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The Moduli Space and Versal Deformationsof Algebraic Structures

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Abstract. In this talk I consider deformations of algebraic structures. The notion of 1-parameter deformation is due to Gerstenhaber. Here I give a generalization of the classical notion by considering deformations with a commutative algebra base, and define the miniversal formal deformation. This notion is necessary to describe non-equivalent deformations with the same infinitesimal part, and to find singular nontrivial deformations with zero infinitesimal part. I use the example of a vector field Lie algebra to demonstrate the computation. Another example which underlines the importance of such general deformations is to consider moduli spaces of Lie algebras. This I also demonstrate on an example.

Key words and phrases: Lie algebra, moduli space, miniversal deformation, orbifold

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1 Introduction

Deforming a given mathematical structure is a tool of fundamental importance in most parts of mathematics and physics. The theory of deformations originated with the problem of classifying all possible pairwise non-isomorphic complex structures on a given differentiable real manifold. The fundamental idea, which should be credited to Riemann, was to introduce an analytic structure.

The notion of local and infinitesimal deformations of a complex analytic manifold first appeared in the work of Kodaira and Spencer (1958). In particular, they proved that infinitesimal deformations can be parametrized by a corresponding cohomology group. The deformation theory of compact complex manifolds was devised by Kuranishi (1965) and Palamodov (1976). Shortly after the work of Kodaira and Spencer, algebro-geometric foundations were systematically developed by M. Artin (1960) and Schlessinger (1968). Formal deformations of arbitrary rings and associative algebras, as well as the related cohomology questions, were first investigated by Gerstenhaber (1964-1968). The notion of deformation was applied to Lie algebras by Nijenhuis and Richardson (1966-68).

In this talk I consider deformations of Lie algebras – although my general theory can be applied and is already applied to other categories like Leibniz algebras, associative algebras, infinity algebras, dialgebras, algebras over quadratic operad etc.

Deformation is one of the tools used to study a specific object, by deforming it into some families of "similar" structure objects. This way we get a richer picture of the original object itself. But there is also another question approached via deformation. Roughly speaking, it is the question, can we equip the set of mathematical structures under consideration (may be up to certain equivalence) with the structure of a topological or geometric space. In other words, does there exist a moduli space for these structures. If so, then for a fixed object deformations of this object should reflect the local structure of the moduli space at the point corresponding to this object.

Example 1. Consider the Lie algebra of 2×2 matrices over a field $\mathbb{K}$ with the usual bracket operation: $[A, B] = AB - BA$. Let A_0 denote the 2×2 identity matrix. Define

$$A_t = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + t \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

We can say that $\{A_t, t \in \mathbb{K}\}$ is a deformation family of A_0.

Example 2. Define the (solvable) 3-dimensional complex Lie algebra with generators e, f, h having nonzero brackets $[h, e] = e$ and $[h, f] = -f$. Define the following deformation:

$$\begin{aligned}[h,e]_t &= e + t\varphi_1(h,e),\\ [h,f]_t &= -f + t\varphi_1(h,f),\\ [e,f]_t &= t\varphi_1(e,f),\end{aligned}$$

where

$$\varphi_1(h,e) = 0, \qquad \varphi_1(h,f) = 0, \qquad \varphi_1(e,f) = 2h.$$

The resulting Lie algebra is isomorphic to $\mathfrak{sl}_2(\mathbb{C})$, for any non-zero t.

Example 3. Consider the following four parameter family of 7-dimensional Lie algebras $\mathcal{G}(\alpha,\beta,\gamma,\delta)$. The non-zero Lie products of $\mathcal{G}(\alpha,\beta,\gamma,\delta)$ are defined by:

$$\begin{array}{ll}[e_1,e_i] = e_{i+1}\ ,\ 2 \le i \le 6 & [e_2,e_5] = (\alpha-\delta)e_7\\ [e_2,e_3] = \alpha e_5 + \beta e_6 + \gamma e_7 & [e_3,e_4] = \delta e_7\\ [e_2,e_4] = \delta e_6 + \beta e_7\ . & \end{array}$$

