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ELLIPTIC SYSTEMS ON RIEMANN SURFACES

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Abstract. Systematic investigation of Elliptic systems on Riemann surface and new results from the theory of generalized analytic functions and vectors are presented. Complete analysis of the boundary value problem of linear conjugation and Riemann-Hilbert monodromy problem for the Carleman-Bers-Vekua irregular systems are given.

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Preface

The idea of using the methods of complex analysis for a wider class of functions then the space of analytic functions takes its origin from the end of 19th century and actually is connected with the period of conception of complex analysis as the independent branch of science. First attempts for the extension of the complex methods were connected with the generalization of Cauchy-Riemann system and the investigation of the properties of the space of solutions such a system (Picard, Hilbert, Carleman, Teodorescou).

Later on, since the 1950-s, after the appearance of the monographs of I.Vekua and L.Bers, generalized analytic functions due to the terminology of Vekua [124] and pseudo-analytic functions due to Bers [11] these problems are the subject of investigation by many scientists. In the frame of Vekua-Bers theory the geometric (topological) properties of the solutions of elliptic systems on the plane became clear: in particular, the natural relation between the pseudoanalytic functions and the quasiconformal mappings was established. Here we mean that Carleman - Vekua and Beltrami equations always exist in parallels.

Concepts of Bers and Vekua are similar in the cases of Bers normalized pair and Carleman-Bers-Vekua regular equations. In particular, the normalized generating pair induces the regular equation and vice versa, the generating pair, corresponding to the regular equation is normalized. In such conditions these approaches are completing each other and we get the functional spaces with richer properties. In other cases when the equation is non-regular or Bers generating pair isn't normalized the corresponding space of the generalized analytic functions is studied less intensively and the theory itself is not as consistent as the Bers-Vekua theory. Besides, other interesting generalized elliptic equations of Cauchy-Riemann equations are known, for example, the equations which give the spaces of poly-analytic or p-analytic functions. The present work is an attempt to fill up this gap at least partially. Its main part is devoted to the study of the solutions of irregular elliptic systems, specifically to the development of the fundamental theorems of the theory of analytic functions and the corresponding boundary value problems in such spaces as well. The remaining part deals with the extension of Bers-Vekua theory on $2n$-elliptic systems, which has been soon developed independently by B.Bojarski [31], [26] and A.Douglis [45] and it is also the subject of the authors' interest. In particular, the geometric interpretation of the matrix Beltrami equation, the problem of the generating pairs, the Riemann-Hilbert monodromy problem etc.

The authors think that the solution of the listed problems is very important for the study of differential - geometric connections of vector bundles and the complex structure of the moduli space, which in their turn are connected with the fundamental problems of the contemporary mathematical physics: topological and conformal field theories, the string and super string theories, supersymmetry, etc.

Further development of the theories of B.Bojarski and A. Douglis is given in the monographs of G. Manjavidze [90] and R. Gilbert and J.Buchanan [49]. Investigation of such non-regular systems is an open problem yet. This work paves the way for the study of these systems.

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1 Introduction

Let α, β, γ, δ be measurable functions of two variables $(x,\ y)$, and $a, b, c, d, e, f \in L_p(U)$, $p > 2$, $U \subset \mathbb{C}$. Consider the system of linear partial differential equations with respect to functions $(u(x, y),\ v(x, y))$of two variables:

$$\begin{cases} \dfrac{\partial v}{\partial y} = \alpha \dfrac{\partial u}{\partial x} + \beta \dfrac{\partial u}{\partial y} + au + bv + e \\ -\dfrac{\partial v}{\partial x} = \gamma \dfrac{\partial u}{\partial x} + \delta \dfrac{\partial u}{\partial y} + cu + dv + f \end{cases} \tag{1.1}$$

The system (1.1) is elliptic if the inequalities

$$\alpha(x, y) > 0, \qquad 4\alpha(x, y)\delta(x, y) - (\beta(x, y) + \gamma(x, y))^2 > 0$$

hold. Along with ellipticity, let us require uniform ellipticity of the system (1.1), which means that for a fixed constant k_0 the functions α, β, γ, δ satisfy the inequality

$$4\alpha\delta - (\beta + \gamma)^2 \geq k_0 > 0.$$

Consider the functions occurring in the system (1.1) as functions of complex variable $z = x + iy$. Suppose $w(z) = u(x, y) + iv(x, y)$, using appropriate simplifications and notations we get the equation

$$\frac{\partial w(z)}{\partial \bar{z}} - \mu_1(z)\frac{\partial w(z)}{\partial z} - \mu_2(z)\overline{\frac{\partial w}{\partial z}} = A(z)w(z) + B(z)\overline{w(z)} + C(z), \tag{1.2}$$

where

$$\mu_1(z) = \frac{2q}{|q|^2 - |1+p|^2}, \qquad \mu_2(z) = -\frac{|q|^2 + (1+p)(1-\bar{p})}{|q|^2 - |1+p|^2},$$

the functions q and p are expressed by the functions α, β, γ, δ as follows:

$$q = \frac{\alpha + \delta + i(\gamma - \beta)}{2}, \qquad p = \frac{\alpha - \delta + i(\gamma + \beta)}{2},$$

whereas the functions A, B, C are obtained from the functions a, b, c, d, e, f by adding and multiplying by constants and by the function $\frac{1}{|q|^2 - |1+p|^2}$. Since the system is uniformly elliptic, the inequality $|\mu_1| + |\mu_2| \leq \mu_0$ holds for any $z \in U$. Moreover, the functions A, B, C are either bounded on U or belong to $L_p(U)$.

The equation (1.2) on the complex plane with natural restrictions on the functions μ_1, μ_2, A, B, contains many well-known equations: Cauchy-Riemann, Beltrami, Carleman-Bers-Vekua, holomorphic disc and other equations, which are obtained from (1.1) or (1.2) by an appropriate choice of the coefficients. All these equations are "deformations" of the Cauchy-Riemann equation and the properties of the space of solution of the corresponding equations are close to the properties of the spaces of analytic functions.

It is well-known, that the system (1.1) is equivalent to the system of equations

$$u_x - V_y + au + bV = 0,$$
$$u_y + V_x + cu + dV = 0,$$

under sufficiently general conditions. This system has the following complex form

$$\frac{\partial w}{\partial \overline{z}} + Aw + B\,\overline{w} = 0, \tag{1.3}$$

which is *Carleman-Bers-Vekua (CBV) equation*, where

$$\frac{\partial}{\partial \overline{z}} = \frac{1}{2}\Big(\frac{\partial}{\partial x} + i\,\frac{\partial}{\partial y}\Big), \quad w = u + iV,$$

and

$$A = \frac{1}{4}(a + d + ic - ib),\ \ B = \frac{1}{4}(a - d + ic + ib).$$

This system was first investigated by Hilbert. Carleman obtained the fundamental property of the solutions of system (1.3) - the uniqueness theorem. Earlier Teodorescu studied the system of the partial type and obtained the general representation of the solutions by means of the analytic functions (see [124] and [11]). This result turned out to be very important in constructing the general theory.

We consider the relationship between the spaces of solutions of the following equations on the one hand and space of analytic functions on the other hand.

1.The Carleman-Bers-Vekua non homogenous equation:

$$\begin{cases} \dfrac{\partial u}{\partial x} - \dfrac{\partial v}{\partial y} = au + bv + f \\ \dfrac{\partial u}{\partial y} + \dfrac{\partial v}{\partial y} = cu + dv + g \end{cases} \tag{1.4}$$

or

$$\frac{\partial U}{\partial \overline{z}} = AU + B\overline{U} + F, \tag{1.5}$$

where

$$A = \frac{1}{4}(a + d) + \frac{i}{4}(c - b), B = \frac{1}{4}(a - d) + \frac{i}{4}(c + b), U = u + iv, F = f + ig.$$

If $F = 0$, then we obtain the homogenous Carleman-Bers-Vekua equation

$$\frac{\partial U}{\partial \overline{z}} = AU + B\overline{U}, \tag{1.6}$$

2.The degenerate Carleman-Bers-Vekua equation is obtained from (1.4) if we assume $a = d,\ c = -b,\ f = g = 0:$

$$\begin{cases} \dfrac{\partial u}{\partial x} - \dfrac{\partial v}{\partial y} = au + bv \\ \dfrac{\partial u}{\partial y} + \dfrac{\partial v}{\partial x} = -bu + av \end{cases} \tag{1.7}$$

the complex form of this system is

$$\frac{\partial U}{\partial \overline{z}} = AU, \tag{1.8}$$

where $A = \frac{1}{2}(a - bi)$, $U = u + iv$.

3. The normal form of Bers-Vekua system is obtained from (1.4) when $a = -d$, $c = b$:

$$\begin{cases} \dfrac{\partial u}{\partial x} - \dfrac{\partial v}{\partial y} = au + bv + f \\ \dfrac{\partial u}{\partial y} + \dfrac{\partial v}{\partial x} = bu - av + g \end{cases} \tag{1.9}$$

the complex form of this system is

$$\frac{\partial U}{\partial \overline{z}} = B\overline{U} + F. \tag{1.10}$$

If $f = g = 0$, we obtain homogenous nonlinear equation (1.6).

4. If $a = b = c = d = e = f = 0$, we obtain the system

$$\begin{cases} \alpha\dfrac{\partial u}{\partial x} + \beta\dfrac{\partial u}{\partial y} - \dfrac{\partial v}{\partial y} = 0 \\ \gamma\dfrac{\partial u}{\partial x} + \delta\dfrac{\partial u}{\partial y} - \dfrac{\partial v}{\partial x} = 0 \end{cases} \tag{1.11}$$

or in the complex form

$$\frac{\partial w(z)}{\partial \overline{z}} - \mu_1(z)\frac{\partial w(z)}{\partial z} - \mu_2(z)\overline{\frac{\partial w}{\partial z}} = 0. \tag{1.12}$$

Particular cases of this equation are a) *Beltrami equation* and b) *holomorphic disc equation.*

5. The space of *polyanalytic functions:* solution space of the equation $\frac{\partial^n w}{\partial^n_{\overline{z}}} = 0$. The functions belong to this space iff $f(z) = \sum_{k=0}^{n-1} h_k(z)\overline{z}^k$, where h_k, $k = 0, 1, ..., n-1$ are analytic functions.

The analysis of elliptic system of Cauchy-Riemann differential equations

$$\begin{aligned} \frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} = 0, \\ \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = 0, \end{aligned} \tag{1.13}$$

is one among the classical concepts of construction of the theory of analytic functions $w = u + i\,v$ of complex variable $z = x + i\,y$. Picard proposed an idea of possibility of construction of analogous theory on the basis of more general elliptic system of first order differential equation (1.1).

The solutions of equation (1.4) are called the *generalized analytic functions.* Therefore, the theory of generalized analytic functions is the meeting point between two sections of analysis - the theory of complex variable analytic functions and the theory of elliptic type differential equations with two independent variables. This theory was developed as independent part of analysis after appearance of monograph of Vekua I. [124], where the long-term investigation of the author and some results of his disciples and followers (B. Bojarski [25], [23], [26], [28], V. Vinogradov[133], [132], I. Danyluk [42], [43] and etc) are presented completely. The foundations of the theory of generalized analytic functions were established in [124]. Approximately in this period Bers [11], independently from I. Vekua suggested the generalization of analytic functions (so called pseudoanalytic functions), based on the modification of the concept of the derivative. Note that, many authors have proposed various generalizations, reducing the system (1.5) to the partial cases, until the complete theory of generalized analytic functions was created (see [20],[90], [91], [13],[135], [136], [7], [123], [91] and etc).

The natural question arises while investigating above mentioned differential systems: how should the solution be understood (definition problem). It is clear, that even for the simplest and fundamental case of the system (1.13) the fulfillment of indicated differential equalities isn't sufficient to obtain the class of functions with needed structure. For the system (1.4) (the equation (1.6)) the situation is more complicated because of the coefficients involved there.

In connection with the above problem I. Vekua's remarkable idea to interpret the derivative $\frac{\partial}{\partial \overline{z}}$ in Sobolev generalized sense turned out to be very interesting and fruitful. Let G be a domain of $z = x + i\,y$-plane. As usual, denote by $C^1(G)$ the class of all complex valued functions of variables x, y with continuous partial derivatives (in classical sense); by $C^1_0(G)$ the subclass of finite functions of the class $C^1(G)$ is denoted. It means, that for every function $\varphi \in C^1_0(G)$ there exists the compact subset K_φ of the set G, such that, outside of K_φ the function φ vanishes, i.e. $\varphi(x, y) = 0$, $(x, y) \in G \setminus K_\varphi$. Let the functions f, g be from the Lebesque class $L^{\text{loc}}_1(G)$ satisfying the equality

$$\iint_G \left(f \cdot \frac{\partial \varphi}{\partial \overline{z}} + g \cdot \varphi \right) dG = 0,$$

for every function $\varphi \in C^1_0(G)$. In this case g is the generalized derivative of f by $\overline{z}$; in addition we preserve the classical notation $g = \frac{\partial f}{\partial \overline{z}}$. The properties of the functions, admitting the generalized derivative by $\overline{z}$, are presented in monograph [124].

Let G be a domain of complex plane. Denote by $D_{\overline{z}}(G)$ the class of all functions given (almost everywhere) in the domain G admitting the derivative in Sobolev generalized sense by $\overline{z}$. The function $w \in D_{\overline{z}}(G)$ and $\frac{\partial w}{\partial \overline{z}} = 0$, modified appropriately on the set of Lebesque measure zero becomes holomorphic i.e. it will be an analytic function of complex variable without any singularities in the domain G. Here and in what follows under singular (precisely isolated singular) point of analytic function we mean the pole and essentially singular point; removable singular point is not considered as a singular point.

Let A and B be functions given (almost everywhere) in the domain G. We say, that the function w *satisfies the equation* (1.6) in the point $z_0 \in G$ or that the same function w is regular in the point z_0 if there exists a positive number ρ, such that, in ρ-neighborhood

$$V_\rho(z_0) = \{z : |z - z_0| < \rho\}$$

of the point z_0, the function $w \in D_{\bar z}(V_\rho(z_0))$ and the equality (1.6) is fulfilled almost everywhere in $V_\rho(z_0)$.

If the function w is regular in every point of G then we say, that w is a regular solution of the equation (1.6) in the domain G. For the class of all possible regular solutions somewhat long but very suitable notation $\mathfrak{A}(A, B, G)$ is introduced. The pair of functions A, B is called generating pair of the class $\mathfrak{A}(A, B, G)$.

In exactly the same way, as the notion of isolated singular point has the most important meaning for the analytic functions, the following analogous notion is principally important for the solution of the equation (1.6).

Let the point $z_0 \in G$; we say, that z_0 is an isolated singular point for the function w, if there exists a perforated ρ-neighborhood $\overset{(0)}{V}_\rho(z_0) = \{z : 0 < |z - z_0| < \rho\}$, $\rho > 0$, of the point z_0 such, that $w \in \mathfrak{A}(A, B, \overset{(0)}{V}_\rho(z_0))$. It should be mentioned especially, that in the point z_0 the function w must not satisfy the equation (1.6). Let for the function w there exists the isolated subset G^*_w of the set G such, that in every point of G, except the points of G^*, w is regular, i.e. $w \in \mathfrak{A}\big(A, B, G \setminus G^*_w\big)$, then we say, that w is a quasiregular solution of the equation (1.6).

It should be also mentioned, that in the points of the set G^*_w the function w must not satisfy the equation. The class of all possible quasiregular solutions of the equation (1.6) is denoted by $\mathfrak{A}^*(A, B, G)$. It is clear, that the following inclusion

$$\mathfrak{A}(A, B, G) \subset \mathfrak{A}^*(A, B, G)$$

holds. In addition, the solution w of the class $\mathfrak{A}^*(A, B, G)$ is contained in the subclass $\mathfrak{A}(A, B, G)$ if and only if the set of its singularities $G^*_w = \varnothing$.

We just introduced the notion of the solution of the equation (1.6) above. We can't say anything about existence yet. In order to get the corresponding results it is necessary to define the coefficients of the equation concretely. The equation (1.6) is regular by the definition, if the domain G is bounded and the coefficients

$$A, B \in L_p(G) \tag{1.14}$$

for some number $p > 2$. For such equations the existence of regular (quasiregular) solutions is completely solved as well as their general representation by means of holomorphic (analytic) functions is obtained [124].

Denote by $\mathfrak{A}^*_0(G)$ the class of all possible analytic functions in G, which may have arbitrary isolated singularities. By $\mathfrak{A}_0(G)$ the subclass of holomorphic functions without any singularities in G is denoted.

Let (1.6) be a regular solution. Then the relation

$$w = \Phi \cdot \exp(\Theta) \tag{1.15}$$

defines the bijective relation between the class $\mathfrak{A}(A,B,G)$ and $\mathfrak{A}_0(G)$, where

$$\begin{gathered} w\in\mathfrak{A}(A,B,G),\quad \Phi\in\mathfrak{A}_0(G),\\ \Theta(z)=\frac{1}{\pi}\iint_G\frac{A(\zeta)}{\zeta-z}\,dG(\zeta)+\frac{1}{\pi}\iint_G\frac{B(\zeta)}{\zeta-z}\,\Lambda(\zeta)\,dG(\zeta),\\ \Lambda(\zeta)=\begin{cases}\dfrac{\overline{w(\zeta)}}{w(\zeta)}, & \text{for}\quad w(\zeta)\neq0,\\ 0, & \text{for}\quad w(\zeta)=0.\end{cases} \end{gathered}\tag{1.16}$$

When the coefficient B is zero almost everywhere the formula (1.15) turns into the above mentioned result of Teodorescu. In other cases the relation (1.15) is nonlinear integral representation of the solution w by itself and holomorphic function.

For the regular equation (1.6) by the relation (1.15) the bijective relation between the class $\mathfrak{A}^*(A,B,G)$ and $\mathfrak{A}_0^*(G)$ is also given. Moreover, the set of singularities G_w^* of the solution w coincides with the set of singularities D_Φ^* of the function Φ.

Investigation and application of the properties of integral transformation of the operator

$$\begin{gathered} T_G: f\ \mapsto\ F,\quad f\in L_p(G),\quad p>2,\\ F(z)=-\frac{1}{\pi}\iint_G\frac{f(\zeta)}{\zeta-z}\,dG(\zeta), \end{gathered}$$

gave the possibility to state, that the structure of regular and quasiregular solutions of the regular equation (1.6) is completely analogous to the structure of holomorphic and analytic (with arbitrary isolated singularities) functions correspondingly. Here one keeps in mind not only the functional properties of the classes $\mathfrak{A}(A,B,G)$, $\mathfrak{A}_0(G)$ $\big(\mathfrak{A}^*(A,B,G)$, $\mathfrak{A}_0^*(G)\big)$ but the analysis of the boundary problems of mathematical physics.

The behavior of the quasiregular solution w in the neighborhood of its singular point $z_0\in G$ is completely analogous to the behavior of the analytic function Φ in the neighborhood of z_0, which, as was mentioned above, is singular for the function Φ as well. As an example, we point out Sokhotsky-Weierstrass-Casorati theorem about the behavior of analytic functions in the neighborhood of essentially singular point, but it should be noted, that as as for as we know, the corresponding result concerning the Picard theorem isn't obtained.

The characteristic feature of the classes $\mathfrak{A}(A,B,G)$ and $\mathfrak{A}^*(A,B,G)$ with the limits $A,B\in L_p(G)$, $p>2$ is their singletypeness. Speaking figuratively the classes $\mathfrak{A}(A,B,G)$ and $\mathfrak{A}^*(A,B,G)$ almost do not react on the variation of generating pair $A,B\in L_p(G)$, $p>2$. We don't mean, that we want to state the validity of equalities

$$\mathfrak{A}(A_1,B_1,G)=\mathfrak{A}(A_2,B_2,G),\mathfrak{A}^*(A_1,B_1,G)=\mathfrak{A}^*(A_2,B_2,G),$$

as soon as $A_1,A_2,B_1,B_2\in L_p(G)$, $p>2$, however, in order to obtain the main properties of the functions of the classes $\mathfrak{A}(A,B,G)$, $\mathfrak{A}^*(A,B,G)$ the condition

$A, B \in L_p(G)$, $p > 2$ is completely sufficient and it isn't necessary to have the concrete form of the coefficients A, B. As it will be shown below, the classes $\mathfrak{A}(A, B, G)$, $\mathfrak{A}^*(A, B, G)$ formed by the coefficients A, B not satisfying the regularity condition (1.14) are sharply contrast and there occur very unexpected phenomena in general.

Natural and also necessary in applications, the extension of the class of regular equations (1.6) is the class of quasiregular equations [125]. Suppose that the coefficients of the equation (1.6) admit the representation of the form

$$A(z) = \sum_{k=1}^{n} f_k(z)\, A_k(z), \quad B_k(z) = \sum_{k=1}^{n} g_k(z)\, B_k(z) \tag{1.17}$$

where $f_k, g_k \in \mathfrak{A}_0^*(G)$, $A_k, B_k \in L_p(G)$, $p > 2$, $k = 1, 2, \ldots, n$ (n is natural number and G, is bounded domain on the complex plane as before). Generally speaking it is obvious, that such equations are not regular. They are called the quasiregular equations. Vekua I. obtained the following general result, which involves the whole class of quasiregular equations [125]. Namely, we obtained the analogous formula of the form (1.15), but in these cases the situation is much complicated.

Let $w \in \mathfrak{A}^*(A, B, G)$, i.e. let w be an arbitrary quasiregular solution of the quasiregular equation (1.6). Then the function

$$\Phi = w \cdot \exp\{-\Theta\} \tag{1.18}$$

is holomorphic in every point of the domain

$$G \setminus G_w^* \setminus \Delta, \tag{1.19}$$

where

$$\Delta = \bigcup_{k=1}^{n} \big(D_{f_k}^* \cup D_{g_k}^*\big),$$

$$\Theta(z) = \sum_{k=1}^{n} \left(\frac{f_k(z)}{\pi} \iint_G \frac{A_k(\zeta)}{\zeta - z}\, dG(\zeta) + \frac{g_k(z)}{\pi} \iint_G \frac{B_k(\zeta)}{\zeta - z}\, \Lambda(\zeta)\, dG(\zeta) \right),$$

G_w^*, $D_{f_k}^*$, $D_{g_k}^*$ are the sets of singularities of the solution w and the functions f_k, g_k correspondingly, the function $\Lambda(\zeta)$ is given by the formula (1.16). Conversely, for every analytic function $\Phi \in \mathfrak{A}_0^*(G)$ there exists the unique quasiregular solution $w \in \mathfrak{A}^*(A, B, G)$ for which the relation (1.18) holds. For singular points of analytic function Φ the following statement is established: in every point of the set $G_w^* \setminus \Delta$ the function Φ has the isolated singularity; the points of Δ maybe singular as well as the holomorphy points (this depends on the concrete form of the coefficients (1.17) and the solution w; both cases are realized by examples) whether they are singular points for the solution w or not.

From the formula (1.18) according to the properties of the functions

$$M(z) = h(z) \iint_G \frac{f(\zeta)}{\zeta - z}\, dG(\zeta), \quad f \in L_p(G), \quad p > 2, h \in \mathfrak{A}_0^*(G),$$

it follows, that every solution $w \in \mathfrak{A}^*(A, B, G)$ is continuous in any point of the domain (1.19).

We spoke only about quasiregular solutions of the quasiregular equation (1.6). Unlike regular equations, equations of such type may not have nontrivial regular solutions in general, i.e. in some cases of the coefficients (1.17) it may happen, that the unique zero function is contained in the class $\mathfrak{A}(A, B, G)$. On the other hand, in some cases of the coefficients (1.17) the class $\mathfrak{A}(A, B, G)$ may be very extensive.

Within the limits of the coefficients (1.17) and in contrast to (1.14) the singletypeness of the classes $\mathfrak{A}^*(A, B, G)$ doesn't exist. More clearly, the generating pairs of the functions (A_1, B_1), (A_2, B_2) may be of the form (1.17) and "differing in the small" from each other - they may be even identical to within the mark $(+, -)$ in front the coefficients - and nevertheless the classes $\mathfrak{A}^*(A_1, B_1, G)$ and $\mathfrak{A}^*(A_2, B_2, G)$ may be principally of the different structure. In contrast of the regular and the quasiregular solutions of the regular equations the regular and the quasiregular solutions of the quasiregular equations may have principally nonholomorphic and nonanalytic structure. Behavior of the quasiregular solutions of the quasiregular equations in the neighborhood of a singular point of the equation may be fundamentally nonanalytic.

The solutions of the regular equations (1.6) can't have the singularities of the pole type of "infinite order" and nontrivial solutions of such equations can't have nonisolated zero and zero of "infinite order" in the points of regularity. The solutions of the irregular equations of the wide class aren't subjected to such exclusions.

The above mentioned general properties of the irregular equations of the form (1.6) make clear the complexity of investigations. These equations were the subject of investigation of various authors. Among them there are basic works of I. Vekua In this direction of generalized analytic functions the most important results were obtained by L. Mikhailov[97], V. Vinogradov [133], Z. Usmanov [123], N. Bliev [20], V. Shmidt [117], R. Saks [113], A. Tungatarov [121], H. Begehr and D. Q. Dai [14], M. Reissig [106], A. Timofeev [119] and others. The references according to this subject are presented in [84], [85], [63], [54], [53].

2 Functional spaces induced from regular Carleman-Bers-Vekua equations

This section contains auxiliary concepts and results needed for our purpose concerning Vekua-Pompej type integral operators, generalized analytic functions and connected boundary value problems. We use the terms and notations from the monographs [124], [99], [13], [108], [10].

2.1 Some functional spaces

Let $G \subset \mathbb{C}$ be a closed set. Denote by $C(G)$ the set of all continuous bounded functions f on G with the norm

$$\|f\|_{C(G)} = \sup_{z\in G} \big|f(z)\big|.$$

$C(G)$ is a Banach space with this norm.

Denote by $H_\alpha(G)$ the set of all functions satisfying the Hölder condition with exponent α, $0 < \alpha \leq 1$

$$\big|f(z_1) - f(z_2)\big| \leq H|z_1 - z_2|^\alpha,$$

where z_1 and z_2 are arbitrary points, belonging to G and H is a positive constant, not depending on the choice of the points z_1 and z_2. Denote by $H(f)$ the greatest lower bound of the numbers H.

It is clear, that

$$H(f) = \sup_{z_1,z_2\in G} \frac{|f(z_1) - f(z_2)|}{|z_1 - z_2|^\alpha},$$

$$\big|f(z_1) - f(z_2)\big| \leq H(f)|z_1 - z_2|^\alpha.$$

Define the norm of the element f of the set $C_\alpha(G)$ by the following formula:

$$\|f\|_{C_\alpha(G)} = \|f\|_{C(G)} + H(f).$$

We understand the measurability of the set and function in the Lebesgue sense.

Let G be a measurable set and let p be a real number, $1 \leq p < \infty$. Denote by $L_p(G)$ the set of all functions $f(z)$, satisfying the condition

$$\iint_G \big|f(z)\big|^p dx\, dy < +\infty.$$

Denote by $L_p(G)$ and $L_p^{\text{loc}}(G)$ the set of all functions $f(z)$, satisfying the conditions

$$\|f\|_{L_p(G)} = \left(\iint_G \big|f(z)\big|^p dx\, dy\right)^{\frac{1}{p}}$$

and

$$f(z) \in L_p(G')$$

respectively, where G' is an arbitrary closed bounded subset of the set G.

Let $1 \leq p < \infty$, $\nu > 0$. Denote by $L_{p,\nu}(\mathbb{C})$ the set of all functions $f(z)$ defined on the complex plane $\mathbb{C}$ and satisfying the conditions

$$f(z) \in L_p(E_1), \quad f_\nu(z) = \frac{1}{|z|^\nu} f\left(\frac{1}{z}\right) \in L_p(E_1),$$

where E_1 is a circle $|z| \leq 1$.

Define the norm of the space $L_{p,\nu}(\mathbb{C})$ by the formula

$$\|f\|_{L_{p,\nu}(\mathbb{C})} = \|f\|_{L_p(E_1)} + \|f_\nu\|_{L_p(E_1)}.$$

Denote by $C^m(G)$ the set of all functions having the continuous partial derivatives with respect to x and y up to and including the order m. Denote by $D_m^0(G)$ the set of all functions $f(z)$ satisfying the following conditions:

1) $f \in C^m(G)$;

2) for every function $f(z) \in D_m^0(G)$) there exists the closed subset G_f of G outside of which $f = 0$;

Introduce the notion of the *generalized derivative in the Sobolev sense.* Assume that $f, g \in L_1^{\text{loc}}(G)$ and f, g satisfy the equalities

$$\iint_G g \frac{\partial \varphi}{\partial \overline{z}} \, dx \, dy + \iint_G f \, \varphi \, dx \, dy = 0, \quad \left(\iint_G g \frac{\partial \varphi}{\partial z} \, dx \, dy + \iint_G f \, \varphi \, dx \, dy = 0 \right),$$

where φ is a function of the class $D_1^0(G)$. In this case we say that the function f is the *generalized derivative* with respect to $\overline{z}$ (with respect to z) of f on the domain G. We denote by $\dfrac{\partial f}{\partial \overline{z}}$ and $\dfrac{\partial f}{\partial z}$ the *generalized derivatives* of the function f respectively $\overline{z}$ and z.

