MAIN ARTICLES

THE BASIC BVPs OF THE THEORY OF ELASTIC BINARY MIXTURES FOR A HALF-PLANE WITH CURVILINEAR CUTS

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Abstract. The first and second boundary value problems of the theory of elastic binary mixtures for a transversally isotropic half-plane with curvilinear cuts are investigated. The solvability of a system of singular integral equations is proved by using the potential method and the theory of singular integral equations.

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Introduction. In the present paper the first and second boundary value problems (BVPs) of elastic binary mixture theory are investigated for a transversally-isotropic half plane with curvilinear cuts. The boundary value problems of elasticity for anisotropic media with cuts were considered in [1, 2]. In this paper we extend this result to BVPs of elastic mixture for a transversally-isotropic elastic body. Here we shall be concerned with the plane problem of elastic binary mixture theory (it is assumed that the second components u_2' and u_2'' of the three-dimensional partial displacement vectors $u'(u_1', u_2', u_3')$ and $u''(u_1'', u_2'', u_3'')$ are equal to zero, while the components u_1', u_3', u_1'', u_3'' depend only on the variables x_1, x_3). A solution of the first BVP is sought in the form of a double-layer potential while a solution of the second BVP is sought in the form of a single-layer potential. For the unknown density we obtain a system of singular integral equations. Using the potential method and the theory of singular integral equations we rigorously prove the solvability of system of singular integral equations, corresponding to the boundary value problems.

The basic homogeneous equations of statics of the transversally isotropic elastic binary mixtures theory in the case of plane deformation can be written in the form [3]

$$C(\partial x)U = \begin{pmatrix} C^{(1)}(\partial x) & C^{(3)}(\partial x) \\ C^{(3)}(\partial x) & C^{(2)}(\partial x) \end{pmatrix} U = 0, \qquad (1)$$

where the components of the matrix $C^{(j)}(\partial x) = \|C^{(j)}_{pq}(\partial x)\|_{2x2}$ are given in the form

$$C_{pq}^{(j)}(\partial x)=C_{qp}^{(j)}(\partial x),\ \ j=1,2,3,\ \ p,q=1,2,\ \ C_{11}^{(j)}(\partial x)=c_{11}^{(j)}\frac{\partial^2}{\partial x_1^2}+c_{44}^{(j)}\frac{\partial^2}{\partial x_3^2},$$
$$C_{12}^{(j)}(\partial x)=(c_{13}^{(j)}+c_{44}^{(j)})\frac{\partial^2}{\partial x_1\partial x_3},\ \ C_{22}^{(j)}(\partial x)=c_{44}^{(j)}\frac{\partial^2}{\partial x_1^2}+c_{33}^{(j)}\frac{\partial^2}{\partial x_3^2},$$

$c_{pq}^{(k)}$-are constants, characterizing the physical properties of the mixture and satisfying certain inequalities caused by the positive definiteness of potential energy. $U=U^T(x)=(u',u'')$ is four-dimensional displacement vector-function, $u'(x)=(u_1',u_3')$ and $u''(x)=(u_1'',u_3'')$ are partial displacement vectors depending on the variables x_1,x_3. Evrerywhere below by "T" we denote transposition.

Let D be the half-plane $x_3<0$ with the boundary $x_3=0$ and suppose that the boundary of the half-plane is fastened. Let's assume that in D we have p curvilinear cuts, $l_j=a_jb_j, j=1,2,...,p$, which are simple relatively nonintersecting open Lyapunov arcs having no common points, and do not intersect the boundary. The positive direction on l_j is chosen from the point a_j to the point b_j. The normal on l_j is direct to the right with respect to the positive motion of direction. $l=\bigcup\limits_{j=1}^{p}l_j$. We suppose that D is filled with binary transversally-isotropic elastic mixture.

We introduce the notation $z=x_1+ix_3$, $\zeta_k=y_1+\alpha_k y_3$, $t_k=t_1+\alpha_k t_3$, $\sigma_k=z_k-\varsigma_k$, $z_k=x_1+\alpha_k x_3$, $t=t_1+t_3$.

The basic boundary value problems of static of the theory of elastic binary mixtures are formulated as follows:

Problem 1. Find a regular solution of the equation (1) in D, when the boundary values of the displacement vector are given on both sides of the $l_j, j=1,2,...,p$ and on the boundary $x_3=0$. Let's also assume, that the principal vector of external force acting on l, stress vector and the rotation at infinity are zero. It is required to define the deformed state of the plane.

If we denote by $U^+(U^-)$ the limits of U on l from the left (right), then the boundary conditions of the problem will take the form

$$U^+(t_0)=f^+(t_0),\ \ U^-(t_0)=f^-(t_0),\ \ t_0\in l, U^-=0,\ \ x_3=0, \tag{2}$$

where f^+, and f^- are the given functions on l satisfying Hölder's conditions on the cuts l_j, having derivatives in the class H^* (for the definitions of the classes H and H^* see[4]) and satisfying the following conditions on the ends a_j and b_j of l_j

$$f^+(a_j)=f^-(a_j),\ \ f^+(b_j)=f^-(b_j).$$

Problem 2. Find a regular solution of the equation (1) in D, when the stress vector is given on both sides of the $l_j, j=1,2,...,p$ and the boundary $x_3=0$ is traction free. In addition it is assumed that the principal vector of external force acting on l, stress vector and the rotation at infinity are zero. The boundary conditions can be written as follows:

$$[TU]^+(t_0) = f^+(t_0),$$
$$[TU]^-(t_0) = f^-(t_0), \quad t_0 \in l, \quad [TU]^- = 0, \quad x_3 = 0, -\infty < x_3 < +\infty, \tag{3}$$

where f^+, and f^- are the given known vector-functions on l of the Holder class H, which have derivatives in the class H^* and satisfying at the ends a_j and b_j of l_j, the conditions

$$f^+(a_j) = f^-(a_j), \quad f^+(b_j) = f^-(b_j).$$

Therefore, it is interesting to study the behavior of the solution of the problem in the neighborhood of cuts.