Consider the deformation $\mathcal{G}(1,0,0,t)$ of $\mathcal{G}(1,0,0,0)$ parametrized by $k[[t]]$. If $\mathcal{G}(\alpha,\beta,$
$\gamma,\delta)$ is isomorphic to $\mathcal{G}(\alpha',\beta',\gamma',\delta')$, then $\frac{\alpha'-\delta'}{\alpha'} = \frac{\alpha-\delta}{\alpha}$. It follows that $\mathcal{G}(1,0,0,\delta)$ is not isomorphic to $\mathcal{G}(1,0,0,\delta')$ if $\delta \ne \delta'$, so this deformation family of $\mathcal{G}(1,0,0,0)$ has nonequivalent elements for each δ. On the other hand, the family $\mathcal{G}(\mu,\mu,\mu,\mu)$ consists of Lie algebras isomorphic to $\mathcal{G}(1,1,1,1)$, and so the Lie algebra $\mathcal{G}(t,t,t,t)$ over $k[[t]]$ is a deformation parameterized by $k[[t]]$ of $\mathcal{G}(0,0,0,0)$ with all the nonzero members of the family being isomorphic to each other.

2 Basic definitions

In my talk I will consider Lie algebras which are widely used in mathematical physics.

Let $\mathcal{L}$ be a Lie algebra with Lie bracket μ_0 over a field $\mathbb{K}$.

a) *Intuitive definition.* A deformation of $\mathcal{L}$ is a one-parameter family $\mathcal{L}_t$ of Lie algebras with the bracket (possibly infinite series)

$$\mu_t = \mu_0 + t\varphi_1 + t^2\varphi_2 + \dots$$

where φ_i are $\mathcal{L}$-valued 2-cochains, i.e. elements of $\mathrm{Hom}_{\mathbb{K}}(\Lambda^2\mathcal{L},\mathcal{L}) = C^2(\mathcal{L};\mathcal{L})$, and $\mathcal{L}_t$ is a Lie algebra for each $t \in \mathbb{K}$. Two deformations, $\mathcal{L}_t$ and $\mathcal{L}'_t$ are equivalent if there exists a linear automorphism $\widehat{\psi}_t = \mathrm{id} + \psi_1 t + \psi_2 t^2 + \dots$ of $\mathcal{L}$ where ψ_i are linear maps over $\mathbb{K}$, i.e. elements of $C^1(\mathcal{L},\mathcal{L})$ such that

$$\mu'_t(x,y) = \widehat{\psi}_t^{-1}\big(\mu_t(\widehat{\psi}_t(x),\widehat{\psi}_t(y))\big) \quad \text{for } x,y \in \mathcal{L}.$$

The Jacobi identity for the algebras $\mathcal{L}_t$ implies that the 2-cochain φ_1 is indeed a cocycle, i.e. $d_2\varphi_1 = 0$. (Here d_i is the differential in the cochain complex.) If

φ_1 vanishes identically, the first non-vanishing φ_i will be a cocycle. If μ'_t is an equivalent deformation with cochains φ'_i, then

$$\varphi'_1 - \varphi_1 = d_1\psi_1,$$

hence every equivalence class of deformations defines uniquely an element of $H^2(\mathcal{L}, \mathcal{L})$. This definition was introduced by Nijenhuis and Richardson [12]. We call a Lie algebra rigid, if it has no nontrivial deformations.

b) *General definition.* Consider now a deformation $\mathcal{L}_t$ not as a family of Lie algebras, but as a Lie algebra over the algebra $\mathbb{K}[[t]]$. The natural generalization is to allow more parameters, or to take in general a commutative algebra over $\mathbb{K}$ with identity as base of a deformation. Let us fix an augmentation $\varepsilon : A \to \mathbb{K}$, $\varepsilon(1) = 1$, and set $\operatorname{Ker}\varepsilon = m$, which is a maximal ideal.

Definition 1. [2] A deformation λ of $\mathcal{L}$ with base (A, m) is a Lie A-algebra structure on the tensor product $A \otimes_{\mathbb{K}} \mathcal{L}$ with bracket $[\ ,\ ]_\lambda$ such that

$$\varepsilon \otimes \mathrm{id} : A \otimes \mathcal{L} \to \mathbb{K} \otimes \mathcal{L} = \mathcal{L}$$

is a Lie algebra homomorphism.