The functions having the generalized derivative with respect to $\overline{z}$ on the domain G compose the *linear space,* which we denote by $D_{\overline{z}}(G)$.

2.2 The Vekua-Pompej type integral operators

Let G be a bounded domain. It is known that if $f \in L_1(G)$ then the integral

$$(T_G f)(z) = -\frac{1}{\pi} \iint_G \frac{f(\zeta)}{\zeta - z}, \, d\zeta \, d\eta, \quad \zeta = \xi + i\eta \tag{2.1}$$

exists for all points z outside of $\overline{G}$, is holomorphic outside of $\overline{G}$ and vanishes at infinity. In addition, $T_G f$ almost everywhere on G exists as the function of the point $z \in G$ and belongs to the class $L_p^{loc}(\mathbb{C})$, where p is an arbitrary number, satisfying the condition $1 \leq p < 2$. $T_G f \in D_{\overline{z}}(G)$ and the equality

$$\frac{\partial T_G f}{\partial \overline{z}} = f(z), \quad z \in G$$

holds.

Analogically to above if $f \in L_{p,2}(\mathbb{C})$, $p > 2$ then there exists the integral

$$(T_{\mathbb{C}}f)(z) = -\frac{1}{\pi}\iint_{\mathbb{C}} \frac{f(\zeta)}{\zeta - z}\, d\zeta\, d\eta, \quad \zeta = \xi + i\eta \tag{2.2}$$

everywhere on the plane $\mathbb{C}$, belongs to $D_{\overline{z}}(\mathbb{C})$ as the function of the point z and the equality

$$\frac{\partial T_E f}{\partial \overline{z}} = f(z), \quad z \in \mathbb{C} \tag{2.3}$$

is valid. It is known also, that if $g \in L_1^{\mathrm{loc}}(G)$ and $\dfrac{\partial g}{\partial \overline{z}} = 0$, $z \in G$ then $g(z)$ is holomorphic in the domain G.

The operator $T_G f$ gives the solution of the equation $\dfrac{\partial g}{\partial \overline{z}} = f$ for $f \in L_1(G)$. It means, that it is possible to express $\dfrac{\partial}{\partial \overline{z}}$-primitive by this operator. In particular, we have the following result.

Theorem 2.1 *If* $\dfrac{\partial g}{\partial \overline{z}} = f$ *and* $f \in L_1(G)$, *then*

$$g(z) = \Phi(z) - \frac{1}{\pi}\iint_{G} \frac{f(\zeta}{\zeta - z}\, d\zeta\, d\eta \equiv \Phi(z) + (T_G f)(z), \quad z \in G \tag{2.4}$$

is valid, where F(z) is a holomorphic function in the domain G*. On the other hand* $f \in L_1(G)$ *and* $\Phi(z)$ *is a holomorphic function in the domain* G *then*

$$g(z) = \Phi(z) + (T_G f)(z) \in D_{\overline{z}}(G)$$

and the equality

$$\frac{\partial g}{\partial \overline{z}} = f(z), \quad z \in G \tag{2.5}$$

is fulfilled.

Let the function $f(z)$ has the generalized derivative with respect to $\overline{z}$ at every point of the domain G, i.e. for every point $z_0 \in G$ there exists the neighborhood of this point inside of which $f(z) \in D_{\overline{z}}(G_0)$. In this case $f \in D_{\overline{z}}(G)$. If $g \in D_{\overline{z}}(G)$, then $g \in D_{\overline{z}}(G_1)$, where G_1 is an arbitrary subdomain of the domain G.

Theorem 2.2 *Let* G *be a bounded domain. If* $f \in L_p(G)$, $p > 2$ *then the function* $g(z) = (T_G f)(z)$ *satisfies the conditions*

$$\left|g(z)\right| \le M_1 \left\|f\right\|_{L_p(G)}, \quad z \in \mathbb{C}, \tag{2.6}$$

$$\left|g(z_1) - g(z_2)\right| \le M_2 \left\|f\right\|_{L_p(G)} |z_1 - z_2|^{\alpha}, \quad \alpha = \frac{p-2}{p}\,, \tag{2.7}$$

where z_1 *and* z_2 *are the arbitrary points of the complex plane and* M_1 *and* M_2 *are the constants. Moreover,* M_1 *depends on* p *and* G, M_2 *depends only on* p.

Therefore, if G is a bounded domain and $f \in L_p(G)$, $p > 2$ then $T_G f \in C_{\frac{p-2}{p}}(\mathbb{C})$ and if $f \in D_{1,p}(G)$, $p > 2$, then $f(z)$ belongs to the class $C_{\frac{p-2}{p}}$ inside of G.

Theorem 2.3 *Let $f \in L_{p,2}(\mathbb{C})$, $p > 2$ then the function $g(z) = (T_{\mathbb{C}} f)(z)$ satisfies the conditions*

$$\big|g(z)\big| \leq M_p \left\|f\right\|_{L_{p,2}(\mathbb{C})}, \tag{2.8}$$

$$\big|g(z_1) - g(z_2)\big| \leq M_p \left\|f\right\|_{L_{p,2}(\mathbb{C})} |z_1 - z_2|^{\frac{p-2}{p}}, \quad z_1, z_2 \in \mathbb{C}, \tag{2.9}$$

Moreover, for the given $R > 1$ there exists the number $M_{p,R}$ such, that

$$\big|g(z)\big| \leq M_{p,R} \left\|f\right\|_{L_{p,2}(\mathbb{C})} |z|^{\frac{2-p}{p}}, \quad \text{if} \quad |z| \geqslant R. \tag{2.10}$$

Therefore, if $f \in L_{p,2}(\mathbb{C})$, $p > 2$, then

$$T_{\mathbb{C}} f \in C_{\frac{p-2}{p}}(\mathbb{C}), \quad p > 2.$$

The operator $\frac{\partial}{\partial \overline{z}}$ satisfies the important rules of the differential operators:

Theorem 2.4 *1) Let $f \in D_{1,p}(G)$, $1 < p < 2$, $g \in D_{1,p'}(G)$, $p' = \dfrac{2p}{3p-2}$, then $fg \in D_{\overline{z}}(G)$. In addition*

$$\frac{\partial}{\partial \overline{z}}(fg) = g(z)\,\frac{\partial f}{\partial \overline{z}} + f(z)\,\frac{\partial g}{\partial \overline{z}}.$$

2) Assume the values of the function $z_* = f(z)$ of the class $D_{\overline{z}}(G)$ belong to the bounded domain G_* and

$$g(z) = \frac{\partial f}{\partial \overline{z}} \in L_p^{\mathrm{loc}}(G), \quad p > 2.$$

Let $\Phi(z_*)$ be a holomorphic function in G_*^0 moreover $\overline{G}_* \subset G_*^0$. In this case the composite function $f_*(z) = \Phi(f(z))$ belongs to $D_{\overline{z}}(G)$ and the equality

$$\frac{\partial f_*(z)}{\partial \overline{z}} = \Phi'(f(z))\,\frac{\partial f(z)}{\partial \overline{z}}$$

is valid.

2.3 The Carleman-Bers-Vekua equation

Consider the Carleman-Bers-Vekua homogeneous equation:

$$Lw \equiv \frac{\partial w}{\partial \overline{z}} + Aw + B\overline{w} = 0, \tag{2.11}$$

where A and B are given functions on $G \subset \mathbb{C}$ and

$$A, B \in L_p^{\mathrm{loc}}(G), \quad p > 2. \tag{2.12}$$

We say that $w(z)$ *satisfies equation* (2.11) in the neighborhood G_0 of the point z_0 if $w \in D_{\overline{z}}(G_0)$ and $Lw \equiv \frac{\partial w}{\partial \overline{z}} + Aw + B\overline{w} = 0$ in G_0.

If w satisfies equation (2.11) in G, then we say that w is a *regular solution* of (2.11) in G, i.e. $w \in D_{\overline{z}}(G)$ and $Lw = 0$ almost everywhere in G.

Denote by $\mathfrak{A}(A, B, G)$ the class of solutions of (2.11) when A, B are regular in G. If $A, B \in L_p^{\mathrm{loc}}(G)$ $(A, B \in L_{p,2}(\mathbb{C}))$, then we write $\mathfrak{A}_p(A, B, G)$ $(\mathfrak{A}_{p,2}(A, B, G))$. The regular solutions of (2.11) in G when $A, B \in L_p^{\mathrm{loc}}(G)$, are called *generalized analytic functions* of the class $\mathfrak{A}_p(A, B, G)$. Regular solution of (2.11) in the domain G, where $A, B \in L_{p,2}(\mathbb{C})$, called is *generalized analytic function* of the class $\mathfrak{A}_{p,2}(A, B, \mathbb{C})$.

Theorem 2.5 *(Main lemma)[124]. Let $w(z)$ be a generalized analytic function of class $\mathfrak{A}_{p,2}(A, B, G)$, $p > 2$ and let*

$$g(z) = \begin{cases} A(z) + B(z)\,\dfrac{\overline{w(z)}}{w(z)}, & \text{if} \quad w(z) \neq 0, \quad z \in G, \\ A(z) + B(z), & \text{if} \quad w(z) = 0, \quad z \in G. \end{cases} \tag{2.13}$$

Then the function

$$\Phi(z) = w(z)\, e^{(T_G g)(z)} \tag{2.14}$$

is holomorphic on G.

If $w(z)$ isn't identically zero, then (2.14) can be rewritten in the following form:

$$w(z) = \Phi(z)\, e^{-T_G(A + B\frac{\overline{w}}{w})(z)}. \tag{2.15}$$

We call the function $\Phi(z)$ in the formula (2.15) the *analytic divisor* of the generalized analytic function $w(z)$.

Theorem 2.6 *Let $\Phi(z)$ be an analytic function in the domain G and t be a fixed point of the extended complex plane, i.e $t \in \mathbb{C}$ or $t = \infty$. Let $A, B \in L_{p,2}(\mathbb{C})$, $p > 2$ then there exists the function $w(z)$ defined on G satisfying the following conditions:*

1) the function $w(z)$ is a regular solution of the equation

$$Lw \equiv \frac{\partial w}{\partial \overline{z}} + Aw + B\overline{w} = 0$$

in the domain G;

2) the function $w_0(z) = \dfrac{w(z)}{\Phi(z)}$ is continuous in G and is continuously extendable on the whole complex plane, moreover $w_0 \in C_{\frac{p-2}{p}}(\mathbb{C})$;

3) $w_0(z) \neq 0$, $z \in \mathbb{C}$;

4) $w_0(t) = 1$;

5) the function $w_0(z)$ is a holomorphic function outside of $\overline{G}$.

The function $w(z)$, satisfying the conditions 1)-5), is unique.

The function $w(z) = \Phi(z)\, w_0(z)$ satisfies the following non-linear integral equation

$$w(z) = \Phi(z)\, e^{T_G(A+B\frac{\overline{w}}{w})(t) - T_G(A+B\frac{\overline{w}}{w})(z)}. \tag{2.16}$$

Therefore, we can consider the operator which corresponds to every analytic function $\Phi(z)$ and to the fixed point t of the extended complex plane the regular solution $w(z,t)$ of the equation $Lw = 0$ in the domain G. We denote this operator by $R_t(\Phi)$.

Denote by $R_t^{A,B}$ the operator which connects to every analytic function on G and to every point t of the extended complex plane the regular solution $w(z,t) = R_t^{A,B}\Phi(z)$ of the equation (2.1) on the domain G, satisfying the following conditions:

1) the function $w_0(z) = \dfrac{w(z)}{\Phi(z)}$ is continuous in the domain G and is continuously extendable on the whole complex plane, moreover $w_0 \in C_{\frac{p-2}{p}}(\mathbb{C})$;

2) $w_0(z) \neq 0$, $z \in \mathbb{C}$;

3) $w_0(t) = 1$;

4) the function $w_0(z)$ is holomorphic outside of $\overline{G}$.

By means of the operator $R_t(\Phi)$ it is possible to construct the solution of the equation $Lw = 0$ the analytic divisor of which is an arbitrary analytic function.

Theorem 2.6 and the formula (2.16) are valid even in case $t = \infty$. Therefore the operator $R_\infty(\Phi)$ corresponds to given arbitrary analytic divisor $\Phi(z)$ the appropriate solution of the equation $Lw = 0$.

Theorem 2.7 *1) If the generalized analytic function of the class $\mathfrak{A}_{p,2}(A,B,\mathbb{C})$, $p > 2$ is bounded on the whole complex plane and is equal to zero in the fixed point z_0 of the extended complex plane, i.e $z_0 \in \mathbb{C}$ or $z_0 = \infty$, then $w(z) = 0$, $\forall\ z \in \mathbb{C}$.*

2) Every function $w(z)$ of the class $\mathfrak{A}_{p,2}(A,B,\mathbb{C})$, $p > 2$, bounded on the whole complex plane, has the following form:

$$w(z) = c\, e^{-T_E(A+B\frac{\overline{w}}{w})(z)}, \quad c = \text{const}. \tag{2.17}$$

We call the functions of the form (2.17) the *generalized constants* of the class $\mathfrak{A}_{p,2}(A,B,\mathbb{C})$. Therefore, arbitrary regular solution of the equation

$$\frac{\partial w}{\partial \overline{z}} + Aw + B\overline{w} = 0, \quad A, B \in L_{p,2}(\mathbb{C}), \quad p > 2$$

bounded on the whole complex plane is referred as the generalized constant.

2.4 The generalized polynomials of the class $\mathfrak{A}_{p,2}(A,B,\mathbb{C})$, $p > 2$

The regular solution of the equation

$$\frac{\partial w}{\partial \overline{z}} + Aw + B\overline{w} = 0$$

on the whole plane, the normal analytic divisor of which is a classical polynomial of order n is called the *generalized polynomial* of order n of the class $\mathfrak{A}_{p,2}(A,B,\mathbb{C})$, $p>2$.

Theorem 2.8 *If 1) $w\in\mathfrak{A}_{p,2}(A,B,\mathbb{C})$, $p>2$ and 2) $w(z)=O(z^n)$, $z\to\infty$, where n is a non-negative integer, then $w(z)$ is the generalized polynomial of order at most n of the class $\mathfrak{A}_{p,2}(A,B,\mathbb{C})$.*

Let

$$\Phi_1(z)=\frac{1}{2(t-z)}\,,\quad \Phi_2(z)=\frac{1}{2i(t-z)}\,, \tag{2.18}$$

where t is a fixed point of the complex plane and let $X_j(z,t)=R_t(\Phi_j(z))$ $(j=1,2)$ be the regular solutions of the equation

$$Lw\equiv\frac{\partial w}{\partial\overline{w}}+Aw+B\overline{w}=0,\quad A,B\in L_{p,2}(\mathbb{C}),\quad p>2$$

on the domain $\mathbb{C}\setminus\{t\}$, corresponding to the functions $\Phi_1(z)$ and $\Phi_2(z)$.

Consider the functions

$$\Omega_1(z,t)=X_1(z,t)+iX_2(z,t),\Omega_2(z,t)=X_1(z,t)-iX_2(z,t). \tag{2.19}$$

We call the functions $\Omega_1(z,t)$ and $\Omega_2(z,t)$ the *main kernels* of the class $\mathfrak{A}_{p,2}(A,B,\mathbb{C})$, $p>2$.

Let $V_{2n}(z,z_0)$ and $V_{2n+1}(z,z_0)$ be the generalized polynomials of the class $\mathfrak{A}_{p,2}(A,B,\mathbb{C})$, $p>2$ corresponding to the normal analytic divisors $(z-z_0)^n$ and $i(z-z_0)^n$, and $V_{2n}(z,z_0)=R^{A,B}_\infty\big((z-z_0)^n\big)$, $V_{2n+1}(z,z_0)=R^{A,B}_\infty\big(i(z-z_0)^n\big)$, $n=0,1,2,\ldots$

Consider the following Carleman-Bers-Vekua equation

$$\frac{\partial w}{\partial\overline{z}}+Aw+B\overline{w}=0,\quad A,B\in L_{p,2}(\mathbb{C}),\quad p>2 \tag{2.20}$$

and its conjugate equation

$$\frac{\partial w'}{\partial\overline{z}}-Aw'-\overline{Bw'}=0. \tag{2.21}$$

We call the functions $V_n(z,z_0)$ the *generalized power functions* of the class $\mathfrak{A}_{p,2}(A,B,\mathbb{C})$, $p>2$. Analogically, we denote by $\mathfrak{A}_{p,2}(-A,-\overline{B},\mathbb{C})$ the generalized power functions of he class $V'_n(z,z_0)$

$$\begin{aligned}V'_{2n}(z,z_0)&=R^{-A,-\overline{B}}_\infty\big((z-z_0)^n\big),\\ V'_{2n+1}(z,z_0)&=R^{-A,-\overline{B}}_\infty\big(i(z-z_0)^n\big).\end{aligned} \tag{2.22}$$

Theorem 2.9 *[69]Let $w(z)$ be a function of the class $\mathfrak{A}_{p,2}(A,B,G)$, $p>2$ where G is the circle $|z-z_0|<R$. Then the following expansion holds on this circle*

$$w(z)=\sum_{n=0}^{\infty} c_n\, V_n(z,z_0),$$

where c_n are defined by the formulas:

$$c_{2n}=\operatorname{Re}\frac{1}{2\pi i}\int_{|t-z_0|=\rho} w(t)\, V'_{-2(n+1)}(t,z_0)\,dt,$$
$$c_{2n+1}=-\operatorname{Re}\frac{1}{2\pi i}\int_{|t-z_0|=\rho} w(t)\, V'_{-2n-1}(t,z_0)\,dt,$$
$$0<\rho<R,\quad n=0,1,2,\dots$$

2.5 Some properties of the generalized power functions

Consider the generalized power functions

$$U_{2k}(z,z_0)=\mathcal{R}^{A,B}_{z_0}((z-z_0)^k), U_{2k+1}(z,z_0)=\mathcal{R}^{A,B}_{z_0}(i(z-z_0)^k), \tag{2.23}$$

$$V_{2k}(z,z_0)=\mathcal{R}^{A,B}_{\infty}((z-z_0)^k), V_{2k+1}(z,z_0)=\mathcal{R}^{A,B}_{\infty}(i(z-z_0)^k), \tag{2.24}$$

of the Carleman-Bers-Vekua equation

$$\partial_{\overline{z}}w+Aw+B\overline{w}=0, A,B\in L_{p,2}, p>2, \tag{2.25}$$

where $z_0\neq\infty$, $k=0,\pm1,\pm2,...$, $\mathcal{R}^{A,B}_{z_0}$ is the operator (see [124], chapter 3,§3) associating to every analytic function φ and the point $z_0\in\overline{\mathbb{C}}$ the solution w of the equation (2.25), satisfying the following conditions

1)the function $\widetilde{w}(z,z_0)=\frac{w(z,z_0)}{\varphi}$ is continuous in the domain, where φ is analytic and continuously extendable on $\mathbb{C}$, moreover $\widetilde{w}\in C_\alpha(\mathbb{C})$, $\alpha=\frac{p-2}{p}$;

2)$\widetilde{w}(z,z_0)\neq0$ on $\overline{\mathbb{C}}$;

3)$\widetilde{w}(z_0,z_0)=1$.

The function φ is called an analytic divisor (with respect to the point z_0) of the function $w=\mathcal{R}^{A,B}_{z_0}(\varphi)$; we call the point z_0 the point of the coordination φ and w. When $z_0=\infty$, the function φ is called the normal analytic divisor of the function w.

These functions are representable in the following form

$$U_k(z,z_0)=(z-z_0)^{[\frac{k}{2}]}\widetilde{U}_k(z,z_0), V_k(z,z_0)=(z-z_0)^{[\frac{k}{2}]}\widetilde{V}_k(z,z_0), \tag{2.26}$$

where $\widetilde{U}_{2k},\widetilde{U}_{2k+1},\widetilde{V}_{2k}$ and $\widetilde{V}_{2k+1}$ are the generalized constants (see [124], chapter 3, §4) of the class $\mathfrak{A}(A,B)$, $B_k(z)=B(z)\frac{(\overline{z}-\overline{z}_0)^k}{(z-z_0)^k}$, satisfying the conditions

$$\widetilde{U}_{2k}(z_0,z_0)=\widetilde{V}_{2k}(\infty,z_0)=lim_{z\to\infty}\widetilde{V}_{2k}(z,z_0)=1 \tag{2.27}$$

$$\widetilde{U}_{2k+1}(z_0, z_0) = \widetilde{V}_{2k+1}(\infty, z_0) = lim_{z\to\infty}\widetilde{V}_{2k+1}(z, z_0) = i. \tag{2.28}$$

Besides $\widetilde{U}_k(*, z_0)$, $\widetilde{V}_k(*, z_0)$ belong to the class $C_{\frac{p-2}{p}}(\mathbb{C})$ and satisfy the inequality

$$M^{-1} \le |\widetilde{U}_k(z, z_0)| \le M, M^{-1} \le |\widetilde{V}_k(z, z_0)| \le M, \tag{2.29}$$

$$z \in \mathbb{C}, k = 0, \pm1, \pm2, ...,$$

where $M = exp\{M_p|||A| + |B|||_{p,2}$, M_p is a constant, depending only on p (see [124], chapter 3, §4).

The generalized power functions U_k and V_k differ from each other only by coordinated points with their analytic divisors.

It is easy to see that the following equalities hold:

$$U_{2k}(z, z_0) = c_{2k,0}V_{2k}(z, z_0) + c_{2k,1}V_{2k+1}(z, z_0) \tag{2.30}$$

$$U_{2k+1}(z, z_0) = c_{2k+1,0}V_{2k}(z, z_0) + c_{2k+1,1}V_{2k+1}(z, z_0) \tag{2.31}$$

where $c_{2k,\alpha}, c_{2k+1,\alpha}, \alpha = 1, 2$ are real constants (z_0 is fixed point), representable by the formulas:

$$c_{2k,0} + ic_{2k,1} = \widetilde{U}_{2k}(\infty, z_0) = -\frac{Im\widetilde{V}_{2k+1}(z_0, z_0) - iIm\widetilde{V}_{2k}(z_0, z_0)}{Im[\widetilde{V}_{2k}(z_0, z_0)\overline{\widetilde{V}_{2k+1}(z_0, z_0)}]} \tag{2.32}$$

$$c_{2k+1,0} + ic_{2k+1,1} = \widetilde{U}_{2k+1}(\infty, z_0) = \frac{Re\widetilde{V}_{2k+1}(z_0, z_0) - iRe\widetilde{V}_{2k}(z_0, z_0)}{Im[\widetilde{V}_{2k}(z_0, z_0)\overline{\widetilde{V}_{2k+1}(z_0, z_0)}]} \tag{2.33}$$

Note, that the denominator in the right-hand sides of equalities (2.32) and (2.33) is not equal to zero. Indeed, assuming the contrary we have

$$\widetilde{V}_{2k}(z_0, z_0) = c\widetilde{V}_{2k+1}(z_0, z_0)$$

where c is a real constant. But the last equality is impossible as the functions $\widetilde{U}_{2k}(*, z_0)$ and $c\widetilde{U}_{2k+1}(*, z_0)$ are the generalized constants of one and the same class $\mathfrak{A}(A, B_k)$, satisfying the conditions

$$\widetilde{V}_{2k}(\infty, z_0) = 1, c\widetilde{V}_{2k+1}(\infty, z_0) = ic$$

and hence they couldn't have the same meanings in any point of the plane.

The equations (2.32) and (2.33) are obviously equivalent to the following equations

$$V_{2k}(z, z_0) = \widehat{c}_{2k,0}U_{2k}(z, z_0) + \widehat{c}_{2k,1}U_{2k+1}(z, z_0), \tag{2.34}$$

$$V_{2k+1}(z, z_0) = \widehat{c}_{2k+1,0}U_{2k}(z, z_0) + \widehat{c}_{2k+1,1}U_{2k+1}(z, z_0), \tag{2.35}$$

where

$$\widehat{c}_{2k,0} = \frac{1}{\Delta_k}c_{2k+1,0}, \widehat{c}_{2k,1} = -\frac{1}{\Delta_k}c_{2k,1}, \widehat{c}_{2k+1,0} = -\frac{1}{\Delta_k}c_{2k+1,0}, \tag{2.36}$$

$$\widehat{c}_{2k+1,1} = \frac{1}{\Delta_k} c_{2k,0},$$

$$\Delta_k \equiv c_{2k,0}c_{2k+1,1} - c_{2k,1}c_{2k+1,0} = [Im(\widetilde{V}_{2k+1}(z, z_0)\overline{\widetilde{V}_{2k}(z, z_0)}]^{-1}.$$

The last equality follows directly from the formulas (2.32),(2.33).

The generalized power functions of the conjugate equation of the equation (2.25)

$$\partial_{\overline{z}} w' - Aw' - \overline{B}\overline{w}' = 0, A, B \in L_{p,2}, p > 2, \tag{2.37}$$

of the lass $\mathfrak{A}(-A, -\overline{B})$ are denoted by U'_k and V'_k, $k = 0, \pm 1, \pm 2, ...$

It is evident, that all relations, established above for the functions U_k and V_k take place for U'_k and V'_k too.

Let us prove the following theorem.

Theorem 2.10 *Let Γ be a piecewise-smooth simple closed curve, surrounding the point $z_0 \neq \infty$. Then the following identities hold*

$$Re \frac{1}{2\pi i} \int_\Gamma U_k(z, z_0) U'_m(z, z_0) dz = I_{k,m}, \tag{2.38}$$

$$Re \frac{1}{2\pi i} \int_\Gamma V_k(z, z_0) V'_m(z, z_0) dz = I_{k,m}, \tag{2.39}$$

where $I_{k,m} = 1 (I_{k,m} = -1)$, if k and m even (odd) numbers and $[\frac{k}{2}] + [\frac{m}{2}] = -1$; if all the remaining cases $I_{k,m} = 0$.

Proof. Denote by $I_{k,m}(U, \Gamma)$ and $I_{k,m}(V, \Gamma)$ the left-hand sides of the identities (2.38) and (2.39). From the Green identity (see [124], chapter 3, §9) it follows that for every $R > 0$

$$I_{k,m}(U, \Gamma) = I_{k,m}(U, \Gamma_R), I_{k,m}(V, \Gamma) = I_{k,m}(V, \Gamma_R), \tag{2.40}$$

where Γ_R is a circle with the radius R and the origin in the point z_0. By virtue of the equalities (2.26) we have

$$I_{k,m}(U, \Gamma_R) = Re \frac{1}{2\pi i} \int_{\Gamma_R} \chi^{(U)}_{k,m}(z, z_0)(z - z_0)^\alpha dz, \tag{2.41}$$

$$I_{k,m}(V, \Gamma_R) = Re \frac{1}{2\pi i} \int_{\Gamma_R} \chi^{(V)}_{k,m}(z, z_0)(z - z_0)^\alpha dz, \tag{2.42}$$

where

$$\chi^{(U)}_{k,m}(z, z_0) = \widetilde{U}_k(z, z_0)\widetilde{U}'_m(z, z_0), \tag{2.43}$$

$$\chi^{(V)}_{k,m}(z, z_0) = \widetilde{V}_k(z, z_0)\widetilde{V}'_m(z, z_0), \tag{2.44}$$

$$\alpha = \left[\frac{k}{2}\right] + \left[\frac{m}{2}\right].$$

The functions $\chi_{k,m}^{(U)}(*, z_0)$ are $\chi_{k,m}^{(V)}(*, z_0)$ the Hölder continuous and are bounded on the whole complex plane.

When $\alpha > -1$ and $\alpha' < -1$ it follows from identities (2.41), (2.42), that

$$lim_{R\to 0} I_{k,m}(U, \Gamma_R) = lim_{R\to 0} I_{k,m}(V, \Gamma_R) = 0$$

and

$$lim_{R\to\infty} I_{k,m}(U, \Gamma_R) = lim_{R\to\infty} I_{k,m}(V, \Gamma_R) = 0$$

respectively.

