ON THE RIEMANN–HILBERT PROBLEM IN WEIGHTED CLASSES OF CAUCHY TYPE INTEGRALS WITH DENSITY FROM $L^{P(\cdot)}(\Gamma)$

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Abstract. We consider the Riemann–Hilbert problem formulated as follows: define a function $\phi \in K^{p(\cdot)}(D;\omega)$ whose boundary values $\phi^+(t)$ satisfy the condition $\mathrm{Re}[(a(t)+ib(t))\phi^+(t)] = c(t)$ a.e. on the Γ. Here D is the finite simply connected domain bounded by a simple closed curve Γ, and $K^{p(\cdot)}(D;\omega)$ is the set of functions $\phi(z)$ representable in the form $\phi(z) = \omega^{-1}(z)(K_\Gamma\varphi)(z)$, where $\omega(z)$ is a weight function and $(K_\Gamma\varphi)(z)$ is a Cauchy type integral whose density φ is integrable with a variable exponent $p(t)$. It is assumed that Γ is a piecewise-Lyapunov curve without zero angles, $\omega(z)$ is an arbitrary power function and $p(t)$ satisfies the Log-Hölder condition. The solvability conditions are established and solutions are constructed. In addition to the weight ω and functions a, b, c, these solutions largely depend both on the values of $p(t)$ at the angular points of Γ and on the values of angles at these points.

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Contents

1. Introduction

Let D be the simply connected domain bounded by a simple closed rectifiable curve Γ; a, b, c be the real functions given on Γ. The Riemann–Hilbert problem is formulated as follows [1, p. 144]1: Find an analytic in D function ϕ from the given class $A(D)$ that possesses boundary values $\phi^+(t)$, $t \in \Gamma$, which satisfy the condition

$$\operatorname{Re}\left[(a(t)+ib(t))\,\phi^+(t)\right] = c(t), \quad t \in \Gamma. \tag{1}$$

This problem is a particular case of the quite general problem posed by Riemann [2] and considered for the first time by Hilbert [3] The survey of the available results on this problem can be found in [1] and [4].

N. Muskhelishvili indicated an effective way of solving problem (1) by reducing it to the well studied problem of linear conjugation [1,§§ 40–43]. This method was generalized in [5] for the case where $A(D)$ is the class of functions representable by a Cauchy type integral with density from the Lebesgue class $L^p(\Gamma)$, $p > 1$, and also $\phi^+(t)$ in (1) is understood as an angular boundary value of the function ϕ at a point t and the equality in (1) is assumed to hold almost everywhere. It should be mentioned that for multi-connected domains the Riemann-Hilbert problem for analytic functions was firstly investigated by D. Kveselava. For generalized analytic functions this problem in multi-connected domains was studied by I. Vekua, B. Bojarski and I. Danilyuk.

In the recent time, an intensive development of the theory of Lebesgue spaces with a variable exponent has made it possible to investigate boundary value problems of analytic functions and mathematical physics formulated in more advantageous terms, taking into account the local behavior of the given functions and the functions we want to define (see, e.g., [6]–[10]).

Let us recall the definition of weighted Lebesgue spaces with a variable exponent Let $t = t(s)$, $0 \le s \le l$, be the equation of a simple rectifiable curve Γ with respect to the arc abscissa. Let, further, $p : \Gamma \to \mathbb{R}$ be a measurable function with the condition.

$$p_- = \operatorname*{ess\,inf}_{t\in\Gamma} p(t) > 1 \quad \text{and} \quad \operatorname*{ess\,sup}_{t\in\Gamma} p(t) = p_+ < \infty.$$

For the measurable, a.e. finite function $\rho = \rho(t)$ we assume

$$L^{p(\cdot)}(\Gamma;\rho) = \left\{ f : \|f\|_{L^{p(\cdot)}(\Gamma;\rho)} < \infty \right\},$$

where

$$\|f\|_{L^{p(\cdot)}(\Gamma;\rho)} = \inf\left\{ \lambda > 0 : \int_0^l \left| \frac{f(t(s))\rho(t(s))}{\lambda} \right|^{p(t(s))} ds \le 1 \right\}. \tag{2}$$

The space $L^{p(\cdot)}(\Gamma;\rho)$ is a Banach space. For the investigation of these spaces see, e.g., [7].