Two deformations of a Lie algebra $\mathcal{L}$ with the same base A are called equivalent (or isomorphic) if there exists a Lie algebra isomorphism between the two copies of $A \otimes \mathcal{L}$ with the two Lie algebra structures, compatible with $\varepsilon \otimes \mathrm{id}$.

A deformation with base A is called *local* if the algebra A is local, and it is called *infinitesimal* if, in addition to this, $m^2 = 0$. For general commutative algebra base, we call the deformation *global*.

c) *Formal deformations.* Let A be a complete local algebra (completeness means that $A = \varprojlim_{n\to\infty} (A/m^n)$, where m is the maximal ideal in A). A formal deformation of $\mathcal{L}$ with base A is a Lie A-algebra structure on the completed tensor product $A \widehat{\otimes} \mathcal{L} = \varprojlim_{n\to\infty} \big((A/m^n) \otimes \mathcal{L}\big)$ s.t.

$$\varepsilon \widehat{\otimes} \mathrm{id} : A \widehat{\otimes} \mathcal{L} \to \mathbb{K} \otimes \mathcal{L} = \mathcal{L}$$

is a Lie algebra homomorphism.

The previous notion of equivalence can be extended to formal deformations in an obvious way.

Example 3. Consider the following four parameter family of 7-dimensional Lie algebras $\mathcal{G}(\alpha, \beta, \gamma, \delta)$. The non-zero Lie products of $\mathcal{G}(\alpha, \beta, \gamma, \delta)$ are defined by:

$$\begin{array}{ll} [e_1, e_i] = e_{i+1}\ ,\ 2 \le i \le 6 & [e_2, e_5] = (\alpha - \delta)e_7 \\ [e_2, e_3] = \alpha e_5 + \beta e_6 + \gamma e_7 & [e_3, e_4] = \delta e_7 \\ [e_2, e_4] = \delta e_6 + \beta e_7\ . & \end{array}$$

Consider the deformation $\mathcal{G}(1,0,0,t)$ of $\mathcal{G}(1,0,0,0)$ parametrized by $k[[t]]$. If $\mathcal{G}(\alpha, \beta,$
$\gamma, \delta)$ is isomorphic to $\mathcal{G}(\alpha', \beta', \gamma', \delta')$, then $\frac{\alpha'-\delta'}{\alpha'} = \frac{\alpha-\delta}{\alpha}$. It follows that $\mathcal{G}(1,0,0,\delta)$ is not isomorphic to $\mathcal{G}(1,0,0,\delta')$ if $\delta \neq \delta'$, so this deformation family of $\mathcal{G}(1,0,0,0)$ has nonequivalent elements for each δ. On the other hand, the family $\mathcal{G}(\mu,\mu,\mu,\mu)$ consists of Lie algebras isomorphic to $\mathcal{G}(1,1,1,1)$, and so the Lie algebra $\mathcal{G}(t,t,t,t)$ over $k[[t]]$ is a deformation parameterized by $k[[t]]$ of $\mathcal{G}(0,0,0,0)$ with all the nonzero members of the family being isomorphic to each other.

d) *Versal formal deformations.* It is known that in the category of algebraic varieties the quotient by a group action does not always exist [10]. Specifically, there is no universal deformation in general of a Lie algebra $\mathcal{L}$ with a commutative algebra base B with the property that for any other deformation of $\mathcal{L}$ with base A there exists a unique homomorphism $f : B \to A$ that induces an equivalent deformation. If such a homomorphism exists (but not unique), we call the deformation of $\mathcal{L}$ with base B *versal*.

The classical one-parameter deformation theory is not satisfactory for studying the versal property of deformations.

For a more general deformation theory of Lie algebras let us introduce the notion of a deformation with base and define a formal versal deformation of a Lie algebra.

Definition 2. [2, 3] A formal deformation η of a Lie algebra $\mathcal{L}$ with a complete local algebra base B is called *miniversal*, if

i) for any formal deformation λ of $\mathcal{L}$ with any complete local base A there exists a homomorphism $f : B \to A$ s.t. the deformation λ is equivalent to the push-out of η by f;

ii) if A satisfies $m^2 = 0$, then f is unique.

