the mapping properties of the corresponding single and double layer potentials. We establish the jump relations for layer potentials and investigate mapping properties of the boundary integral operators generated by the layer potentials and their co-normal derivatives. We introduce a special class, $Z(\Omega^-)$ of vectors bounded at infinity in the case of an unbounded domain Ω^- and show that the layer potentials belong to this class. On the basis of these results we derive a general integral representation formula for solutions of the equations of thermostatics in unbounded domains. These results play a crucial role in the proof of the corresponding existence theorems.

Keywords: Thermoelasticity, Uniqueness theorems, Layer potentials.

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Let $\Omega^+ \in \mathbb{R}^3$ be a bounded domain and $S = \partial\Omega^+$. For simplicity, we assume that S is a $C^{2,\alpha}$-smooth surface with $0 < \alpha \leqslant 1$. We set $\overline{\Omega^+} = \Omega^+ \cup S$ and $\Omega^- = \mathbb{R}^3 \setminus \overline{\Omega^+}$.

The basic governing equations of the linear thermoelastostatics read as (for details see [1], [2], [5], [6]):

$$c_{kjpq}\partial_j\partial_q u_p(x) - \beta_{kj}\partial_j\vartheta(x) = 0, \quad k = 1,2,3, \quad x \in \Omega^\pm, \tag{1}$$

$$\lambda_{pq}\partial_p\partial_q\vartheta(x) = 0, \quad x \in \Omega^\pm, \tag{2}$$

where $c_{kjpq} = c_{pqkj} = c_{jkpq}$ are elastic constants, $\lambda_{pq} = \lambda_{qp}$ are heat conduction coefficients, $u = (u_1, u_2, u_3)^\top$ is the displacement vector, ϑ is the temperature distribution function, $\beta_{pq} = \beta_{qp}$ are the material constant, describing the coupling of mechanical and thermal fields, $\partial_j = \partial/\partial x_j$, $\partial = \nabla = (\partial_1, \partial_2, \partial_3)$. Throughout the paper summation over repeated indexes is meant from one to three if not otherwise stated.

In order to rewrite the above equations in matrix form, let us introduce the

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notation

$$
\begin{aligned}
U =&(u_1, u_2, u_3, \vartheta)^\top = (u, \vartheta)^\top,\\
C(\partial) &= \big[C_{kp}(\partial)\big]_{3\times 3}, \qquad C_{kp}(\partial) = c_{kjpq}\partial_j\partial_q,\\
\Lambda(\partial) &= \lambda_{pq}\partial_p\partial_q.
\end{aligned}
$$

The system (1)-(2) can be then rewritten as

$$
A(\partial)U(x) = 0, \qquad x \in \Omega^\pm, \tag{3}
$$

where

$$
A(\partial) := \big[A_{kj}(\partial)\big]_{4\times 4} = \begin{bmatrix} C(\partial) & [-\beta_{kj}\partial_j]_{3\times 1} \\ [0]_{1\times 3} & \Lambda(\partial) \end{bmatrix}_{4\times 4},
$$

Denote by $\Gamma(x) = \big[\Gamma_{kj}(x)\big]_{4\times 4}$ the fundamental matrix of the operator $A(\partial_x)$,

$$
A(\partial_x)\Gamma(x-y) = I_4\,\delta(x-y).
$$

Here and in what follows I_k stands for the $k \times k$ unit matrix.

The fundamental matrix $\Gamma(x)$ can be constructed explicitly and is written in the form

$$
\Gamma(x) = \mathcal{F}^{-1}_{\xi\to x}\big[A^{-1}(-i\xi)\big],
$$

where $\mathcal{F}^{-1}{}_{\xi\to x}$ is the inverse Fourier transform[1] and $A^{-1}(-i\xi)$ is the matrix inverse to $A(-i\xi)$,

$$
A^{-1}(-i\xi) = \frac{1}{\det A(-i\xi)} A^{(c)}(-i\xi).
$$

Here $A^{(c)}(-i\xi)$ is the matrix of co-factors of the matrix $A(-i\xi)$,

$$
A(-i\xi) = \begin{bmatrix} \big[C_{kjpq}(-i\xi_j)(-i\xi_q)\big]_{3\times 3} & \big[-\beta_{kj}(-i\xi_j)\big]_{3\times 1} \\ \big[0\big]_{1\times 3} & \lambda_{pq}(-i\xi_j)(-i\xi_q) \end{bmatrix}_{4\times 4},
$$