The first BVP for the half-plane with curvilinear cuts. We seek for a solution of the problem in the form [5],

$$U(z) = \frac{1}{\pi} Im \sum_{k=1}^{4} E_{(k)} \int_l \frac{\partial ln(t_k - z_k)}{\partial s}[g(s) + ih(s)]ds +$$
$$+\frac{1}{\pi} Im \sum_{k=1}^{4} \sum_{j=1}^{4} E_{(k)} \overline{E_j} \int_l \frac{\partial ln(t_k - z_k)}{\partial s}[g(s) - ih(s)]ds + \sum_{k=1}^{p} V_j(z), \tag{4}$$

where $E_{(k)} = \|A_{pq}^{(k)} A^{-1}\|_{4x4}$ denotes a special matrix that reduces the first BVP to a Fredholm integral equation of second order, $A^{(k)} = \|A_{pq}^{(k)}\|_{4x4}$. The elements of the matrix $E_{(k)}$ and the matrix A^{-1} are defined as follows

$$A^{-1} = \frac{1}{\Delta_1 \Delta_2} \begin{pmatrix} A_{33}\Delta_2 & 0 & -A_{13}\Delta_2 & 0 \\ 0 & A_{44}\Delta_1 & 0 & -A_{24}\Delta_1 \\ -A_{13}\Delta_2 & 0 & A_{11}\Delta_2 & 0 \\ 0 & -A_{24}\Delta_1 & 0 & A_{22}\Delta_1 \end{pmatrix},$$

$$\begin{aligned}
&A_{ij} = \sum_{k=1}^{4} A_{ij}^{(k)}, A_{ij}^{(k)} = A_{ji}^{(k)}, \\
&A_{11}^{(k)} = d_k[c_{11}^{(2)} q_4 + \alpha_k^2 t_{11} + \alpha_k^4 t_{12} + c_{44}^{(2)} q_3 \alpha_k^6], \\
&A_{22}^{(k)} = d_k[q_1 c_{44}^{(2)} + \alpha_k^2 t_{44} + \alpha_k^4 t_{42} + c_{33}^{(2)} q_4 \alpha_k^6], \\
&A_{33}^{(k)} = d_k[c_{11}^{(1)} q_4 + \alpha_k^2 t_{33} + \alpha_k^4 t_{23} + c_{44}^{(1)} q_3 \alpha_k^6], \\
&A_{44}^{((k))} = d_k[q_1 c_{44}^{(1)} + \alpha_k^2 t_{55} + \alpha_k^4 t_{52} + c_{33}^{(1)} q_4 \alpha_k^6], \\
&A_{12}^{(k)} = d_k \alpha_k [v_{12} + v_{11} \alpha_k^2 + v_{13} \alpha_k^4], \\
&A_{14}^{(k)} = d_k \alpha_k [w_{11} + w_{12} \alpha_k^2 + w_{13} \alpha_k^4], \quad k = 3,\, 4, \\
&A_{23}^{(k)} = d_k \alpha_k [v_{22} \alpha_k + v_{21} \alpha_k^2 + v_{23} \alpha_k^4], \\
&A_{34}^{(k)} = d_k \alpha_k [w_{24} + w_{14} \alpha_k^2 + w_{34} a_k^4], \quad k = 3,\, 4,\, 5,\, 6, \\
&A_{24}^{(k)} = d_k[-q_1 c_{44}^{(3)} + \alpha_k^2 t_{66} + \alpha_k^4 t_{62} - c_{33}^{(3)} q_4 \alpha_k^6], \\
&A_{13}^{(k)} = d_k[-q_4 c_{11}^{(3)} + \alpha_k^2 t_{22} + \alpha_k^4 t_{13} - c_{33}^{(3)} q_3 \alpha_k^6],
\end{aligned} \tag{5}$$

$$\Delta_1=\sqrt{a_1a_2a_3a_4}\Delta_2,\quad \Delta_2=A_{22}A_{44}-A_{24}^2,\ \ \Delta_1=A_{11}A_{33}-A_{13}^2,$$

$$t_{11}=c_{11}^{(2)}\delta_{22}+c_{44}^{(2)}q_4-\alpha_{13}^{(3)2}c_{44}^{(2)}+2c_{44}^{(3)}\alpha_{13}^{(2)}\alpha_{13}^{(3)}-\alpha_{13}^{(2)2}c_{44}^{(1)},$$

$$t_{12}=c_{44}^{(2)}\delta_{22}+c_{11}^{(2)}q_3-\alpha_{13}^{(3)2}c_{33}^{(2)}+2c_{33}^{(3)}\alpha_{13}^{(2)}\alpha_{13}^{(3)}-\alpha_{13}^{(2)2}c_{33}^{(1)},$$

$$t_{22}=-c_{11}^{(3)}\delta_{22}-c_{44}^{(3)}q_4+\alpha_{13}^{(1)}\alpha_{13}^{(3)}c_{44}^{(2)}-c_{44}^{(3)}(\alpha_{13}^{(2)}\alpha_{13}^{(1)}+\alpha_{13}^{(3)2})+\alpha_{13}^{(2)}\alpha_{13}^{(3)}c_{44}^{(1)},$$