In the past decades, much attention has been paid to infinite dimensional Lie algebras, mainly because of their applications in mathematical physics. There are basically two kinds of infinite dimensional objects which are intensively studied: Lie algebras of geometric origin, like vector fields on a smooth manifold, and the so called Kac-Moody algebras, the theory of which is closely related to the theory of finite dimensional semisimple Lie algebras. Among the infinite dimensional Lie algebras - as in finite dimension - the hardest to deal with are the nilpotent ones. Any classification, cohomology or deformation result for those is really valuable.

Formal deformations are deformations with a complete local algebra base. A deformation with a commutative (non-local) algebra base gives a much richer picture of deformation families, depending on the augmentation of the base algebra. If we identify the base of deformation – which is a commutative algebra of functions – with a smooth manifold, an augmentation corresponds to choosing a point on the manifold. So choosing different points should in general lead to different deformation situations. In infinite dimension there is no tight relation between global and formal deformations.

In finite dimension global deformations coincide with formal deformations, so we can use cohomology theory. Here cohomology and versal deformations make it possible to get a geometric description of the moduli space of a certain type of algebraic objects in a given dimension. This feature is completely new and underlines the importance of those invariants.

3 Applications of deformation theory in physics

In physics, the mathematical theory of deformations is a powerful tool to construct new theories of physical reality from known ones. The concepts of *symmetry* and *deformations* are considered to be two fundamental guiding principles for further developing physical theory. Nowadays the infinite-dimensional case is in the center of interest, and also deformations of higher algebraic structures play a prominent role.

Some examples:

Quantum groups, deformation of Hopf algebras, q-deformed physics, fuzzy spaces, quantum systems as deformations of classical systems.

(1) The deformation quantization of symplectic and Poisson manifolds, in particular also the question to find subalgebras for which the deformation quantization converges, furthermore the behaviour of deformation quantizations under reduction by a group action, Drinfeld associators.

(2) Deformed Geometry and Gravity, with the help of fuzzy space geometries, large N limits of Yang-Mills matrix models, Anti-de-Sitter space time.

(3) Quantum Field Theory, in particular the deformation of the local observable algebra, renormalisation and regularisation of QFT, family of Dirac operators.

4 Versal formal deformations

Using Schlessinger's general set-up (1968, [13]) one can prove, that for complete local algebra base deformations, under some minor restriction, there exists a miniversal deformation:

Theorem 1 *[3] Let $\mathcal{L}$ be a Lie algebra. Assume that the space $H^2(\mathcal{L},\mathcal{L})$ is finite-dimensional. Then there exists a versal formal deformation of $\mathcal{L}$, and the base of this versal deformation is formally embedded into $H^2(\mathcal{L},\mathcal{L})$, i.e. it can be described in $H^2(\mathcal{L},\mathcal{L})$ by a finite system of formal equations.*

Another question is how to construct such a deformation. I underlined a construction for the versal deformation in [2], using Harrison cohomology of commutative algebras [11]. The construction is parallel to the general constructions in deformation theory, like Palamodov, Illusie, Laudal, Goldman-Millson, Kontsevich. The procedure needs a proper theory of Massey operations in the cohomology, and an algorithm for computing all the possible ways for a given infinitesimal deformation to extend to a formal deformation.

There is a confusion in the literature when one tries to describe all nonequivalent deformations of a given Lie algebra. There were several attempts to work out an appropriate theory for solving this basic problem in deformation theory, but none of them were completely adequate. In particular, the following questions remained open:

1) How many non-equivalent deformations have the same infinitesimal part?

2) Are there any singular nontrivial deformations, i.e. deformations with zero infinitesimal part?

The versal deformation theory answers both questions.

Let $W^{\mathrm{pol}} = W_1$ be the Lie algebra of vector fields on the line with polynomial coefficients $f(x)\frac{d}{dx}$. This Lie algebra has an additive algebraic basis

$$e_i = x^{i+1}\frac{d}{dx}, \qquad i \geq -1.$$

In this basis the bracket operation is

$$[e_i, e_j] = (j - i)e_{i+j}.$$

Let us introduce the subalgebra L_i, $i \geq 0$ of $\mathcal{W}$ which is generated by the basis elements $\{e_i, e_{i+1}, \dots\}$. Let us investigate the subalgebra L_1, the Lie algebra of polynomial vector fields in $\mathbb{C}$ with trivial 1-jet at 0. The Lie algebra L_1 is naturally graded, the weight of e_i equals i. With this grading L_1^{pol} is a graded Lie algebra: $L_1^{\mathrm{pol}} = \bigoplus_{m=1}^{\infty} L_1^{(m)}$.