In this paper, the Riemann–Hilbert problem is considered in the class of functions representable in the form $\phi(z) = \omega^{-1}(z)(K_\Gamma\varphi)(z)$, where $(K_\Gamma\varphi)(z)$ is a Cauchy type integral with density from the class $L^{p(\cdot)}(\Gamma)$, and $\omega(z)$ is an arbitrary function of the form

$$\omega(z) = \prod_{k=1}^{\nu} (z - t_k)^{\alpha_k}, \quad t_k \in \Gamma, \quad \alpha_k \in \mathbb{R}. \tag{3}$$

We call the set of all such functions ϕ the weighted class of Cauchy type integral with density from $L^{p(\cdot)}(\Gamma)$ and denote it by $K^{p(\cdot)}(D;\omega)$ as different from the set of Cauchy type integrals with density from $L^{p(\cdot)}(\Gamma;\omega)$ denoted by $K^{p(\cdot)}(\Gamma;\omega)$ [9]. When a weight function $\omega \in W^{p(\cdot)}(\Gamma)$, i.e. a singular Cauchy operator is continuous in $L^{p(\cdot)}(\Gamma;\omega)$, we show that $K^{p(\cdot)}(D;\omega)$ and $K^{p(\cdot)}(\Gamma;\omega)$ coincide for the wide class of curves Γ and functions $p(t)$ (see Theorem 1 and its corollary below). Thus the results obtained in the paper extend to the case of the problem considered in the class $K^{p(\cdot)}(\Gamma;\omega)$, $\omega \in W^{p(\cdot)}(\Gamma)$. In [12], problem (1) is considered for an arbitrary power weight when $p(t) = p = const$.

Below problem (1) is considered in classes $K^{p(\cdot)}(D;\omega)$, where D is the finite domain bounded by simple piecewise-Lyapunov curve with angular points A_k, at which the angle values with respect to D are equal to $\pi\nu_k$, $0 < \nu_k \le 2$. The weight ω is assumed to be an arbitrary power function of form (3), while the coefficients a, b are piecewise-Hölder with the condition $\inf(a^2(t)+b^2(t)) > 0$ and $c(t)\omega(t) \in L^{p(\cdot)}(\Gamma)$. Of the function $p(t)$ it is required that it satisfy the Log-Hölder condition. Under these assumptions, we obtain a complete picture of the solvability – the conditions for the problem to be solvable are derived and solutions are constructed. These conditions, the number of linearly independent solutions and solutions largely depend both on the values of $p(t)$ at the angular points of Γ and on the angle values at these points.

2. Some Definitions and Auxiliary Statements

We denote by $C_L(A_1,\dots,A_i;\nu_1,\dots,\nu_i)$ the set of simple closed piecewise-Lyapunov curves Γ with angular points A_k, whose angle values with respect to the finite domain D bounded by Γ are equal to $\nu_k\pi$, $k=\overline{1,i}$.

Let $z = z(w)$ be a conformal mapping of the circle $U = \{w : |w| < 1\}$ onto D, and $w = w(z)$ be its inverse function. Assume $\gamma = \{\tau : |\tau| = 1\}$, $\tau_k = w(t_k)$, $a_k = w(A_k)$. It is known that

$$z(w) - z(a_k) = (w-a_k)^{\nu_k} z_{0,k}(w), \quad z'(w) = (w-a_k)^{\nu_k-1} z_{1,k}, \tag{4}$$

where $z_{0,k}$, $z_{1,k}$ are nonzero continuous functions [14] belonging to the Hölder class ([15], see also [13, p. 155]).

Definition 1. A real function $p(t)$ given on Γ belongs to the class $Q(\Gamma)$ if:
(i) there exists a constant A such that

$$\forall t_1,t_2 \in \Gamma \quad |p(t_1)-p(t_2)| < \frac{A}{|\ln|t_1-t_2||}\,; \tag{5}$$

$$(ii) \qquad p_- = \min_{t\in\Gamma} p(t) > 1. \tag{6}$$

Proposition 1. *Let* $\Gamma \in C_L(A_1,\dots,A_i;\nu_1,\dots,\nu_i)$, $0<\nu_k\le 2$, $k=\overline{1,i}$, $p(t)\in Q(\Gamma)$, *then the function* $\ell(\tau) = p(z(\tau))$ *belongs to* $Q(\gamma)$ (*see* [10, Lemma 1]).