Note that $\det A(-i\xi)$ is a homogeneous polynomial of order 8 in variables ξ_1, ξ_2, ξ_3, and, moreover, $\det A(-i\xi) \neq 0$ for $\xi \in \mathbb{R}^3 \setminus \{0\}$. It is also evident that the entries of the matrix $A^{(c)}(-i\xi)$ are also homogeneous polynomials and at the origin and

[1] For absolutely integrable functions f and g the direct and inverse Fourier transforms are defined as follows $\mathcal{F}_{x\to\xi}[f(x)] = \int_{\mathbb{R}^3} f(x)\,e^{i\,x\cdot\xi}\,dx$ and $\mathcal{F}^{-1}{}_{\xi\to x}[g(\xi)] = \int_{\mathbb{R}^3} g(\xi)\,e^{-i\,x\cdot\xi}\,d\xi$ respectively.

at infinity they have the following asymptotic behaviour:

$$
\begin{aligned}
&A^{(c)}_{kj}(-i\xi) = \mathcal{O}(|\xi|^6), \qquad k,j=\overline{1,3},\\
&A^{(c)}_{j4}(-i\xi) = \mathcal{O}(|\xi|^5), \qquad A^{(c)}_{4j}(-i\xi) = 0, \qquad j=\overline{1,3},\\
&A^{(c)}_{44}(-i\xi) = \mathcal{O}(|\xi|^6).
\end{aligned}
$$

In addition, we see that the $A^{(c)}_{j4}(-i\xi)$ for $j=\overline{1,3}$, are odd polynomials in ξ. Consequently, the entries

$$
K_j(\xi) := \frac{A^{(c)}_{j4}(-i\xi)}{\det A(-i\xi)}, \qquad j=\overline{1,3},
$$

are odd functions in ξ: $K_j(-\xi) = -K_j(\xi)$, $j=\overline{1,3}$. Therefore for the functions $K_j(\xi)$ the cancelation Tricomi condition

$$
\int\limits_{|\xi|=1} K_j(\xi)dS = 0, \qquad j=\overline{1,3}. \tag{4}
$$

holds, whence it follows that the generalized inverse Fourier transforms of these functions $\widetilde{K}_j(x) \equiv \mathcal{F}^{-1}_{\xi\to x}[K_j(\xi)]$, understood in the Cauchy principal value sense, satisfy the same type cancelation Tricomi condition (see [4, Ch. 2, Proposition 2.16])

$$
\int\limits_{|x|=1} \widetilde{K}_j(x)dS = 0, \qquad j=\overline{1,3}. \tag{5}
$$

It is well known that the generalized Fourier transform (inverse Fourier transform) of a homogeneous function of order $-r<0$ is a homogenous function of order $-3+r$ provided $0<|r|<3$ (see [4, Ch. 2, Proposition 2.13]).

Therefore the entries of the fundamental matrix $\Gamma(x)$ are homogeneous functions in x and in a neighbourhood of the origin and infinity have the following asymptotic behaviour

$$
\Gamma(x) = \begin{bmatrix} \big[\mathcal{O}(|x|^{-1})\big]_{3\times 3} & \big[\mathcal{O}(1)\big]_{3\times 1} \\ [0]_{1\times 3} & \mathcal{O}\big(|x|^{-1}\big) \end{bmatrix}_{4\times 4},
$$

i.e., the entries $\Gamma_{kj}(x)$ for $k,j=\overline{1,3}$, or $k=j=4$, are homogeneous function of order -1, while the entries $\Gamma_{k4}(x)$ for $1\leqslant k\leqslant 3$ are homogeneous functions of zero order.

Moreover, form (4) and (5) it follows that

$$
\int\limits_{|x|=1} \Gamma_{k4}(x)dS = 0, \qquad k=1,2,3.
$$

Now, let us introduce the single and double layer potentials

$$V(h)(x) = V_S(h)(x) := \int_S \Gamma(x-y)\, h(y)\, dS_y, \qquad x \in \mathbb{R}^3 \setminus S, \tag{6}$$

$$W(h)(x) = W_S(h)(x) := \int_S \big[Q\big(\partial_y, n(y)\big)\Gamma^\top(x-y)\big]^\top h(y)\, dS_y, \quad x \in \mathbb{R}^3 \setminus\ S, \tag{7}$$