$$t_{13}=-c_{44}^{(3)}\delta_{22}-c_{11}^{(3)}q_3+\alpha_{13}^{(1)}\alpha_{13}^{(3)}c_{33}^{(2)}-c_{33}^{(3)}(\alpha_{13}^{(2)}\alpha_{13}^{(1)}+\alpha_{13}^{(3)2})+\alpha_{13}^{(2)}\alpha_{13}^{(3)}c_{33}^{(1)},$$

$$t_{52}=t_{33}-c_{11}^{(1)}\delta_{22}+c_{33}^{(1)}\delta_{11},\ \ t_{62}=t_{22}-c_{11}^{(3)}\delta_{22}+c_{33}^{(3)}\delta_{11},$$

$$t_{42}=t_{11}-c_{11}^{(2)}\delta_{22}+c_{33}^{(2)}\delta_{11},$$

$$t_{44}=c_{44}^{(2)}\delta_{11}+c_{33}^{(2)}q_1-\alpha_{13}^{(3)2}c_{11}^{(1)}+2c_{11}^{(3)}\alpha_{13}^{(2)}\alpha_{13}^{(3)}-\alpha_{13}^{(3)2}c_{11}^{(2)},$$

$$t_{66}=-c_{44}^{(3)}\delta_{11}-c_{33}^{(3)}q_1+\alpha_{13}^{(2)}\alpha_{13}^{(3)}c_{11}^{(1)}-c_{11}^{(3)}(\alpha_{13}^{(2)}\alpha_{13}^{(1)}+\alpha_{13}^{(3)2})+\alpha_{13}^{(1)}\alpha_{13}^{(3)}c_{11}^{(2)},$$

$$t_{23}=c_{44}^{(1)}\delta_{22}+c_{11}^{(1)}q_3-\alpha_{13}^{(1)2}c_{33}^{(2)}+2c_{33}^{(3)}\alpha_{13}^{(1)}\alpha_{13}^{(3)}-\alpha_{13}^{(3)2}c_{33}^{(1)},$$

$$t_{33}=c_{11}^{(1)}\delta_{22}+c_{44}^{(1)}q_4-\alpha_{13}^{(1)2}c_{44}^{(2)}+2c_{44}^{(3)}\alpha_{13}^{(1)}\alpha_{13}^{(3)}-\alpha_{13}^{(3)2}c_{44}^{(1)},$$

$$t_{55}=c_{44}^{(1)}\delta_{11}+c_{33}^{(1)}q_1-\alpha_{13}^{(1)2}c_{11}^{(2)}+2c_{11}^{(3)}\alpha_{13}^{(1)}\alpha_{13}^{(3)}-\alpha_{13}^{(3)2}c_{11}^{(1)},$$

$$\begin{aligned}v_{11}=\ &\alpha_{13}^{(2)}(\alpha_{13}^{(2)}\alpha_{13}^{(1)}-\alpha_{13}^{(3)2})-\alpha_{13}^{(1)}(c_{44}^{(2)2}+c_{11}^{(2)}c_{33}^{(2)})-\alpha_{13}^{(2)}(c_{44}^{(3)2}+c_{11}^{(3)}c_{33}^{(3)})\\&+\alpha_{13}^{(3)}(2c_{44}^{(2)}c_{44}^{(3)}+c_{11}^{(2)}c_{33}^{(3)}+c_{11}^{(3)}c_{33}^{(2)}),\end{aligned}$$

$$\begin{aligned}w_{12}=\ &-\alpha_{13}^{(2)}(\alpha_{13}^{(2)}\alpha_{13}^{(1)}-\alpha_{13}^{(3)2})-\alpha_{13}^{(3)}(c_{44}^{(1)}c_{44}^{(2)}+c_{11}^{(2)}c_{33}^{(1)}+c_{44}^{(3)2}+c_{11}^{(3)}c_{33}^{(3)})\\&+\alpha_{13}^{(2)}(c_{44}^{(1)}c_{44}^{(3)}+c_{11}^{(3)}c_{33}^{(3)})+\alpha_{13}^{(1)}(c_{44}^{(2)}c_{44}^{(3)}+c_{11}^{(2)}c_{33}^{(3)}),\end{aligned}$$

$$\begin{aligned}v_{21}=\ &-\alpha_{13}^{(3)}(\alpha_{13}^{(2)}\alpha_{13}^{(1)}-\alpha_{13}^{(3)2})-\alpha_{13}^{(3)}(c_{44}^{(2)}c_{44}^{(1)}+c_{11}^{(3)}c_{33}^{(3)}+c_{44}^{(3)2}+c_{11}^{(1)}c_{33}^{(2)})\\&+\alpha_{13}^{(2)}(c_{44}^{(1)}c_{44}^{(3)}+c_{11}^{(1)}c_{33}^{(3)})+\alpha_{13}^{(1)}(c_{44}^{(2)}c_{44}^{(3)}+c_{11}^{(3)}c_{33}^{(2)}),\end{aligned}$$