Definition 2. We denote by $\mathcal{R}$ the set of pairs $(\Gamma; p(t))$, for which operator

$$S_\Gamma : f \to S_\Gamma f, \quad (S_\Gamma f)(t) = \frac{1}{\pi i}\int_\Gamma \frac{f(\tau)\,d\tau}{\tau - t}\,, \quad t\in\Gamma, \tag{7}$$

is continuous in $L^{p(\cdot)}(\Gamma)$.

Definition 3. $W^{p(\cdot)}(\Gamma)$ is the set of all those weight functions ω, which the operator $T : f \to \omega S_\Gamma(\omega^{-1}f)$ is continuous in $L^{p(\cdot)}(\Gamma)$.

Proposition 2. ([12]). *If $p \in Q(\Gamma)$, then a pair $(\Gamma; p(t))$ belongs to $\mathcal{R}$ if and only if Γ is a regular curve, i.e. for the measure defined by the arc abscissa of the set $\Gamma \cap B(z;r)$ we have $\sup\limits_{r>0,\, z\in\Gamma} \frac{|\Gamma\cap B(z;r)|}{r} < \infty$, where $B(z;r)$ is the circle with center at the point z and of radius r.*

Since piecewise-smooth curves are regular, a pair $(\Gamma; p(t))$, where Γ is a piecewise-smooth curve and $p \in Q(\Gamma)$, belongs to $\mathcal{R}$.

Definition 4. We denote by $K^{p(\cdot)}(D;\omega)$ the set functions ϕ, analytic in D, representable in the form

$$\phi(z) = \frac{1}{\omega(z)} \frac{1}{2\pi i} \int\limits_{\Gamma} \frac{\varphi(t)\,dt}{t-z} = \omega^{-1}(z)(K_\Gamma\varphi)(z)\,, \quad z \in D, \quad \varphi \in L^{p(\cdot)}(\Gamma), \tag{8}$$

and by $K^{p(\cdot)}(\Gamma;\omega)$ the set of functions, analytic in D, representable in the form

$$\phi(z) = \frac{1}{2\pi i} \int\limits_{\Gamma} \frac{f(t)\,dt}{t-z}\,, \quad z \in D, \quad f \in L^{p(\cdot)}(\Gamma;\omega). \tag{9}$$

Lemma 1. *If a pair $(\Gamma; p(\cdot))$ belongs to $\mathcal{R}$, then for almost all t from Γ each function $\phi \in K^{p(\cdot)}(D;\omega)$ has an angular boundary value $\phi^+(t)$ and*

$$\phi^+(t)\omega(t) \in L^{p(\cdot)}(\Gamma), \quad \text{i.e.} \quad \phi^+ \in L^{p(\cdot)}(\Gamma;\omega). \tag{10}$$

Definition 5. An analytic function ϕ in the simply connected domain D bounded by a simple rectifiable curve Γ belongs to the Smirnov class $E^q(D)$, $q > 0$, if

$$\sup_{\rho\in(0,1)} \int\limits_{\Gamma_\rho} |\phi(z)|^q\,|dz| = \sup_{\rho\in(0,1)} \int\limits_{|w|=\rho} |\phi(z(w))|^q |z'(w)|\,|dw| < \infty,$$

where Γ_ρ is the image of the circle $|w| = \rho$ for a conformal mapping U onto D.

Lemma 2. *If D is the Smirnov domain, $\phi \in E^\delta(D)$, $\delta > 0$, and $\phi^+ \in L^{p(\cdot)}(\Gamma;\rho)$, where $\inf p > 1$ and $\rho^{-1} \in L^{p'(\cdot)}(\Gamma)$, $p'(t) = \frac{p(t)}{p(t)-1}$, then $\phi \in K^{p(\cdot)}(\Gamma;\rho)$.*

Proposition 3. ([12]). *If Γ is a regular curve and $p \in Q(\Gamma)$, then the function $\omega(t) = \prod\limits_{k=1}^{\nu} |t-t_k|^{\alpha_k}$, $t_k \in \Gamma$, $\alpha_k \in \mathbb{R}$, belongs to $W^{p(\cdot)}(\Gamma)$ if and only if $-\frac{1}{p(t_k)} < \alpha_k < \frac{1}{p'(t_k)}$.*

3. Conditions of the Coincidence of the Classes $K^{p(\cdot)}(D;\omega)$ and $K^{p(\cdot)}(\Gamma;\omega)$

Theorem 1. *Let a pair $(\Gamma; p(\cdot))$ belong to $\mathcal{R}$, $p_- > 1$, $\omega^{-1}(z) \in E^\delta(D)$ and $\omega^+ \in W^{p(\cdot)}(\Gamma)$. Then the equality*

$$K^{p(\cdot)}(D;\omega) = K^{p(\cdot)}(\Gamma;\omega) \tag{11}$$

is fulfilled.