where $\Gamma(x-y) = \big[\Gamma_{kj}(x-y)\big]_{4\times 4}$ is the fundamental matrix of the operator $A(\partial_x)$, and

$$Q(\partial, n) := \begin{bmatrix} \big[T(\partial,n)\big]_{3\times 3} & [0]_{3\times 1} \\ [0]_{1\times 3} & \lambda(\partial, n) \end{bmatrix}_{4\times 4},$$

$$T(\partial, n) := \big[T_{kp}(\partial, n)\big]_{3\times 3} = [c_{kjpq} n_j \partial_q]_{3\times 3}, \qquad \lambda(\partial, n) = \lambda_{pq} n_q\, \partial_p.$$

Note that for regular vector-functions $U = (u, \vartheta)^\top \in [C^2(\overline{\Omega^+})]^4$ and $U^* = (u^*, \vartheta^*)^\top \in [C^2(\overline{\Omega^+})]^4$ the following Green's formulae hold

$$\int_{\Omega^+} A(\partial)U \cdot U^* dx = -\int_{\Omega^+} E(U, U^*) dx + \int_{\partial\Omega^+} \big\{B(\partial, n)U\big\}^+ \cdot \{U^*\}^+ dS,$$

$$\int_{\Omega^+} \Big\{A(\partial)U \cdot U^* - U \cdot A^*(\partial)U^*\Big\} dx = \int_{\partial\Omega^+} \Big\{\{B(\partial, n)U\}^+ \cdot \{U^*\}^+ - \{U\}^+ \cdot \{Q(\partial, n)U^*\}^+\Big\} dS, \tag{8}$$

where $A^*(\partial)$ is the operator adjoint to $\big[A^\top(-\partial)\big]$,

$$A^*(\partial) := \big[A^\top(-\partial)\big] = \begin{bmatrix} C(\partial) & [0]_{3\times 1} \\ \big[\beta_{kj}\partial_j\big]_{1\times 3} & \Lambda(\partial) \end{bmatrix}_{4\times 4},$$

$B(\partial, n)$ is the boundary differential operator

$$B(\partial, n) = \begin{bmatrix} T(\partial, n) & [-\beta_{kj} n_j]_{3\times 1} \\ [0]_{1\times 3} & \lambda(\partial, n) \end{bmatrix}_{4\times 4},$$

and

$$E(U, U^*) = c_{kjpq}\, \partial_p u_q\, \partial_k u_j^* - \beta_{kj}\, u_4\, \partial_j u_k^* + \lambda_{pq}\, \partial_q u_4\, \partial_p u_4^*.$$

Here and in the sequel, the symbols $\{\,\cdot\,\}^\pm$ denote the limits on $\partial\Omega^\pm$ from $\Omega^\pm$ respectively and the central dot stands for the scalar product of two vectors.

From Green's identity (8) by standard arguments one can derive the following general integral representation of solutions to the homogeneous equation

$A(\partial)U(x) = 0$ in Ω^+,

$$W\big(\{U\}^+\big)(x) - V\big(\{BU\}^+\big)(x) = \begin{cases} U(x) & \text{for } x \in \Omega^+, \\ 0 & \text{for } x \in \Omega^-. \end{cases} \tag{9}$$

Next we introduce a special class of vector functions in an unbounded exterior domain Ω^- which plays a crucial role in the study of exterior boundary value problems.

Definition 1: A vector function $U = (u, \vartheta)^\top$ is said to belong to the class $Z(\Omega^-)$ if it is continuous in a neighbourhood of infinity and satisfies the following asymptotic conditions

$$u(x) = \mathcal{O}(1), \quad k = 1,2,3, \qquad \vartheta(x) = \mathcal{O}(|x|^{-1}), \quad |x| \to \infty, \tag{10}$$

$$\lim_{R\to\infty} \frac{1}{4\pi R^2} \int\limits_{\Sigma(0,R)} u(x)\, d\Sigma(0,R) = 0, \tag{11}$$

where $\Sigma(0,R)$ is a sphere, centered at the origin and radius R.

Further we analyze properties of the above introduced layer potentials.

Theorem 2: *The single and doubel layer potentials solve the homogeneous equation*

$$A(\partial)U(x) = 0 \quad in \quad \mathbb{R}^3 \setminus S \tag{12}$$

and belong to the class $Z(\Omega^-)$.