$$\begin{aligned}w_{14}=\ &\alpha_{13}^{(1)}(\alpha_{13}^{(2)}\alpha_{13}^{(1)}-\alpha_{13}^{(3)2})+\alpha_{13}^{(3)}(2c_{44}^{(1)}c_{44}^{(3)}+c_{11}^{(3)}c_{33}^{(1)}+c_{44}^{(3)2}+c_{11}^{(1)}c_{33}^{(3)})\\&-\alpha_{13}^{(1)}(c_{44}^{(3)2}+c_{11}^{(3)}c_{33}^{(3)})-\alpha_{13}^{(2)}(c_{44}^{(1)2}+c_{11}^{(1)}c_{33}^{(1)}),\end{aligned}$$

$$v_{12}=-\alpha_{13}^{(1)}c_{44}^{(2)}c_{11}^{(2)}+\alpha_{13}^{(3)}(c_{11}^{(2)}c_{44}^{(3)}+c_{11}^{(3)}c_{44}^{(2)})-\alpha_{13}^{(2)}c_{11}^{(3)}c_{44}^{(3)},$$

$$v_{13}=-\alpha_{13}^{(1)}c_{44}^{(2)}c_{33}^{(2)}+\alpha_{13}^{(3)}(c_{33}^{(2)}c_{44}^{(3)}+c_{33}^{(3)}c_{44}^{(2)})-\alpha_{13}^{(2)}c_{33}^{(3)}c_{44}^{(3)},$$

$$w_{11}=\alpha_{13}^{(1)}c_{44}^{(3)}c_{11}^{(2)}-\alpha_{13}^{(3)}(c_{11}^{(2)}c_{44}^{(1)}+c_{11}^{(3)}c_{44}^{(3)})+\alpha_{13}^{(2)}c_{11}^{(3)}c_{44}^{(1)},$$

$$w_{13}=\alpha_{13}^{(1)}c_{44}^{(2)}c_{33}^{(3)}-\alpha_{13}^{(3)}(c_{33}^{(1)}c_{44}^{(2)}+c_{33}^{(3)}c_{44}^{(3)})+\alpha_{13}^{(2)}c_{44}^{(3)}c_{11}^{(1)},$$

$$v_{22}=\alpha_{13}^{(1)}c_{44}^{(2)}c_{11}^{(3)}-\alpha_{13}^{(3)}(c_{11}^{(1)}c_{44}^{(2)}+c_{11}^{(3)}c_{44}^{(3)})+\alpha_{13}^{(2)}c_{11}^{(1)}c_{44}^{(3)},$$

$$v_{23}=\alpha_{13}^{(1)}c_{44}^{(3)}c_{33}^{(2)}-\alpha_{13}^{(3)}(c_{33}^{(2)}c_{44}^{(1)}+c_{33}^{(3)}c_{44}^{(3)})+\alpha_{13}^{(2)}c_{33}^{(3)}c_{44}^{(1)},$$

$$w_{24}=-\alpha_{13}^{(1)}c_{44}^{(3)}c_{11}^{(3)}+\alpha_{13}^{(3)}(c_{11}^{(3)}c_{44}^{(1)}+c_{11}^{(1)}c_{44}^{(3)})-\alpha_{13}^{(2)}c_{11}^{(1)}c_{44}^{(1)},$$

$$w_{34} = -\alpha_{13}^{(1)} c_{44}^{(3)} c_{33}^{(3)} + \alpha_{13}^{(3)} (c_{33}^{(1)} c_{44}^{(3)} + c_{33}^{(3)} c_{44}^{(1)}) - \alpha_{13}^{(2)} c_{33}^{(1)} c_{44}^{(1)},$$

α_k, $k = 1, 2, 3, 4$, are the roots of the characteristic equation ([6]). g and h are the unknown real vectors from the Holder class that have derivatives in the class H^*. We write

$$V_j(z) = \frac{1}{\pi} \operatorname{Im} \sum_{k=1}^{4} A^{(k)} \frac{(z_k - b_j^{(k)}) ln(z_k - b_j^{(k)}) - (z_k - a_j^{(k)}) ln(z_k - a_j^{(k)})}{b_j^{(k)} - a_j^{(k)}} K^j,$$

$$a_j^{(k)} = \operatorname{Re} a_j + \alpha_k \operatorname{Im} a_j, \quad b_j^{(k)} = \operatorname{Re} b_j + \alpha_k \operatorname{Im} b_j.$$

$K^{(j)}, j = 1, ..., p$, are the unknown real constant vectors to be defined later on.

The vector $V_j(z)$ satisfies the following conditions:

1. $V_j(z)$ has the logarithmic singularity at infinity

$$V_j = \frac{1}{\pi} \operatorname{Im} \sum_{k=1}^{4} A^{(k)} (-\ln\, z_k + 1) K^j + O(z_k^{-1}).$$

2. Under V_j is supposed a branch, which is uniquely defined on the cut plane along l_j.

3. V_j is continuously continued function on l_j from the left and right, including the end points a_j and b_j, i. e., we have the equalities

$$V_j^+(a_j) = V_j^-(a_j), \quad V_j^+(b_j) = V_j^-(b_j),$$

$$V_j^+ - V_j^- = 2 \operatorname{Re} \sum_{k=1}^{4} A^{(k)} \frac{\tau_k - a_j^{(k)}}{b_j^{(k)} - a_j^{(k)}} K^j, \quad j = 1, ..., p.$$