Corollary. *Let Γ be a regular curve, $\omega(z)$ be given by equality (7), and at the points t_k the curve Γ have the one-sided tangents forming a nonzero angle. If $p \in Q(\Gamma)$ and $\omega \in W^{p(\cdot)}(\Gamma)$, then equality (11) is fulfilled.*

4. Reduction of Problem (1) to a Linear Conjugation Problem

Let D be the simply connected domain bounded by the curve $\Gamma \subset C_L(A_1,\dots,A_i;\nu_1,\dots,\nu_i)$, $0<\nu_k\leq 2$, $k=\overline{1,i}$; $a(t)$, $b(t)$ be the piecewise-Hölder functions with the condition $\inf(a^2(t)+b^2(t))>0$, Furthermore, let $\omega(z)$ be a weight function of form (3), $p(t)\in Q(\Gamma)$ and let $c(t)\omega(t)\in L^{p(\cdot)}(\Gamma)$.

Using these assumptions, we will consider the Riemann–Hilbert problem formulated as follows: find a function $\phi(z)\in K^{p(\cdot)}(D;\omega)$ whose angular boundary values $\phi^+(t)$, $t\in\Gamma$, satisfy relation (1) a.e. on Γ.

Let $\phi(z)$ be a solution of the problem posed and

$$\Psi(w)=\phi(z(w))=\frac{1}{\omega(z(w))}\int\limits_{\Gamma}\frac{\varphi(t)\,dt}{t-z(w)}\,,\quad \varphi\in L^{p(\cdot)}(\Gamma). \tag{12}$$

Then $\Psi(w)$ satisfies the boundary condition

$$\operatorname{Re}\left[(A(\tau)+iB(\tau))\,\Psi^+(\tau)\right]=C(\tau),\quad \tau\in\gamma, \tag{13}$$

where $A(\tau)=a(z(\tau))$, $B(\tau)=b(z(\tau))$, $C(\tau)=c(z(\tau))$.

Assuming that

$$G(\tau)=-\left[A(\tau)-iB(\tau)\right]\left[A(\tau)+iB(\tau)\right]^{-1},\ \ c_1(\tau)=2C(\tau)\left[A(\tau)+iB(\tau)\right]^{-1},$$

we give from (12) that

$$\Psi^+(\tau)=G(\tau)\overline{\Psi^+(\tau)}+c_1(\tau),\quad \tau\in\gamma, \tag{14}$$

where the coefficient $G(\tau)$ is a piecewise-Hölder function. Let $b_1,b_2,\dots,b_\lambda$ be its discontinuity points, then $|G(\tau)|=1$ for $\tau\neq b_k$, and $|G(b_k\pm)|=1$.

Let $G(b_k-)[G(b_k+)]^{-1}=\exp(2\pi i u_k)$. If

$$r_k(w)=\begin{cases}(w-b_k)^{u_k}, & |w|<1,\\ \overline{\left(\frac{1}{\overline{w}}-b_k\right)^{u_k}}, & |w|>1.\end{cases}\qquad R_k(\tau)=\frac{r_k^+(\tau)}{r_k^-(\tau)},r(w)=\prod_{k=1}^{\lambda}r_k(w). \tag{15}$$

then function $G_1(\tau)=G(\tau)\prod\limits_{k=1}^{\lambda}R_k(\tau)$ is Hölder-continuous on γ and different from zero. Let $X_1(w)$ be a canonical function for $G_1(\tau)$, i.e.

$$X_1(w)=\begin{cases}C\exp\left(\frac{1}{2\pi i}\int_\gamma\frac{\ln\widetilde{G}_1(\tau)\,d\tau}{\tau-w}\right), & |w|<1,\\ C(w-w_0)^{-\varkappa_1}\exp\left(\frac{1}{2\pi i}\int_\gamma\frac{\ln\widetilde{G}_1(\tau)\,d\tau}{\tau-w}\right), & |w|>1,\ \ |w_0|<1,\end{cases} \tag{16}$$

where C is an arbitrary constant, $\widetilde{G}_1(\tau)=G_1(\tau)(\tau-w_0)^{-\varkappa_1}$ and $\varkappa_1=\operatorname{ind}\widetilde{G}_1(\tau)=(2\pi)^{-1}[\arg\widetilde{G}_1(\tau)]_\Gamma$.