Therefore, by virtue of (2.40) we get

$$I_{k,m}(U, \Gamma) = I_{k,m}(U, \Gamma) = 0.$$

Let now $\alpha = -1$. Consider three different cases separately.

a) k and m are even numbers. Then from (2.43),(2.44),(2.27), (2.28) we have:

$$\chi_{k,m}^{(U)}(z_0, z_0) = lim_{z\to\infty} \chi_{k,m}^{(V)}(z, z_0) = 1.$$

Taking into account these equations, from (2.41), (2.42) we obtain

$$lim_{R\to 0} I_{k,m}(U, \Gamma_R) = lim_{R\to\infty} I_{k,m}(V, \Gamma_R) = 1.$$

Hence, in the considered case the following identity

$$I_{k,m}(U, \Gamma) = I_{k,m}(V, \Gamma) = 1$$

holds.

b) k and m are odd numbers. Then from (2.26)

$$\chi_{k,m}^{(U)}(z_0, z_0) = lim_{z\to\infty} \chi_{k,m}^{(V)}(z_0, z_0) = -1$$

and as in the above case

$$I_{k,m}(U, \Gamma) = I_{k,m}(V, \Gamma) = -1;$$

c) k and m are numbers with different frequency. In this case

$$\chi_{k,m}^{(U)}(z_0, z_0) = lim_{z\to\infty} \chi_{k,m}^{(V)}(z_0, z_0) = i$$

and therefore

$$lim_{R\to 0} I_{k,m}(U, \Gamma_R) = lim_{R\to o} I_{k,m}(V, \Gamma) = 0.$$

Hence, by virtue of (2.41) we have

$$I_{k,m}(U, \Gamma) = I_{k,m}(V, \Gamma) = 0.$$

The theorem is proved.

In the particular case, when $B=0$ in (2.24) we have

$$U_{2k}(z,z_0)=(z-z_0)^k e^{\omega(z)-\omega(z_0)}, U_{2k+1}(z,z_0)=iU_{2k}(z,z_0);$$
$$V_{2k}(z,z_0)=(z-z_0)^k e^{\omega(z)}, V_{2k+1}(z,z_0)=iV_{2k}(z,z_0);$$
$$U'_{2k}(z,z_0)=(z-z_0)^k e^{\omega(z_0)-\omega(z)}, U'_{2k+1}(z,z_0)=iU'_{2k}(z,z_0);$$
$$V'_{2k}(z,z_0)=e^{-\omega(z)}, V'_{2k+1}(z,z_0)=iV'_{2k}(z,z_0),$$

where $\omega=\frac{1}{p}\iint_{\mathbb{C}}\frac{A(\xi}{\xi-z}d\sigma_\xi$. From, the above identities we get the following formula

$$\frac{1}{2\pi i}\int_\gamma (z-z_0)^\alpha=\begin{cases}1, \text{ if } \alpha=-1\\ 0, \text{ if } \alpha\neq-1.\end{cases}$$

2.6 The problem of linear conjugation for generalized analytic functions

Consider the Carleman-Bers-Vekua equation

$$\frac{\partial w}{\partial\overline{z}}+Aw+B\overline{w}=0, \quad A,B\in L_{p,2}(\mathbb{C}), \quad p>2. \tag{2.45}$$

Let Γ be a smooth closed curve, bounding the finite domain D^+ and the infinite domain D^-. We say that the function w is a *piecewise regular solution* of the equation (2.45) with the boundary curve Γ if w is a regular solution of the equation (2.45) in the domains D^+ and D^- and is continuously extendable from both sides on Γ (with respect to the chosen direction on Γ.)

Let Γ be a smooth closed curve. Let $G(t)$ and $g(t)$ be the given functions of the class $H_\alpha(\Gamma)$, $0<\alpha\leq1$. Moreover $G(t)\neq0$ everywhere on Γ. Consider the problem of linear conjugation for the generalized analytic function:

Find a piecewise regular solution of the equation (2.45) *with the boundary curve* Γ, *satisfying the following conditions:*

$$w^+(t)=G(t)\,w^-(t)+g(t), \quad t\in\Gamma, \tag{2.46}$$

$$w(z)=O(z^N), \quad z\to\infty, \tag{2.47}$$

where N is given integer.

Let $X(z)$ be a canonical solution of the homogeneous linear conjugation problem [99]

$$\varphi^+(t)=G(t)\,\varphi^-(t), \quad t\in\Gamma$$

for the analytic function. Consider the self-conjugate Carleman-Bers-Vekua equations

$$\frac{\partial V}{\partial\overline{z}}+AV+B_1\overline{V}=0, \tag{2.48}$$
$$\frac{\partial V'}{\partial\overline{z}}-AV'-\overline{B}_1\,\overline{V'}=0, \tag{2.49}$$

where $B_1(z) = B(z)\,\dfrac{\overline{X(z)}}{X(z)}$.

Let $\Omega_1(z,t)$, $\Omega_2(z,t)$ be the main kernels of class $\mathfrak{A}_{p,2}(A, B_1, \mathbb{C})$ and let V_k, V_k' be the generalized power functions of the classes $\mathfrak{A}_{p,2}(A, B_1, \mathbb{C})$ and $\mathfrak{A}_{p,2}(-A, -\overline{B}_1, \mathbb{C})$ correspondingly

$$V_{2k}(z) = R_\infty^{A,B_1}(z^k), V_{2k+1}(z) = R_\infty^{A,B_1}(i\,z^k), V_{2k}'(z) = R_\infty^{-A,-\overline{B}_1}(z^k),$$

$$V_{2k+1}'(z) = R_\infty^{-A,-\overline{B}_1}(i\,z^k), \qquad k = 0, 1, 2, \ldots$$

The following theorems are valid

Theorem 2.11 *1) Let $\chi + N \geqslant -1$, where χ is the index of the function $G(t)$ on Γ ($\chi = \dfrac{1}{2\pi}\left[\arg G(t)\right]_\Gamma$). Then the general solution of the problem (2.45), (2.46), (2.47) is given by the formula*

$$w(z) = \frac{X(z)}{2\pi i}\int_\Gamma \Omega_1(z,t)\frac{g(t)}{X^+(t)}\,dt - \Omega_2(z,t)\frac{\overline{g(t)}}{\overline{X^+(t)}}\,d\bar{t} + X(z)\,V_{\chi+N}(z),$$

where $V_{\chi+N}(z)$ is a generalized polynomial of order at most $\chi + N$ of the class $\mathfrak{A}_{p,2}(A, B_1, \mathbb{C})$. It is supposed, that $V_{-1}(z) \equiv 0, \quad z \in \mathbb{C}$.

2) Let $\chi + N \leq -2$. Then the necessary and sufficient solvability conditions for the problem (2.45), (2.46), (2.47) are the following conditions

$$\operatorname{Im}\int_\Gamma v_k'(t)\frac{g(t)}{X^+(t)}\,dt = 0, \qquad k = 0, 1, 2, \ldots, 2(-N-\chi) - 3. \tag{2.50}$$

If the conditions (2.50) are fulfilled then the solution of the problem is given by the formula:

$$w(z) = \frac{X(z)}{2\pi i}\int_\Gamma \Omega_1(z,t)\frac{g(t)}{X^+(t)}\,dt - \Omega_2(z,t)\frac{\overline{g(t)}}{\overline{X^+(t)}}\,d\bar{t}.$$

For detailed analysis see [21], [25], [69].

3 Beltrami equation

In this section the review of important results of B.Bojarski is given. We follow B.Bojarski papers [24], [27] and prove that the quasiregular mappings given by the (normalized) principal solution of the linear Beltrami equation (3.1) and the principal solution of the quasilinear Beltrami equation are inverse to each other. This basic fact is deduced from the Liouville theorem for generalized analytic functions. It essentially simplifies the known proofs of the measurable Riemann mapping theorems and its holomorphic dependence on parameters.

The first global, i.e. defined in the full complex plane $\mathbb{C}$ and expressed by an explicit analytical formula, solution of the Beltrami equation

$$w_{\overline{z}} - q(z)w_z = 0 \tag{3.1}$$

was given by Vekua [124].

In this paper the equation (3.1) is considered with compactly supported $q(z)$, $q(z) \equiv 0$ for $|z| > R$, for some finite R, satisfying the uniform ellipticity condition

$$|q(z)| \leq q_0 < 1, q_0 \text{ constant.} \tag{3.2}$$

Vekua considers the class of solutions of (3.1), represented by the Cauchy complex potential $T\omega$ in the form

$$\omega(z) = -\frac{1}{\pi}\int_C \frac{\omega(z)d\sigma_\zeta}{\zeta - z} + \phi(z) \equiv T\omega + \phi(z) \tag{3.3}$$

where $\omega(\zeta)$ is a complex density, $\omega \in L_p(\mathbb{C}), p > 1$, and $\phi(z)$ is an entire function.

The function $w = w(z)$ is a $W^{1,p}_{loc}(\mathbb{C})$ solution of (3.1) iff the density ω is a solution of the singular integral equation

$$\omega - q(z)S\omega = q(z)\phi'(z) \equiv h(z), \tag{3.4}$$

with the singular integral

$$S\omega = -\frac{1}{\pi}\int_C \frac{\omega(z)}{(\zeta - z)^2}d\sigma_\zeta \tag{3.5}$$

understood in the sense of the Cauchy principal value.

It was probably Vekua who first introduced the singular integral operator S to the study of elliptic equations in the plane (see [27], [32]. It appeared in connection with the study of general boundary value problems, specifically the Poincaré boundary value problem, in the theory of generalized analytic functions, which was defined and developed in Vekua's famous paper [127] (see also [124] and B.Bojarski's Ph.D. dissertation [25], [26]).

Later the operator S was called the Hilbert transform.

The main role of the operator S in the Vekua school was to transform the derivative $w_{\overline{z}}$ into w_z for compactly supported smooth functions $w \in C_0^\infty(\mathbb{C})$,

$$S(w_{\overline{z}}) = w_z = \frac{\partial}{\partial z}T(w_{\overline{z}}).$$

Since for $w \in C_0^\infty(\mathbb{C})$, the entire function $\phi(z)$ in (3.3) reduces to $\phi(z) \equiv 0$, integration by parts then gives $||w_{\overline{z}}||_{L^2} = ||w_z||_{L^2}$ and

$$||Sw_{\overline{z}}||_{L^2} = ||w_z||_{L^2} \tag{3.6}$$

where $||.||_{L^2}$ denotes the L^2 norm of square integrable functions. As a result, S extends as a unitary isometry to the Hilbert space $L^2(\mathbb{C})$.

Lemma 3.1 *For arbitrary measurable dilatation $q(z)$, satisfying (3.2), the integral equation (3.10)(see below) has a unique solution in $L^2(\mathbb{C})$ given by the formula*

$$\omega = (1 - qS)^{-1}h. \tag{3.7}$$

This means that the differential Beltrami equation (3.1) with the compactly supported coefficient $q = q(z)$ has a unique solution in the Sobolev space $W^{1,2}_{loc}(\mathbb{C})$, admitting a holomorphic extension of the form (3.3) outside the support of q.

Lemma 3.2 *For compactly supported C^∞-smooth dilatation $q(z)$ the weak $W^{1,2}_{loc}(\mathbb{C})$ solutions of the Beltrami equation (3.1) are C^∞-smooth.*

The proof of Lemma 3.1 is rather direct, relying on the classical tools of standard potential theory and is described in detail in Vekua's book [124].

By the Calderón-Zygmund theorem [38] the operator S acts also as a bounded operator in $L^p(\mathbb{C})$ for each p, $1 < p < 1$, and its norm $\mathcal{A}_p$ is continuous at $p = 2$. Thus

$$\mathcal{A}_p q_0 < 1 \text{ for } 2 \leq p < 2 + \epsilon \tag{3.8}$$

and the equation $\omega - qS\omega = h$ is uniquely solvable

$$\omega = (I - qS)^{-1}h, \omega \in L^p, \tag{3.9}$$

for any $h \in L_p$ and p satisfying (3.8), what we henceforth assume. In particular, for any measurable dilatation $q(z)$ the L^2 solution ω of equation (3.4) is actually in some $L^p, p > 2$. Thus, in other words, the $W^{1,2}_{loc}(\mathbb{C})$ solutions of (3.1) belong to $W^{1,2}_{loc}(\mathbb{C}), p > 2$. In particular, they are continuous (α-Hölder,$\alpha = 1 - \frac{2}{p} > 0$).

For $\phi(z) \equiv z, h(z) \equiv q(z)$, formula (3.3) gives a particular solution of the Beltrami equation (3.1)

$$w \equiv z - \frac{1}{\pi} \int_C \frac{\omega(\zeta)}{\zeta - z} d\sigma_\zeta, \tag{3.10}$$

where ω is the unique solution of the equation

$$\omega - q(z)S\Omega = q(z). \tag{3.11}$$

We call (3.10) the *principal solution* of the Beltrami equation. A fundamental issue of the theory of elliptic equations and planar quasiconformal mappings was the understanding that the formulae (3.10)-(3.11) give a *univalent solution* of the uniformly elliptic Beltrami equation (3.1)-(3.2) realizing a homeomorphic quasiconformal mapping of the complex plane with the assigned measurable complex dilatation $q(z)$ (the Measurable Riemann Mapping Theorem). The existence of $W^{1,2}_{loc}(\mathbb{C})$ solutions was clear from the outset and the problem essentially reduced to the L^2 isometry of the operator S and the classical properties of the complex potential $T : L^2 \to W^{1,2}_{loc}(\mathbb{C})$, described in [127], [126]. The idea of applying the Calderón-Zygmund theorem [38] and, thus, extending the range of admissible parameters p to the interval $2 - \epsilon < p < 2 + \epsilon$ for some positive ϵ, due to B.Bojarski [21],(see also [124], [24], [27], [32]), immediately allowed us to consider $W^{1,2}_{loc}(\mathbb{C})$ solutions, $p > 2$, and, by the Sobolev imbedding theorems, or classical properties of the complex potentials $T\omega$, α-Hölder continuous solutions with $\alpha = 1 - \frac{2}{p} > 0$.

Proposition 3.3 *The Beltrami equation (3.1) with an arbitrary measurable dilatation $q(z)$, satisfying (3.2) and compactly supported, always admits the solution of the form (3.10) in the Sobolev class $W^{1,2}_{loc}(\mathbb{C})$, $p > 2$. Moreover, the norms $||w_z - 1||_{L_p}$, $||w_{\overline{z}}||_{L_p}$ of this solution are uniformly bounded by quantities, depending only on q_0 in (3.2) and $||q||_{L_p}$ (or the support of $|q|$).*

Not necessarily homeomorphic solutions of the Beltrami equations are known as *quasiregular mappings*. By formulas (3.3) and (3.4) above they are relatively easy to construct. The proof that univalent solutions exist at all, the more so, that the solutions (3.10) are homeomorphisms onto, is much more subtle.

Proposition 3.4 *[27] (see also [124]) 1. If the dilatation $q(z)$ is sufficiently smooth, then the mapping (3.10) is a homeomorphism onto, i.e., it is a quasiconformal mapping of the complex plane.*

2. For arbitrary measurable dilatation $q(z)$, satisfying condition (3.2), the formulae (3.10)-(3.11) realize a quasiconformal mapping of the complex plane with the assigned dilatation almost everywhere.

In view of the approximating procedure, described in [27] it is, obviously, enough to consider the Beltrami equation (3.1) with dilatation $q(z)$ of arbitrary high smoothness (even C^∞). In [27] considered, parallel to equation (3.1), the quasilinear equation for the mappings $z = z(w)$ of the image plane C_w in (3.1) to the source plane C_z

$$\frac{\partial z}{\partial \overline{w}} + q(z)\overline{\frac{\partial z}{\partial w}} = 0. \tag{3.12}$$

We call it the *conjugate (quasilinear) Beltrami equation* (or holomorphic disc equation [40], see section 4). Now we are interested in a particular solution of (3.12) of the form

$$\psi(w) = w + T\widetilde{\omega} \equiv w - \frac{1}{\pi}\int_C \frac{\widetilde{\omega}(\zeta)}{\zeta - w} d\sigma_\zeta \tag{3.13}$$

with $\widetilde{\omega} \in L^p$ for some $p > 2$.

(3.13) is a solution of (3.12) of the Sobolev class $W^{1,2}_{loc}(\mathbb{C})$ iff the complex density $\widetilde{\omega}$ is a solution of the singular integral equation

$$\widetilde{\omega} + \widetilde{q}(w)\overline{S\widetilde{\omega}} = -\widetilde{q}(w) \tag{3.14}$$

with $q(w) \equiv q(\psi(w))$. Hence $\psi(w) - w$ is in the class $W^{1,p}$ and $\psi(w)$ is the solution of the conjugate Beltrami equation

$$\frac{\partial \psi}{\partial \overline{w}} + \widetilde{q}(w)\overline{\frac{\partial \psi}{\partial w}} = 0 \tag{3.15}$$

with $\widetilde{q}(w)$ at least Hölder continuous with exponent $\alpha = 1 - \frac{2}{p} > 0$. In the terminology adopted above the mapping $\psi(w)$ is a quasiregular mapping of the complex plane C_w into the plane C_z.

Considered as an operator equation for the unknown density $\widetilde{\omega}(w)$, (3.14) is a highly nonlinear operator equation. However, its solvability in L^p spaces is easily controlled.

Lemma 3.5 *The quasilinear conjugate Beltrami equation with smooth dilatation $q(z)$ always admits a solution of type (3.13) in some $W^{1,2}_{loc}(\mathbb{C})$. Equivalently, the nonlinear equation (3.14) always admits a solution $\widetilde{\omega}$ in $L_p(\mathbb{C}_w)$ (compactly supported) for some $p > 2$.*

The solution (3.13) of (3.12) is unique.

Lemma 3.6 *Let $w = w(z)$, in $W^{1,2}(\mathbb{C})$ be a (generalized) solution of the equation*

$$w_{\overline{z}} - q(z)w_z = Aw \tag{3.16}$$

with the coefficient $q(z)$: measurable, compactly supported and satisfying uniform ellipticity condition (3.2), and $A \in L^p(\mathbb{C})$ for some $p > 2$. For simplicity assume also that A is compactly supported. If w vanishes at ∞, i.e. $|z||w(z)| < C$ for all z, then $w \equiv 0$.

For details see [21] and [124].

Corollary 3.7 *The conclusion of Lemma 3.6 holds also for mappings $w = w(z)$ in $W^{1,2}(\mathbb{C}), w(\infty) = 0$, satisfying the inequality*

$$|w_{\overline{z}} - q_1(z)w_z - q_2(z)\overline{w_z}| \leq A(z)|w(z)| \tag{3.17}$$

if the coefficients q_1, q_2 have compact support and satisfy the uniform ellipticity condition

$$|q_1(z)| + |q_2(z)| \leq q_0 < 1, q_0\text{-}const. \tag{3.18}$$

and $A \in L_p(\mathbb{C}), p > 2$, vanishes for $|z|$ large enough.

The important concept of generalized analytic function, corresponding to the system (3.16), and discussed in [126] and [124] under the term: generalized constants (or generalized units), is also useful in the global theory of the Beltrami equation (3.1).

Lemma 3.8 *In the conditions of Lemma 3.6 the equation (3.16) has a unique solution, defined in the full complex plane* $v = v(z), z \in \mathbb{C}$, *regular at* $z \to \infty$, *and such that* $v(\infty) = \infty$. *This solution does not vanish for any* $z \in \mathbb{C}$,

$$v(z) \neq 0.$$

Lemma 3.8, as Lemma 3.6 above, could be also referred to [21].

Corollary 3.9 *The derivative* w_z *of the principal solution (3.10) of the Beltrami equation with smooth dilatation* $q(z)$ ($q(z) \in W^{1,p}, p > 2$, *is enough) is a generalized constant for equation (3.16). In particular,*

$$w_z \equiv 1 + S\omega \neq 0, \text{ for all } z \in \mathbb{C} \tag{3.19}$$

Corollary 3.7 immediately implies the following

Proposition 3.10 *In the conditions of Proposition 3.4 the principal (quasiregular) solution (3.10) is a local homeomorphism.*

In [124] Vekua deduced Proposition 3.4 from Proposition 3.10 by appealing to the "argument principle" for local homeomorphisms of the complex plane. It was also well known that the *monodromy theorem* for open mappings of the Riemann sphere S^2 or the closed plane $\widehat{\mathbb{C}}$ may also be used to deduce Proposition 3.3 from Proposition 3.10.

Let us now consider the Beltrami equation (3.1) with a smooth compactly supported dilatation $q(z)$ and the conjugate Beltrami quasilinear equation (3.12).

Lemma 3.11 *Let* $\chi = \chi(z)$ *be the normalized (principal) solution (3.10) of equation (3.1) and* $\psi = \psi(w)$ *the principal solution (3.13) of the quasilinear equation (3.12). Consider the composed mappings*

$$\widetilde{\phi}(w) = \chi \circ \psi(w), \widetilde{\phi} : \mathbb{C}_w \to \mathbb{C}_w, \ \phi(w) = \chi \circ \psi(w), \phi : \mathbb{C}_w \to \mathbb{C}_w, \tag{3.20}$$

Then $\widetilde{\phi} = \widetilde{\phi}(w)$ *is a solution of the Cauchy-Riemann equation*

$$\frac{\partial \widetilde{\phi}}{\partial \overline{w}} = 0 \tag{3.21}$$

and ϕ *satisfies the inequality*

$$|\phi_{\overline{z}} - \widetilde{q}(z)(\phi_z - \overline{\phi}_z)| \leq A(z)|\phi(z) - z| \tag{3.22}$$

with a bounded, compactly supported function $A(z)$ *and*

$$\widetilde{q}(z) \equiv \frac{q(z)}{1 + |q(z)|^2}.$$

Lemma 3.11 and its proof have a nice geometric interpretation in terms of Lavrentiev fields (characteristics), see below, and also [34] and [124], [108].

Proposition 3.12 *The normalized solutions (3.10) and (3.13) of the smooth Beltrami equation and the conjugate quasilinear equation are homeomorphisms of the complex planes $C_z \to C_w$ inverse to each other, i.e. the formulas hold*

$$\chi(w) \circ \psi(w) \equiv w \text{ and } \psi \circ \chi(z) \equiv z. \tag{3.23}$$

Corollary 3.13

$$J_\chi \cdot J_\psi \equiv 1 \tag{3.24}$$

where $J_\chi = |\chi_z|^2 - |\chi_{\overline{z}}|^2$ and $J_\psi = |\psi_w|^2 - |\psi_{\overline{w}}|^2$ are the Jacobians. In particular,

$$J_\chi \neq 0 \text{ and } J_\psi \neq 0 \tag{3.25}$$

at every point. Actually $J_\chi \geq c_\chi > 0$ and $J_\psi \geq c_\psi > 0$ for positive constants (in general, dependent on the mapping).

(3.25) is also a direct consequence of Lemma 3.1 above.

The work of Vekua and his school on the solutions of the Beltrami equation yielded much more than the previous methods due to Lichtenstein [79], Lavrentiev [75],[76] or Morrey [98], where, in various forms, the Riemann mapping theorem for QC-maps was proved (see [24], [33].

The explicit representation formulas of Vekua's school and related a priori estimates for global mapping problems, created a powerful and flexible tool and a method to attack many local and global problems, inaccessible in any preceding theory. The study of quasiconformal extensions of holomorphic univalent functions and of the theory of deformations of planar quasiconformal mappings is hardly conceivable without these tools. They serve as a solid foundation for the development of important applications of the theory inside as well as outside the planar elliptic partial differential equations theory. The long list of the first ones starts with the deep results of Vinogradov and Danilyuk on basic boundary value problems for general elliptic equations and generalized analytic functions described in Vekua's monograph [124]. For the latter, i.e. applications outside the genaralized analytic function, it is enough to mention the deep and beautiful ideas and constructions of the Ahlfors-Bers school in the theory of Teichmüller spaces, moduli spaces and Kleinian groups or the results in complex holomorphic dynamics [2] (the 2006 edition).

It is necessary to stress here that the explicit formulas (3.10) and (3.19) written in the form

$$w_{\overline{z}} = \omega = (1 - qS)^{-1}q$$

and

$$w_z - 1 = S\omega = S(1 - qS)^{-1}q$$

show that the derivatives $w_{\overline{z}}$ and w_z of the principal solution (3.10) depend holomorphically, in the general functional sense, on the complex dilatation q. This functional dependence, naturally, implies that, if the dilatation $q(z)$ itself depends on some parameters t, holomorphically, real analytically, smoothly or just continuously, then the principal solutions (3.10) depend holomorphically, smoothly... . etc., as the case may be, on these parameters.

In [124] the existence of homeomorphic solutions of the complex Beltrami equation is also discussed in the compactified complex plane $\widehat{\mathbb{C}}$, identified with the Riemann sphere S^2. In this case, for the general measurable dilatation satisfying only the condition (3.2), the homeomorphic solution cannot be in general represented by formula (3.10). However, as shown in [124], the principal homeomorphism can be constructed by the composition of two homeomorphisms of type (3.10), obtained by splitting the complex dilatation $q(z) = q_1 + q_2$ with $q_1(z)$ and $q_2(\frac{1}{z})$ compactly supported, and a simple natural change of variables.

Besides, the behaviour of the complex dilatation $q_w = \frac{w_{\overline{z}}}{w_z}$ under composition of quasiconformal mappings $f = w \circ v^{-1}$ is discussed in [21] and the simple, but important, formula

$$q_f = \{\frac{q_w - q_v}{1 - \overline{q}_v q_w}\frac{v_z}{\overline{v_z}}\} \circ v^{-1}$$

appears and is used, at some crucial points, in [21].

Consider the convex set $\sum$ of mappings of the form (3.13), parametrized by the densities $\widetilde{\omega} \in L^p(\mathbb{C}_w)$ for some fixed admissible $p > 2$. For $z = z(w) \in \sum$ consider the principal solution $\psi(w)$ of the conjugate linear Beltrami equation

$$\frac{\partial \psi}{\partial \overline{w}} + \widetilde{q}(z)\overline{\frac{\partial \psi}{\partial w}} = 0. \tag{3.26}$$

with $\widetilde{q}(w) \equiv q(z(w))$

This defines the nonlinear map $\psi = F(z)$ of $\sum$ into $\sum$. Since (3.26) is again a Beltrami equation in the w-plane, with the same uniform ellipticity estimate as (3.1), Lemma 3.1 and Proposition 3.3 hold and a priori estimates follow. Hence F is compact and the fixed point of F is the required solution of the quasilinear equation (3.12).

Remarks. The concept of the principal solution of form (3.10) or its slight generalization

$$w(z) = az - \frac{1}{\pi}\int_C \frac{\omega(z)}{\zeta - z} d\sigma_\zeta, \quad a\text{-complex constant}, \tag{3.27}$$

is meaningful for the general Beltrami equation

$$w_{\overline{z}} + q(z)w_z - q_1(z)\overline{w}_z = 0 \tag{3.28}$$

with the uniform ellipticity condition

$$|q(z)| + |q_1(z)| \leq q_0 < 1, \quad q_0\text{- const.} \tag{3.29}$$

These equations correspond to Lavrentiev's quasiconformal mappings [75], [76], with "two pairs of characteristics" [124], [125], [28], [33], and in Vekua's school they have been considered from the outset [28], [27], [124], [103].

The infinitesimal geometric meaning of a differentiable transformation $w = w(z)$ at a point z_0 is defined by the linear tangent map

$$Dw(z)(\xi) = w_z(z_0)\xi + w_{\overline{z}}\overline{\xi} \tag{3.30}$$

It transforms ellipses in the tangent plane at z_0 into ellipses in the tangent plane at the image point $w(z_0)$.

Ellipses centred at z are defined up to a similarity transformation by the ratio $p \geq 1$ of their semiaxes and, if $p > 1$, the angle $\theta \mathtt{mod} \pi$ between major axis and the positive z-axis, and denoted by $\mathcal{E}(p, \theta, z)$ or $\mathcal{E}_h(p, \theta, z)$ where h is the length of the minor axis. The pair (p, θ) is called the characteristic of the infinitesimal ellipse, and the family $\mathcal{E}_h(p, \theta, z), h > 0, z \in G$, is a field of infinitesimal ellipses (Lavrentiev field). A homeomorphism $w = w(z)$ is said to map the infinitesimal ellipse $\mathcal{E}(p, \theta, z)$ onto $\mathcal{E}(p_1, \theta_; z)$ if the tangent map $Dw(z)$ transforms $\mathcal{E}(p, \theta; z)$ onto $\mathcal{E}(p_1, \theta_1; z)$.

Analytically this is described in terms of the components $w_{\overline{z}}$ and w_z in the tangent map Dw (3.30) by the general Beltrami equation (3.28) where the coefficients q and q_1 are determined by the invertible formulas

$$q(z) = -\frac{p - p^{-1}}{p + p^{-1} + p_1 + p_1^{-1}} e^{2i\theta}, q_1(z) = -\frac{p_1 - p_1^{-1}}{p + p^{-1} + p_1 + p_1^{-1}} e^{2i\theta_1}. \tag{3.31}$$

In particular, the solutions of the Beltrami equation (3.1) ($q_1 \equiv 0$) map the field of ellipses $\mathcal{E}(p_1, \theta_; z)$ into infinitesimal circles ($p_1 \equiv 1$) whereas the conjugate Beltrami equations (3.12), (3.15) map the infinitesimal discs ($p \equiv 1$) into ellipses ($p_1 \geq 1$).

The density $\omega(\zeta)$ of the principal solution (3.27) satisfies the singular integral equation

$$\omega - qS\omega - q_1\overline{S}\omega = aq + \overline{a}q_1 \tag{3.32}$$

which is uniquely solvable and its L^2 solutions are necessarily in L_p for some $p > 2$.