It is easily to check $U^- = 0$, $x_3 = 0$, and

$$\int_{-\infty}^{+\infty} (TU)^- ds + \int_l [(TU)^+ - (TU)^-] ds = 0.$$

To define the unknown density, we obtain by virtue of (4)-(2), the following system of singular integral equation of the normal type

$$2g(\tau) = f^- - f^+ - \operatorname{Re} \sum_{j=1}^{p} \sum_{k=1}^{4} A^{(k)} \frac{\tau_k - a_j^{(k)}}{b_j^{(k)} - a_j^{(k)}} K^j,$$

$$\frac{1}{\pi} \int_l \frac{h(t)dt}{t - t_0} + \frac{1}{\pi} \int_l K(t_0, t) dt = \Omega(t_0), \quad t_0 \in l, \tag{6}$$

where

$$K(t_0, t) = -iE \frac{\partial \theta}{\partial s} +$$

$$+\mathrm{Re}\sum_{k=1}^{4}E_{(k)}\frac{\partial}{\partial s}\ln(1+\lambda_k\frac{\overline{\overline{t}}-\overline{t_0}}{t-t_0}), -\mathrm{Re}\sum_{k=1}^{4}\sum_{q=1}^{4}E_{(k)}\overline{E_q}\frac{\partial}{\partial s}\ln(\overline{t_q}--t_{k0}),$$

$$\Omega(t_0)=\frac{1}{2}(f^{+}+f^{-})-\frac{1}{2}\sum_{j=1}^{p}(V_j^{+}+V_j^{-})$$

$$-\frac{1}{\pi}\int_l[E\frac{\partial\theta}{\partial s(t)}+\mathrm{Im}\sum_{k=1}^{4}E_{(k)}\frac{\partial}{\partial s(t)}\ln(1+\lambda_k\frac{\overline{t}-\overline{t_0}}{t-t_0})$$

$$+\mathrm{Im}\sum_{k=1}^{4}\sum_{q=1}^{4}E_{(k)}\overline{E_q}\frac{\partial}{\partial s(t)}\ln(\overline{t_q}-t_{k0})]g(t)ds,$$

$$\lambda_k=\frac{1+i\alpha_k}{1-i\alpha_k},\theta=\arg(t-t_0),\quad t_{k0}=\mathrm{Re}\,t_0+\alpha_k\,\mathrm{Im}\,t_0,\quad t=y_1+iy_3.$$

Thus we defined vector g on l. It is not difficult to verify, that $g\in H$, $g'\in H^*$, $g(a_j)=g(b_j)=0$, $\Omega\in H$, $\Omega'\in H^*$. Formula (6) is a system of singular integral equations of the normal type with respect to the vector h. The points a_j and b_j are nonsingular ones, while the summary index of the class h_{2p} is equal to $-4p$ (for the definition of the class h_{2p} see [4]).

If the solution of equation (6) on the class h_{2p} exists , it will satisfy Holder's condition on l, vanishing on the points a_j and b_j and having to derivatives in the class H^*.

Let's prove that the homogeneous equation corresponding to the system of equation (6) only has a trivial solution h_0 in the class h_{2p}. Let's assume the contrary. Let h_0 be nontrivial solution of the homogeneous system in the class h_{2p} and construct the potential

$$U_0(z)=\frac{1}{\pi}\mathrm{Re}\sum_{k=1}^{4}E_{(k)}\int_l\frac{\partial\ln(t_k-z_k)}{\partial s}[g(t)+ih(t)]ds$$

$$+\frac{1}{\pi}\mathrm{Im}\sum_{k=1}^{4}\sum_{j=1}^{4}E_{(k)}\overline{E_j}\int_l\frac{\partial\ln(\overline{t_j}-z_k)}{\partial s}[g(t)-ih(t)]ds.$$

It is clear $U_0^{+}=U_0^{-}=0$, $t\in l$ and on the basis of uniqueness theorem $U_0=0$, $z\in D$. Therefore $TU_0=0$, $z\in D$, and

$$(TU_0)^{+}-(TU_0)^{-}=2A^{-1}\frac{\partial h_0}{\partial s}=0.$$

Consequently $h_0=0$, since $h_0(a_j)=0$, that completes the proof. Thus the corresponding adjoint homogeneous equation has 4p linearly independent solution σ_j, $j=1,...,4p$ in the adjoined class and the conditions of solvability are

$$\int_l\Omega\sigma_j ds=0.$$

From last equality we get $4p$ algebraic equation with respect to K_j. On the basis of the uniqueness theorem it is not difficult to prove that the last system is solvable.

The second BVP for the half-plane with curvilinear cuts. We seek for a solution of the above formulated second BVP for the half-plane in the form [5]

$$U(z)=\frac{1}{\pi}\operatorname{Im}\sum_{k=1}^{4}R^{T}_{(k)}iL(\int\limits_{l}\ln{(t_k-z_k)}[g(s)+ih(s)]ds$$
$$\sum_{j=1}^{4}\overline{L_{(k)}iL}\int\limits_{l}\ln{(z_k-\overline{t_j})}[g(s)-ih(s)]ds),$$

where g and h are unknown real vector-functions.

$$\begin{aligned}
&R^{(k)}=\|R^{(k)}_{pq}\|_{4x4},\quad p,\,q=1,2,3,4,\\
&R^{(k)}_{1j}=\alpha_k(c^{(1)}_{44}A^{(k)}_{1j}+c^{(3)}_{44}A^{(k)}_{j3})+c^{(1)}_{44}A^{(k)}_{j2}+c^{(3)}_{44}A^{(k)}_{j4},\\
&R^{(k)}_{2j}=-\alpha_k^{-1}R^{(k)}_{1j},\\
&R^{(k)}_{3j}=\alpha_k(c^{(3)}_{44}A^{(k)}_{1j}+c^{(2)}_{44}A^{(k)}_{j3})+c^{(3)}_{44}A^{(k)}_{j2}+c^{(2)}_{44}A^{(k)}_{j4},\\
&R^{(k)}_{4j}=-\alpha_k^{-1}R^{(k)}_{3j},\quad j=1,2,3,4,
\end{aligned}\tag{7}$$