Let

$$X(w)=X_1(w)r(w)=X_1(w)\prod_{k=1}^{\lambda}r_k(w).$$

Following [1, pp. 145, 146]1 it is assumed that

$$\Omega(w)=\begin{cases}\Psi(w), & |w|<1,\\ \overline{\Psi\left(\frac{1}{\overline{w}}\right)}, & |w|>1,\end{cases} \tag{17}$$

where, as above, $\Psi(w)=\phi(z(w))$.

Then $\Omega^-(\tau)=\overline{\Psi^+(\tau)}$ and the latter boundary condition takes the form

$$\Omega^+(\tau)[X^+(\tau)]^{-1}=\Omega^-(\tau)[X^-(\tau)]^{-1}+c_2(\tau),\quad c_2(\tau)=c_1(\tau)[X^+(\tau)]^{-1}. \tag{18}$$

It should be as well noted that if for an analytic function $f(z)$ in $\mathbb{C}\setminus\gamma$ we set

$$f_*(w)=\overline{f\Big(\frac{1}{\overline{w}}\Big)},\quad |w|\neq 1,$$

then f_* is analytic in $\mathbb{C}\setminus\gamma$ and $(f_*)_*=f$.

From definition (17) we see that $\Omega_*(w)=\Omega(w)$. Therefore we should look for those solutions Ω of problem (18) for which the latter condition is fulfilled.

5. The Properties of the Function Ω for $\phi\in K^{p(\cdot)}(D;\omega)$

We set

$$T=\{\tau_k:\tau_k=w(t_k)\},\quad A=\{a_k:a_k=w(A_k)\},\quad B=\{b_k\},$$

where $w=w(z)$ is the inverse function to $z(w)$, t_k are numbers from weigh (3), A_k are angular points of Γ and b_k are the discontinuity points of the function G.

Among the points τ_k, a_k, b_k some may coincide.

Let us renumber the points from $T\cup A\cup B$ so as to have

$$\begin{cases} w_1=\tau_1=a_1=b_1,\dots,\ w_\mu=\tau_\mu=a_\mu=b_\mu,\\ w_{\mu+1}=\tau_{\mu+1}=a_{\mu+1},\dots,\ w_{\mu+r}=\tau_{\mu+r}=a_{\mu+r},\\ w_{\mu+r+1}=\tau_{\mu+r+1}=b_{\mu+1},\dots,\ w_{\mu+r+q}=\tau_{\mu+r+q}=b_{\mu+q},\\ w_{\mu+r+q+1}=a_{\mu+r+1}=b_{\mu+q+1},\dots,\ w_{\mu+r+q+p}=a_{\mu+r+p}=b_{\mu+q+p},\\ w_{\mu+r+q+p+1}=\tau_{\mu+r+q+1},\dots,\ w_{\mu+r+q+p+m}=\tau_{\mu+r+q+m},\\ w_{\mu+r+q+p+m+1}=a_{\mu+r+p+1},\dots,\ w_{\mu+r+q+p+m+n}=a_{\mu+r+p+n},\\ w_{\mu+r+q+p+m+n+1}=b_{\mu+q+p+1},\dots,\ w_{\mu+r+q+p+m+n+s}=b_{\mu+q+p+s}. \end{cases}\tag{19}$$

According to the adopted numbering of points τ_k, a_k, b_k, we have

$$\Psi^+(\tau)=\prod_{k=1}^{j}(\tau-w_k)^{-\delta_k}\Psi_0(\tau),\quad \Psi_0(\tau)\in L^{\ell(\cdot)}(\gamma),\tag{20}$$

where

$$\delta_k=\begin{cases} \alpha_k\nu_k+\frac{\nu_k-1}{\ell(a_k)}+u_k, & k=\overline{1,\mu},\\ \alpha_k\nu_k+\frac{\nu_k-1}{\ell(a_k)}, & k=\overline{\mu+1,\mu+r},\\ \alpha_k+u_{k-r}, & k=\overline{\mu+r+1,\mu+r+q},\\ \frac{\nu_{k-q}-1}{\ell(a_{k-q})}+u_{k-r}, & k=\overline{\mu+r+q+1,\mu+r+q+p},\\ \alpha_{k-p}, & k=\overline{\mu+r+q+p+1,\mu+r+q+p+m},\\ \frac{\nu_{k-q-m}-1}{\ell(a_{k-q-m})}, & k=\overline{\mu+r+q+p+m+1,\mu+r+q+p+m+n},\\ u_{k-r-m-n}, & k=\overline{\mu+r+q+p+m+n+1,\mu+r+q+p+m+n+s}. \end{cases}\tag{21}$$