Proof: We start with the single layer potential and show that the vector-function $V(h)(x)$ solves equation (12) for $x \notin S$:

$$\begin{aligned}
\big[A(\partial_x)V(h)(x)\big]_k &= A_{kj}(\partial_x)\big[V(h)(x)\big]_j \\
&= A_{kj}(\partial_x)\Big[\int\limits_S \Gamma_{jp}(x-y)h_p(y)dS_y\Big] \\
&= \int\limits_S A_{kj}(\partial_x)\Gamma_{jp}(x-y)h_p(y)dS_y = 0, \qquad k = \overline{1,4},
\end{aligned}$$

since $A_{kj}(\partial_x)\Gamma_{jp}(x-y) = \big[A(\partial_x)\Gamma(x-y)\big]_{kp} = 0, \quad k,p = \overline{1,4}, \quad x \neq y.$

Similarly for the double layer potential we have:

$$\begin{aligned}\big[A(\partial_x)W(h)(x)\big]_k &= A_{kj}(\partial_x)[W(h)(x)]_j\\ &= A_{kj}(\partial_x)\Big[\int_S \big[Q\big(\partial_y,n(y)\big)\Gamma^\top(x-y)\big]^\top h(y)dS_y\Big]_j\\ &= A_{kj}(\partial_x)\int_S \Big[Q(\partial_y,n(y))\Gamma^\top(x-y)\Big]_{pj} h_p(y)dS_y\\ &= A_{kj}(\partial_x)\int_S Q_{pm}\big(\partial_y,n(y)\big)\Gamma_{jm}(x-y)h_p(y)dS_y\\ &= \int_S Q_{pm}\big(\partial_y,n(y)\big)A_{kj}(\partial_x)\Gamma_{jm}(x-y)h_p(y)dS_y = 0,\\ &\qquad k=\overline{1,4}.\end{aligned}$$

To prove the second part of the theorem, let us use the asymptotic property of the fundamental matrix $\Gamma(x-y)$. It can be shown that if y belongs to a compact set, say S, and $|x|$ is sufficiently large, then the following relation

$$\big[Q(\partial_y,n(y))\Gamma^\top(x-y)\big]^\top = \begin{bmatrix}\big[\mathcal{O}(|x|^{-2})\big]_{3\times3} & \big[\mathcal{O}(|x|^{-1})\big]_{3\times1}\\ \big[0\big]_{1\times3} & \mathcal{O}(|x|^{-2})\end{bmatrix}_{4\times4},$$

holds. Therefore

$$\big[W(h)(x)\big]_k = \begin{cases}\mathcal{O}(|x|^{-1}), & k=\overline{1,3},\\ \mathcal{O}(|x|^{-2}), & k=4,\end{cases} \qquad \text{as} \quad |x|\to\infty,$$

whence the inclusion $W(h)\in Z(\Omega^-)$ follows immediately.

To prove the same type inclusion for the single layer potential we proceed as follows. First, let us note that $\Gamma(x-y) = \Gamma(x) + \mathcal{O}\big(|x|^{-1}\big)$ for $y\in S$ and $|x|$ sufficiently large, and we have

$$\begin{aligned}V_k(x) &= \sum_{p=1}^4\int_S \Gamma_{kp}(x-y)h_p(y)dS_y = \sum_{p=1}^4\int_S \Big[\Gamma_{kp}(x)+\mathcal{O}(|x|^{-1})\Big]h_p(y)dS_y\\ &= \sum_{p=1}^4 \Gamma_{kp}(x)\int_S h_p(y)dS_y + \mathcal{O}(|x|^{-1})\\ &= \sum_{p=1}^3 \Gamma_{kp}(x)\int_S h_p(y)dS_y + \Gamma_{k4}(x)\int_S h_4(y)dS_y + \mathcal{O}(|x|^{-1})\\ &= \Gamma_{k4}(x)\int_S h_4(y)dS + \mathcal{O}(|x|^{-1}), \qquad k=\overline{1,3},\\ V_4(x) &= \mathcal{O}(|x|^{-1}).\end{aligned}$$