$A^{(k)}_{pq}$ are given by (5) and $\overline{L^{(k)T}L}$ denotes the complex conjugate matrix of $L^{(k)T}L$,

$$L=\frac{1}{\Delta_1\Delta_2}\begin{pmatrix} L_{33}\Delta_2 & 0 & -L_{13}\Delta_2 & 0\\ 0 & L_{44}\Delta_1 & 0 & -L_{24}\Delta_1\\ -L_{13}\Delta_2 & 0 & L_{11}\Delta_2 & 0\\ 0 & -L_{24}\Delta_1 & 0 & L_{22}\Delta_1\end{pmatrix},\tag{8}$$

$$L_{11}=-\Delta q_4[a_{44}B_1+(b_{11}+2a_{34})A_1+a_{33}D_1],$$
$$A_1=-B_0m_3,\quad B_1=B_om_1,$$
$$L_{13}=\Delta q_4[a_{24}B_1+(-b_{33}+a_{14}+a_{23})A_1+a_{13}D_1],$$
$$C_1=-\frac{A_!+B_1m_2}{\alpha_1\alpha_2\alpha_3\alpha_4},$$
$$L_{22}=-\Delta q_4[a_{44}C_1+(b_{11}+2a_{34})B_1+a_{33}A_1],$$
$$D_1=-A_1m_2-B_1\alpha_1\alpha_2\alpha_3\alpha_4,$$
$$L_{24}=\Delta q_4[a_{24}C_1+(-b_{33}+a_{14}+a_{23})B_1+a_{13}A_1],$$
$$\Delta_1=L_{11}L_{33}-L^2_{13},$$
$$L_{33}=-\Delta q_4[a_{22}B_1+(b_{22}+2a_{12})A_1+a_{11}D_1],$$
$$\Delta_2=L_{22}L_{44}-L^2_{24},$$
$$L_{44}=-\Delta q_4[a_{22}C_1+(b_{22}+2a_{12})B_1+a_{11}A_1],$$
$$\Delta_2=[b_4(m_1m_3-2\sqrt{a_1a_2a_3a_4})+q_4\Delta m_0]q_4\Delta B_0>0,$$
$$m_0=(a_{11}a_{44}+a_{33}a_{22}-2a_{13}a_{24}),\quad \Delta_1=\sqrt{a_1a_2a_3a_4}\Delta_2,$$

$$m_1 = \sum_{k=1}^{4} \alpha_k, m_2 = \alpha_1\alpha_2 + \alpha_1\alpha_3 + \alpha_1\alpha_4 + \alpha_2\alpha_3 + \alpha_2\alpha_4 + \alpha_3\alpha_4,$$
$$m_3 = \alpha_1\alpha_2\alpha_3 + \alpha_1\alpha_2\alpha_4 + \alpha_1\alpha_3\alpha_4 + \alpha_2\alpha_3\alpha_4,$$
$$B_0^{-1} = (\alpha_1 + \alpha_2)(\alpha_1 + \alpha_3)(\alpha_1 + \alpha_4)(\alpha_2 + \alpha_3)(\alpha_2 + \alpha_4)(\alpha_3 + \alpha_4).$$

Between the coefficients a_{pq}, b_{pp} and $c_{pq}^{(j)}$ there are the relations