For a real number x we assume $x=[x]+\{x\}$, where $0\le\{x\}<1$. For all k we require that

$$\{\delta_k\}\neq\frac{1}{\ell'(w_k)},\tag{22}$$

and let

$$\gamma_k=\begin{cases}[\delta_k], & \text{if }\ \{\delta_k\}<\frac{1}{\ell'(w_k)},\\ [\delta_k]+1, & \text{if }\ \{\delta_k\}>\frac{1}{\ell'(w_k)}.\end{cases}\tag{23}$$

Then

$$-\frac{1}{\ell(w_k)}<\delta_k-\gamma_k<\frac{1}{\ell'(w_k)}.\tag{24}$$

Setting

$$Q(w)=\prod_{k=1}^{j}(w-w_k)^{\gamma_k},\quad k=\overline{1,j}, \tag{25}$$

we have

$$|Q(\tau)\Psi^+(\tau)(X^+(\tau))^{-1}|\in L^{\ell(\cdot)}(\gamma;\rho), \tag{26}$$

where

$$\rho(\tau)=\prod_{k=1}^{j}(\tau-w_k)^{\delta_k-\gamma_k}. \tag{27}$$

By virtue of (24) and using Proposition 3 we conclude that $\rho(\tau)\in W^{\ell(\cdot)}(\gamma)$.

Remark. By the assumptions made for Γ and G (i.e. by conditions (22)) we obtain

$$\rho(\tau)\sim w(z(\tau))r(\tau)|z'(\tau)|^{\frac{1}{\ell(\tau)}}Q^{-1}(\tau). \tag{28}$$

Here sign $\varphi\sim\psi$ denote that $0<\inf\left|\frac{\varphi}{\psi}\right|\le\sup\left|\frac{\psi}{\varphi}\right|\le\infty$.

Lemma 3. *The following inclusion holds*

$$R(w)\equiv Q(w)\phi(z(w))[X(w)]^{-1}\in K^{\ell(\cdot)}(\gamma;\rho). \tag{29}$$

Theorem 2. *If $\phi\in K^{p(\cdot)}(D;\omega)$ and $Q(w)$ is the meromorphic function defined by equality* (25)*, then inclusion (29) holds provided that conditions* (22) *are fulfilled. Conversely, if (29) holds, then $\phi(z)\in K^{p(\cdot)}(D;\omega)$.*

The function $R(w)=Q(w)\Omega(w)X^{-1}(w)$ is holomorphic in U^- (complemented by U) everywhere except, perhaps, the point $z=\infty$ and has, at that point, order

$$\varkappa=\varkappa_0+\varkappa_1,$$

where $\varkappa_0$ is the order of $Q(w)$, while $\varkappa_1$ is the order of $X^{-1}(w)$.

Definition 6. The set of functions F representable in the form

$$F(w)=\frac{1}{2\pi i}\int\limits_{\gamma}\frac{f(\tau)\,d\tau}{\tau-w}+P_n(w),\quad f\in L^{\ell(\cdot)}(\gamma;\rho),\quad |w|\ne 1, \tag{30}$$

where P_n is some polynomial of order n, is denoted by $K^{\ell(\cdot)}(\gamma;\rho;n)$. We consider this class for negative n, too, assuming that in that case $P_n\equiv 0$ and $(K_\gamma f)(w)$ has zero of order $(-n)$.

Theorem 3. *If $\phi(z)\in K^{p(\cdot)}(D;\omega)$, $\Psi(w)=\phi(z(w))$, and the functions $\Omega(w)$ and $Q(w)$ are defined in $\mathbb{C}\setminus\gamma$ by formulas* (17) *and* (25)*, then the function*

$$F(w)=Q(w)\Omega(w)X^{-1}(w),\quad |w|\ne 1, \tag{31}$$

belongs to $K^{\ell(\cdot)}(\gamma;\rho,\varkappa)$.