Proposition 3.14 *The equation (3.28) has always a unique principal solution of the form (3.27). If $a \neq 0$ then the principal solution realizes a homeomorphic quasiconformal mapping of the full complex plane $\overline{\mathbb{C}}$.*

For $a = 0$ the principal solution is identically $\equiv 0$ (Liouville theorem).

4 The pseudoanalytic functions

The pseudoanalytic functions in the sense of Bers are defined by generating pairs (F, G), where complex functions F and G satisfy the Hölder condition and the inequality

$$Im(\overline{F}G) > 0. \tag{4.1}$$

Every complex function w can be expressed in the following form

$$w(z) = \varphi(z)F(z) + \psi(z)G(z),$$

where φ and ψ are real functions. The (F, G)-derivative $\dot{w}(z_0)$ at the point $z_0 \in U$ is defined by

$$\dot{w}(z_0) \equiv \frac{d_{(F,G)}w(z_0)}{dz} := lim_{z\to z_0} \frac{w(z) - \varphi(z_0)F(z) - \psi(z_0)G(z)}{z - z_0}.$$

A function w for which the above limit exists for all $z_0 \in U$ is called an (F, G)-pseudoanalytic function or *pseudoanalytic* in the sense of Bers in U.

Connection between the coefficients of the equation (1.6) and the generating pair is carried out by the following identities

$$A = \frac{\overline{F}\partial_{\overline{z}}G - \overline{G}\partial_{\overline{z}}F}{F\overline{G} - G\overline{F}}, B = \frac{G\partial_{\overline{z}}F - F\partial_{\overline{z}}G}{F\overline{G} - G\overline{F}}. \tag{4.2}$$

The theory of pseudo analytic functions bears the same relationship to the general theory of elliptic equations as the theory of analytic functions does to that of the Couchy-Riemann equation. This follows essentially from the following theorem:

Theorem 4.1 *The function $w(z) = \varphi(z)F(z) + \psi(z)G(z)$, possessing continuous first partial derivatives, is (F, G)-pseudo-analytic if and only if $\frac{1}{2}(w_x + iw_y) \equiv w_{\overline{z}} = Aw + B\overline{w}$ where as above*

$$A = \frac{\overline{F}\partial_{\overline{z}}G - \overline{G}\partial_{\overline{z}}F}{F\overline{G} - G\overline{F}}, B = \frac{G\partial_{\overline{z}}F - F\partial_{\overline{z}}G}{F\overline{G} - G\overline{F}}.$$

that is, if and only if $F\varphi_z + G\psi_{\overline{z}} = 0$.

The (F, G)-integral of a continuous function $W(z)$, taken over a rectifiable arc Γ leading from z_0 to z_1, is defined by

$$\int_\Gamma W d_{(F,G))} z = F(z_1) Re \int_\Gamma \frac{\overline{G}W}{F\overline{G} - \overline{F}G} dz - G(z_1) \int_\Gamma \frac{\overline{F}W}{F\overline{G} - \overline{F}G} dz.$$

Theorem 4.2 *A continuous function is (F, G)-integrable (i.e., its (F, G)-integral vanishes over all closed curves, homologous to zero in U) if and only if it is an (F, G)-derivative.*

By definition, a generating pair (F_1, G_1) is called a *successor* of (F,G) if (F, G)-derivatives are (F_1, G_1)-pseudo-analytic, and (F_1, G_1)-pseudo-analytic functions are (F, G)-integrable. (F, G) is then called a *predecessor* of (F_1, G_1).

Proposition 4.3 *Any generating pair $(F, G) \equiv (F_0, G_0)$ may be imbedded in a generating sequence of pairs (F_ν, G_ν), such that each (F_ν, G_ν), $\nu - 0, \pm 1, \pm 2, ...$, is a successor of $(F_{\nu+1}, G_{\nu+1})$.*

With respect to such a generating sequence, it may be shown that there are certain pseudo-analytic functions, called (global) *formal powers*, denoted by $Z_\nu^{(r)}(a, z_0; z)$, which have like the usual powers $a(z - z_0)^r$, where r is a real rational number and a and z_0 are complex numbers.

In terms of these formal powers, numerous results analogous to those of function theory, e.g. Taylor and Laurent expansions for single and multiple valued functions, Runge's approximation theorem, etc., can be established.

If (F, G) is a normal generating pair (see [11] pp.65 or [124], §6) then the space of pseudoanalytic functions coincides with the space of the generalized analytic functions, obtained from the regular equation (1.6) whose coefficients are evaluated from the correspondence (4.2). Note that to pass from coefficients of the Carlemann-Bers-Vekua equation to the generating pair is a nontrivial problem, because it is necessary to solve a system of nonlinear integral equations (4.2). Only the existence theorem is known (see [11], theorem 16.1). It means that for a normal generating pair there exists a regular equation of type (1.6). Only in some particular cases it is possible to construct corresponding generating pairs exactly [73].

Proposition 4.4 *[11] If (F, G) is a generating pair, then the functions F and G are solutions of the equation (1.6) with coefficients defined by (4.2).*

This proposition follows from the formulas for partial derivatives of the functions F and G with respect to z and $\overline{z}$, respectively:

$$\partial_{\overline{z}} F = -AF - B\overline{F}, \partial_{\overline{z}} G = -AG - B\overline{G}.$$

From this it follows that there always exists the space of pseudoanalytic functions, containing a given admissible function; moreover it is possible to construct the spaces $\mathfrak{A}_{p,2}(A, B, U)$ and $\mathfrak{A}_{p,2}(A_1, B_1, U)$ with nonempty intersection. Indeed, if (F, G_1) ie the generating pair with characteristic coefficients ([11], p.5) A and B and (F, G_2) is other generating pair not equivalent ([11], p.37) to (F, G_1) with characteristic coefficients A_1, B_1, then F is a common element of the spaces $\mathfrak{A}_{p,2}(A, B, U)$ and $\mathfrak{A}_{p,2}(A_1, B_1, U)$.

Proposition 4.5 *Let f be a real value positive function. $(f, \frac{i}{f})$-pseodoanalytic functions of first kind are p-analytic functions with $p = f^2$ and corresponding pseodoanalytic function of second kind satisfies the Beltrami equation with coefficient $\frac{f^2+1}{f^2-1}$.*

Proposition 4.6 *If Q is a positive real valued function, then $(Q, i/Q)$ defines (1.10) type homogeneous*

$$\frac{\partial w}{\partial \overline{z}} = \frac{Q_{\overline{z}}}{Q}\overline{w}$$

equation.

Proposition 4.7 *The generating pair of the equation (1.8) is (e^{-Q}, ie^{-Q}), where Q is $\frac{\partial}{\partial \overline{z}}$-primitive of A.*

Proofs of this propositions follows from formula (4.2).

4.1 Relation between Beltrami and holomorphic disc equations

In this section we give detailed analysis of the theory of pseudoanalytic functions in the light of Beltrami equation and holomorphic disc equation and prove the equivalence of these equations.

Let (F, G) be a normalized generating pair on complex space $\mathbb{C}$ [11] it means that 1) $F, G \in C_{\frac{p-2}{p}}, p > 2$; 2)$F_{\overline{z}}, G_{\overline{z}} \in L_{p,2}(\mathbb{C}) \cap C_\beta$, $0 < \beta < 1$; 3)$Im(\overline{F}(z)G(x)) \geq K_0 > 0$, $K_0 = const$, $z \in \mathbb{C}$. As above, every function W, at every point, is uniquely represented by $F(z), G(z)$ in the following form

$$W(z) = \varphi(z)F(z) + \psi(z)G(z), \tag{4.3}$$

where φ, ψ are real functions.

Let $W(z)$ be (F, G)-pseudoanalytic in $\mathbb{C}$, then it is known that $W(z)$ is the solution of the Carlemann-Bers-Vekua equation

$$W_{\overline{z}} = AW + B\overline{W}, \tag{4.4}$$

where A and B may be calculated by formula (4.2).

From the pseudoanalyticity it follows also, that there exist continuations of the partial derivatives $\varphi_z, \varphi_{\overline{z}}, \psi_z, \psi_{\overline{z}}$ and

$$F\varphi_{\overline{z}} + G\psi_{\overline{z}} = 0.$$

Consider the function

$$\omega(z) = \varphi(z) + i\psi(z).$$

Then

$$2(F\varphi_{\overline{z}} + G\psi_{\overline{z}}) = (F - iG)(\varphi_{\overline{z}} + i\psi_{\overline{z}}) + (F + iG)(\varphi_{\overline{z}} - i\psi_{\overline{z}}) =$$

$$= (F - iG)(\varphi + i\psi)_{\overline{z}} + (F + iG)(\varphi - i\psi)_{\overline{z}} = (F - iG)\omega_{\overline{z}} + (F + iG)\overline{\omega}_{\overline{z}} = 0.$$

Hence it follows, that

$$\omega_{\overline{z}}(F - iG) + \overline{\omega_z}(F + iG) = 0. \tag{4.5}$$

Lemma 4.8 $F(z) - iG(z) \neq 0$.

Indeed,

$$|F(z) - iG(z)|^2 = (F(z) - iG(z)(\overline{F(z) - iG(z)}) = (F(z) - iG(z)(\overline{F(z)} + i\overline{G(z)}) =$$

$$= |F(z)|^2 + |G(z)|^2 - i(\overline{F(z)}G(z) - F(z)\overline{G(z)}) =$$

$$= |F(z)|^2 + |G(z)|^2 + 2Im(\overline{F(z)}G(z)) \geq 2K_0 > 0, \tag{4.6}$$

when $|F(z)|^2 > 0, |G(z)|^2 > 0, Im(\overline{F(z)}G(z)) \geq K_0$ for every $z \in \mathbb{C}$. The lemma is proved.

From lemma 4.8 and (4.5) it follows, that

$$\Rightarrow \omega_{\overline{z}} + \overline{\omega_z}\frac{F + iG}{F - iG} = 0. \tag{4.7}$$

Denote by $q(z) = -\frac{F(z)+iG(z)}{F(z)-iG(z)}$.

Lemma 4.9 $|q(z)| \leq q_0 < 1$, $z \in \mathbb{C}$.

Step 1.

$$|q(z)|^2 = \frac{|F(z) + iG(z)|^2}{|F(z) - iG(z)|^2} = \frac{(F(z) + iG(z))\overline{(F(z) + iG(z))}}{(F(z) - iG(z))\overline{(F(z) - iG(z))}} \Rightarrow$$

$$\Rightarrow \frac{|F(z)|^2 + |G(z)|^2 - 2Im(\overline{F(z)}G(z))}{|F(z)|^2 + |G(z)|^2 + 2Im(\overline{F(z)}G(z))} < 1, \tag{4.8}$$

when $Im(\overline{F(z)}G(z)) \geq K_0 > 0$, $z \in \mathbb{C}$.

Step 2. The function F, G satisfies Carlemnan-Bers-Vekua equation

$$F_{\overline{z}} = aF + b\overline{F}, G_{\overline{z}} = aG + b\overline{G}, \tag{4.9}$$

when $F \in C_{\frac{p-1}{p}}(\mathbb{C})$, $a, b \in L_{p,2}(\mathbb{C})$ we obtain $aF + b\overline{F} \in L_{p,2}(\mathbb{C})$. From (4.9) it follows,that

$$F(z) = \Phi(z) + T_{\mathbb{C}}(aF + b\overline{F})(z), \tag{4.10}$$

where $\Phi(z)$ is an entire function. From $F(z), T_{\mathbb{C}}(aF + b\overline{F})(z) \in C_{\frac{p-2}{p}}(\mathbb{C})$, it follows that $\Phi(z) \in C_{\frac{p-2}{p}}(\mathbb{C})$. By Liouville theorem we obtain $\Phi(z) = const$, therefore $\Phi(z) = C, z \in \mathbb{C}$. From this and (4.10) we obtain

$$F(z) = C + T_{\mathbb{C}}(aF + b\overline{F})(z). \tag{4.11}$$

When $T_{\mathbb{C}}(aF + b\overline{F})(\infty) = 0$, from (4.11) it follows, that $F(\infty) = C$. In a similar way we obtain $G(\infty) = C_1$.

When $Im(\overline{F(z)}G(z)) \geq K_0$, therefore

$$Im(\overline{F(\infty)}G(\infty)) \geq K_0 \tag{4.12}$$

and from (4.8) and (4.12) we obtain

$$\Rightarrow |q(\infty)|^2 = \frac{|F(\infty)|^2 + |G(\infty)|^2 - 2Im(\overline{F(\infty)}G(\infty))}{|F(\infty)|^2 + |G(\infty)|^2 + 2Im(\overline{F(\infty)}G(\infty))} < 1. \tag{4.13}$$

From (4.8) and (4.13) it follows, that

$$|q(z)| < 1, z \in \mathbb{C}, |q(\infty)| < 1,$$

therefore $|q(z) \leq q_0 < 1$, $z \in \mathbb{C}$.

Proposition 4.10 *There exists the function $\widetilde{q}(z)$, such that ω is the solution of Beltrami equation with the coefficient $\widetilde{q}(z)$.*

Introduce the function $\widetilde{q}(z)$:

$$\widetilde{q}(z) = \begin{cases} q(z)\dfrac{\overline{\partial_z \omega}}{\partial_z \omega}, \text{when } \partial_z \omega \neq 0, \\ \qquad 0, \text{when } \partial_z \omega = 0. \end{cases} \tag{4.14}$$

and consider the equation

$$\partial_{\overline{z}}\omega - q(z)\frac{\overline{\partial_z \omega}}{\partial_z \omega} = 0.$$

From (4.14) itfollows that ω satisfies the equation

$$\partial_{\overline{z}}\omega - \widetilde{q(z)}\partial_z \omega = 0. \tag{4.15}$$

It is clear, that

$$|\widetilde{q(z)}| = |q(z)\frac{\overline{\partial_z \omega}}{\partial_z \omega}| = |q(z)||\frac{\overline{\partial_z \omega}}{\partial_z \omega}| = |q(z)| \leq q_0 < 1. \tag{4.16}$$

From (4.15) and (4.16) it follows, that $\omega(z)$ is solution of the Beltrami equation

$$\partial_{\overline{z}}h - \widetilde{q(z)}\partial_z h = 0. \tag{4.17}$$

In the area $U \subset \mathbb{C}$ the function ω represented as $\omega(z) = \Psi(W(z))$, where $W(z)$ is complete homeomorphism of the equation (4.17) and $\Psi(\zeta)$ analytic on $W(U)$ function.

4.2 The periodicity of the space of generalized analytic functions

Let $F(z), G(z)$ be two complex valued Hölder continuous functions, defined in some domain such that $Im(\overline{F}G) > 0$. A function $w = \phi F + \psi G$, where ϕ and ψ are real,is called (F, G) pseudo-analytic, if $\phi_{\overline{z}}F + \psi_{\overline{z}}G = 0$. The function $\dot{w} = \phi_z F + \psi_z G$ is called the (F, G) derivatives of w. Every generating pair (F, G) has a successor

(F_1, G_1) such that (F, G) derivatives are (F_1, G_1) pseudo-analytic. The successor is not uniquely determined. A generating pair (F, G) is said to have *minimum period* n if there exists generating pairs (F_i, G_i) such that $(F_0, G_0) = (F, G)$, (F_{i+1}, G_{i+1}) is a successor of (F_i, G_i) and $(F_n, G_n) = (F_0, G_0)$. If such n does not exist, (F, G) is said to have minimum period ∞.

It is known, that w is pseudonalytic iff w satisfies the following Carleman-Bers-Vekua equation

$$w_{\overline{z}} = aw + b\overline{w}, \tag{4.18}$$

where the function $a(z, \overline{z}), b(z, \overline{z})$ expressed by the generating pair (F, G) through the following identity (see also (4.2))

$$a = \frac{\overline{G}F_{\overline{z}} - \overline{F}G_{\overline{z}}}{F\overline{G} - \overline{F}G}, b = \frac{FG_{\overline{z}} - GF_{\overline{z}}}{F\overline{G} - \overline{F}G}. \tag{4.19}$$

Define also the the quantities

$$A = \frac{\overline{G}F_z - \overline{F}G_z}{F\overline{G} - \overline{F}G}, B = \frac{FG_z - GF_z}{F\overline{G} - \overline{F}G}. \tag{4.20}$$

The (F, G)-derivative $\dot{w}$ satisfies the following Carleman-Bers-Vekua equation

$$\dot{w}_{\overline{z}} = a\dot{w} - B\overline{\dot{w}} \tag{4.21}$$

The functions a, b, A, B are called the characteristic coefficients of the generating pair (F, G).

Proposition 4.11 *[104] Given functions a, b, A, B are characteristic coefficients of the generating pair if and only if they satisfy the system of differential equations*

$$A_{\overline{z}} = a_z + b\bar{b} - B\overline{B}, \quad B_{\overline{z}} = b_z + (\overline{a} - A)b + (a - \overline{A})B. \tag{4.22}$$

Proposition 4.12 *[104] 1) The space $\Omega(a, b)$ have period one iff there exist a function A_0 satisfying the equation*

$$A_{0\overline{z}} = a, \quad (A_0 - \overline{A}_0) = \overline{a} - a + \frac{1}{b}(b_z + b_{\overline{z}}) \tag{4.23}$$

2)The space $\Omega(a, b)$ have period two iff there exist the functions A_0, A_1, B_0 satisfying the system of equations

$$A_{0\overline{z}} = a_z + b\bar{b} - B_0\overline{B}_0, \quad B_{0\overline{z}} = b_z + (\overline{a} - A_0)b + (a - \overline{A}_0)B_0, \tag{4.24}$$

$$A_{1\overline{z}} = a_z + b\bar{b} - B_0\overline{B}_0, \quad B_{0z} = b_{\overline{z}} + (A_1 - \overline{a})B_0 + (\overline{A}_1 - a)b. \tag{4.25}$$

Proposition 4.13 *Let (F, G) be a generating pair of (4.18), then the generating pair of the adjoint equation*

$$w_{\overline{z}} = -aw - \overline{B}\overline{w}, \tag{4.26}$$

is

$$F^* = \frac{2\overline{G}}{F\overline{G} - \overline{F}G}, \quad G^* = \frac{2\overline{F}}{F\overline{G} - \overline{F}G}, \tag{4.27}$$

We prove, that the characteristic coefficients, induced from adjoint generating pair (F^*, G^*), are equal to $-a$ and $-\overline{B}$.

Indeed,

$$\frac{\overline{G}^* F^*_{\overline{z}} - \overline{F}^* G^*_{\overline{z}}}{F^* \overline{G}^* - \overline{F}^* G^*} = \frac{\frac{2F}{\overline{D}}(\frac{2\overline{G}}{D})_{\overline{z}} - \frac{2G}{\overline{D}}(\frac{2\overline{F}}{D})_{\overline{z}}}{\frac{2\overline{G}}{D}\frac{2F}{\overline{D}} - \frac{2G}{\overline{D}}\frac{2\overline{F}}{D}} = \frac{\frac{4F}{\overline{D}}(\frac{\overline{G}_{\overline{z}}}{D} - \frac{\overline{G}}{D^2}D_{\overline{z}}) - \frac{4G}{\overline{D}}(\frac{\overline{F}_{\overline{z}}}{D} - \frac{\overline{F}}{D^2}D_{\overline{z}})}{\frac{4}{D\overline{D}}(F\overline{G} - \overline{F}G)} =$$

$$= \frac{F\overline{G}_{\overline{z}} - G\overline{F}_{\overline{z}}}{D} - \frac{F\overline{G} - \overline{F}G}{D^2}D_{\overline{z}}, \tag{4.28}$$

where $D = F\overline{G} - \overline{F}G, D = -\overline{D}, D_{\overline{z}} = F_{\overline{z}}\overline{G} + F\overline{G}_{\overline{z}} - \overline{F}_{\overline{z}}G - \overline{F}G_{\overline{z}}$. From (4.28) we have

$$a_1 = \frac{-F_{\overline{z}}\overline{G} - F\overline{G}_{\overline{z}} - \overline{F}_{\overline{z}}G - \overline{F}G_{\overline{z}} + \overline{F}_{\overline{z}}G + \overline{F}G_{\overline{z}}}{D} = -\frac{F_{\overline{z}}\overline{G} - \overline{F}G_{\overline{z}}}{D} \Longrightarrow a = -a_1$$

Analogously as above

$$b_{1(F^*,G^*)} = \frac{F^* G^*_{G_{\overline{Z}}} - G^* F^*_{\overline{z}}}{F^* \overline{G}^* - \overline{F}^* G^*} = \frac{\frac{G}{\overline{D}}\overline{G}(\frac{\overline{F}}{D})_{\overline{z}} - \frac{G}{\overline{D}}\overline{F}(\frac{\overline{G}}{D})_{\overline{z}}}{\frac{G}{\overline{D}}} =$$

$$= \frac{\overline{D}}{D}(\frac{\overline{GF_{\overline{z}}}}{D} - \frac{\overline{GF}}{D^2}D_{\overline{z}} - \frac{\overline{FG_{\overline{z}}}}{D} + \frac{\overline{GF}}{D^2}D_{\overline{z}}) = -\frac{\overline{GF_{\overline{z}}} - \overline{FG_{\overline{z}}}}{D},$$

therefore

$$\bar{b}_1 = -\frac{GF_z - FG_z}{D} \Longrightarrow b_1 = -\overline{B}.$$

By definition [11] the pseudo-analytic functions corresponding to (4.18) satisfy the following holomorphic disc equation

$$\omega_{\overline{z}} = q(z)\overline{\omega}_z, \quad \text{where} \quad q(z) = \frac{F + iG}{F - iG} \tag{4.29}$$

Proposition 4.14 *[37] Holomorphic disc equation, corresponding to (4.21) is*

$$\omega_{\overline{z}} = -\overline{q(z)}\overline{\omega}_z.$$

Indeed, coefficient of holomorphic disc equation, corresponding to (4.21) expressed by the generating pair (F^*, G^*) of (4.21) as

$$q_1 = \frac{F^* + iG^*}{F^* - iG^*} \Rightarrow q_1 = \frac{\frac{2\overline{G}}{D} + i\frac{2\overline{F}}{D}}{\frac{2\overline{G}}{D} - i\frac{2\overline{F}}{D}} = \frac{\overline{G} + i\overline{F}}{\overline{G} - i\overline{F}} \Rightarrow q_1 = -\overline{q}.$$

Proposition 4.15 *[37] If system (4.18) has the period one, then system (4.21) also has period one.*

The proof immediately follows from the proof of the preceding proposition.

Proposition 4.16 *The generating pair of the space $\Omega(a,0)$ is (f,if), where $f \neq 0$ and is solution of the equation $f_{\overline{z}} = -af$.*

Indeed,

$$Im(\overline{f}if) = i|f|^2; (F\overline{G} - \overline{F}G) = f(-i\overline{f}) - \overline{f}(if) = -2i|f|^2.$$

$$a_{(f,if)} = \frac{\overline{f}if_{\overline{z}} - f_{\overline{z}}(-i\overline{f})}{-2i|f|^2} = -\frac{f_{\overline{z}}}{f}, b_{(f,if)} = \frac{fif_{\overline{z}} - f_{\overline{z}}if}{-2i|f|^2} = 0.$$

Consider the particular cases of this theorem. When f is constant, or is complex analytic, we obtain the space of holomorphic functions $\Omega(0,0)$.

Proposition 4.17 *If f is real and $f \neq 0$, then $(f, \frac{i}{f})$ generates the space $\Omega(0,b)$.*

The proof is obtained from direct calculation:

$$Im(f\frac{i}{f}) = 1 > 0, \text{ because } \overline{f} = f; a_{(f,\frac{i}{f})} = \frac{-f(\frac{if_{\overline{z}}}{f^2}) - f_{\overline{z}}(-\frac{i}{f})}{-2i} = 0;$$

$$b_{(f,\frac{i}{f})} = -\frac{-f(\frac{if_{\overline{z}}}{f^2}) - f_{\overline{z}}(\frac{i}{f})}{-2i} = \frac{f_{\overline{z}}}{f};$$

Proposition 4.18 *[37] From $\omega \in \Omega(a,0)$ it follows, that $\dot{\omega} \in \Omega(a,0)$.*

By proposition 4.16 the generating pair of the space $\Omega(a,0)$ is $(f, \frac{i}{f})$. The function $\dot{\omega}$ satisfies the equation (4.21), therefore it is necessary to calculate B from (4.20). It is easy, that $B = 0$.

In case, when the functions F, G are complex analytic, then from (4.19) it follows, that we obtain the space of holomorphic functions $\Omega(0,0)$, but this space is not "isomorphic" to the space of holomorphic functions generated by the pair $(1,i)$, because it follows from (4.20), B isnot equal to zero. From this follows, that this space has period $N > 1$. It is shown [104], that period of this space is equal to 2.

Proposition 4.19 *[104] 1)There exists real analytic function b in the a neighborhood of the origin, such that the space $\Omega(0,b)$ has minimum period infinity.*

2) For each positive integer N there exists a real analytic function b in the neighborhood of the origin, such that the space $\Omega(0,b)$ has minimum period N.

L. Bers obtained the necessary and sufficient condition that $\Omega(a,b)$ generated by (F,G) has the period one and proved that this condition is identity $\frac{F}{G} = \tau(y)$. L. Bers also proved, that if $\frac{F}{G} = \sigma(x)$, then the minimum period is at most two.

Remark. A. Markushevich observed that every system of linear partial differential equations

$$c_i u_x + d_i v_x = a_i u_y + b_i v_y, \ i = 1,2 \tag{4.30}$$

with sufficiently smooth coefficients $a_1(x,y),...,d_2(x,y)$ can be written in the form $\frac{\partial c_i}{\partial x} = \frac{\partial a_i}{\partial y}, \frac{\partial d_i}{\partial x} = \frac{\partial b_i}{\partial y}$, $i = 1,2$. In this case the integrals

$$U = \int (a_2u + b_2v)dx + (c_2u + d_2v)dy, V = \int (a_1u + b_1v)dx + (c_1u + d_1v)dy \quad (4.31)$$

are path-independent and (u,v) satisfy a system $(4.31)_1$ which is of the same form as (see [102]). System (4.30) is said to be embedded into a cycle if there exists a sequence of systems $(4.31)_1$, $(4.31)_2$, $(4.31)_3$,... such that $(4.31)_i$, is related to $(4.31)_{i+1}$ as (4.30) was related to $(4.31)_1$. The cycle is said to be of finite order n if $(4.31)_n$, is equivalent to (4.30), of infinite order if such n doesn't exist. In [80]-[81](see [102]) M. Lukomskaya (a) proves that every (4.30) can be embedded into a cycle of infinite order, and (b) gives necessary and sufficient conditions in order that the minimum order n_{min} of a cycle beginning with (4.30) be 1. In [11] is stated as an open problem the question on the existence of systems with $n_{min} > 2$. Note that for elliptic systems the natural setting for this problem is the theory of pseudo-analytic functions [11] and finely result in this direction gives M. Protter [104] solving the so called *periodicity problem* for pseudoanalytic functions.

4.3 Almost complex structure

Let X be a two-dimensional connected smooth manifold. By definition two complex atlases U and V are equivalent if their union is a complex atlas. A complex structure on X is an equivalence class of complex atlases. A Riemann surface is a connected surface with a complex structure. A differential 1-form on X with respect to a local coordinate z can be represented in the form $\omega = \alpha dz + \beta d\bar{z}$. Thus, ω has bidegree (1,1) and is a sum of the forms $\omega^{1,0} = \alpha dz$ and $\omega^{0,1} = \beta d\bar{z}$ of bidegree (1,0) and (0,1)respectively. The change of local coordinate $z \to iz$ induces on the differential forms the mapping given by $\omega \to i(\alpha dz - \beta d\bar{z}) = i\omega^{1,0} - i\omega^{0,1}$. Denote by J the operator defined on 1-forms by the rule $J\omega = i\omega^{1,0} - i\omega^{0,1}$. This operator does not depend on the change of the local coordinate z and $J^2 = -1$, where 1 denotes the identity operator. Therefore, the splitting $\Lambda^1 = \Lambda^{1,0} + \Lambda^{0,1}$ is the decomposition of the space of differential 1-forms into eigenspaces of $J : T^*(X)_{\mathbb{C}} \to T^*(X)_{\mathbb{C}}$. On the tangent space TX the operator J acts via $\omega(Jv) = (J\omega)(v)$, for every vector field $v \in TX$. If $z = x + iy$ and taking $v = \frac{\partial}{\partial x}$, one has

$$dz(Jv) = idz(\frac{\partial}{\partial x}) = i = dz(\frac{\partial}{\partial y}) \Rightarrow J\frac{\partial}{\partial x} = \frac{\partial}{\partial y}, J\frac{\partial}{\partial y} = -\frac{\partial}{\partial x}.$$

It means that on the basis $(\frac{\partial}{\partial x}, \frac{\partial}{\partial y})$ of TX the operator J is given by $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.

Therefore, by the complex structure defined from local coordinates defines the operator $J : T^*(X)_{\mathbb{C}} \to T^*(X)_{\mathbb{C}}$, with the property $J^2 = -1$. This operator is called *almost complex structure.*

Conversely, let X be a smooth surface and let $J : T_x(X) \to T_x(X)$, $x \in X$, be such an operator, i. e. $J^2 = -1$. The pair (X, J) is called a pseudoanalytic surface.