Using Feigin-Fuchs spectral sequence [1], some results of Feigin and Fuchs on cohomology with coefficients in tensor field modules and Goncharova's result on trivial coefficient cohomology [9], I was able to compute the 1- and 2-dimensional cohomology space of L_1:

Theorem 2 *[3] For* $q > 0$, $H^q_{(m)}(L_1; L_1) \cong H^{q-1}_{(m)}(L_2; \mathbb{C})$. *The cohomology space* $H^q(L_1; L_1)$ *has dimension* $2q - 1$ *and is generated by elements of weight* $-\frac{3q^2-q}{2} + i$ *where* $i = 1, 2, \dots, 2q - 1$.

In particular, the cohomology space $H^1(L_1; L_1)$ *is of dimension* 1 *and has weight* 0*; the space* $H^2(L_1; L_1)$ *is three-dimensional with generators* α, β, γ *of weight* $-2, -3$ *and* -4*, while* $\dim H^3(L_1; L_1) = 5$ *with generators of weight* $-7, -8, -9, -10$ *and* -11.

Identifying explicit cocycles, we can compute the Massey products of those. They are responsible for extending a deformation to higher order. The result is the following:

Theorem 3 *[3] In the case of L_1 the Massey products $\underbrace{\langle\alpha,\alpha,\dots,\alpha\rangle}_{i}$ are zero for all i, the brackets $[\beta,\beta],[\alpha,\beta]$ and $[\alpha,\gamma]$ are trivial, while $[\gamma,\gamma]$ and $[\beta,\gamma]$ are not. The only nontrivial 3-products are $\langle\beta,\beta,\beta\rangle$ and $\langle\alpha,\beta,\beta\rangle$. The higher operations are either not defined or they are trivial.*

The proof of this Theorem follows from computing all the defined Massey brackets and showing that some of them are nontrivial, while others are not. The nontrivial Massey brackets give the equations for the parameter space of the versal deformation.

This way we can give the complete description of all nonequivalent formal deformations for the Lie algebra L_1.

Let us now define three real deformations of the Lie algebra L_1 with the brackets

$$[e_i,e_j]^1_t=(j-i)(e_{i+j}+te_{i+j-1});$$
$$[e_i,e_j]^2_t=\begin{cases}(j-i)e_{i+j} & \text{if } i,j>1,\\ (j-i)e_{i+j}+tje_j, & \text{if } i=1;\end{cases}$$
$$[e_i,e_j]^3_t=\begin{cases}(j-i)e_{i+j} & \text{if } i,j\neq 2\\ (j-i)e_{i+j}+tje_j, & \text{if } i=2.\end{cases}$$

These deformations have infinitesimal deformations of weight $-1,-1$ and -2. Denote the three Lie algebra families by $L_1^{(1)}$, $L_1^{(2)}$ and $L_1^{(3)}$.

Theorem 4 *[3] The Lie algebra families $L_1^{(1)}$, $L_1^{(2)}$ and $L_1^{(3)}$ are nontrivial and pairwise non-isomorphic.*

Later we analized the case L_2 with Post [7] and computed all deformations.

Based on my versal deformation construction and on my example L_1, Fuchs and I worked out a detailed straightforward form of the construction of versal deformation, convenient for explicit computations in [4].

The starting point in the construction is to explicitly give the universal infinitesimal deformation, which we then extend step by step, with the help of Massey operations. In the one-dimensional base extensions we use Harrison cohomology of commutative algebras. In [4] we also provide a scheme for computing the base of a miniversal deformation of a Lie algebra convenient for practical use.

c) Moduli space of Lie algebras

In finite dimension, cohomology and versal deformations make it possible to get a geometric description of the moduli space of a certain type of algebraic objects in a given dimension. This feature is completely new and underlines the importance of those invariants.