6. Solution of the Riemann–Hilbert Problem

We multiply equality (18) by $Q(w)$ and rewrite (18) as

$$F^+(\tau) - F^-(\tau) = c_2(\tau)Q(\tau), c_2(\tau) = 2C(\tau)[A(\tau)+iB(\tau)]^{-1}[X^+(\tau)]^{-1}. \qquad (32)$$

The solution of this problem is to be sought for in the class $K^{\ell(\cdot)}(\gamma;\rho,\varkappa)$.

According to Theorem 2, if $F(w) \in K^{\ell(\cdot)}(\gamma;\rho,\varkappa)$, then $\phi(z) = F(w(z)) \times X(w(z))Q^{-1}(w(z))$ is a function of the class $K^{p(\cdot)}(D;\omega)$. Thus we need a solution $\Omega(w)$ of problem (18) representable by the form $\Omega(w) = F(w)X(w)Q^{-1}(w)$, where $F(w)$ is the solution of (32) from $K^{\ell(\cdot)}(\gamma;\rho,\varkappa)$ and

$$\left(\frac{FX}{Q}\right)_*(w) = \left(\frac{FX}{Q}\right)(w), \quad |w| \neq 1. \qquad (33)$$

If this condition is fulfilled, then by the restricting of the function $\Omega(w) = F(w) \times X(w)Q^{-1}(w)$ on U we find the function $\Psi(w) = \phi(z(w)) = F(w)X(w)Q^{-1}(w)$ and, eventually, obtain

$$\phi(z) = F(w(z))X(w(z))Q^{-1}(w(z)) \qquad (34)$$

The function $\phi(z)$ is a solution of problem (1) in the class $K^{p(\cdot)}(D;\omega)$ by virtue of Theorems 2 and 3.

Now we may formulate the main results.

Let 1) D be the finite simply connected domain bounded by the curve $\Gamma \in C_L(A_1,\dots,A_i;\nu_1,\dots,\nu_i)$, $0 < \nu_k \le 2$; 2) $\omega(z) = \prod\limits_{k=1}^{\nu}(z-t_k)^{\alpha_k}$, $t_k \in \Gamma$, $\alpha_k \in \mathbb{R}$; 3) $a(t)$, $b(t)$ be piecewise-Hölder functions with the condition $\inf(a^2(t)+b^2(t)) > 0$ such that $G(t) = -[a(t)-ib(t)][a(t)+ib(t)]^{-1}$ has discontinuity points B_k, $k=\overline{1,\lambda}$, and also $G(B_k-)[G(B_k+)]^{-1} = \exp 2\pi i u_k$, $u_k \in \mathbb{R}$; 4) $p(t)$ be a function from the class $Q(\Gamma)$ given on Γ and $\ell(\tau) = p(z(\tau))$.

Assume that $\tau_k = w(t_k)$, $a_k = w(A_k)$, $b_k = w(B_k)$. Further, let $c_2(\tau) = 2C(\tau)[A(\tau)+iB(\tau)]^{-1}[X^+(\tau)]^{-1}$, $F_c(w) = (K_\gamma c_2 Q)(w)$ and let

$$\widetilde{\Omega}_c(w) = \frac{1}{2}(\Omega_c(w) + (\Omega_c)_*(w)), \qquad (35)$$

where $\Omega_c(w) = F_c(w)X(w)Q^{-1}(w)$.

Theorem 4. *Let*

i) *the points τ_k, a_k, b_k be numbered according to* (19), *the numbers δ_k be defined by equalities* (21) *and $\{\delta_k\} \neq [\ell'(w_k)]^{-1}$, while the integer numbers γ_k be chosen with condition* (23);

ii) *$Q(w)$ be a meromorphic function defined by equality* (25) *and the order of $Q(w)$ at infinity be equal to $\varkappa_0$;*

iii) *$\rho(\tau)$ be the weight function given by equality* (27);

iv) *the functions $r(w)$ and $X_1(w)$ be given by equalities* (15), (16); *$X(w) = r(w)X_1(w)$ so that $X(w)$ has, at infinity, order $(-\varkappa_1)$;*

v) *$c(t)\omega(t) \in L^{p(\cdot)}(\Gamma)$.*

Assume that $\varkappa = \varkappa_0 + \varkappa_1$. Then

a) *if $\varkappa < 0$, then for problem* (1) *to be solvable in the class $K^{p(\cdot)}(D;\omega)$ it is necessary and sufficient that conditions*

$$\int\limits_\Gamma \frac{c(t)Q(w(t))w^k(t)w'(t)}{X^+(w(t))(a(t)+ib(t))}dt = 0, \quad k = \overline{0,|\varkappa|}, \qquad (36)$$

be fulfilled and, upon their fulfillment, (1) *has a unique solution given by equality* $\phi(z) = \widetilde{\Omega}_c(w(z))$.