As above, by duality it is possible to define J on 1-forms on X. The the space of 1-forms Λ^1 decomposes into eigenspaces corresponding to the eigenvalues $\pm i$ of J and $\Lambda^1 = \Lambda_J^{1,0} + \Lambda_J^{0,1}$. In particular, $J\Lambda_J^{1,0} = i\Lambda_J^{1,0}$ and $J\Lambda_J^{0,1} = -i\Lambda_J^{0,1}$.

Let f be a smooth function, then $df \in \Lambda^1$ and decomposes by bidegree as $df = \partial_J f + \overline{\partial}_J f$, where $\partial_J f := (df)_J^{1,0}$ and $\overline{\partial}_J f := (df)_{\underline{J}}^{1,0}$. By definition, f is J-holomorphic, if it satisfies the Cauchy-Riemann equation $\overline{\partial}_J f$.

Let (X, J) be a pseudoanalytic surface. In the neighborhood of every point $x \in X$ it is possible to change the local coordinate in such a way that dz will be of $(1,0)_J$-type. Then the decomposition of dz by bidegree is $dz = \omega + \overline{\delta}$, where ω, δ are forms of bidegree $(1,0)_J$. Because the fibre of $T_J^{1,0}X$ is a one-dimensional complex space, we have $\delta = \mu\overline{\omega}$, where μ is some smooth function $\mu(0) = 0$. From this it follows, that

$$dz = \omega + \mu\overline{\omega} \text{ and } d\overline{z} = \overline{\omega} + \overline{\mu}\omega. \tag{4.32}$$

Therefore, for every smooth function f in the neighborhood of $x \in X$ we have

$$df = (\partial f + \overline{\mu}\overline{\partial} f)\omega + (\overline{\partial} f + \mu\partial f)\overline{\omega} = \partial_J f + \overline{\partial}_J f$$

From this it follows, that f is J-holomorphic iff $\overline{\partial}_J f = 0$, i.e.

$$\overline{\partial} f + \mu\partial f = 0. \tag{4.33}$$

The equation (4.33) is called the Beltrami equation. Thus a smooth function defined on a pseudoanalytic surface (X, J) is J-holomorphic iff it satisfies the Beltrami equation (5.26).

Suppose f is J-holomorphic and let $f = \varphi + i\psi$, where φ and ψ are real-valued functions. Consider the complex-valued function w defined by the identity $w = \varphi F + \psi G$, where F, G are complex-valued Hölder continuous functions satisfying the condition $Im(\overline{F}G) > 0$.

Theorem 4.20 *The function $w = \varphi F + \psi G$ is (F, G)-pseudo-analytic.*

Indeed, $w = \varphi F + \psi G = \frac{iG-F}{2} f + \frac{-iG-F}{2}\overline{f}$, from which it follows, that f is a solution of the Beltrami equation

$$(iG - F)\overline{\partial f} - (iG + F)\partial f = 0$$

iff w is a solution of the Carleman-Bers-Vekua equation

$$\overline{\partial} w + \frac{\overline{F}\overline{\partial} G - \overline{\partial}\overline{F}\overline{G}}{F\overline{G} - F\overline{G}} w + \frac{F\overline{\partial} G - \overline{\partial} F G}{F\overline{G} - F\overline{G}}\overline{w} = 0.$$

In $D \subset \mathbb{C}$ every metric has the form $\lambda|dz + \mu d\overline{z}|$, where $\lambda > 0$ and the complex function μ satisfies $|\mu| < 1$, from which it follows, that J is defined uniquely by the 1-form $\omega = dz + \mu d\overline{z}$ on D with properties $J\omega = i\omega$, $J\overline{\omega} = -i\omega$. The forms of this type are forms of bidegree (1,0) with respect to J (the space of such forms has been denoted above by $\Lambda_J^{1,0}$). If $\delta \in \Lambda_J^{1,0}$, then $\delta = \alpha\omega + \beta\overline{\omega}$ and it is proportional to ω. Therefore J is determined uniquely up to a constant multiplier $(1,0)_J$ by the form ω. Functions holomorphic with respect to J have differentials proportional to ω. Indeed, if $df + iJ(df) = 0$, then $J(df) = idf$ and from the representation $df = \alpha\omega + \beta\overline{\omega}$ we obtain, that $\beta\overline{\omega} = 0$. Since $df = \alpha\omega + \beta\overline{\omega}$, in $D \subset \mathbb{C}$ the Cauchy-Riemann equation with respect to J with base form $\omega = dz+\mu d\overline{z}$ can be represented as the Beltrami equation $\overline{\partial}f = \mu\partial f$. This equation has a solution f such that it is a biholomorphic map from (D, J) to $f(D), J_{st}$, where J_{st} is the standard conformal structure on $\mathbb{C}$.

Therefore we have proved the following proposition.

Proposition 4.21 *[40] On simply connected areas there exists only one complex structure and conformal structures are in one-to-one correspondence with complex functions μ with $|\mu| < 1$.*

From this proposition and Theorem 4.20 follows the proposition

Proposition 4.22 *[40] There exists a one-to-one correspondence between the space of conformal structures and the space of generalized analytic functions on each simply connected open area of the complex plane.*

4.4 The holomorphic discs equation

Let $\mathbb{D}$ be the unit disc in the complex plane $\mathbb{C}$ with the standard complex structure J_{st} and the coordinate function ζ. J_{st} is uniquely determined by the form $d\zeta \in \Lambda^{1,0}_{J_{st}}$. The map $\phi : \mathbb{D} \to X$ of class C^1 is holomorphic iff $\psi^*\Lambda^{1,0}_J(X) \subset \Lambda^{1,0}(\mathbb{D})$. Let z be another coordinate function on $\mathbb{D}$. We study a local problem, therefore, without loss of generality, it is possible to consider ϕ as mapping from $(\mathbb{D}, J_{st})$ to $(\mathbb{C}_z, J)$, where the complex structure J is defined by $dz = \omega + \mu\overline{\omega}$, $\overline{\omega} \in \Lambda^{1,0}_J$. Therefore we have

$$\zeta \to z = z(\zeta), z(0) = 0.$$

From (4.32) we obtain

$$\omega = \frac{dz - \mu d\overline{z}}{1 - |\mu|^2}.$$

The form ω is J-holomorphic, which means that the form

$$z^*(dz - \mu d\overline{z}) = (\partial_\zeta z - \mu\partial_\zeta\overline{z})d\zeta + (\partial_{\overline{\zeta}} z - \mu\partial_{\overline{\zeta}}\overline{z})d\overline{\zeta}$$

has bidegree $(1, 0)$ on $\mathbb{D}$, therefore

$$\partial_{\overline{\zeta}} z - \mu\partial_{\overline{\zeta}}\overline{z} = 0.$$

From this after using the identity $\partial_{\overline{\zeta}}\overline{z}$ we obtain

$$\partial_{\overline{\zeta}} z = \mu(z)\overline{\partial_\zeta} z. \tag{4.34}$$

The obtained expression is called the *equation of holomorphic disc*. It is known that f satisfies this equation iff f^{-1} satisfies the corresponding Beltrami equation (see [55]).

Proposition 4.23 *[55] If $\omega = u + iv$ satisfies the equation $\frac{\partial w(z)}{\partial \bar{z}} + \mu(z)\overline{\frac{\partial w(z)}{\partial z}} = 0$, $|\mu| < 1$ and a and b are holomorphic functions such that $\mu = \frac{a-b}{a+b}$, then $W = au + ibv$ is holomorphic.*

Indeed,

$$\frac{\partial}{\partial \bar{z}}(a\frac{\omega + \overline{\omega}}{2} + ib\frac{\omega - \overline{\omega}}{2}) = \frac{a}{2}(\omega_{\bar{z}} + \overline{\omega}_{\bar{z}}) + \frac{b}{2}(\omega_{\bar{z}} - \overline{\omega}_{\bar{z}}) = \omega_{\bar{z}}(\frac{a+b}{2}) + \overline{\omega}_{\bar{z}}(\frac{a-b}{2}),$$

therefore if ω is a solution of the equation $\omega_{\bar{z}} + \frac{a-b}{a+b}\overline{\omega_z} = 0$, then $\partial_{\bar{z}} W = 0$.

From this proposition it follows in particular, that W is (a, ib)-pseudo-analytic.

5 Irregular Carleman-Bers-Vekua equation with weak singularity at infinity

In [129] I. Vekua is interested in the behavior of the solutions of the equation

$$\frac{\partial w}{\partial \overline{z}} + A(z)w + B(z)\overline{w} = 0 \tag{5.1}$$

in the neighborhood of isolated singularity of the coefficients. It is known that in the case in which $A, B \in L_{p,2}(\mathbb{C}), p > 2$, every solution of (5.1) is expressible in the form

$$w(z) = \Phi(z)e^{-T(A+B\overline{w}w^{-1})},$$

where $\Phi(z)$ is analytic and $Tf = \frac{1}{\pi}\iint_{\mathbb{C}} f(\zeta)(\zeta - z)^{-1}d\xi d\eta$; if A and B are merely quasi-summable, i.e., $A_1 = \varphi^{-1}A$ and $B_1 = \psi^{-1}B$ are in $L_{p,2}(\mathbb{C}), p > 2$, for some analytic functions $\varphi(z)$ and $\psi(z)$ with arbitrary singularities (isolated in $\mathbb{C}$,) then every solution of (5.1) is expressible in the form

$$w(z) = \Phi(z)e^{\varphi(z)\omega(z)+\psi(z)\chi(z)} \tag{5.2}$$

where $\Phi(z)$ is analytic, $\omega = -T(A_1)$ and $\chi = -T(B_1\overline{w}w^{-1})$. In [129] the main statement is: *for a given analytic function Φ, (5.2) is a solution of (5.1) whenever the function $\chi(z)$ satisfies the equation*

$$\chi = T_0(\chi), \tag{5.3}$$

where $T_0(\chi) = -T[B_* e^{-2iIm(\psi\chi)}]$ and $B_* = B_1\overline{\Phi}\Phi^{-1}e^{-2iIm\varphi T(A_1)}$; a fixed point argument yields the existence of a solution of (5.3). Vekua then uses the representation (5.2) to study the behavior of solutions of (5.1) near arbitrary isolated singularities of A and B.

In this section the concept of $\dfrac{\partial}{\partial \overline{z}}$ primitive of the function of the class $L_p^{\text{loc}}(E)$, $p > 2$, is introduced and its existence is proved. Some properties of this primitive are established. The classes of the functions $J_0(\mathbb{C})$ and $J_1(\mathbb{C})$ are introduced and studied.

In the terminology of Vekua, generalized analytic functions of class $w \in D_{\overline{z}}(U)$ are solutions of the first order elliptic partial differential equation given in a complex form as follows

$$\partial_{\overline{z}}w + Aw + B\overline{w} = 0, \tag{5.4}$$

where $D_{\overline{z}}(U)$ (respectively, $D_z(U)$) denotes the linear space of functions defined on $U \subset \mathbb{C}$ which are differentiable in the Sobolev sense with respect to $\overline{z}$ (respectively, with respect to z). Till now, except for some exceptions (see [125], [13], [133],[132] [84], [106], [20], [123], [97]) the object of systematic investigation has been the equation (5.4) when the coefficients A and B satisfy the following regularity conditions

$$A, B \in L_p(U), p > 2, \tag{5.5}$$

or

$$A, B \in L_{p,2}(\mathbb{C}), p > 2. \tag{5.6}$$

In this cases the equation (5.4) is called the regular Carleman-Bers-Vekua equation.

From the representation of solutions of the equation (5.4) by the integral operator

$$T_U g = -\frac{1}{\pi} \int \int_U \frac{g(\zeta) d\xi d\eta}{\zeta - z}, \zeta = \xi + i\eta$$

of the form

$$w(z) = \Phi(z) e^{-T_U g}, \tag{5.7}$$

where

$$g(z) = \begin{cases} A(z) + B(z)\,\dfrac{\overline{w(z)}}{w(z)}, & \text{if} \quad w(z) \neq 0, \quad z \in U, \\ A(z) + B(z), & \text{if} \quad w(z) = 0, \quad z \in U. \end{cases} \tag{5.8}$$

(the so called *similarity principle*), follow many properties of generalized analytic functions similar to properties of classical analytic functions. It is known that the majority of fundamental theorems on analytic functions extend to generalized analytic functions, too. If the function $\Phi(z)$ in the representation (5.7) is analytic, we will call the corresponding generalized analytic function $w(z)$ *pseudoanalytic*.

As it is well-known if $A, B \in L_{p,2}(\mathbb{C}), p > 2$ then the equation is called *regular equation* and the solutions are the *generalized analytic functions* (*pseudoanalytic*) [124], [11]. Denote by $\Omega(A, B)$ the space of solutions of the equation for the fixed coefficients A and B. This space is linear space over the field of real numbers and for every pair of functions from $L_{p,2}(\mathbb{C}), p > 2$ the most fundamental theorems of the theory of analytic functions on every $\Omega(A, B)$ are extended. In view of the theory of functions these spaces are much the same.

We call Carleman-Bers-Vekua equation *irregular* [125] if both functions A and B or at least one of them doesn't belong to $L_{p,2}(\mathbb{C}), p > 2$. In this case the analytic properties of the classes $\Omega(A, B)$ are different. In other words, *for irregular equations the dependence of the functional classes $\Omega(A, B)$ on the coefficients A and B is rigid.* In particular, when A and B are constants (in general complex) and $w(z) = O(z^N), z \to \infty$, then 1) $dim_{\mathcal{R}}\Omega(A, B) = 0$ if $|A| < |B|$; 2) $dim_{\mathcal{R}}\Omega(A, B) = N + 1$, if $|A| = |B|$; 3) $dim_{\mathcal{R}}\Omega(A, B) = 2(N + 1)$, if $|A| > |B|$ [133],[132].

As noted in the work of I. Vekua [129], the necessity of investigation of Carleman-Bers-Vekua equation is motivated by the problems of mechanics [124] on one hand and by the construction of complete theory of generalized analytic functions on the other [133],[132] [20]. The same is the purpose of recent publications [10], [14], [54], [53], [51], [50], [63], [64], [84], [108] (see also the references of these works).

$\dfrac{\partial}{\partial \overline{z}}$-primitive of the function of the class $L_p^{\text{loc}}(\mathbb{C})$, $p > 2$. In this section we prove the existence theorem of $\dfrac{\partial}{\partial \overline{z}}$-primitive (see [129]) for the functions of the classes $L_{p,2}^{loc}(\mathbb{C}), p > 2$. In the next sections, using this theorem, we consider the spaces of solutions of Carleman-Bers-Vekua irregular equation, investigate their properties, some interesting irregular equations and related spaces of generalized analytic functions.

As, well known, for every function $a \in L_{p,2}(\mathbb{C})$, $p > 2$, using the integral

$$A(z) = -\frac{1}{\pi} \iint_{\mathbb{C}} \frac{a(\zeta)}{\zeta - z}\, d\xi\, d\eta, \quad \zeta = \xi + i\eta \tag{5.9}$$

we can construct a $\frac{\partial}{\partial \overline{z}}$-primitive [125] on the whole plane with respect to a generalized derivative $\frac{\partial}{\partial \overline{z}}$ in the Sobolev sense [125]. It means that we consider Carleman-Bers-Vekua equations with irregular coefficients, hence it is necessary to investigate the problem of existence of $\frac{\partial}{\partial \overline{z}}$-primitives of functions not belonging to the class $L_{p,2}(\mathbb{C})$, $p > 2$. Note that the integral (5.9) is meaningless for such functions. We obtain a solution, complete in a certain sense, of the above-mentioned problem for functions of the class $L_p^{\mathrm{loc}}(\mathbb{C})$, $p > 2$. In particular, the existence of Hölder-continuous $\frac{\partial}{\partial \overline{z}}$-primitive is established.

The following theorem is valid.

Theorem 5.1 *Every function $a(z)$ of the class $L_p^{\mathrm{loc}}(\mathbb{C})$, $p > 2$, has $\frac{\partial}{\partial \overline{z}}$-primitive function $Q(z)$ on the whole complex plane satisfying the Hölder condition with the exponent $\frac{p-2}{p}$ on each compact subset of the complex plane $\mathbb{C}$; moreover if $q(z)$ is one $\frac{\partial}{\partial \overline{z}}$- primitive of the function $a(z)$ then all $\frac{\partial}{\partial \overline{z}}$-primitives of this function are given by the formula*

$$Q(z) = q(z) + \Phi(z), \tag{5.10}$$

where $\Phi(z)$ is an arbitrary entire function.

Proof. Let $a(z)$ be an arbitrary function $L_p^{\mathrm{loc}}(\mathbb{C})$, $p > 2$. Consider two sequences of positive numbers R_n and R'_n, $n = 1, 2, 3, \ldots$, satisfying the conditions

$$R'_1 < R_1; \quad R_{n-1} < R'_n < R_n, \quad n = 2, 3, \ldots \tag{5.11}$$

$$\lim_{n\to\infty} R_n = +\infty. \tag{5.12}$$

Consider also the following sequences of the domains:

$$G_1 = \big\{z : |z| < R_1\big\}, \quad G_n = \big\{z : R_{n-1} < |z| < R_n\big\}, \quad n = 2, 3, \ldots,$$
$$G'_n = \big\{z : |z| < R'_n\big\}, \quad n = 1, 2, 3, \ldots, \quad D_1 = G_1,$$
$$D_n = \big\{z : |z| < R_n\big\}, \quad n = 2, 3, \ldots$$

and the boundaries of G_n

$$\gamma_n = \big\{z : |z| = R_n\big\}, \quad n = 1, 2, 3, \ldots$$

Construct the sequences of the functions:

$$g_n(t) = -\frac{1}{\pi}\iint_{G_{n+1}} \frac{a(\zeta)}{\zeta - t}\, d\xi\, d\eta + \frac{1}{\pi}\iint_{G_n} \frac{a(\zeta)}{\zeta - t}\, d\xi\, d\eta, \tag{5.13}$$
$$\zeta = \xi + i\eta, \quad t \in \gamma_n, \quad n = 1, 2, 3, \ldots$$
$$F_k(z) = \frac{1}{2\pi i}\int_{\gamma_k} \frac{g_k(t)}{t - z}\, dt, \tag{5.14}$$
$$z \in E \setminus \gamma_k, \quad k = 1, 2, 3 \ldots$$

It is clear that

$$g_n(t) = \big(T_{G_{n+1}}a\big)(t) - \big(T_{G_n}a\big)(t),$$

where by definition $(T_G a))(t) = -\frac{1}{\pi}\int\int_G \frac{a(\zeta)}{\zeta - t} d\xi d\eta$ for every bounded domain $G \in \mathbb{C}$, $\zeta = \xi + i\eta$ (see section 2.2).

Since $a \in L_p^{\mathrm{loc}}(\mathbb{C})$, $p > 2$, we have

$$T_{G_n}a,\, T_{G_{n+1}}a \in C_{\frac{p-2}{p}}(\gamma_n), \quad n = 1, 2, 3, \ldots$$

Hence it is evident, that every function

$$g_n(t) \in C_{\frac{p-2}{p}}(\gamma_n) \tag{5.15}$$

and every function $F_k(z)$ is holomorphic for each $z \in \mathbb{C}\setminus\gamma_k$. Since the function $F_k(z)$ is holomorphic on the circle D_k, it can be expanded into a Taylor series on D_k

$$F_k(z) = \sum_{j=0}^{\infty} C_j^{(k)} z^j, \quad |z| < R_k. \tag{5.16}$$

Consider the sequence of positive numbers ε_k, $k = 1, 2, 3, \ldots$,for which the series $\sum_{k=1}^{\infty} \varepsilon_k$ converges.

Since the Taylor-series (5.16) uniformly converges on the closed circle $\overline{G'_k} \subset D_k$, there exist the natural numbers N_k such that the inequality

$$\left| F_k(z) - \sum_{j=0}^{n} C_j^{(k)} z^j \right| < \varepsilon_k, \quad z \in \overline{G'_k} \tag{5.17}$$

holds for every natural $n > N_k$. In particular, if we assume that $n = N_k + 1$, then for completely defined polynomials

$$f_k(z) = \sum_{j=0}^{N_k+1} C_j^{(k)} z^j, \quad k = 1, 2, 3, \ldots \tag{5.18}$$

the inequality

$$\big| F_k(z) - f_k(z) \big| < \varepsilon_k, \quad z \in \overline{G'_k}, \quad k = 1, 2, 3, \ldots \tag{5.19}$$

holds.

Consider the function

$$\Phi(z) = \sum_{k=1}^{\infty} \big(F_k(z) - f_k(z)\big), \quad z \in \mathbb{C} \setminus \bigcup_{k=1}^{\infty} \gamma_k. \tag{5.20}$$

Let us fix arbitrary natural number n and prove that $\Phi(z)$ is a holomorphic function in the domain G_n. Indeed,

$$\Phi(z) = \sum_{k=1}^{n} \big(F_k(z) - f_k(z)\big) + \sum_{k=n+1}^{\infty} \big(F_k(z) - f_k(z)\big), \quad z \in G_n. \tag{5.21}$$

from (5.19) we obtain the inequalities

$$\begin{array}{l} \big|F_{n+1}(z) - f_{n+1}(z)\big| < \varepsilon_{n+1}, \quad z \in \overline{G'_{n+1}} \\ \big|F_{n+2}(z) - f_{n+2}(z)\big| < \varepsilon_{n+2}, \quad z \in \overline{G'_{n+2}} \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \big|F_{n+l}(z) - f_{n+l}(z)\big| < \varepsilon_{n+l}, \quad z \in \overline{G'_{n+l}} \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \end{array} \tag{5.22}$$

(5.21) implies that the inclusion

$$G_n \subset \overline{G'_{n+1}} \subset \overline{G'_{n+2}} \subset \ldots \subset \overline{G'_{n+l}} \subset \ldots \tag{5.23}$$

is valid. Thus from (5.19) we have

$$\big|F_{n+j}(z) - f_{n+j}(z)\big| < \varepsilon_{n+j}, \quad z \in G_n, \quad j = 1, 2, 3, \ldots \tag{5.24}$$

Every function $F_k(z)$, $k = 1, 2, 3, \ldots$, is holomorphic on the D_k, when $k \geqslant n+1$, $G_n \subset D_k$. Hence the function $F_k(z)$ is holomorphic on G_n. The function $f_k(z)$ is holomorphic on the whole plane, as it is a polynomial. Consequently, $F_k(z) - f_k(z)$ is holomorphic on the domain G_n.

From the inequalities (5.24) we get that the series $\sum\limits_{k=n+1}^{\infty} (F_k(z) - f_k(z))$ is uniformly convergent on G_n. By virtue of Weierstrass first theorem the series $\sum\limits_{k=n+1}^{\infty} (F_k(z) - f_k(z))$ is holomorphic function on the domain G_n.

Since the functions $F_k(z)$, $k = 1, 2, 3, \ldots$ are holomorphic functions in every point z except the points of the curve γ_k, then the first summand on the right-hand side of the inequality (5.21) $\sum\limits_{k=1}^{n} (F_k(z) - f_k(z))$ is a holomorphic function on the domain G_n. From here the function $\Phi(z)$ is holomorphic on the domain G_n. Since

$$\bigcup_{n=1}^{\infty} G_n = \mathbb{C} \setminus \bigcup_{k=1}^{\infty} \gamma_k, \tag{5.25}$$

$\Phi(z)$ is a holomorphic function at every point z, where $z \in \mathbb{C} \setminus \bigcup\limits_{k=1}^{\infty} \gamma_k$.

Along with the function $\Phi(z)$, let us consider one more function which is defined on the set $\bigcup\limits_{n=1}^{\infty} G_n$ by

$$Q(z) = \Phi(z) - \frac{1}{\pi} \iint_{G_n} \frac{a(\zeta)}{\zeta - z}\, d\xi\, d\eta, \quad z \in G_n, \zeta = \xi + i\eta. \tag{5.26}$$

Let z be an arbitrary point from the set $\bigcup\limits_{n=1}^{\infty} G_n$, then there exists the unique natural number n, such that $z \in G_n$. Denote by

$$H(z) = -\frac{1}{\pi} \iint_{G_n} \frac{a(\zeta)}{\zeta - z}\, d\xi\, d\eta, \quad z \in G_n, \quad \zeta = \xi + i\eta. \tag{5.27}$$

By means of the functions $\Phi(z)$ and $H(z)$ construct the function

$$Q(z) = \Phi(z) + H(z), \quad z \in \mathbb{C} \setminus \cup \gamma_n. \tag{5.28}$$

$Q(z)$ is a continuous function in every point of the set $\bigcup\limits_{n=1}^{\infty} G_n$. Indeed, if $z \in G_n$, then $H(z) = \big(T_{G_n} a\big)(z) \in C_{\frac{p-2}{p}}(\mathbb{C})$.

The function $\Phi(z)$ is a continuous on the domain G_n.Therefore $Q(z)$ is continuous on the domain G_n, to.

Let us prove, that the function $Q(z)$ is continuously extentable on the whole complex plane $\mathbb{C}$. For this, we fix an arbitrary natural number n and consider the left-hand $Q^+(t_0)$ and right-hand $Q^-(t_0)$ limits of the function $Q(z)$ in an arbitrary point $t_0 \in \gamma_n$.

Taking into account that the interior domain of the contour γ_n contains the domain G_n and the exterior domain contains the domain G_{n+1}, we represent the limits as

$$Q^+(t_0) = \lim_{\substack{z \to t_0 \\ z \in G_n}} Q(z), \tag{5.29}$$

$$Q^-(t_0) = \lim_{\substack{z \to t_0 \\ z \in G_{n+1}}} Q(z). \tag{5.30}$$

To calculate the limits (5.29),(5.30) we write the function $Q(z)$ in the form

$$\begin{aligned} Q(z) = \sum_{k=1}^{n-1} \big(F_k(z) - f_k(z)\big) + \big(F_n(z) - f_n(z)\big) + \\ + \sum_{k=n+1}^{\infty} \big(F_k(z) - f_k(z)\big) - \frac{1}{\pi} \iint_{G_n} \frac{a(\zeta)}{\zeta - z}\, d\xi\, d\eta, \quad z \in G_n, \end{aligned} \tag{5.31}$$

$$Q(z) = \sum_{k=1}^{n-1} \big(F_k(z) - f_k(z)\big) + \big(F_n(z) - f_n(z)\big) +$$

$$+\sum_{k=n+1}^{\infty}\big(F_k(z)-f_k(z)\big)-\frac{1}{\pi}\iint_{G_{n+1}}\frac{a(\zeta)}{\zeta-z}\,d\xi\,d\eta,\quad z\in G_{n+1}. \tag{5.32}$$

Each function $F_k(z)$, where $k\neq n$, is continuous on the curve γ_n, as the Cauchy -type integral. Therefore the sum $\sum\limits_{k=1}^{n-1}\big(F_k(z)-f_k(z)\big)$ is continuous on the curve γ_n. In case $k\geqslant n+1$, the function $F_k(z)$ is holomorphic in the domain G'_{n+1}. i.e. the function $F_k(z)-f_k(z)$ is holomorphic on G'_{n+1}. The series $\sum\limits_{k=n+1}^{\infty}\big(F_k(z)-f_k(z)\big)$ is uniformly convergent on the domain G'_{n+1}. By virtue of Weirstrass first theorem about the holomorphic functions, the sum $\sum\limits_{k=n+1}^{\infty}\big(F_k(z)-f_k(z)\big)$ is holomorphic on the domain G'_{n+1}. Hence, the sum $\sum\limits_{k=n+1}^{\infty}\big(F_k(z)-f_k(z)\big)$ is continuous on the curve γ_n.

It follows from the formula (5.13), that

$$g_n(t)=\big(T_{G_{n+1}}a\big)(t)-\big(T_{G_n}a\big)(t).$$

Since $a\in L_p^{\text{loc}}(\mathbb{C})$, $p>2$ ([125],theorem 1.19) we have

$$\big(T_{G_n}a\big)(t),\big(T_{G_{n+1}}a\big)(t)\in C_{\frac{p-2}{p}}(\mathbb{C}),\quad t\in\mathbb{C}.$$

Therefore $g_n(t)\in C_{\frac{p-2}{p}}(\mathbb{C})$.