Let me give an example. Consider the variety of 4-dimensional Lie algebras on $\mathbb{C}^4$. It turnes out that the moduli space of this variety is essentially an orbifold given by the natural action of the symmetric group Σ_3 on the complex projective space $\mathbb{P}^2(\mathbb{C})$. In addition, there are two exceptional complex projective lines, one of which has an action of the symmetric group Σ_2. Finally, there are 6 exceptional points. The moduli space is glued together by the miniversal deformations, which determine the elements that one may deform to locally, so deformation theory determines the geometry of the space. The exceptional points play a role in refining the picture of how this space is glued together. By orbifold, we mean essentially a topological space factored out by the action of a group. In the case of $\mathbb{P}^n$, there is a natural action of Σ_{n+1} induced by the natural action of Σ_{n+1} on $\mathbb{C}^{n+1}$. An orbifold point is a point which is fixed by some element in the group. In the case of Σ_{n+1} acting on $\mathbb{P}^n$, points which have two or more coordinates with the same value are orbifold points, but there are some other ones, such as the point $(1 : -1) = (-1 : 1)$.

We get similar, but of course, simpler picture for 3-dimensional complex Lie algebras [5]

In the classical theory of deformations, a deformation is called a *jump deformation* if there is a 1-parameter family of deformations of a Lie algebra structure such that every nonzero value of the parameter determines the same deformed Lie algebra, which is not the original one. There are also deformations which move along a family, meaning that the Lie algebra structure is different for each value of the parameter. There can be multiple parameter families as well.

In the picture we assembled, both of these phenomena arise. Some of the structures belong to families and their deformations simply move along the family to which they belong. If there is a jump deformation from an element to a member of a family, then there will always be deformations from that element along the family as well, although they will typically not be jump deformations. In addition, there are sometimes jump deformations either to or from the exceptional points, so these exceptional points play an interesting role in the picture of the moduli space.

In classical Lie algebra theory, the cohomology of a Lie algebra is studied by considering a differential on the dual space of the exterior algebra of the underlying vector space, considered as a cochain complex. If V is the underlying vector space on which the Lie algebra is defined, then its exterior algebra $\bigwedge V$ has a natural $\mathbb{Z}_2$-graded coalgebra structure as well. In this language, a Lie algebra is simply a quadratic odd codifferential on the exterior coalgebra of a vector space [5]. An odd codifferential is simply an odd coderivation whose square is zero. The space L of coderivations has a natural $\mathbb{Z}$-grading $L = \bigoplus L_n$, where L_n is the subspace of coderbluez-utilsivations determined by linear maps $\phi : \bigwedge^n V \to V$. A Lie algebra is a codifferential in L_2, in other words, a quadratic codifferential.

The space of coderivations has a natural structure of a $\mathbb{Z}_2$-graded Lie algebra. The condition that a coderivation d is a codifferential can be expressed in the form $[d, d] = 0$. The coboundary operator $D : L \to L$ is given simply by the rule $D(\varphi) = [d, \varphi]$ for $\varphi \in L$; the fact that $D^2 = 0$ is a direct consequence of the fact that d is an odd codifferential. Moreover, $D(L_n) \subseteq L_{n+1}$, which means that the cohomology $H(d) = \ker D / \operatorname{Im} D$ has a natural decomposition as a $\mathbb{Z}$-graded space: $H(d) = \prod H^n(d)$, where

$$H^n(d) = \ker(D : L_n \to L_{n+1}) / \operatorname{Im}(D : L_{n-1} \to L_n).$$

The Lie algebra structures are codifferentials in L_2. In order to represent a codifferential d as a matrix, we choose the following order for the increasing pairs $I = (i_1, i_2)$ of indices:

$$\{(1,2), (1,3), (2,3), (1,4), (2,4), (3,4)\},$$

and denote the ith element of this ordered set by $S(i)$. Using this order and the Einstein summation convention, we can express

$$d = a^i_j \varphi^{S(j)}_i.$$

We summarize our results and give the Lie bracket operations in standard terminology in the Table below.