$$a_{11}\Delta = c_{11}^{(2)} q_3 - c_{33}^{(1)} c_{13}^{(2)2} + 2c_{13}^{(2)} c_{13}^{(3)} c_{33}^{(3)} - c_{33}^{(2)} c_{13}^{(2)2} > 0,$$
$$a_{12}\Delta = c_{13}^{(2)}(c_{13}^{(1)} c_{13}^{(2)} - c_{13}^{(3)2}) - c_{13}^{(1)} c_{11}^{(2)} c_{33}^{(2)} - c_{13}^{(2)} c_{11}^{(3)} c_{33}^{(3)} + c_{13}^{(3)}(c_{11}^{(2)} c_{33}^{(3)} + c_{11}^{(3)} c_{33}^{(2)}),$$
$$a_{13}\Delta = -c_{11}^{(3)} q_3 + c_{33}^{(2)} c_{13}^{(1)} c_{13}^{(3)} + c_{33}^{(1)} c_{13}^{(2)} c_{13}^{(3)} - c_{33}^{(3)}(c_{13}^{(1)} c_{13}^{(2)} + c_{13}^{(3)2}),$$
$$a_{14}\Delta = -c_{13}^{(3)}(c_{13}^{(1)} c_{13}^{(2)} - c_{13}^{(3)2}) + c_{13}^{(1)} c_{11}^{(2)} c_{33}^{(3)} + c_{13}^{(2)} c_{11}^{(3)} c_{33}^{(1)} - c_{13}^{(3)}(c_{11}^{(2)} c_{33}^{(1)} + c_{11}^{(3)} c_{33}^{(3)}),$$
$$a_{23}\Delta = -c_{13}^{(3)}(c_{13}^{(1)} c_{13}^{(2)} - c_{13}^{(3)2}) + c_{13}^{(1)} c_{11}^{(3)} c_{33}^{(2)} + c_{13}^{(2)} c_{11}^{(1)} c_{33}^{(3)} - c_{13}^{(3)}(c_{11}^{(1)} c_{33}^{(2)} + c_{11}^{(3)} c_{33}^{(3)}),$$
$$a_{22}\Delta = c_{33}^{(2)} q_1 - c_{11}^{(1)} c_{13}^{(2)2} + 2c_{13}^{(2)} c_{13}^{(3)} c_{11}^{(3)} - c_{11}^{(2)} c_{13}^{(2)2} > 0,$$
$$a_{24}\Delta = -c_{33}^{(3)} q_1 + c_{11}^{(2)} c_{13}^{(1)} c_{13}^{(3)} + c_{13}^{(2)} c_{13}^{(3)} c_{11}^{(1)} - c_{11}^{(3)}(c_{13}^{(1)} c_{13}^{(2)} + c_{13}^{(3)2}),$$
$$a_{33}\Delta = c_{11}^{(1)} q_3 - c_{33}^{(2)} c_{13}^{(1)2} + 2c_{13}^{(1)} c_{13}^{(3)} c_{33}^{(3)} - c_{33}^{(1)} c_{13}^{(3)2} > 0,$$
$$a_{34}\Delta = c_{13}^{(1)}(c_{13}^{(1)} c_{13}^{(2)} - c_{13}^{(3)2}) - c_{13}^{(2)} c_{11}^{(1)} c_{33}^{(1)} - c_{13}^{(1)} c_{11}^{(3)} c_{33}^{(3)} + c_{13}^{(3)}(c_{11}^{(1)} c_{33}^{(3)} + c_{11}^{(3)} c_{33}^{(1)}),$$
$$a_{44}\Delta = c_{33}^{(1)} q_1 - c_{11}^{(2)} c_{13}^{(1)2} + 2c_{13}^{(1)} c_{13}^{(3)} c_{11}^{(3)} - c_{11}^{(1)} c_{13}^{(3)2} > 0,$$
$$\Delta = (c_{11}^{(1)} a_{11} + c_{11}^{(2)} a_{33} + 2c_{11}^{(3)} a_{13})\Delta - q_1 q_3 + (c_{13}^{(1)} c_{13}^{(2)} - c_{13}^{(3)2}) > 0,$$
$$b_{jj} = c_{44}^{(j)} q_4^{-1} > 0, \quad j = 1, 2, 3.$$

$$L^{(k)} = \begin{pmatrix} \alpha_k^2 L_{22}^{(k)} & -\alpha_k L_{22}^{(k)} & \alpha_k^2 L_{24}^{(k)} & -\alpha_k L_{24}^{(k)} \\ -\alpha_k L_{22}^{(k)} & L_{22}^{(2)} & -\alpha_k L_{24}^{(k)} & L_{24}^{(k)} \\ \alpha_k^2 L_{24}^{(k)} & -\alpha_k L_{24}^{(k)} & \alpha_k^2 L_{44}^{(k)} & -\alpha_k L_{44}^{(k)} \\ -\alpha_k L_{24}^{(k)} & L_{24}^{(k)} & -\alpha_k L_{44}^{(k)} & L_{44}^{(k)} \end{pmatrix},$$

$$L_{22}^{(k)} = -\Delta q_4 d_k[a_{44} + \alpha_k^2(b_{11} + 2a_{34}) + a_{33}\alpha_k^4],$$
$$L_{24}^{(k)} = \Delta q_4 d_k[a_{24} + \alpha_k^2(-b_{33} + a_{14} + a_{23}) + a_{13}\alpha_k^4],$$
$$L_{44}^{(k)} = -\Delta q_4 d_k[a_{22} + \alpha_k^2(b_{22} + 2a_{12}) + a_{11}\alpha_k^4].$$

For the stress vector we get

$$T(\partial x, n)U(z) = \frac{1}{\pi}\operatorname{Im}\sum_{k=1}^{4} L_{(k)} iL[\int_l \frac{\partial \ln(t_k - z_k)}{\partial s}[g(t) + ih(t)]ds +$$
$$\frac{1}{\pi}\operatorname{Im}\sum_{j=1}^{4} \overline{L_{(j)} iL}\int_l \frac{\partial \ln(z_k - \overline{t_j})}{\partial s}[g(t) - ih(t)]ds].$$

It can be shown that the vector TU satisfies the condition $(TU)^{-}=0$, automatically;

$$[T(\partial x,n)U(z)]^{-}=\frac{1}{\pi}\operatorname{Im}\sum_{k=1}^{4}L_{(k)}iL[\int_{l}\frac{g(t)+ih(t))}{x_1-t_j}ds$$
$$+\frac{1}{\pi}\operatorname{Im}\sum_{j=1}^{4}\overline{L_{(j)}iL}\int_{l}\frac{g(t)-ih(t)}{x_1-t_j}ds]=0.$$

But on the arc ends $l_j, j=1,2,...,p$, we have

$$[T(\partial x,n)U(z)]^{\pm}=\mp g(t_0)+\frac{1}{\pi}\operatorname{Im}\sum_{k=1}^{4}L_{(k)}iL[\int_{l}\frac{\partial\ln{(t_{k0}-t_k)}}{\partial s}[g(t)+ih(t)]ds$$
$$+\frac{1}{\pi}\operatorname{Im}\sum_{j=1}^{4}\overline{L_{(j)}iL}\int_{l}\frac{\partial\ln{(t_{k0}-\overline{t_j})}}{\partial s}[g(t)-ih(t)]ds]=f^{\pm}(t_0).$$