Since $F_n(z)=\dfrac{1}{2\pi i}\displaystyle\int_{\gamma_n}\frac{g_n(t)}{t-z}\,dt$, by the Sokhotsky-Plemely formulas we get

$$\begin{aligned}F_n^+(t_0)&=\frac{1}{2}\,g_n(t_0)+\frac{1}{2\pi i}\int_{\gamma_n}\frac{g_n(t)}{t-t_0}\,dt,\\ F_n^-(t_0)&=-\frac{1}{2}\,g_n(t_0)+\frac{1}{2\pi i}\int_{\gamma_n}\frac{g_n(t)}{t-t_0}\,dt.\end{aligned} \tag{5.33}$$

By (5.29)-(5.33) and the above reasoning we get

$$Q^+(t_0)=\sum_{k=1}^{n-1}\big(F_k(t_0)-f_k(t_0)\big)+\frac{1}{2}\,g_n(t_0)+$$

$$+\frac{1}{2\pi i}\int_{\gamma_n}\frac{g_n(t)}{t-t_0}\,dt-f_n(t_0)+\sum_{k=n+1}^{\infty}\big(F_k(t_0)-f_k(t_0)\big)-\frac{1}{\pi}\iint_{G_n}\frac{a(\zeta)}{\zeta-t_0}\,d\xi\,d\eta, \tag{5.34}$$

$$Q^-(t_0)=\sum_{k=1}^{n-1}\big(F_k(t_0)-f_k(t_0)\big)-\frac{1}{2}\,g_n(t_0)+$$

$$+\frac{1}{2\pi i}\int_{\gamma_n}\frac{g_n(t)}{t-t_0}\,dt-f_n(t_0)+\sum_{k=n+1}^{\infty}\big(F_k(t_0)-f_k(t_0)\big)-\frac{1}{\pi}\iint_{G_{n+1}}\frac{a(\zeta)}{\zeta-t_0}\,d\xi\,d\eta. \tag{5.35}$$

Applying formulas (5.34) and (5.35) we have

$$Q^+(t_0) - Q^-(t_0) = g_n(t_0) - \frac{1}{\pi}\iint_{G_n} \frac{a(\zeta)}{\zeta - t_0}\,d\xi\,d\eta + \frac{1}{\pi}\iint_{G_{n+1}} \frac{a(\zeta)}{\zeta - t_0}\,d\xi\,d\eta = 0.$$

Consequently, the function $Q(z)$ is continuously extendable on whole complex plane.

Let us prove, that the function $Q(z)$ is $\dfrac{\partial}{\partial \overline{z}}$ primitive of the function $a(z)$.

Consider arbitrary domain G_n, $n = 1, 2, 3, \ldots$. On this domain the function $Q(z)$ is representable by the following form:

$$Q(z) = \Phi(z) - \frac{1}{\pi}\iint_{G_n} \frac{a(\zeta)}{\zeta - z}\,d\xi\,d\eta = \Phi(z) + \big(T_{G_n} a\big)(z), \quad z \in G_n. \tag{5.36}$$

It is evident that G_n on $\dfrac{\partial \Phi}{\partial \overline{z}} = 0$.

Since $a \in L_p(G_n)$, $p > 2$, by virtue of the theorem 1.14 from [125] we have

$$\frac{\partial}{\partial \overline{z}}\big(T_{G_n} a\big)(z) = a(z), \quad z \in G_n.$$

Using the equality (5.36) we get the following equality on the domain G_n:

$$\frac{\partial Q}{\partial \overline{z}} = a(z), \quad z \in G_n. \tag{5.37}$$

The function $Q(z)$ is a continuous function on the whole complex plane. It is clear that $\mathbb{C} = (\bigcup_{n=1}^{\infty} G_n) \cup (\bigcup_{n=1}^{\infty} \gamma_n)$.

By equality (5.37) and the above reasoning we have that the equality

$$\frac{\partial Q}{\partial \overline{z}} = a(z), \quad z \in \mathbb{C} \tag{5.38}$$

is fulfilled on the whole plane.

We obtain that the constructed function $Q(z)$ is $\dfrac{\partial}{\partial \overline{z}}$-primitive of the function $a(z)$ on the whole plane.

Let us prove that the function $Q(z)$ satisfies the Hölder condition with the exponent $\dfrac{p-2}{p}$ on each compactum.

Consider arbitrary compact $D \subset \mathbb{C}$. Consider also the bounded domain G, which contains D. It is easy to see that

$$\frac{\partial Q}{\partial \overline{z}} = a(z), \quad z \in G.$$

Since $a \in L_p(G)$, $p > 2$ by virtue of the theorem 1.16 from [125] the following equality

$$Q(z) = K(z) + \big(T_G\, a\big)(z), \quad z \in G, \tag{5.39}$$

is valid, where $K(z)$ is a holomorphic function on G.

Using the theorem 1.19 from [125], we have

$$\big(T_G\, a\big)(z) \in C_{\frac{p-2}{p}}(\mathbb{C}).$$

Since the function $K(z)$ is a holomorphic on the domain G, we have

$$K(z) \in C_{\frac{p-2}{p}}(D).$$

It follows from the equality (5.39) that $Q(z) \in C_{\frac{p-2}{p}}(D)$.

Let $q(z)$ be one of $\dfrac{\partial}{\partial \overline{z}}$-primitives of the function $a(z)$ and let $\Phi(z)$ be an arbitrary entire function. Consider the function $Q(z) = q(z) + \Phi(z)$. Then the equality

$$\frac{\partial Q}{\partial \overline{z}} = \frac{\partial q}{\partial \overline{z}} + \frac{\partial \Phi}{\partial \overline{z}} = \frac{\partial q}{\partial \overline{z}} = a(z), \quad z \in \mathbb{C},$$

is valid since $\dfrac{\partial \Phi}{\partial \overline{z}} = 0$.

Let $Q(z)$ be an arbitrary $\dfrac{\partial}{\partial \overline{z}}$-primitive of the function $a(z)$. Then the equality $\dfrac{\partial Q}{\partial \overline{z}} = a(z)$, $z \in \mathbb{C}$ is fulfilled.

Since $\dfrac{\partial q}{\partial \overline{z}} = a(z)$, $z \in \mathbb{C}$, the following equality

$$\frac{\partial (Q(z) - q(z))}{\partial \overline{z}} = \frac{\partial Q}{\partial \overline{z}} - \frac{\partial q}{\partial \overline{z}} = 0, \quad z \in \mathbb{C},$$

is fulfilled. From here the function $\Phi(z) = Q(z) - q(z)$ by virtue of the theorem 1.5 from [125] holomorphic on the whole plane, i.e. $\Phi(z)$ is an entire function. The theorem is proved.

5.1 The functional spaces induced from irregular Carleman-Bers-Vekua equations

As it was proved earlier, by every function $a \in L_p^{\rm loc}(\mathbb{C})$, $p > 2$, has $\dfrac{\partial}{\partial \overline{z}}$primitive. Introduce subclasses of the class $L_p^{\rm loc}(\mathbb{C})$, $p > 2$, elements of which have $\dfrac{\partial}{\partial \overline{z}}$ primitives, satisfying certain additional asymptotic conditions, needed in the sequel. In the present section we define these classes and prove their properties.

Denote by $J_0(\mathbb{C})$ the set of functions $a \in L_p^{\rm loc}(\mathbb{C})$, $p > 2$ for which there exists $\dfrac{\partial}{\partial \overline{z}}$-primitive $Q(z)$ satisfying the following condition

$$\operatorname{Re} Q(z) = O(1), \quad z \in \mathbb{C}. \tag{5.40}$$

Denote by $J_1(\mathbb{C})$ the set of the functions $a \in L_p^{\text{loc}}(\mathbb{C})$, $p > 2$, for which there exists $\dfrac{\partial}{\partial \overline{z}}$ primitive $Q(z)$, satisfying the following conditions

$$z^n \exp\left\{Q(z)\right\} = O(1), \quad z \in \mathbb{C}, \tag{5.41}$$

for every natural number n.

Theorem 5.2 *The function $a(z)$ of the class $L_p^{\text{loc}}(\mathbb{C})$, $p > 2$, belongs to the class $J_1(\mathbb{C})$ if and only if its $\dfrac{\partial}{\partial \overline{z}}$-primitive exists and satisfies the condition*

$$\lim_{z\to\infty} z^k \exp\left\{Q(z)\right\} = 0, \tag{5.42}$$

for every natural number k.

Proof. Let $a \in J_1(\mathbb{C})$ and let $Q(z)$ be $\dfrac{\partial}{\partial \overline{z}}$ primitive of the function $a(z)$, participating in the definition of the class $J_1(\mathbb{C})$. Consider arbitrary fixed integer $k \in Z$. Consider the natural number n satisfying the condition $n > k$. By definition of the class $J_1(\mathbb{C})$ we have

$$z^n \exp\left\{Q(z)\right\} = O(1), \quad z \in E.$$

Then we get

$$\lim_{z\to\infty} \left|z^k \exp\left\{Q(z)\right\}\right| = \lim_{z\to\infty} |z|^k \left|\exp\left\{Q(z)\right\}\right| =$$
$$= \lim_{z\to\infty} \frac{1}{|z|^{n-k}} \left|z^n \exp\left\{Q(z)\right\}\right| = O,$$

since $\displaystyle\lim_{z\to\infty} \frac{1}{|z|^{n-k}} = 0$, and the function $\left|z^n \exp\{Q(z)\}\right|$ is a bounded function on the whole plane.

Let $Q(z)$ be $\dfrac{\partial}{\partial \overline{z}}$-primitive of the function $a \in L_p^{\text{loc}}(\mathbb{C})$, $p > 2$, satisfying the condition (5.42) for any $k \in Z$. Then it is evident, that the condition (5.42) will be fulfilled for any n, i.e.

$$\lim_{z\to\infty} z^n \exp\{Q(z)\} = 0. \tag{5.43}$$

Since $Q(z)$ is a continuous function on the plane $\mathbb{C}$, therefore the function $z^n \exp\{Q(z)\}$ is continuous on the whole plane. Then applying the condition (5.43) we conclude, that the function $z^n \exp\{Q(z)\}$ is a bounded function on $\mathbb{C}$, i.e. $a \in J_1(\mathbb{C})$.

Proposition 5.3 *Let $a_1(z), a_2(z) \in J_1(\mathbb{C})$. Then $a_1(z) + a_2(z)$ also belongs to the class $J_1(\mathbb{C})$.*

Proof. Let $Q_1(z)$ and $Q_2(z)$ be $\dfrac{\partial}{\partial\overline{z}}$-primitives of the functions $a_1(z)$ and $a_2(z)$ respectively, involved in the definition of the class $J_1(\mathbb{C})$. Then the following equalities

$$\lim_{z\to\infty} z^k \exp\big\{Q_1(z)+Q_2(z)\big\} = \lim_{z\to\infty} z^k \exp\big\{Q_1(z)\big\}\big\{\exp Q_2(z)\big\} =$$
$$= \lim_{z\to\infty} z^k \exp\big\{Q_1(z)\big\} \lim_{z\to\infty}\big\{\exp Q_2(z)\big\} = 0, \tag{5.44}$$

are valid for every integer $k\in Z$. Since $\lim\limits_{z\to\infty} z^k\exp\big\{Q_1(z)\big\}=0$, $\lim\limits_{z\to\infty}\exp\big\{Q_2(z)\big\}=0$. We obtain, that for the function $a_1(z)+a_2(z)$ there exists $\dfrac{\partial}{\partial\overline{z}}$-primitive $Q_1(z)+Q_2(z)$ such that the equation (5.44) is fulfilled. i.e.$a_1(z)+a_2(z)\in J_1(\mathbb{C})$.

Proposition 5.4 *Let $a(z)\in J_1(\mathbb{C})$ and let α be a arbitrary positive number. Then $\alpha a(z)\in J_1(\mathbb{C})$.*

Proof. Let $k\in Z$ be some integer. Assume, that the function $Q(z)$ is $\dfrac{\partial}{\partial\overline{z}}$-primitive of the function $a(z)$, involved in the definition of the class $J_1(\mathbb{C})$. Then the following equations are valid

$$\lim_{z\to\infty}\big|z^k e^{\alpha Q(z)}\big| = \lim_{z\to\infty}|z^k|\big|e^{\alpha Q(z)}\big| = \lim_{z\to\infty}|z|^k e^{\mathrm{Re}\,\alpha Q(z)} =$$
$$= \lim_{z\to\infty}|z|^k e^{\alpha\,\mathrm{Re}\,Q(z)} = \lim_{z\to\infty}\Big[|z|^{\frac{k}{\alpha}} e^{\mathrm{Re}\,Q(z)}\Big]^{\alpha} =$$
$$= \lim_{z\to\infty}\Big[|z|^{\frac{k}{\alpha}}|e^{Q(z)}|\Big]^{\alpha} = 0^{\alpha} = 0. \tag{5.45}$$

Since $\alpha>0$ and $\lim\limits_{z\to\infty}|z|^{\frac{k}{\alpha}}\big|e^{Q(z)}\big|=0$. we get,

$$\lim_{z\to\infty} z^k e^{\alpha Q(z)} = 0, \tag{5.46}$$

for any $k\in Z$.

We obtain, that there exists $\dfrac{\partial}{\partial\overline{z}}$-primitive $\alpha Q(z)$ of the function $\alpha a(z)$ satisfying the condition (5.46), i.e. $\alpha a(z)\in J_1(\mathbb{C})$.

Proposition 5.5 *Assume, that the functions $a_1(z)$ and $a_0(z)$ satisfy the conditions $a_1(z)\in J_1(\mathbb{C})$, $a_0(z)\in J_0(\mathbb{C})$. Then $a_1(z)+a_0(z)\in J_1(\mathbb{C})$.*

Proof. Let $Q_1(z)$ and $Q_0(z)$ be $\dfrac{\partial}{\partial\overline{z}}$-primitives of the functions $a_1(z)$ and $a_0(z)$ involved in the definitions of the classes $J_1(\mathbb{C})$ and $J_0(\mathbb{C})$ respectively. The following equations hold for every $k\in Z$

$$\lim_{z\to\infty}\big|z^k\exp\{Q_1(z)+Q_0(z)\}\big| = \lim_{z\to\infty}\big|z^k e^{Q_1(z)}e^{Q_0(z)}\big| =$$
$$= \lim_{z\to\infty}\big|z^k e^{Q_1(z)}\big|\big|e^{Q_0(z)}\big| = \lim_{z\to\infty}\big|z^k e^{Q_1(z)}\big|e^{\mathrm{Re}\,Q_0(z)} = 0. \tag{5.47}$$

As far as $\lim_{z\to\infty} \left|z^k e^{Q_1(z)}\right| = 0$ and the function $e^{\operatorname{Re} Q_0(z)}$ is bounded on the whole plane, the following equality

$$\lim_{z\to\infty} z^k \exp\left\{Q_1(z) + Q_0(z)\right\} = 0 \tag{5.48}$$

is valid.

We obtain that there exists $\dfrac{\partial}{\partial \overline{z}}$ primitive $Q_1(z) + Q_0(z)$ of the function $a_1(z) + a_0(z)$ satisfying the equality (5.48). i.e. $a_1(z) + a_0(z) \in J_1(\mathbb{C})$.

Theorem 5.6 *The class $J_0(\mathbb{C})$ is a linear space over the field of real numbers. Moreover, for arbitrary real $p > 2$ the following inclusion*

$$L_{p,2}(\mathbb{C}) \subset J_0(\mathbb{C}) \tag{5.49}$$

holds.

Proof. Let $a_1(z), a_2(z) \in J_0(\mathbb{C})$. Suppose, that $Q_1(z)$ and $Q_2(z)$ are $\dfrac{\partial}{\partial \overline{z}}$-primitives of the function $a_1(z)$ and $a_2(z)$ involved in the definition of the class $J_0(\mathbb{C})$.

Since $\operatorname{Re} Q_1(z) = O(1)$, $z \in \mathbb{C}$, $\operatorname{Re} Q_2(z) = O(1)$, $z \in \mathbb{C}$, then it is evident, that

$$\operatorname{Re}\big(Q_1(z) + Q_2(z)\big) = \operatorname{Re} Q_1(z) + \operatorname{Re} Q_2(z) = O(1), \quad z \in \mathbb{C}. \tag{5.50}$$

We get that there exists $\dfrac{\partial}{\partial \overline{z}}$-primitive$Q_1(z) + Q_2(z)$ of the function $a_1(z) + a_2(z)$ such that the condition (5.50) is fulfilled, i.e. $a_1(z) + a_2(z) \in J_0(\mathbb{C})$.

Let $a(z) \in J_0(\mathbb{C})$, and let α be arbitrary real number, $Q(z)$ is $\dfrac{\partial}{\partial \overline{z}}$-primitive of the function $a(z)$, involved in the definition of the class $J_0(\mathbb{C})$. Since $\operatorname{Re} Q(z) = O(1)$, $z \in \mathbb{C}$, it is clear that

$$\operatorname{Re} \alpha Q(z) = \alpha \operatorname{Re} Q(z) = O(1), \quad z \in \mathbb{C}. \tag{5.51}$$

We obtain, that there exists $\dfrac{\partial}{\partial \overline{z}}$-primitive $\alpha a(z)$of the function $\alpha Q(z)$, satisfying the condition (5.51), i.e. $\alpha a(z) \in J_0(\mathbb{C})$.

Hence we proved, that the class $J_0(\mathbb{C})$ is a linear space over the filed of real numbers.

Let us prove, that

$$L_{p,2}(\mathbb{C}) \subset J_0(\mathbb{C}), \quad p > 2.$$

Let $a(z) \in L_{p,2}(\mathbb{C})$. Consider the following function

$$Q(z) = \big(T_E a\big)(z) = -\frac{1}{\pi} \iint_E \frac{a(\zeta)}{\zeta - z}\, d\xi\, d\eta, \quad \zeta = \xi + i\eta.$$

By virtue of the theorem 1.14 from [125], we have

$$\frac{\partial Q}{\partial \overline{z}} = \big(T_E a\big)_{\overline{z}} = a(z) \tag{5.52}$$

and from the theorem 1.11 from [125], we have that the function $Q(z) = (T_E a)(z)$ is bounded function on the whole plane. Therefore $\operatorname{Re} Q(z)$ is a bounded on the whole plane:

$$\operatorname{Re} Q(z) = O(1), \quad z \in \mathbb{C}. \tag{5.53}$$

From the above mentioned we conclude, that there exists $\frac{\partial}{\partial \overline{z}}$-primitive $Q(z)$ of $a(z) \in L_{p,2}(\mathbb{C})$, satisfying the condition (5.53). i.e. $a \in J_0(\mathbb{C})$. The theorem is proved.

Theorem 5.7 *The following equality*

$$J_0(\mathbb{C}) \cap J_1(\mathbb{C}) = \varnothing \tag{5.54}$$

holds.

Proof. Suppose the contrary. Assume that the condition (5.54) is not fulfilled. Then there exists the function $a(z)$ such that

$$a \in J_0(\mathbb{C}) \cap J_1(\mathbb{C}).$$

and there exists $\frac{\partial}{\partial \overline{z}}$ primitives $Q_0(z)$ and $Q_1(z)$ of $a(z)$, satisfying the conditions:

$$\operatorname{Re} Q_0(z) = O(1), \operatorname{Re} Q_1(z) = O(1), \quad z \in \mathbb{C}. \tag{5.55}$$

Since $a \in J_1(\mathbb{C})$, then there exists $\frac{\partial}{\partial \overline{z}}$-primitives $Q_1(z)$ of $a(z)$, satisfying the condition

$$\lim_{z \to \infty} e^{Q_1(z)} = 0. \tag{5.56}$$

It follows from (5.56) that

$$\lim_{z \to \infty} \left| e^{Q_1(z)} \right| = \lim_{z \to \infty} e^{\operatorname{Re} Q_1(z)} = 0. \tag{5.57}$$

Consider the function

$$\Phi(z) = Q_1(z) - Q_0(z).$$

Since $\frac{\partial Q_0(z)}{\partial \overline{z}} = a(z)$, $\frac{\partial Q_1(z)}{\partial \overline{z}} = a(z)$, then it is easy to see, that

$$\frac{\partial \Phi}{\partial \overline{z}} = \frac{\partial}{\partial \overline{z}}(Q_1(z) - Q_0(z)) = \frac{\partial}{\partial \overline{z}} Q_1(z) - \frac{\partial}{\partial \overline{z}} Q_0(z) = 0.$$

Hence we have, that $\Phi(z)$ is a holomorphic function on the whole plane, i.e. $\Phi(z)$ is an entire function.

Consider the following entire function $h(z) = e^{\Phi(z)}$ which has no zeros on the complex plane $\mathbb{C}$.

From (5.55) and (5.57) we have

$$\lim_{z\to\infty} |h(z)| = \lim_{z\to\infty} \left|e^{\Phi(z)}\right| = \lim_{z\to\infty} e^{\operatorname{Re}\Phi(z)} = \\ = \lim_{z\to\infty} e^{\operatorname{Re}(Q_1(z)-Q_0(z))} = \lim_{z\to\infty} e^{\operatorname{Re}Q_1(z)} e^{-\operatorname{Re}Q_0(z)} = 0. \tag{5.58}$$

Consequently

$$\lim_{z\to\infty} h(z) = 0. \tag{5.59}$$

As far as $h(z)$ is an entire function, it follows from the equation (5.59) that $h(z)$ is identically equal to zero on the whole plane, i.e. $h(z) \equiv 0$, $z \in \mathbb{C}$. This contradiction proves the theorem.

Proposition 5.8 *The function $a(z)$, satisfying the condition*

$$a(z) \in J_1(\mathbb{C}), \quad -a(z) \in J_1(\mathbb{C}) \tag{5.60}$$

doesn't exist.

Proof. Suppose the opposite. Let $a(z)$ be a function satisfying (5.60), i.e. $a(z) \in J_1(\mathbb{C})$, $-a(z) \in J_1(\mathbb{C})$. Then from Proposition 5.5 we have $a(z) + (-a(z)) \equiv 0$, i.e. an identically zero function belongs $J_1(\mathbb{C})$.

$\dfrac{\partial}{\partial \overline{z}}$ primitive of the zero function is entire function. Therefore, it is clear, that $\dfrac{\partial}{\partial \overline{z}}$ primitive $Q(z)$ of the identically zero function involved in the definition of the class $J_1(\mathbb{C})$ is an entire function. Then we have

$$\lim_{z\to\infty} e^{Q(z)} = 0.$$

Taking into account, that $e^{Q(z)}$ is an entire function, by virtue of Liouville theorem we get

$$e^{Q(z)} \equiv 0, \quad z \in \mathbb{C}.$$

So we get the opposite. The proposition is proved.

Proposition 5.9 *Let $f(z)$ be an arbitrary entire function. Then $\overline{f(z)} \in J_0(\mathbb{C})$.*

Proof. As it is known every $\dfrac{\partial}{\partial z}$-primitive of the function $f(z)$ is given by the following formula:

$$h(z) = \int_{z_0}^{z} f(t)\,dt + C,$$

where c is arbitrary complex number. For the function $h(z)$ the following equality $\dfrac{\partial h}{\partial z} = f(z)$ is valid. Let $g(z)$ be $\dfrac{\partial}{\partial z}$ primitive of the function $f(z)$, i.e. the following equality $\frac{\partial g}{\partial z} = f(z)$ holds. It is evident that

$$\frac{\partial \overline{g}}{\partial \overline{z}} = \overline{\frac{\partial g}{\partial z}} = \overline{f(z)}. \tag{5.61}$$

Hence $\overline{g(z)}$ is one of the $\dfrac{\partial}{\partial \overline{z}}$-primitives of the function $\overline{f(z)}$.

It is evident also, that

$$\frac{\partial g}{\partial \overline{z}} = 0. \tag{5.62}$$

Consider the function

$$Q(z) = \overline{g(z)} - g(z) = -2i \operatorname{Im} g(z).$$

From (5.61) and (5.62) we have

$$\frac{\partial Q}{\partial \overline{z}} = \frac{\partial}{\partial \overline{z}}\big(\overline{g(z)} - g(z)\big) = \frac{\overline{g(z)}}{\partial \overline{z}} - \frac{\partial g(z)}{\partial \overline{z}} = \overline{f(z)}. \tag{5.63}$$

We get from the quality (5.63), that $Q(z)$ is $\dfrac{\partial}{\partial \overline{z}}$-primitive of the function $\overline{f(z)}$. On the other hand

$$\operatorname{Re} Q(z) = \operatorname{Re}(-2i \operatorname{Im} g(z)) = 0, \quad z \in \mathbb{C}.$$

Therefore, $\operatorname{Re} Q(z)$ is a bounded function on the whole plane $\mathbb{C}$ and $\overline{f(z)} \in J_0(\mathbb{C})$. The proposition is proved.

Proposition 5.10 *Let $f(z)$ be an entire function and $a(z) \in J_1(\mathbb{C})$. Then*

$$a(z) + \overline{f(z)} \in J_1(\mathbb{C}). \tag{5.64}$$

Proof. From the Proposition 5.9 we have ‘ $\overline{f(z)} \in J_0(\mathbb{C})$. Since $a(z) \in J_1(\mathbb{C})$, therefore from Proposition 5.5 we have

$$a(z) + \overline{f(z)} \in J_1(\mathbb{C}).$$

The theorem is proved.

Proposition 5.11 *The class $L_{p,2}(\mathbb{C}), p > 2$ is a proper subset of the class $J_0(\mathbb{C})$, i.e.*

$$\begin{aligned} &L_{p,2}(\mathbb{C}) \subset J_0(\mathbb{C}), \\ &L_{p,2}(\mathbb{C}) \neq J_0(\mathbb{C}), \quad p > 2. \end{aligned} \tag{5.65}$$

Proof. The first relation (5.65) was already proved above (see theorem 5.6). In order to prove the second relation let us seek the function of the class $J_0(\mathbb{C})$, not belonging to the class $L_{p,2}(\mathbb{C})$. Such functions are all the function of the following form $\overline{\Phi(z)}$, where $\Phi(z)$ is arbitrary non-zero entire function. In fact, from the Proposition 5.9 we have $\overline{\Phi(z)} \in J_0(\mathbb{C})$.

Prove, that $\overline{\Phi(z)} \notin L_{p,2}(\mathbb{C})$, $p > 2$. If we assume the contrary, we get that for some non-zero entire function the following inclusion

$$\overline{\Phi(z)} \in L_{p,2}(\mathbb{C}), \quad p > 2,$$

takes place, therefore,

$$\Phi(z) \in L_{p,2}(\mathbb{C}), \quad p > 2. \tag{5.66}$$

Let us expand the function $\Phi(z)$ in Teylor series in the neighborhood of the point $z = 0$

$$\Phi(z) = \sum_{k=0}^{\infty} c_k z^k, \quad z \in \mathbb{C}. \tag{5.67}$$

By virtue (5.66), from the definition of the class $L_{p,2}(\mathbb{C})$ we have

$$\Phi_2(z) = \frac{1}{|z|^2}\,\Phi\Big(\frac{1}{z}\Big) = \frac{1}{|z|^2}\sum_{k=0}^{\infty}\frac{c_k}{z^k} \in L_p(E_1), \quad p > 2, \tag{5.68}$$

where $E_1 = \{z : |z| \le 1\}$ is unit circle of the complex plane.

Consider the function

$$H_m(z) = \Big(\frac{z}{|z|}\Big)^m \frac{1}{|z|^2}\sum_{k=0}^{\infty}\frac{c_k}{z^k}, \quad z \neq 0, \tag{5.69}$$

where $m = 0, 1, 2, 3, \ldots,$ arbitrary non-positive number. Since $\Big|\Big(\frac{z}{|z|}\Big)^m\Big| = 1$, then from (5.68) we have

$$H_m(z) \in L_p(E_1), \quad p > 2. \tag{5.70}$$

Since $L_p(E_1) \subset L_1(E_1)$, then from (5.70) we have

$$H_m(z) \in L_1(E_1). \tag{5.71}$$

It follows from (5.71), that there exists the finite limit

$$\iint_{E_1} H_m(z)dx\,dy = \lim_{\varepsilon\to 0}\iint_{G_\varepsilon^0} H_m(z)dx\,dy, \tag{5.72}$$

where $G_\varepsilon^0 = \{z : \varepsilon \le |z| \le 1\}$.

Using the polar coordinates

$$x = r\cos\varphi, \quad y = r\sin\varphi, \quad z = re^{i\varphi}, \quad \varepsilon \le r \le 1, \quad 0 \le \varphi \le 2\pi.$$

Let's verify, that for arbitrary non-positive integer $m = 0, 1, 2, 3, \ldots$ there exists the finite limit:

$$\begin{aligned}
\lim_{\varepsilon\to 0}\iint_{G_\varepsilon^0} H_m(z)dx\,dy &= \lim_{\varepsilon\to 0}\iint_{G_\varepsilon^0}\Big(\frac{z}{|z|}\Big)^m\frac{1}{|z|^2}\sum_{k=0}^{\infty}\frac{c_k}{z^k}dx\,dy = \\
&= \lim_{\varepsilon\to 0}\sum_{k=0}^{\infty}\iint_{G_\varepsilon^0}\Big(\frac{z}{|z|}\Big)^m\frac{1}{|z|^2}\frac{c_k}{z^k}dx\,dy = \lim_{\varepsilon\to 0}\sum_{k=0}^{\infty}\int_0^{2\pi}d\varphi\int_\varepsilon^1\frac{e^{im\varphi}}{r^2}\,\frac{c_k}{r^k e^{ik\varphi}}\,r\,dr = \\
&= \lim_{\varepsilon\to 0}\sum_{k=0}^{\infty}\int_0^{2\pi}e^{i(m-k)\varphi}d\varphi\int_\varepsilon^1\frac{c_k}{r^{k+1}}\,dr = \lim_{\varepsilon\to 0}2\pi\int_\varepsilon^1\frac{c_m}{r^{m+1}}\,dr = 2\pi\lim_{\varepsilon\to 0}\int_\varepsilon^1\frac{c_m}{r^{m+1}}\,dr.
\end{aligned} \tag{5.73}$$

Since for every non-positive integer m, $m = 0, 1, 2, 3, \dots,$

$$\lim_{\varepsilon\to 0}\int_{\varepsilon}^{1}\frac{dr}{r^{m+1}} = \infty.$$

Then it comes from the equality (5.73), that for every non-positive integer m, $m = 0, 1, 2, 3, \dots,$ $c_m = 0$.