Whence

$$
\begin{aligned}
\lim_{R\to\infty}\frac{1}{4\pi R^2}&\int\limits_{\Sigma(0,R)} V_k(x)\,d\Sigma(0,R)=\\
&=\lim_{R\to\infty}\frac{1}{4\pi R^2}\int\limits_{\Sigma(0,R)}\Big\{\Gamma_{k4}(x)\int\limits_S h_4(y)dS+\mathcal{O}(|R|^{-1})\Big\}d\Sigma(0,R)\\
&=\lim_{R\to\infty}\frac{1}{4\pi R^2}\Big\{\int\limits_{\Sigma(0,R)}\Gamma_{k4}(x)d\Sigma(0,R)\int\limits_S h_4(y)dS+\int\limits_{\Sigma(0,R)}\mathcal{O}(|R|^{-1})d\Sigma(0,R)\Big\}\\
&=\lim_{R\to\infty}\frac{1}{4\pi R^2}\int\limits_{\Sigma(0,R)}\mathcal{O}(|R|^{-1})d\Sigma(0,R)\\
&=\lim_{R\to\infty}\frac{1}{4\pi}\int\limits_{\Sigma(0,1)}\mathcal{O}(|R|^{-1})\,d\Sigma(0,1)=0,
\end{aligned}
$$

which completes the proof. □

Next we show that the general integral representation formula of solutions to the homogeneous equation $A(\partial)U(x)=0$, a counterpart of (9), holds also for an exterior domain Ω^- in the class of vector-functions, satisfying the asymptotic properties (10)-(11). To this purpose, let us write the integral representation formula for the bounded domain $\Omega^-_R:=\Omega^-\cap B(0,R)$, where R is a sufficiently large positive number, such that $\overline{\Omega^+}\subset B(0,R):=\{x\in\mathbb{R}^3\,:\,|x|<R\}$,

$$
U(x)=-W_S(\{U\}^-_S)+V_S(\{BU\}^-_S)+\Phi_R(x),\qquad x\in\Omega^-_R, \tag{13}
$$

$$
0=-W_S(\{U\}^-_S)+V_S(\{BU\}^-_S)+\Phi_R(x),\qquad x\in\Omega^+\cup\big[\mathbb{R}^3\setminus\overline{B(0,R)}\big]; \tag{14}
$$

here V_S and W_S are the single and double layer potentials defined by formulas (6) and (7) respectively, while

$$
\Phi_R(x):=W_{\Sigma_R}\big(\{U\}^+_{\Sigma_R}\big)(x)-V_{\Sigma_R}\big(\{BU\}^+_{\Sigma_R}\big)(x) \tag{15}
$$

with V_{Σ_R} and W_{Σ_R} being again the single and double layer potentials with the integration surface $\Sigma_R=\partial\,B(0,R)$.

From equality (15) it follows that

$$
A(\partial)\Phi_R(x)=0,\qquad x\notin\Sigma_R. \tag{16}
$$

Moreover, from (13) and (14) we have

$$
\begin{aligned}
\Phi_R(x)&=U(x)+W_S\big(\{U\}^-_S\big)-V_S\big(\{BU\}^-_S\big),\qquad x\in\Omega^-_R,\\
\Phi_R(x)&=W_S\big(\{U\}^-_S\big)-V_S\big(\{BU\}^-_S\big),\qquad x\in\Omega^+\cup\big[\mathbb{R}^3\setminus\overline{B(0,R)}\big].
\end{aligned}
$$

Which implies that for sufficiently large numbers $R_1 < R_2$,

$$\Phi_{R_1}(x) = \Phi_{R_2}(x) \quad \text{for} \quad |x| < R_1 < R_2. \tag{17}$$

Therefore, for arbitrary $x \in \mathbb{R}^3$ the following limit exists

$$\Phi(x) := \lim_{R\to\infty} \Phi_R(x) = \begin{cases} U(x) + W_S\big(\{U\}_S^-\big) - V_S\big(\{BU\}_S^-\big), & x \in \Omega^-, \\ W_S\big(\{U\}_S^-\big) - V_S\big(\{BU\}_S^-\big), & x \in \Omega^+. \end{cases} \tag{18}$$

Consequently,

$$A(\partial)\Phi(x) = 0, \qquad x \in \Omega^+ \cup \Omega^-.$$

On the other hand, from (17) we get

$$\Phi(x) = \lim_{R\to\infty} \Phi_R(x) = \Phi_{R_1}(x)$$

for arbitrary $x \in \mathbb{R}^3$ with $R_1 > |x|$ and $\overline{\Omega^+} \subset B(0, R_1)$. From (15) and (16) then we conclude

$$A(\partial)\Phi(x) = 0, \qquad x \in \mathbb{R}^3. \tag{19}$$

At the same time from (18) we have

$$\Phi \in Z(\mathbb{R}^3), \tag{20}$$

i.e., Φ_k, $k = \overline{1,3}$, are bounded in $\mathbb{R}^3$, while $\Phi_4(x) = \mathcal{O}(|x|^{-1})$ and

$$\lim_{R\to\infty} \frac{1}{4\pi R^2} \int_{\Sigma_R} \Phi(x) d\Sigma_R = 0, \tag{21}$$

since $U \in Z(\Omega^-)$ and $W_S, V_S \in Z(\Omega^-)$ due to Theorem 2.