b) *For* $\varkappa \geq 0$, *problem* (1) *is certainly solvable and all its solutions are given by equality*

$$\Phi(z) = \Omega(w(z)) = \widetilde{\Omega}_c(w(z)) + X(w(z))Q^{-1}(w(z))P_{\varkappa}(w(z)), \tag{37}$$

where $P_{\varkappa} = h_0 + h_1 w + \cdots + h_{\varkappa} w^{\varkappa}$ *and*

$$\overline{h}_k = A h_{\varkappa - k}, \quad k = \overline{0, \varkappa}, A = (-1)^{\kappa_0} \prod_{k=1}^{j} w_k^{\gamma_k}. \tag{38}$$

7. Some Particular Cases

I. The Riemann–Hilbert problem with Hölder coefficients $a(t)$, $b(t)$ in the class $K^{p(\cdot)}(\Gamma;\omega)$ for $\omega \in W^{p(\cdot)}(\Gamma)$. Let

$$\begin{aligned} \varkappa_0 &= N\left\{a_k \overline{\in} \cup \{\tau_k\} : \nu_k > \ell(a_k)\right\} \\ &\quad + N\left\{\tau_k = a_k : \frac{\ell(a_k)}{1+\alpha_k \ell(a_k)} < \nu_k < \frac{2\ell(a_k)}{1+\alpha_k \ell(a_k)}\right\}. \end{aligned} \tag{39}$$

Where $N(E)$ denote a number of the element of the set E.

We have the

Corollary. *If problem* (1) *is considered in the class* $K^{p(\cdot)}(\Gamma;\omega)$, $\omega \in W^{p(\cdot)}(\Gamma)$ *and* $a(t)$, $b(t)$ *belongs to the Hölder class, the number* $\varkappa_0$ *in Theorem* 3 *is calculated by equality (39).*

II. The Dirichlet problem in the weighted Smirnov class. Let $a(t) = 1$, $b(t) = 0$, $\omega(z) = \prod\limits_{k=1}^{\nu} (z - t_k)^{\alpha_k}$, $\omega(t) \in W^{p(\cdot)}(\Gamma)$, $c(t) \in L^{p(\cdot)}(\Gamma;\omega)$.We deal with the Dirichlet problem: define a function u for which

$$\begin{cases} \Delta u = 0, \quad u = \operatorname{Re} \phi, \quad \phi \in K^{p(\cdot)}(\Gamma;\omega), \\ u^+(t) = c(t), \quad t \in \Gamma, \quad c(t)\omega(t) \in L^{p(\cdot)}(\Gamma). \end{cases} \tag{40}$$

Then $r(w) = 1$, $X_1(w) = \begin{cases} -i, & |w| < 1, \\ i, & |w| > 1 \end{cases}$ (we need this to have $(X_1)_*(w) = X_1(w)$). $\varkappa = \varkappa_0 \geq 0$ and according to Corollary of Theorem 4 $\varkappa_0$ is calculated by formula (39).

III. The Dirichlet problem in the Smirnov class (i.e. problem (40) for $\omega \equiv 1$). From condition (22) we obtain $\nu_k \neq p(A_k)$, $k = \overline{1, i}$. The order $\varkappa_0$ of $Q(w)$ at infinity is equal to the number of angular points for which $\nu_k > p(A_k)$.

Let $i = 1$, $\nu > p(A_1) = \ell(a_1)$ and $c(t) = 0$. In the case the problem

$$\begin{cases} \Delta u = 0, \quad u \in \operatorname{Re} K^{p(\cdot)}(\Gamma), \quad p \in Q(\Gamma), \quad \Gamma \in C_L(A_1, \nu), \\ u^+(t) = 0, \quad t \in \Gamma, \end{cases}$$

has a solution

$$u(z) = s \operatorname{Re} \frac{w(z) + w(A_1)}{w(z) - w(A_1)}$$

depending on one real parameter.

If $\nu < p(A_1)$, then the problem has only a trivial solution.

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