Hence $\Phi(z)$ is identically zero function $\Phi \equiv 0$. We get the contradiction, i.e. the non-zero entire function $\Phi(z)$ doesn't exist, satisfying the condition (5.66). The proposition is proved.

Examples. We construct some other examples of the functions of the class $J_0(\mathbb{C})$, not belonging to the class $L_{p,2}(\mathbb{C})$, $p > 2$.

The classes $J_0(\mathbb{C})$, $J_1(\mathbb{C})$ are introduced and studied in [63]. The following functions are introduced in [129]:

$$A_k(z) = (-1)^k z, \quad D_k(z) = (-1)^k z e^{|z|^2}, \quad z \in \mathbb{C}, \quad k = 1, 2. \tag{5.74}$$

Note, that these functions play a very important role in the theory of Carleman-Bers-Vekua irregular equations.

At it is easy to see

$$A_1 \in J_1(\mathbb{C}), \quad A_2 \notin J_1(\mathbb{C}), \quad D_1 \in J_1(\mathbb{C}), \quad D_2 \notin J_1(\mathbb{C}). \tag{5.75}$$

Indeed, we prove that $A_1 \in J_1(\mathbb{C})$, $A_1(z) = -z$. $A_1(z)$ is $\dfrac{\partial}{\partial \overline{z}}$-primitive of the function A_1 take the function $Q(z) = -z\,\overline{z} = -|z|^2$. Then it is evident, that the following equality

$$\lim_{z\to\infty}\left|z^k\, e^{Q(z)}\right| = \lim_{z\to\infty}\frac{|z|^k}{e^{|z|^2}} = 0$$

is fulfilled for every integer k. Therefore $A_1(z) \in J_1(\mathbb{C})$.

The function $A_2(z) = z$ doesn't belong to the class $J_1(\mathbb{C})$ (see Proposition 5.8). Verify, that the function $D_1(z) = -z\, e^{|z|^2}$ belongs to the class $J_1(\mathbb{C})$. Let us take as $\dfrac{\partial}{\partial \overline{z}}$-primitive the function $Q(z) = e^{-|z|^2}$ of the function $D_1(z)$. Then

$$\lim_{z\to\infty}\left|z^k\, e^{Q(z)}\right| = \lim_{z\to\infty}\frac{|z|^k}{e^{e^{|z|^2}}} = 0$$

for every integer k. Hence $D_1(z) \in J_1(\mathbb{C})$. The function $D_2(z) = z\, e^{|z|^2}$ is not the function of the class $J_1(\mathbb{C})$ (see Proposition5.8).

It is clear that the function $A_1(z) = -z$ is the particular case of the following function

$$A_{\lambda,\nu}(z) = \lambda |z|^{\nu}\, e^{i \arg z}, \quad z \in \mathbb{C}, \tag{5.76}$$

where ν is a real number and λ is a complex number, satisfying the condition

$$\nu > -1, \quad \operatorname{Re}\lambda < 0. \tag{5.77}$$

Indeed, when $\lambda = -1$, $\nu = 1$ then $A_{-1,1}(z) = -|z|\, e^{i \arg z} = -z = A_1(z)$.

It can be easily checked that $A_{\lambda,\nu} \in J_1(\mathbb{C})$. In fact it is evident, that $A_{\lambda,\nu} \in L_p^{\mathrm{loc}}(\mathbb{C})$, $p > 2$.

Consider the function

$$Q_{\lambda,\nu}(z) = \frac{2\lambda}{\nu+1}\, |z|^{\nu+1}, \quad z \in \mathbb{C} \tag{5.78}$$

which is $\dfrac{\partial}{\partial \overline{z}}$ -primitive of the function $A_{\lambda,\nu}(z)$. It can be checked directly if we rewrite the operator $\dfrac{\partial}{\partial \overline{z}}$ in polar coordinates $z = r\, e^{i\varphi}$ as follows

$$\frac{\partial}{\partial \overline{z}} = \frac{e^{i\varphi}}{2}\Big(\frac{\partial}{\partial r} + \frac{i}{r}\frac{\partial}{\partial \varphi}\Big), \tag{5.79}$$

$$\frac{\partial}{\partial \overline{z}}\frac{2\lambda}{\nu+1}\,|z|^{\nu+1} = \frac{2\lambda}{\nu+1}\frac{\partial}{\partial \overline{z}}\,|z|^{\nu+1} = \frac{2\lambda}{\nu+1}\frac{e^{i\varphi}}{2}\Big(\frac{\partial}{\partial r}\, r^{\nu+1} + \frac{i}{r}\frac{\partial}{\partial \varphi}\, r^{\nu+1}\Big) =$$

$$= \frac{2\lambda}{\nu+1}\frac{e^{i\varphi}}{2}(\nu+1) r^{\nu} = \lambda\, r^{\nu}\, e^{i\varphi} = \lambda |z|^{\nu}\, e^{i \arg z}.$$

Let k be an arbitrary integer. Then we get

$$\lim_{z\to\infty} \big|z^k\, e^{Q_{\lambda,\nu}(z)}\big| = \lim_{z\to\infty} |z|^k \big|e^{\frac{2\lambda}{\nu+1}|z|^{\nu+1}}\big| =$$

$$= \lim_{z\to\infty} |z|^k\, e^{\mathrm{Re}\,\frac{2\lambda}{\nu+1}|z|^{\nu+1}} =$$

$$= \lim_{z\to\infty} |z|^k\, e^{\frac{2}{\nu+1}|z|^{\nu+1}\,\mathrm{Re}\,\lambda} = \lim_{z\to\infty} \frac{|z|^k}{e^{-\frac{2}{\nu+1}|z|^{\nu+1}\,\mathrm{Re}\,\lambda}} = 0.$$

Hence we conclude that the condition (5.42) is fulfilled. Therefore,

$$A_{\lambda,\nu} \in J_1(\mathbb{C}), \quad \nu > -1, \quad \mathrm{Re}\,\lambda < 0.$$

Let us prove that for $\mathrm{Re}\,\lambda = 0$, $\lambda \neq 0$, $\nu > -1$, the function $A_{\lambda,\nu}(z)$ satisfies the condition

$$A_{\lambda,\nu} \in J_0(\mathbb{C}), \quad A_{\lambda,\nu} \notin L_{p,2}(\mathbb{C}), \quad p > 2.$$

Indeed, $A_{\lambda,\nu} \in L_p^{\mathrm{loc}}(\mathbb{C})$, $p > 2$,

$$\mathrm{Re}\, Q_{\lambda,\nu}(z) = \mathrm{Re}\, \frac{2\lambda}{\nu+1}\, |z|^{\nu+1} = \frac{2}{\nu+1}\, |z|^{\nu+1}\, \mathrm{Re}\,\lambda = 0.$$

Hence condition (5.40) is verified and $A_{\lambda,\nu} \in J_0(\mathbb{C})$.

Let us prove, that $A_{\lambda,\nu} \notin L_{p,2}(\mathbb{C})$, $p > 2$. Consider the following function

$$\left|\frac{1}{|z|^2}\,\lambda \frac{1}{|z|^{\nu}}\, e^{i \arg z}\right| = \frac{\lambda}{|z|^{\nu+2}} \notin L_p(E_1),$$

where $E_1 = \{z : |z| \leq 1\}$. Since $\nu+2 > 1$, $(\nu+2)p > p > 2$, we have $A_{\lambda,\nu} \notin L_{p,2}(\mathbb{C})$, $p > 2$.

Now we construct the functions of sufficiently general form, including the Vekua function D_1 as a particular case.

Let n be nonnegative integer and let $\nu, \delta_k,\ k = 0, 1, 2, \dots, n,\ \nu, \delta_k > 0,\ k = 0, 1, 2, \dots, n$ be positive numbers. Consider the functions

$$h_0(z) = \exp\left\{\delta_0 |z|^\nu\right\}, \quad h_1(z) = \exp\left\{\delta_1 h_0(z)\right\},$$
$$h_2(z) = \exp\left\{\delta_2 h_1(z)\right\}, \dots\ h_n(z) = \exp\left\{\delta_n h_{n-1}(z)\right\},$$
$$q(z) = -\frac{\nu\, e^{i \arg z}}{2}\, |z|^{\nu-1} \prod_{k=0}^{n} h_k(z).$$

Let us prove that

$$q(z) \in J_1(\mathbb{C}).$$

It is clear that $q(z) \in L_p^{\mathrm{loc}}(\mathbb{C})$, $p > 2$.

A function defined as

$$Q(z) = -\frac{1}{\prod\limits_{k=0}^{n} \delta_k}\, h_n(z)$$

is a $\dfrac{\partial}{\partial \overline{z}}$-primitive of the function $q(z)$. Indeed if we use the formula (5.79) we get

$$\frac{\partial Q}{\partial \overline{z}} = -\frac{1}{\prod\limits_{k=0}^{n} \delta_k} \frac{\partial h_n}{\partial \overline{z}} = -\frac{1}{\prod\limits_{k=0}^{n} \delta_k} \frac{\partial}{\partial \overline{z}} \exp\{\delta_n\, h_{n-1}(z)\} = -\frac{1}{\prod\limits_{k=0}^{n} \delta_k}\, e^{\delta_n\, h_{n-1}(z)} \delta_n \frac{\partial h_{n-1}}{\partial \overline{z}} =$$
$$= -\frac{1}{\prod\limits_{k=0}^{n} \delta_k}\, \delta_n\, h_n(z) \frac{\partial h_{n-1}}{\partial \overline{z}} = -\frac{1}{\prod\limits_{k=0}^{n} \delta_k}\, \delta_n\, h_n(z)\, \delta_{n-1}\, h_{n-1}(z) \frac{\partial h_{n-2}}{\partial \overline{z}} =$$
$$- \frac{1}{\prod\limits_{k=0}^{n} \delta_k}\, \delta_n\, \delta_{n-1} \cdots \delta_1\, h_n(z)\, h_{n-1}(z) \cdots h_1(z) \cdot \quad \cdot \exp\{\delta_0 |z|^\nu\} \frac{\partial}{\partial \overline{z}}\, \delta_0\, |z|^\nu =$$
$$= \frac{1}{\prod\limits_{k=0}^{n} \delta_k} \delta_n \delta_{n-1} \cdots \delta_1\, h_n(z)\, h_{n-1}(z) \cdots h_1(z)\, h_0(z) \cdot \delta_0 \frac{e^{i\varphi}}{2} \cdot \frac{\partial}{\partial r}\, r^\nu =$$
$$= -\frac{\nu}{2}\, e^{i \arg z} |z|^{\nu-1} \prod_{k-0}^{n} h_k(z) = q(z).$$

Therefore the function $Q(z)$ is $\dfrac{\partial}{\partial \overline{z}}$-primitive of the function $q(z)$.

Let k be an arbitrary integer. Then we have

$$\lim_{z\to\infty} \left| z^k \exp Q(z) \right| = \lim_{z\to\infty} |z|^k \exp\left\{ -\frac{1}{\prod\limits_{k=0}^{n} \delta_k}\, h_n(z) \right\} = 0.$$

Therefore the condition (5.42) is fulfilled. Hence it follows that $q(z) \in J_1(\mathbb{C})$.

Assume $n = 0$, $\delta_0 = 1$, $\nu = 2$ in the expression of the function $q(z)$, we obtain

$$q(z) = -e^{i \arg z}|z|h_0(z) = -ze^{|z|^2} = D_1(z).$$

Consequently, we get important properties for the classes of functions $J_0(\mathbb{C})$ and $J_1(\mathbb{C})$ and we can construct various representations of these classes as well.

5.2 The Liouville type theorems for Carleman-Bers-Vekua irregular equations

Let N be a given non-negative integer. Denote by $\Omega(N)$ the space of all regular solutions of the equation (2.11) in the complex plane $\mathbb{C}$ satisfying the condition

$$w(z) = O(z^N), \quad z \to \infty. \tag{5.80}$$

It is clear, that $\Omega(N)$ is an $\mathcal{R}$-linear space and as is known if the coefficients of the equation (2.11) satisfy the regularity condition (2.12) on $\mathbb{C}$, then

$$\dim_{\mathcal{R}} \Omega(N) = 2N + 2.$$

Indeed, in case a regular solution of the equation (2.11) satisfies the condition (2.12) on the whole plane, it is therefore a generalized polynomial of order at most N of the class $u_{p,2}(A, B, \mathbb{C})$, so in our case the class $\Omega(N)$ coincides with the class of the generalized polynomials of order at most N. This space is a linear space on $\mathcal{R}$ with the dimension equal to $2N + 2$. A basis of this space is

$$v_{2n}(z) = R_\infty^{A,B}(z^n), \quad v_{2n+1}(z) = R_\infty^{A,B}(iz^n), \quad n = 0, 1, 2, \ldots, N.$$

In fact, every generalized polynomial w of order at most N of the class $u_{p,2}(A, B, \mathbb{C})$ can be expanded into the following finite sum [124]:

$$w(z) = \sum_{n=0}^{2N+1} c_n \, v_n(z),$$

where the coefficients c_n are real numbers, and moreover, this expansion is unique.

By definition the *generalized constants* are bounded solutions of the equation (2.11) [124]. Therefore, $\Omega(0)$ is the space of generalized constants and $\dim_{\mathcal{R}} \Omega(0) = 2$.

In this section we investigate these cases of the coefficients of the equation (2.11) where B is always regular, i.e. belongs to $L_{p,2}(\mathbb{C}), p > 2$. The irregularity of (5.26) is concentrated in the coefficient A in the sense that $A \in J_j(\mathbb{C})$, $j = 0$ or 1. We calculate dimension of the vector space $\Omega(N)$ for arbitrary nonnegative integer N. We will call propositions of this *Liouville type theorems* for irregular Carleman-Bers-Vekua equations.

Let us prove in advance the following theorem, often needed in the sequel. This theorem is an extension of a result of Vekua from [125] to irregular equations.

Theorem 5.12 *Consider the Carleman-Bers-Vekua equation (2.11) with the coefficients*

$$A \in L_p^{\mathrm{loc}}(\mathbb{C}), \quad B \in L_{p,2}(\mathbb{C}), \quad p > 2. \tag{5.81}$$

Let $Q(z)$ be a $\dfrac{\partial}{\partial \overline{z}}$-primitive of the coefficient $A(z)$ on the whole complex plane. Then, if $w(z)$ is a regular solution of the equation (2.11) on the whole plane $\mathbb{C}$, then the function defined by

$$w^*(z) = w(z)\, e^{Q(z)}, \quad z \in \mathbb{C}, \tag{5.82}$$

is a regular solution of the following equation on the whole plane

$$\frac{\partial w^*}{\partial \overline{z}} + B^* \overline{w^*} = 0. \tag{5.83}$$

where $B^*(z) = B(z)\, e^{2i \operatorname{Im} Q(z)}$.

Conversely, if $w^*(z)$ is a regular solution of the regular equation then the function (5.82) is a regular solution of the equation (2.11) on the whole plane, i.e. the relation (5.82) establishes a bijection between the regular solutions of the equations (2.11) and (5.83).

Proof. Let $w(z)$ be a regular solution of the equation (2.11). Then by means of (2.11) and (5.82) we have the chain of equalities:

$$0 = \frac{\partial w}{\partial \overline{z}} + Aw + B\,\overline{w} = e^{-Q(z)}\Big(\frac{\partial w^*}{\partial \overline{z}} + B(z)\, e^{2i \operatorname{Im} Q(z)}\, \overline{w^*(z)}\Big) =$$

$$= e^{-Q(z)}\Big(\frac{\partial w^*}{\partial \overline{z}} + B^*(z)\, \overline{w^*(z)}\Big).$$

From this equality it follows, that

$$\frac{\partial w^*}{\partial \overline{z}} + B^*\, \overline{w^*(z)} = 0,$$

i.e. w^* is a regular solution of the equation (5.83).

Let $w^*(z)$ be a regular solution of the equation (5.83). Then it follows from the equalities (5.83) and (5.82) that

$$0 = \frac{\partial w^*}{\partial \overline{z}} + B^*(z)\, \overline{w^*(z)} = e^{Q(z)}\Big(\frac{\partial w}{\partial \overline{z}} + A(z)\, w(z) + B^*(z)\, e^{-2i \operatorname{Im} Q(z)}\, \overline{w(z)}\Big) =$$

$$= e^{Q(z)}\Big(\frac{\partial w}{\partial \overline{z}} + A(z)\, w(z) + B(z)\, \overline{w(z)}\Big).$$

It follows from this equality, that

$$\frac{\partial w}{\partial \overline{z}} + A(z)\, w(z) + B(z)\, \overline{w(z)} = 0.$$

i.e. the function w is a regular solution of the equation (5.26).

One has $B \in L_{p,2}(\mathbb{C})$, $p > 2$ and $|e^{2i \operatorname{Im} Q(z)}| = 1$, therefore $B^* \in L_{p,2}(\mathbb{C})$, $p > 2$. The theorem is proved.

Theorem 5.13 *Let the coefficients of the equation (2.11) satisfy the condition*

$$A \in J_1(C), \quad B \in L_{p,2}(\mathbb{C}), \quad p > 2. \tag{5.84}$$

Then for arbitrary nonnegative integer N

$$\dim \Omega(N) = 0. \tag{5.85}$$

Proof. Let $w \in \Omega(N)$, and $Q(z)$ be the $\dfrac{\partial}{\partial \overline{z}}$-primitive of the function $A(z)$, which participates in the definition of the class $J_1(\mathbb{C})$. Since w satisfies the condition

$$w(z) = O(z^N), \quad z \to \infty,$$

it follows that, by virtue of the Theorem 5.12 the function $w^*(z)$, defined by formula (5.82), is a regular solution of the equation (5.83) and

$$w^*(z)\, e^{-Q(z)} = O(z^N), \quad z \to \infty. \tag{5.86}$$

The condition (5.86) implies, that

$$w^*(z) = O\big(z^N\, e^{Q(z)}\big), \quad z \to \infty. \tag{5.87}$$

Since the function $Q(z)$ is the $\dfrac{\partial}{\partial \overline{z}}$-primitive of the function $A(z)$, which participates in the definition of the class $J_1(\mathbb{C})$, one has

$$\lim_{z \to \infty} z^N\, e^{Q(z)} = 0. \tag{5.88}$$

It follows from (5.87) and (5.88), that

$$\lim_{z \to \infty} w^*(z) = 0, \tag{5.89}$$

i. e. $w^*(z)$ is a regular solution of the regular equation (5.83). Since the condition (5.89) is fulfilled, one has by means of the Liouville theorem, concerning the generalized analytic functions, that $w^*(z) \equiv 0$ for arbitrary $z \in \mathbb{C}$. Then from the equality (5.82) one has $w(z) \equiv 0$ for arbitrary $z \in \mathbb{C}$. Therefore $\dim \Omega(N) = 0$.

Theorem 5.14 *Let the coefficients of the equation (2.11) satisfy the condition*

$$-A \in J_1(C), \quad B \in L_{p,2}(\mathbb{C}), \quad p > 2. \tag{5.90}$$

Then for arbitrary nonnegative integer N

$$\dim \Omega(N) = \infty. \tag{5.91}$$

Proof. Let $w \in \Omega(N)$ and let $-Q(z)$ be the $\dfrac{\partial}{\partial \overline{z}}$-primitive of the function $-A(z)$, which participates in the definition of the class $J_1(\mathbb{C})$. Then the function $w^*(z)$, defined by the formula (5.82), is a regular solution of the equation (5.83) and satisfies (5.87). On the contrary, if $w^*(z)$ is a regular solution of the regular equation (5.83) and satisfies (5.87), then the function $w(z)$ defined by the formula (5.82) is a regular solution of the equation (2.11) and satisfies (5.87), i.e. $w \in \Omega(N)$. Since the function w^* is a regular solution of the regular equation (5.83), by virtue of the main Lemma concerning the generalized analytic functions, there exists an entire function $\Phi^*(z)$, such that the condition

$$w^*(z) = \Phi^*(z)\, e^{-T_{\mathbb{C}}(B^* \frac{\overline{w^*}}{w^*})(z)}. \tag{5.92}$$

is fulfilled.

We conclude from the conditions (5.87) and (5.92), that

$$\Phi^*(z)\, e^{-T_{\mathbb{C}}(B^* \frac{\overline{w^*}}{w^*})(z)} = O\big(z^N\, e^{Q(z)}\big). \tag{5.93}$$

It follows from (5.93), that

$$\Phi^*(z) = O\big(z^N\, e^{Q(z)}\, e^{T_{\mathbb{C}}(B^* \frac{\overline{w^*}}{w^*})(z)}\big), \quad z \to \infty. \tag{5.94}$$

Since $B^* \in L_{p,2}(\mathbb{C})$, $p > 2$, and the function $\dfrac{\overline{w^*}}{w^*}$ is bounded on the whole plane, one has

$$B^* \frac{\overline{w^*}}{w^*} \in L_{p,2}(\mathbb{C}), \quad p > 2.$$

It means that the function $T_{\mathbb{C}}\Big(B^* \dfrac{\overline{w^*}}{w^*}\Big)(z)$ is bounded on the whole plane $\mathbb{C}$. Therefore it follows from (5.94), that

$$\Phi^*(z) = O\big(z^N\, e^{Q(z)}\big), \quad z \to \infty. \tag{5.95}$$

Conversely, if the entire function $\Phi^*(z)$ satisfies the condition (5.95), then the function $w^*(z)$ defined from the equation (5.94) is a regular solution of the equation (5.83) and satisfies the condition (5.87).

Every polynomial $\Phi^*(z) = a_0 z^n + a_1 z^{n-1} + \cdots + a_n$, $a_j \in \mathbb{C}$, $j = 0, 1, 2, \ldots, n$, satisfies the condition (5.95).

Indeed this follows from the equalities:

$$\begin{aligned}\lim_{z\to\infty} \Phi^*(z)\, z^{-N}\, e^{-Q(z)} &= \lim_{z\to\infty} (a_0 z^n + a_1 z^{n-1} + \cdots + a_n)\, z^{-N}\, e^{-Q(z)} = \\ &= a_0 \lim_{z\to\infty} z^{n-N}\, e^{-Q(z)} + a_1 \lim_{z\to\infty} z^{n-1-N}\, e^{-Q(z)} + \cdots + \\ &+ a_n \lim_{z\to\infty} z^{-N}\, e^{-Q(z)} = 0\end{aligned}$$

Here we used the fact, that $-Q(z)$ is the $\frac{\partial}{\partial \overline{z}}$-primitive of the function $-A(z)$, which participates in the definition of the class $J_1(\mathbb{C})$, therefore

$$\lim_{z\to\infty} z^{n-N}\, e^{-Q(z)} = 0, \quad \lim_{z\to\infty} z^{n-1-N}\, e^{-Q(z)} = 0, \dots$$
$$\lim_{z\to\infty} z^{-N}\, e^{-Q(z)} = 0.$$

Hence every generalized polynomial corresponding to the equation (5.83) satisfies the condition (5.87). Since the space of all generalized polynomials corresponding to the equation (5.83) is infinitedimensional, the space of solutions of the equation (5.83) satisfying the condition (5.87) is infinitedimensional.

Let $\{w_j^*\}$, $j = 1, 2, 3, \dots$, be an infinite system of linearly independent functions in this space. Let us prove that

$$w_j(z) = e^{-Q(z)}\, w_j^*(z), \quad j = 1, 2, 3, \dots$$

is the infinite system of linearly independent functions in the space $\Omega(N)$. Since $w_j^*(z)$ is a regular solution of the equation (5.83) and satisfies the condition (5.87), it follows that the function $w(z)$ is a regular solution of the equation (2.11) by means of the Theorem 5.13 and satisfies the condition $w_j(z) = O(z^N)$, i.e. $w_j(z) \in \Omega(N)$.

Let

$$\sum_{j=1}^{n} c_j\, w_j(z) = 0, \quad z \in E, \quad c_j \in R, \quad j = 1, 2, \dots, n.$$

Then we have

$$\sum_{j=1}^{n} c_j\, e^{-Q(z)}\, w_j^*(z) = 0, \quad z \in \mathbb{C}.$$

It follows from this equality, that

$$\sum_{j=1}^{n} c_j\, w_j^*(z) = 0.$$

Since the system of functions $\{w_j^*(z)\}$, $j = 1, 2, \dots, n$, is linearly independent, one has $c_j = 0$, $j = 1, 2, \dots, n$. i.e. we obtain, that the system of functions $\{w_j(z)\}$, $j = 1, 2, 3, \dots$, is linearly independent.

Consequently, the space $\Omega(N)$ is infinitedimensional.

Theorem 5.15 *Let the coefficients of the equation (2.11) satisfy the condition*

$$A \in J_0(\mathbb{C}), \quad B \in L_{p,2}(\mathbb{C}), \quad p > 2, \tag{5.96}$$

Then for arbitrary nonnegative integer N

$$\dim \Omega(N) = 2N + 2. \tag{5.97}$$

Proof. Let $w \in \Omega(N)$, and let $Q(z)$ be the $\dfrac{\partial}{\partial \overline{z}}$-primitive of the function $A(z)$, which participates in the definition of the class $J_0(\mathbb{C})$. Then the function defined by the formula (5.82), is a regular solution of the equation (5.83) and satisfies the condition (5.87).

From (5.87) we have

$$\big|w^*(z)\big| = O\big(|z|^N \, |e^{Q(z)}|\big), \quad z \to \infty.$$

or

$$\big|w^*(z)\big| = O\big(|z|^N \, e^{\operatorname{Re} Q(z)}\big), \quad z \to \infty.$$

Since the function $ReQ(z)$ is bounded on the whole complex plane, one has

$$\big|w^*(z)\big| = O\big(|z|^N\big), \quad z \to \infty.$$

Therefore

$$w^*(z) = O\big(z^N\big), \quad z \to \infty. \tag{5.98}$$

Conversely, if the function $w^*(z)$ is a regular solution of the equation (5.83) and satisfies the condition (5.98), then the function $w(z)$ defined by the formula (5.82) is a regular solution of the equation (2.11) and satisfies the condition (5.80), i.e. $w \in \Omega(N)$.

But the space of all regular solutions of the equation (5.83) satisfying the condition (5.98) is the space of generalized polynomials corresponding to the regular equation (5.83) of order at most N.

As is well, known the dimension of the space of generalized polynomials corresponding to the regular equation (5.83) of order at most N is equal to $2N+2$.

Let $\{w_j^*(z)\}$, $j = 1, 2, \dots, 2N+2$ be a basis of this space.

Let us prove that

$$w_j(z) = e^{-Q(z)} \, w_j^*(z), \quad j = 1, 2, \dots, 2N+2$$

is the basis of the space $\Omega(N)$. Since the function $w_j^*(z)$ is a regular solution of the equation (5.83) and satisfies the condition (5.98), using the Theorem 5.13 we have, that the function $w(z)$ is a regular solution of the equation (2.11) and satisfies the condition $w_j(z) = O(z^N)$, $z \to \infty$, i.e. $w_j \in \Omega(N)$.

Let

$$\sum_{j=1}^{2N+2} c_j \, w_j(z) = 0, \quad c_j \in R, \quad j = 1, 2, \dots, 2N+2, \quad z \in \mathbb{C}.$$

Then we get

$$\sum_{j=1}^{2N+2} c_j \, e^{-Q(z)} \, w_j^*(z) = 0, \quad z \in \mathbb{C}.$$

This equality implies, that

$$\sum_{j=1}^{2N+2} c_j \, w_j^*(z) = 0, \quad z \in \mathbb{C}.$$

Since the system of the functions $\{w_j^*(z)\}$, $j = 1, 2, \ldots, 2N+2$, is linearly independent, we have $c_j = 0$, $j = 1, 2, \ldots, 2N+2$. Therefore the system of the functions $\{w_j(z)\}$, $j = 1, 2, \ldots, 2N+2$ is linearly independent.