From last equation we get

$$2g(\tau)=f^{-}-f^{+},$$
$$\begin{aligned}&\frac{1}{\pi}\operatorname{Re}\sum_{k=1}^{4}P_{(k)}\Big[\int_{l}\frac{\partial\ln{(t_{k0}-t_k)}}{\partial s}h(t)dt\\&-\sum_{j=1}^{4}\overline{P_{(j)}}\int_{l}\frac{\partial\ln{(t_{k0}-\overline{t_j})}}{\partial s}h(t)dt\Big]=\Omega(t_0),\end{aligned}\tag{9}$$

where

$$\Omega(t_0)=\frac{1}{2}(f^{-}+f^{+})-\frac{1}{\pi}\operatorname{Im}\sum_{k=1}^{4}P_{(k)}[\int_{l}\frac{\partial\ln{(t_{k0}-t_k)}}{\partial s}g(t)dt$$
$$-\sum_{j=1}^{4}\overline{P_{(j)}}\int_{l}\frac{\partial\ln(t_{k0}-\overline{t_j})}{\partial s}g(t)dt].$$

We must require in addition that the solution h of the system of singular integral equations (9) satisfies the condition

$$\int_{l}h(t)ds=RL\int_{l}(f^{-}-f^{+})ds,\tag{10}$$

where $\sum\limits_{k=1}^{4}R^{(k)T}=-E+iR.$

We seek for the solution of the system (9) in the class h_0 ([4]). Therefore, the total index in the class h_0 is 4p.

Let's prove that the adjoint homogeneous system corresponding to the system (9) has only the trivial solution in the adjoint class. The adjoint homogeneous system has the form

$$\begin{aligned}&\frac{1}{\pi}\operatorname{Re}\sum_{k=1}^{4} iLL_{(k)}\int_{l}\frac{\partial \ln(t_k-t_{k0})}{\partial s}\nu(t)ds\\&-\sum_{k=1}^{4}\sum_{j=1}^{4}\overline{LiL_{(j)}iLL_k}\int_{l}\frac{\partial \ln(\overline{t_j}-t_{k0})}{\partial s}\nu(t)ds=0.\end{aligned}\tag{11}$$

Note, that the solution of the system (11) will be vector, satisfying the Holder's conditions, vanishing at the end points a_j and b_j, and having derivative in the class H^* ([4]).

Multiplying the system (11) by nonsingular matrix $a=L^{-1}$ given by (8) and taking into account the identity $aLL_{(k)}=P_{(k)}a$, we obtain

$$\begin{aligned}&\frac{1}{\pi}\operatorname{Re}\sum_{k=1}^{4} P_{(k)}\int_{l}\frac{\partial \ln(t_k-t_{k0})}{\partial s}a\nu(t)ds\\&-\operatorname{Re}\sum_{k=1}^{4}\sum_{j=1}^{4}\overline{P_{(j)}P_k}\int_{l}\frac{\partial \ln(\overline{t_j}-t_{k0})}{\partial s}a\nu(t)ds=0.\end{aligned}\tag{12}$$

Let ν_0 be the nontrivial solution of the system (12) and construct the potential

$$\begin{aligned}U_0(z)=&\frac{1}{\pi}\operatorname{Re}\sum_{k=1}^{4} R_{(k)}^{T}iL\int_{l}\frac{\partial \ln(t_k-z_k)}{\partial s}a\nu(t)ds\\&-\operatorname{Re}\sum_{k=1}^{4}\sum_{j=1}^{4}\overline{P_{(j)}P_k}\int_{l}\frac{\partial \ln(\overline{t_j}-t_k)}{\partial s}a\nu(t)ds.\end{aligned}\tag{13}$$

Then at every point of l, except may be a_j and b_j

$$[T(\partial t,n)U_0]^{\pm}=0$$

and on the basis of the uniqueness theorem we have $U_0=0$.

Finally, from equality: $U_0^{+}-U_0^{-}=-2a\nu(t_0)=0,\quad t_0\in l$, it follows, that $\nu_0=0$, which contradicts our assumption.

Thus, the system (9) is solvable in the class h_0 for the arbitrary right-hand side and the solution depended on the 4p arbitrary constants. These constants are fixed by the conditions (10), given the system of algebraic 4p linear equations.

Let's prove that the determinant of this system is not zero. Indeed, let's take the homogeneous system, corresponding to the conditions $f^{\pm}=0$. Supposing the solution $K_1^{(0)},...,K_{4p}^{(0)}$ nontrivial, we construct the potential

$$U_0(z)=\frac{1}{\pi}\operatorname{Re}\sum_{k=1}^{4} R_{(k)}^{T}L\int_{l}\ln(z_k-t_k)h^{(0)}ds,\tag{14}$$

where $h^{(0)}$ is a linear combinations of solutions $h^{(j)}$

$$h^{(0)} = \sum_{k=1}^{4p} K^{(j)} h^{(j)},$$

and $h^{(j)}$ are linearly independent solutions of the homogeneous equation corresponding to (9). $h^{(j)}$has satisfy the following condition

$$\int_{l_j} h^{(0)} ds = 0.$$

Then the potential (14) is regular at infinity and by the uniqueness theorem $h^{(0)} = 0$. But we have the following equality

$$\left(\frac{\partial u}{\partial s}\right)^{+} - \left(\frac{\partial u}{\partial s}\right)^{-} = Lh^{(0)} = 0.$$

Whence we conclude that $K^{(j)} = 0$, which contradict the assumption. Thus the solvability of the problem is proved.

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