From relations (19)-(21) we deduce that

$$\Phi(x) = 0, \qquad \forall x \in \mathbb{R}^3.$$

Indeed, from relations (19)-(20) by the Fourier transform we get

$$A(-i\xi)\widehat{\Phi}(\xi) = 0, \qquad \xi \in \mathbb{R}^3,$$

where $\widehat{\Phi}(\xi)$ is a generalized vector-function from the Schwartz space of tempered distributions. Since the determinant $\det A(-i\xi)$ is nonsingular for $\xi \in \mathbb{R}^3 \setminus \{0\}$, it follows that the support of $\widehat{\Phi}(\xi)$ is the origin $\xi = 0$. Consequently, the vector function $\widehat{\Phi}$ is a linear combination of the Dirac distribution and its derivatives,

$$\widehat{\Phi}(\xi) = \sum_{|\alpha|\leqslant M} C_\alpha \delta^{(\alpha)}(\xi),$$

where $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ is a multi-index with $|\alpha| = \alpha_1 + \alpha_2 + \alpha_3$, M is a nonnegative integer, while $\delta^{(\alpha)}$ stands for the α-th order derivative of δ. Therefore the vector function $\Phi(x)$ is a polynomial in x,

$$\Phi(x) = \sum_{|\alpha| \leqslant M} C_\alpha x^\alpha, \qquad x \in \mathbb{R}^3.$$

Further, since $\Phi \in Z(\mathbb{R}^3)$ and the condition (21) holds, we finally conclude

$$\Phi(x) = 0, \qquad x \in \mathbb{R}^3.$$

Now, passing to the limit in (13) as $R \to \infty$, we get the general integral representation formula of solutions to the equation $A(\partial)U = 0$ in Ω^- of the space $Z(\Omega^-)$,

$$-W\big(\{U\}^-\big) + V\big(\{BU\}^-\big) = \begin{cases} U(x) & \text{for } x \in \Omega^-, \\ 0 & \text{for } x \in \Omega^+. \end{cases} \tag{22}$$

Here V and W are the single and double layer potentials defined by equalities (6) and (7).

The mapping properties of the layer potentials are described by the following assertions.

Theorem 3: *Let $S \in C^{k+1,\alpha}$ with $k \geqslant 1$ and $0 < \beta < \alpha \leqslant 1$. Then the following operators are continuous*

$$V \; : \; C^{k,\beta}(S) \to C^{k+1,\beta}(\overline{\Omega^\pm}), \qquad W \; : \; C^{k,\beta}(S) \to C^{k,\beta}(\overline{\Omega^\pm}).$$

Proof: It is word for word of the proof of the corresponding theorems in [2], [3]. □

Theorem 4: *Let $S \in C^{2,\alpha}$ with $0 < \beta < \alpha \leqslant 1$, $h \in [C^{0,\beta}(S)]^4$, and $g \in [C^{1,\beta}(S)]^4$. Then the following relations hold for all points $x \in S$:*

$$\{V(h)(x)\}^\pm = \mathcal{H}(h)(x),$$

$$\{B(\partial_x, n(x))V(h)(x)\}^\pm = [\mp 2^{-1} I_4 + \mathcal{K}]h(x), \tag{23}$$

$$\{W(h)(x)\}^\pm = [\pm 2^{-1} I_4 + \mathcal{N}]h(x), \tag{24}$$

$$\{B(\partial_x, n(x))W(g)(x)\}^+ = \{B(\partial_x, n(x))W(g)(x)\}^- =: \mathcal{L}g(x),$$

where $\mathcal{H}$ is a weakly singular integral operator, $\mathcal{K}$ and $\mathcal{N}$ are singular integral *operators, while $\mathcal{L}$ is a singular integro-differential operator,*

$$\mathcal{H}h(x) := \int_S \Gamma(x-y)h(y)dS_y,$$

$$\mathcal{K}h(x) := \int_S [B(\partial_x, n(x))\Gamma(x-y)]h(y)dS_y,$$

$$\mathcal{N}h(x) := \int_S [Q(\partial_y, n(y))\Gamma^\top(x-y)]^\top h(y)dS_y,$$

$$\mathcal{L}g(x) := \lim_{\Omega^\pm \ni z \to x \in S} B(\partial_z, n(x)) \int_S [Q(\partial_y, n(y))\Gamma^\top(x-y)]^\top g(y)dS_y.$$