Type	Brackets
$d_1(\lambda : \mu)$	$[e_2, e_3] = e_3, [e_1, e_4] = (\lambda + \mu)e_1,$
	$[e_2, e_4] = \lambda e_2, [e_3, e_4] = e_2 + \mu e_3$
$d_3(\lambda : \mu : \nu)$	$[e_1, e_4] = \lambda e_1, [e_2, e_4] = e_1 + \mu e_2, [e_3, e_4] = e_2 + \nu e_3$
$d_3(\lambda : \mu)$	$[e_1, e_4] = \lambda e_1, [e_2, e_4] = \lambda e_2, [e_3, e_4] = e_2 + \mu e_3$
d_1	$[e_2, e_4] = e_1$
$d_1^\sharp$	$[e_2, e_3] = e_1, [e_1, e_4] = 2e_1, [e_2, e_4] = e_2, [e_3, e_4] = e_3$
d_2^*	$[e_1, e_2] = e_1, [e_3, e_4] = e_2$
$d_2^\sharp$	$[e_1, e_2] = e_1, [e_3, e_4] = e_3$
d_3	$[e_1, e_2] = e_3, [e_1, e_3] = e_2, [e_2, e_3] = e_1$
d_3^*	$[e_1, e_4] = e_1, [e_2, e_4] = e_2, [e_3, e_4] = e_3$

Table 1: Table of Lie bracket operations

In Table 2, we give a classification of the Lie algebras according to their cohomology. Note that for the most part, elements from the same family have the same cohomology. In fact, the decomposition of the codifferentials into families was strongly influenced by the desire to associate elements with the same pattern of cohomology in the same family. This is why our family $d_3(\lambda:\mu:\nu)$ was not chosen to be the diagonal matrices. Similar considerations influenced our selection of the family $d_3(\lambda:\mu)$.

In Figure 1 below, we give a pictorial representation of the moduli space. The big family $d_3(\lambda:\mu:\nu)$ is represented as a plane, although in reality it is $\mathbb{P}^2/\Sigma_3$. The families $d_1(\lambda:\mu)$, $d_3(\lambda:\mu)$ and the three subfamilies $d_3(\lambda:\mu:0)$, $d_3(\lambda:\lambda:\mu)$ and $d_3(\lambda:\mu:\lambda+\mu)$ are represented by circles, mainly to reflect that the three subfamilies of the big family

Type	H^1	H^2	H^3	H^4
d_3	1	0	1	1
$d_2^\sharp$	0	0	0	0
$d_1(1:-1)$	2	2	2	1
$d_1(1:0)$	1	2	1	0
$d_1(\lambda:\mu)$	1	1	0	0
$d_1^\sharp$	3	3	0	0
$d_3(1:-1:0)$	3	5	5	2
$d_3(\lambda:\mu:\lambda+\mu)$	2	3	1	0
$d_3(\lambda:\mu:0)$	3	3	1	0
$d_3(\lambda:\mu:-\lambda-\mu)$	2	2	1	1
$d_3(\lambda:\mu:\nu)$	2	2	0	0
$d_3(1:0)$	5	7	3	0
$d_3(0:1)$	6	6	2	0
$d_3(1:2)$	4	5	1	0
$d_3(1:-2)$	4	4	1	1
$d_3(\lambda:\mu)$	4	4	0	0
d_1	8	13	10	3
d_2^*	4	6	5	2
d_3^*	8	8	0	0

Table 2: Table of the cohomology

Example 1.

In Figure 1 below, we give a pictorial representation of the moduli space. The big family $d_3(\lambda:\mu:\nu)$ is represented as a plane, although in reality it is $\mathbb{P}^2/\Sigma_3$. The families $d_1(\lambda:\mu)$, $d_3(\lambda:\mu)$ and the three subfamilies $d_3(\lambda:\mu:0)$, $d_3(\lambda:\lambda:\mu)$ and $d_3(\lambda:\mu:\lambda+\mu)$ are represented by circles, mainly to reflect that the three subfamilies of the big family intersect in more than one point, because they each represent not a single $\mathbb{P}^1$, but several copies of $\mathbb{P}^1$ which are identified under the action of the symmetric group.

In the picture, jump deformations from special points are represented by curly arrows. The jump deformations from the small family $d_3(\lambda:\mu)$ to $d_3(\lambda:\lambda:\mu)$ and the jump deformations from $d_3(\lambda:\mu:\lambda+\mu)$ to $d_1(\lambda:\mu)$ are represented by cylinders. The jump deformations from the family $d_3(\lambda:\mu:0)$

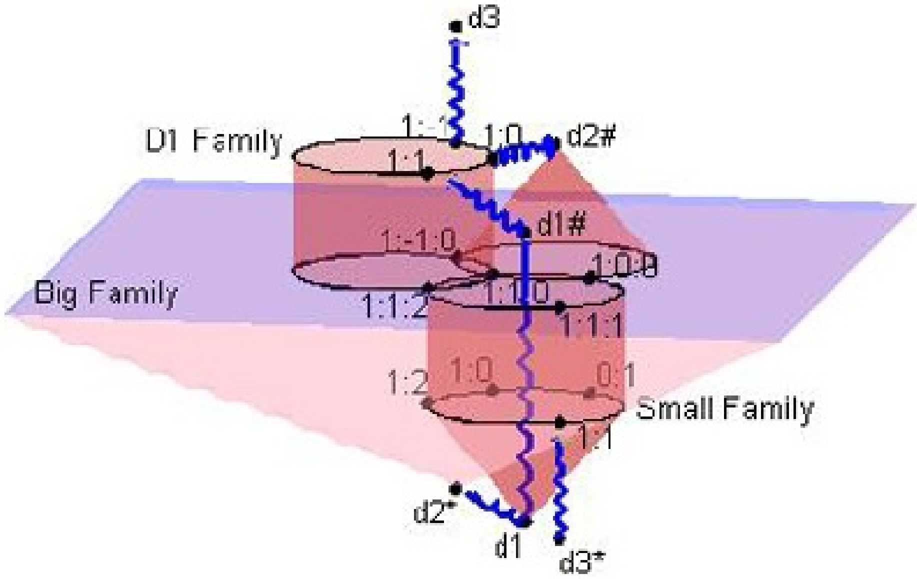

Figure 1: The moduli space of 4 dimensional Lie algebras

to $d_2^\sharp$ and those from d_1 to the small family are represented by cones. Finally, the jump deformations from d_2^* to the big family are represented by an inverted pyramid shape. All jump deformations are either in an upward or a horizontal direction.

The picture tries to capture the order of precedence of the deformations. For example, in the picture, you can trace a path of jump deformations from d_1 to $d_3(1:0)$ to $d_3(1:1:0)$ to $d_1(1:0)$ to $d_2^\sharp$.

The computation of the equivalence classes of non-isomorphic Lie algebra structures in a vector space V determines the elements of the moduli space of Lie algebra structures on V, but is only the first step in the classification of these structures. When classifying the algebras, there are different ways of dividing up the structures according to families; therefore, it is desirable to have a rationale for the division. In this paper, we have shown that there is a natural way to divide up the moduli space into families, using cohomology as a guide to the division, and versal deformations as a tool to refine the analysis.

The four dimensional Lie algebras can be decomposed into families, each of which is naturally an orbifold. If one takes into account the information about jump deformations, the division we have given is uniquely determined. The elements of the family which contain a Lie algebra structure d are precisely those Lie algebras which can be obtained as smooth deformations of d, but which are not smooth deformations of any Lie algebra structure d' which is a jump deformation of d. This rule allows us to distinguish between the algebra d_3^* and $d_3(1:1)$, for example. Even though d_3^* has smooth deformations to the family $d_3(\lambda:\mu)$, it also has a jump deformation to $d_3(1:1)$, which has smooth deformations to the same family. Thus $d_3(1:1)$, which has no jump deformations to any element which has smooth deformations to the family, is the element which belongs to the family.

According to this system, there is one two-parameter family, two one-parameter families, and six singleton elements, giving rise to a two-dimensional orbifold, two one-dimensional orbifolds, and six one-dimensional orbifolds. The jump deformations provide maps between the families which either are smooth maps of orbifolds (or suborbifolds as in the case of the map $d_3(\lambda : \mu : \lambda+\mu) \to d_1(\lambda : \mu)$), or, in the case of some of the singletons, identify the element with a whole family.

The cohomology of a Lie algebra determines the tangent space to the Lie algebra, but the tangent space does not contain enough information to give a good local description of the moduli space. The relations on the base of the versal deformation determine the manner in which the moduli space contacts the tangent space. It is clear that the cohomology is not sufficient to get an accurate picture of the moduli space. Versal deformations provide important detail that characterizes the moduli space completely.

Remark 1 *In the case of real Lie algebras instead of projective spaces, one gets spheres [6].*

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