Proof: The mapping and jump relations are proved by standard arguments (see [1], [2], [3]). We present here a very simple proof of the so called Liapunov-Tauber theorem (6).

Let $U := W(g)$ with $g \in [C^{1,\alpha}(S)]^4$. Evidently $U \in [C^{1,\alpha}(\overline{\Omega^\pm})]^4 \cap Z(\Omega^-)$ and it satisfies the homogeneous equation (3). Therefore we can write a general integral representation formulas (9) and (22) for the vector-function U in $\Omega^\pm$. By adding these formulas termwise we get

$$U(x) = W([U]_S)(x) - V([BU]_S)(x), \qquad x \in \Omega^+ \cup \Omega^-, \tag{25}$$

where $[\Psi]_S$ denotes the jump of a function Ψ across the surface S, $[\Psi]_S := \{\Psi\}^+ - \{\Psi\}^-$. Note that due to the relations (24) we have

$$[U]_S = [W(g)]_S = \{W(g)\}^+ - \{W(g)\}^- = g.$$

Therefore from (25) it follows that

$$W(g)(x) = W(g)(x) - V([BW(g)]_S)(x), \qquad x \in \Omega^+ \cup \Omega^-,$$

i.e., $V([BW(g)]_S)(x) = 0$ for $x \in \Omega^+ \cup \Omega^-$. Let us set $\Phi := [BW(g)]_S$. Then evidently $V(\Phi)(x) = 0$ for all $x \in \Omega^+ \cup \Omega^-$ and in view of (23) we deduce

$0 = \{BV(\Phi)\}^- - \{BV(\Phi)\}^+ = \Phi = [BW(g)]_S = \{BW(g)\}^+ - \{BW(g)\}^-$, and finally we get $\{BW(g)(x)\}^+ = \{BW(g)(x)\}^-$ for $x \in S$ which completes the prof. □

Theorem 5: *Let $S \in C^{k+1,\alpha}$ with $k \geqslant 1$ and $0 < \beta < \alpha \leqslant 1$. Then the operators*

$$\begin{aligned} \mathcal{H} &: C^{k,\beta}(S) \to C^{k+1,\beta}(S), \\ \mathcal{K} &: C^{k,\beta}(S) \to C^{k,\beta}(S), \\ \mathcal{N} &: C^{k,\beta}(S) \to C^{k,\beta}(S), \\ \mathcal{L} &: C^{k+1,\beta}(S) \to C^{k,\beta}(S), \end{aligned}$$

are continuous.

Proof: It is word for word of the proof of the corresponding theorems in [2], [3]. □

Remark 1 : More detailed analysis, based on the results in the references [2] and [3], shows that $\mathcal{H}$ is a pseudodifferential operator of order -1, $\mathcal{K}$ and $\mathcal{N}$ are pseudodifferential operator of order 0, while $\mathcal{L}$ is a pseudodifferential operator of order 1. Moreover, the operators, considered in the Bessel potential spaces $H_2^{\pm\frac{1}{2}}(S)$,

$$\begin{aligned} \mathcal{H} &: [H_2^{-\frac{1}{2}}(S)]^4 \to [H_2^{\frac{1}{2}}(S)]^4, \\ \pm 2^{-1} I_4 + \mathcal{K} &: [H_2^{-\frac{1}{2}}(S)]^4 \to [H_2^{-\frac{1}{2}}(S)]^4, \\ \pm 2^{-1} I_4 + \mathcal{N} &: [H_2^{\frac{1}{2}}(S)]^4 \to [H_2^{\frac{1}{2}}(S)]^4, \\ \mathcal{L} &: [H_2^{\frac{1}{2}}(S)]^4 \to [H_2^{-\frac{1}{2}}(S)]^4, \end{aligned}$$

are Fredholm with zero index.

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