Let us prove that every function $w(z)$ from the space $\Omega(N)$ is representable by a linear combination of the system of the functions $\{w_j(z)\}$, $j = 1, 2, \ldots, 2N+2$. Indeed, as $w^*(z)$ defined by (5.82), is a generalized polynomial, corresponding to the equation (5.83) of order at most N, there exist real numbers $d_1, d_2, \ldots, d_{2N+2}$, such that the following equality holds

$$w^*(z) = \sum_{j=1}^{2N+2} d_j\, w_j^*(z).$$

This yields

$$e^{-Q(z)}\, w^*(z) = \sum_{j=1}^{2N+2} d_j\, e^{-Q(z)}\, w_j^*(z).$$

Hence

$$w(z) = \sum_{j=1}^{2N+2} d_j\, w_j(z),$$

i.e. the function $w(z)$ is representable by the linear combination of functions $w_1(z)$, $w_2(z)$, ..., $w_{2N+2}(z)$. The theorem is proved.

Examples. Here we give examples of irregular Carleman-Bers-Vekua equations and prove, that the dimension of the spaces of generalized constants may be zero, as well as infinity. In [125] the examples of irregular equations $\frac{\partial w}{\partial \overline{z}} + zw = 0$ are considered and $\frac{\partial w}{\partial \overline{z}} - zw = 0$ and various essential properties of the corresponding solutions spaces are pointed out. We consider Vekua examples from the point of view of our approach as $z, -z \in J_j(\mathbb{C})$. Before that we will prove a simple auxiliary result which is applied further on.

Proposition 5.16 *The solution of the equation on the whole plane*

$$\frac{\partial w}{\partial \overline{z}} + Aw = 0, \tag{5.99}$$

where $A \in L_p^{\text{loc}}(\mathbb{C})$, $p > 2$, has the form

$$w(z) = \Phi(z)\, e^{-Q(z)}, \tag{5.100}$$

where $Q(z)$ is one of the $\dfrac{\partial}{\partial \overline{z}}$-primitives of the function $A(z)$ and $\Phi(z)$ is an arbitrary entire function.

First let us prove, that every function w, which can be represented in the form (5.100), where $\Phi(z)$ is an entire function, is a solution of the equation (5.99).

This follows from the following equalities:

$$\frac{\partial w}{\partial \overline{z}} = -\Phi(z)\, e^{-Q(z)} \frac{\partial Q}{\partial \overline{z}} = -A(z)\, w(z).$$

Let us prove now, that if w is a regular solution of the equation (5.99) then it is an representable by (5.100).

Consider the function $\Phi(z) = w(z)\, e^{Q(z)}$. We have

$$\frac{\partial \Phi}{\partial \overline{z}} = \frac{\partial}{\partial \overline{z}}\, w(z)\, e^{Q(z)} = e^{Q(z)} \Big(\frac{\partial w}{\partial \overline{z}} + A(z)\, w(z) \Big) = 0.$$

From this it follows, that the function $\Phi(z)$ is an entire function.

Now let us give the examples.

1.Consider the equation

$$\frac{\partial w}{\partial \overline{z}} - zw = 0 \tag{5.101}$$

and let us prove, that $dim\Omega(N) = 0$.

Indeed, the $\dfrac{\partial}{\partial \overline{z}}$-primitive of the function $A(z) = -z$ is the function $Q(z) = -z\,\overline{z} = -|z|^2$. Therefore by (5.100)

$$w(z) = \Phi(z)\, e^{|z|^2}. \tag{5.102}$$

Let $w \in \Omega(N)$. Then the condition (5.80) is fulfilled. It follows from the condition (5.80), that $\Phi(z)\, e^{|z|^2} = O(z^N), \quad z \to \infty$; from this it follows that $\Phi(z) = O\big(z^N\, e^{-|z|^2}\big), \quad z \to \infty$. Since $\lim\limits_{z\to\infty} |z|^N\, e^{-|z|^2} = \lim\limits_{z\to\infty} \frac{|z|^N}{e^{|z|^2}} = 0$, one has $\lim_{z\to\infty} \Phi(z) = 0$.

It is evident, that by virtue of the Liouville theorem $\Phi(z) \equiv 0$, $z \in \mathbb{C}$. So we have $w(z) = 0$, i.e. $\Omega(N) = \{0\}$.

2. Let us prove that $\dim \Omega(N) = \infty$ for the equation

$$\frac{\partial w}{\partial \overline{z}} + zw = 0. \tag{5.103}$$

The $\dfrac{\partial}{\partial \overline{z}}$-primitive of the function $A(z) = z$ is the function $Q(z) = z\,\overline{z} = |z|^2$. It follows from (5.100) that the general regular solution of the equation (5.103) on the whole plane is representable in the following form:

$$w(z) = \Phi(z)\, e^{-|z|^2}, \tag{5.104}$$

where $\Phi(z)$ is arbitrary entire function.

Let us prove, that every function of the form (5.104) belongs to the corresponding space $\Omega(N)$ of the equation (5.103), where $\Phi(z)$ is an arbitrary entire function of order ρ_Φ which is less than 2 (see [125]).

In fact, consider the real number β satisfying the condition $\rho_\Phi < \beta < 2$. Then it is clear from the definition of the order of the function, that $\Phi(z) = O\big(e^{|z|^\beta}\big), \quad z \to \infty$.

It follows from this that $\Phi(z) = O\big(|z|^N e^{|z|^2}\big), \quad z \to \infty$. Therefore $\Phi(z)\, e^{-|z|^2} = O\big(|z|^N\big)$ and $w(z) = O\big(|z|^N\big), \quad z \to \infty$, i.e. $w \in \Omega(N)$.

It is evident, that the space of all those entire functions whose orders are less than 2 is infinitedimensional (for example, every classical polynomial belongs to this space). Hence the space of the functions of the form (5.104), where $\Phi(z)$ is arbitrary entire function of order less than 2 is infinitedimensional too. Since such functions belong to $\Omega(N)$, this space is infinitedimensional.

The example of an irregular Carleman-Bers-Vekua equation with $dim\Omega(0) = 1$ is given in [109].

5.3 The generating triple

In this subsection we introduce a generating pair, corresponding to solution space of the irregular Carleman-Bers-Vekua equation by analogy to the Bers generating pair in the regular case.

We consider the following irregular Carleman-Bers-Vekua system

$$w_{\overline{z}} + Aw + B\overline{w} = 0, \tag{5.105}$$

where $A \in L_p^{loc}(\mathbb{C}), B \in L_{p,2}(\mathbb{C}), p > 2$.

Let $Q(z)$ be a $\partial_{\overline{z}}$-primitive of $A(z)$. By definition, it means that $Q_{\overline{z}} = A(z)$ on $\mathbb{C}$. Let (F_1, G_1) be the generating pair [11] of the following regular system

$$w_{1\overline{z}} + B_1\overline{w}_1 = 0,$$

where $B_1(z) = B(z)e^{2iImQ(z)}$. Therefore (F_1, G_1) is a generating pair for the class $u_{p,2}(0, B_1)$ such that a) $F_1, G_1 \in C_{\frac{p-2}{2}}(\mathbb{C})$ b) $F_{1\overline{z}}, G_{1,\overline{z}} \in L_{p,2}(\mathbb{C})$ and c) there exists $K_0 > 0$, such that $Im(\overline{F}_1(z)G_1(z)) \geq K_0 > 0$.

It is known, that the functions $F(z) = F_1(z)e^{-Q(z)}$ and $G(z) = G_1(z)e^{-Q(z)}$ are the solutions of (5.105). It is clear that $F, G \in C_{\frac{p-2}{p}}(\mathbf{C})$, $F_{\overline{z}}, G_{\overline{z}} \in L_p^{loc}(\mathbf{C})$ and it is easy to check that $Im(\overline{F(z)e^{Q(z)}}G(z)e^{Q(z)}) > K_0$, it means, that (F, G) is *is generating pair*. From this it follows, that the functions F and G satisfy the following identity

$$F_{\overline{z}} + AF + B\overline{F} = 0, \quad G_{\overline{z}} + AG + B\overline{G} = 0. \tag{5.106}$$

Consider (5.106) as a linear system of equations with respect to $A(z)$ and $B(z)$. The determinant of this system is equal to $-2Im(\overline{F}G) \neq 0$. Therefore,

$$A = \frac{\overline{G}F_{\overline{z}} - \overline{F}G_{\overline{z}}}{G\overline{F} - F\overline{G}}, \quad B = \frac{FG_{\overline{z}} - GF_{\overline{z}}}{G\overline{F} - F\overline{G}}.$$

We call (F, G, Q) a *generating triple* of the irregular system by analogy with the Bers generating pair of the pseudoanalytic functions [11]. Using this concept it is possible to define irregular pseudoanalytic functions similar to regular ones.

5.4 The problem of linear conjugation for some classes of Carleman-Bers-Vekua equation

In this section we investigate the problem of linear conjugation for the CBV equation

$$\frac{\partial w}{\partial \overline{z}} + Aw + B\overline{w} = 0, \tag{5.107}$$

on the whole complex plane $\mathbb{C}$, where

$$A \in L_p^{\text{loc}}(\mathbb{C}), \quad B \in L_{p,2}(\mathbb{C}), \quad p > 2. \tag{5.108}$$

Let Γ be a smooth closed curve with inner domain D^+ and outer domain D^-. Let $G(t)$ and $g(t)$ be the given functions of the class $H_\alpha(\Gamma)$, $0 < \alpha \leq 1$. In addition let $G(t) \neq 0$ everywhere on Γ.

Consider the following problem:

Find a piecewise regular solution of the equation (5.107) *with the boundary curve* Γ *satisfying the conditions:*

$$w^+(t) = G(t)\, w(t) + g(t), \quad t \in \Gamma, \tag{5.109}$$

$$w(z) = O\big(z^N e^{-Q(z)}\big), \quad z \to \infty, \tag{5.110}$$

where N is a given integer, Q(z) *is* $\dfrac{\partial}{\partial \overline{z}}$*-primitive of the function $A(z)$.*

As it was proved in section 2, the condition (5.108) provides the existence of $\dfrac{\partial}{\partial \overline{z}}$-primitive and every $\dfrac{\partial}{\partial \overline{z}}$-primitive is Hölder-continuous function on each compact of the plane.

Note, that the boundary value problem (5.107),(5.109), (5.110) is studied in case when A and B are satisfying the regularity condition, i.e. $A, B \in L_{p,2}(\mathbb{C}), \quad p > 2$.

Consider the CBV regular self-conjugated equations:

$$\frac{\partial V}{\partial \overline{z}} + B_1 \overline{V} = 0, \tag{5.111}$$

$$\frac{\partial V'}{\partial \overline{z}} - \overline{B}_1 \overline{V'} = 0, \tag{5.112}$$

where $B_1(z) = B(z)\, \dfrac{\overline{X(z)}}{X(z)}\, e^{2i\,\mathrm{Im}\, Q(z)}$ and $X(z)$ is the canonical solution of the problem of linear conjugation

$$X^+(t) = G(t)\, X^-(t), \quad t \in \Gamma.$$

for holomorphic functions.

The following theorems are valid.

Theorem 5.17 *Let $\chi + N \geqslant -1$, where $\chi = \frac{1}{2\pi}\left[\arg G(t)\right]_\Gamma$ is the index of the function $G(t)$ on the curve Γ. Then the general solution of the problem (5.107), (5.109), (5.110) is given by the formula*

$$w(z) = \left[\frac{X(z)}{2\pi i}\int_\Gamma \Omega_1(z,t)\frac{g_1(t)}{X^+(t)}\,dt - \Omega_2(z,t)\overline{\frac{g_1(t)}{X^+(t)}}\,d\bar{t} + X(z)\,V_{\chi+N}(z)\right]e^{-Q(z)},$$

where $g_1(t) = g(t)\,e^{Q(t)}$, $t \in \Gamma$. $V_{\chi+N}(z)$ is the generalized polynomial of the class $u_{p,2}(0, B_1, \mathbb{C})$ of order at most $\chi + N$ and it is assumed that $V_{-1}(z) \equiv 0$, $z \in \mathbb{C}$.

Theorem 5.18 *Let $\chi + N \leq -2$, then the necessary and sufficient solvability conditions for the problem (5.107), (5.109), (5.110) are the following conditions*

$$\operatorname{Im}\int_\Gamma v_k'(t)\,\frac{g_1(t)}{X^+(t)}\,dt = 0, \quad k = 0, 1, 2, \ldots 2(-N-\chi)-3. \tag{5.113}$$

If the condition (5.113) is fulfilled then the solution of the problem (5.107), (5.108), (5.110) is given by the formula:

$$w(z) = \left[\frac{X(z)}{2\pi i}\int_\Gamma \Omega_1(z,t)\frac{g_1(t)}{X^+(t)}\,dt - \Omega_2(z,t)\overline{\frac{g_1(t)}{X^+(t)}}\,d\bar{t}\right]e^{-Q(z)}.$$

Proof. Let the function $w(z)$ be a solution of the problem (5.107), (5.109), (5.110). Then by means of the Theorem 2.12, the function $w^*(z)$ defined by the formula (5.82) is a regular solution of the regular equation (5.83) in the domains D^+ and D^-.

Since $Q(z)$ is a continuous function on the whole complex plane then from (5.82) we have

$$w^\pm(t) = \lim_{\substack{z\to t\\ z\in D^\pm}} w^*(z)\,e^{-Q(z)} = \lim_{z\to t} e^{-Q(z)} \lim_{\substack{z\to t\\ z\in D^\pm}} w^*(z) = \\ = e^{-Q(t)}(w^*)^\pm(t), \quad t \in \Gamma,$$

i.e.

$$w^+(t) = e^{-Q(t)}(w^*)^+(t), \quad t \in \Gamma, \tag{5.114}$$

$$w^-(t) = e^{-Q(t)}(w^*)^-(t), \quad t \in \Gamma. \tag{5.115}$$

It follows from the conditions (5.109), (5.114), (5.115), that

$$e^{-Q(t)}(w^*)^+(t) = G(t)\,e^{-Q(t)}(w^*)^-(t) + g(t), \quad t \in \Gamma. \tag{5.116}$$

From (5.116) we conclude, that

$$(w^*)^+(t) = G(t)(w^*)^-(t) + g_1(t), \quad t \in \Gamma. \tag{5.117}$$

Since $g \in H_\alpha(\Gamma)$, $Q \in H_{\frac{p-2}{p}}(\Gamma)$ then $g_1 \in H_\beta(\Gamma)$, where $\beta = \min\Big(\alpha, \frac{p-2}{p}\Big)$. From the conditions (5.82) and (5.110) we have

$$e^{-Q(z)} w^*(z) = O\big(z^N e^{-Q(z)}\big), \quad z \to \infty. \tag{5.118}$$

Therefore

$$w^*(z) = O(z^N), \quad z \to \infty. \tag{5.119}$$

We proved, that if $w(z)$ is a solution of the problem (5.107), (5.109), (5.110) then the function $w^*(z)$ defined by (5.82) is a solution of the problem (5.83), (5.117), (5.119).

Let us prove the converse. Let the function $w^*(z)$ be a solution of the problem (5.83), (5.117), (5.119). Then by virtue of the Theorem 2.12 the function $w(z)$ defined by (5.82) is a regular solution of the equation (5.107) in the domains D^+ and D^-.

From condition (5.117) follows (5.116); from conditions (5.114)-(5.116) follows (5.109); from (5.118) follows (5.119) and from (5.110) follows (5.118). It means that if $w^*(z)$ is a solution of the problem (5.83), (5.117), (5.119), then the function $w(z)$ defined by (5.82) is a solution of the problem (5.107), (5.109), (5.110). Hence the problem (5.107), (5.109), (5.110) is equivalent to the problem (5.83). Let $\chi + N \geqslant -1$, then by means of the Theorem 1.20 the general Solution of the problem (5.83), (5.117), (5.119) is given by the formula:

$$w^*(z) = \frac{X(z)}{2\pi i} \int_\Gamma \Omega_1(z,t) \frac{g_1(t)}{X^+(t)}\, dt - \Omega_2(z,t) \overline{\frac{g_1(t)}{X^+(t)}}\, d\bar{t} + X(z)\, V_{\chi+N}(z),$$
$$z \in D^+, \quad z \in D^-,$$

where $V_{\chi+N}(z)$ is a generalized polynomial of order at most $\chi + N$ of the class $u_{p,2}(0, B_1, \mathbb{C})$. Here $V_{-1}(z) \equiv 0$. Hence it follows that the general solution of the problem (5.107), (5.109), (5.110) is given by the formula:

$$w(z) = w^*(z)\, e^{-Q(z)} = \left[\frac{X(z)}{2\pi i} \int_\Gamma \Omega_1(z,t) \frac{g_1(t)}{X^+(t)}\, dt - \right.$$
$$\left. - \Omega_2(z,t) \overline{\frac{g_1(t)}{X^+(t)}}\, d\bar{t} + X(z)\, V_{\chi+N}(z)\right] e^{-Q(z)}.$$

Let $\chi + N \le -2$; then by means of the Theorem 1.21 the necessary and sufficient solvability conditions for the problem (5.83), (5.117), (5.119) is the fulfillment of the following conditions:

$$\operatorname{Im} \int_\Gamma v_k'(t) \frac{g_1(t)}{X^+(t)}\, dt = 0, \quad k = 0, 1, 2, \ldots 2(-N - \chi) - 3.$$

these are the conditions (5.113). If the conditions (5.113) are fulfilled then the solution of the problem (5.83), (5.117), (5.119) is given by the formula:

$$w^*(z) = \frac{X(z)}{2\pi i} \int_\Gamma \Omega_1(z,t) \frac{g_1(t)}{X^+(t)}\, dt - \Omega_2(z,t) \overline{\frac{g_1(t)}{X^+(t)}}\, d\bar{t}.$$

Since the problem (5.107), (5.109), (5.110) is equivalent of the problem (5.83), (5.117), (5.119) then the necessary and sufficient condition for the problem (5.107), (5.109), (5.110) to be solvable is the fulfillment of the condition (5.113). If the condition (5.113) is fulfilled then the solution of the problem (5.107), (5.109), (5.110) is given by the formula:

$$w(z) = w^*(z)\, e^{-Q(z)} =$$
$$= \left[\frac{X(z)}{2\pi i} \int_\Gamma \Omega_1(z,t) \frac{g_1(t)}{X^+(t)}\, dt - \Omega_2(z,t) \frac{\overline{g_1(t)}}{\overline{X^+(t)}}\, d\bar{t} \right] e^{-Q(z)}.$$

The theorems are proved.

Let the coefficients of the equation (5.107) satisfy the conditions:

$$A \in L_p^{\mathrm{loc}}(\mathbb{C}), \quad A \in J_0(\mathbb{C}), \quad B \in L_{p,2}(\mathbb{C}), \quad p > 2. \tag{5.120}$$

Since $A \in J_0(\mathbb{C})$ then there exists $\dfrac{\partial}{\partial \bar{z}}$-primitive $Q(z)$ of the function $A(z)$ such that the function $ReQ(z)$ is a bounded function on the whole plane. Consider the problem (5.107), (5.109), (5.110) for such a function $Q(z)$. Then the condition (5.110) will be equivalent to following classical condition:

$$w(z) = O(z^N), \quad z \to \infty. \tag{5.121}$$

Indeed, assume that the condition (5.110) is fulfilled. Then

$$|w(z)| = O\big(|z|^N\, |e^{-Q(z)}|\big), \quad z \to \infty.$$

From here we have

$$|w(z)| = O\big(|z|^N\, |e^{-\operatorname{Re} Q(z)}|\big), \quad z \to \infty.$$

Since $\operatorname{Re} Q(z)$ is bounded on the whole plane then $|w(z)| = O\big(|z|^N\big), \quad z \to \infty$ or $w(z) = O(z^N)$. From the condition (5.121) there follows the condition (5.110). Therefore if the conditions (5.120) are fulfilled, the problem (5.107), (5.109), (5.110) is the generalization of the classical problem.

6 Singular points of solutions of some elliptic systems on the plane

6.1 Degenerate elliptic systems

As was mentioned above, I. Vekua's scientific interest was concentrated on construction of the theory of generalized analytic functions and its applications in geometry and in the theory of elastic shells. I. Vekua systematically indicated the necessity of investigation of irregular equations. Let us now consider the following equation

$$\frac{\partial w}{\partial \overline{z}} + \frac{a(z)}{f(z)}\, w + \frac{b(z)}{g(z)}\, \overline{w} = 0, \tag{6.1}$$

in some domain G of z-plane; $a, b \in L_p(G)$, $p > 2$; f and g are analytic functions on G, they may have zeros of arbitrary order and essential singularities. I. Vekua called these functions analytic regularizators of the coefficients of the equation (6.1).

One of the fundamental results (and important tool of investigation of this equation) of the theory of generalized analytic functions is the general representation of solution by the analytic functions. Precisely for any $w(z)$ there exists a function $\Phi(z)$ analytic in G, such that

$$w(z) = \Phi(z)\, \exp\{\Omega(z)\}, \tag{6.2}$$

where

$$\Omega(z) = \frac{1}{\pi f(z)} \iint_G \frac{a(s)}{\zeta - z}\, dG(\zeta) + \frac{1}{\pi g(z)} \iint_G \frac{b(s)}{\zeta - z}\, \frac{\overline{\omega(\zeta)}}{\omega(\zeta)}\, dG(\zeta). \tag{6.3}$$

For regular coefficients the conversion of this relation is given in I. Vekua's famous monograph [124], by the given analytic function $\Phi(z)$ the solution $w(z)$ is constructed. For general case this important result was also generalized by himself.

In regular case this relation completely reveals the properties of generalized analytic functions however even if one of the functions f and g has essentially singular point then nothing meaningful is known on behavior of the solution of the equation (6.1) in the neighborhood of this point. It is unknown how to use the relation in this case too.

Much more is known in case when f and g have zeros but do not have essential singularities. This type of equations are called Carleman-Vekua equations with polar singularities.

Consider typical and important in applications the following Carleman-Vekua equation with polar singularities

$$|z|^{\nu}\, \frac{\partial w}{\partial \overline{z}} + a(z)\, w + b(z)\, \overline{w} = 0, \tag{6.4}$$

where the real number $\nu > 0$, $a, b \in L_p(G)$, $p > 2$ and G contains some neighborhood of $z = 0$ except this point (perforated neighborhood of $z = 0$). For these equations (differing form the regular case $\nu = 0$) it case take place very unexpected phenomena take place.

Very important is I. Vekua's emotional attitude to these problems, which he expressed as follows: "some simple examples show the complicate character of these problems" [125].

To make it clear let's consider the following examples:

$$|z|^{\nu}\frac{\partial}{\partial \overline{z}} + \varepsilon(\cos\varphi + i\sin\varphi)w = 0, \tag{6.5}$$

where $\nu > 1$, $\varepsilon = \pm 1$.

It is easy to show that the solutions of this equation in the neighborhood of $z = 0$ have essentially different behavior for $\varepsilon = 1$ and $\varepsilon = -1$. It follows that the problem of construction of general theory of such singular equations is very different and indeterminable however the validity of the following proposition about the structure of solutions of these equations under general assumptions for given ν, a, b is proved: every solution $w(z)$ of the equation (6.4) satisfying the condition

$$w(z) = O(\Psi(z)), \quad z \to 0 \tag{6.6}$$

for some analytic in the domain G function $\Psi(z)$ is identically zero; every function $\Psi(z)$ satisfying the condition

$$\Psi(z) = O(w(z)), \quad z \to 0 \tag{6.7}$$

in the domain G is identically zero for some solution $w(z)$.

From above the following conclusion holds: the structure of solutions of Carleman-Bers-Vekua equations with polar singularities is principally nonanalytic.

We have obtained correct statement and complete analysis of boundary value problems for a sufficiently wide class of equations of such type. They are first order singular equations. The equations of higher order undoubtly are of much theoretical and practical interest. In this connection let's consider the following system

$$\sum_{k=0}^{m} z^{\nu_k} A_k \frac{\partial^k w}{\partial \overline{z}^k} = 0, \tag{6.8}$$

where m, ν are given natural numbers, A_k $(0 \leq k \leq m)$ are given complex square $n \times n$-matrices. Under the solution of this system we mean a vector-function $w = (w_1, \ldots, w_n)$ of the class $C^m(G)$ satisfying the system (6.8) at every point of G. Note, that G is, as above, perforated neighborhood of $z = 0$. Assume that

$$\det A_0 \neq 0, \quad \det A_m \neq 0, \quad A_k \cdot A_j = A_j \cdot A_k, \quad 0 \leq j, k \leq m. \tag{6.9}$$

Construct all possible polynomials of the form

$$\tau_m \zeta^m + \tau_{m-1}\zeta^{m-1} + \cdots + \tau_1 \zeta + \tau_0 = 0, \tag{6.10}$$

where the coefficients τ_k are some eigenvalues of the matrix A_k, $(0 \leq k \leq m)$. Denote by Δ the set of all complex roots of these polynomials and introduce a number $\delta_0 = \min\limits_{\zeta\in\Delta} |\zeta|$, obviously $\delta_0 > 0$.

Along with the solution $w(z)$ of the system (6.8) construct its characteristic function

$$T_w(\rho) = \max_{0\leq\varphi\leq 2\pi} \sum_{k=1}^{n} \sum_{p=0}^{m-1} \left| \frac{\partial^p \omega}{\partial \overline{z}^p} (\rho\, e^{i\varphi}) \right|, \quad \rho > 0. \tag{6.11}$$

The following theorem holds:

Theorem 6.1 *Let $\nu \geqslant 2$ and $\Psi(z)$ be some analytic function in G. Let the solution $w(z)$ of the system satisfy the condition*

$$T_w(z) = O\bigg(|\Psi(z)| \exp\left\{ \frac{\delta}{|z|^\delta} \right\} \bigg), \quad z \to 0. \tag{6.12}$$

where δ is some number and $\sigma < \nu - 1$.

Then the solution $w(z)$ is identically zero vector-function. Moreover when the condition (6.12) is fulfilled $w(z)$ is also trivial if

$$\sigma = \nu - 1, \quad \delta < \delta_0 \cos \pi\beta, \quad \beta = \max\left\{ \nu, \frac{\nu - 3}{2\nu - 2} \right\}. \tag{6.13}$$

Note that, in particular, where $\nu = 2$ for this system we succeeded to state correct boundary value problem to make its complete analysis.

6.2 Quasilegular Carleman-Bers-Vekua equations

In the present section the structures of the solutions of some important classes of singular elliptic systems on the plane are investigated. In particular, it is proved, that the solutions of such systems have principally nonanalytic behavior in the neighborhood of fixed singular points. These results gave the possibility to state correctly the boundary value problems and make their complete analysis.

To illustrate the possible structures of the solutions of quasiregular equation (1.6) consider the following simplest one

$$\frac{\partial w}{\partial \overline{z}} + \frac{\lambda \cdot \exp(i\,\varphi)}{r^2}\, w = 0, \tag{6.14}$$

where the complex number $\lambda \neq 0$, r, φ - are polar coordinates of the variable z, $z = r\exp(i\varphi)$ and the main thing is, that the domain G contains the origin $z = 0$. It is clear, that (6.14) is an irregular equation. Checking directly we get, that the function w is contained in the class

$$S_\lambda \equiv \mathfrak{A}^* \left(\frac{\lambda \cdot \exp(i\,\varphi)}{r^2}, 0, G \right) \tag{6.15}$$

if and only if w has the form

$$w(z) = \Phi(z) \cdot \exp\left\{\frac{2\lambda}{r}\right\}, \tag{6.16}$$

where the function $\Phi \in \mathfrak{A}_0^*(G)$. It follows from (6.16), that the classes S_λ aren't of single type. In order to explain what has been said note, that in case $\operatorname{Re}\lambda = 0$ the module of every function of the class S_λ coincides with the module of analytic function in the domain G. When $\operatorname{Re}\lambda > 0$ in the class S_λ there are neither nontrivial bounded in the neighborhood of the point $z = 0$ functions nor the functions with power growth; i.e. the functions admitting estimate

$$w(z) = O\left(\frac{1}{|z|^\sigma}\right), \quad z \to 0, \tag{6.17}$$

for some real number $\sigma > 0$. When $\operatorname{Re}\lambda < 0$ in S_λ there exists an extensive subclass, every function of which more rapidly then arbitrary positive power of $|z|$ while $z \to 0$.

When $\operatorname{Re}\lambda > 0$, there aren't nontrivial regular solutions of the equation (6.14) in the point $z = 0$. Indeed, if the function w satisfies the equation (6.14) in $z = 0$, then it has the form (6.16), where Φ is an analytic function in some neighborhood $V_\rho(0) = \{z : |z| < \rho\}$, $\rho > 0$, of the point $z = 0$. It is clear, that the function (6.16) can't satisfy the following condition

$$\iint_{V_\rho(0)} |w(z)|\, dx\, dy < +\infty$$

for every number $\rho > 0$ and for a nontrivial analytic function Φ. When $\operatorname{Re}\lambda \le 0$ the equation (6.14) has an extensive class of regular solutions in $z = 0$. When $\operatorname{Re}\lambda < 0$ the formula (6.16) gives the regular in $z = 0$ solution and if the function Φ has the pole of arbitrary power and even in the case when Φ has the essential singular point in $z = 0$, but hasn't rapidly exponential growth in its neighborhood. This growth measure is limited by the multiplier $\exp\left\{\frac{2\lambda}{r}\right\}$. When $\operatorname{Re}\lambda < 0$ the regular (of sufficiently wide class) solutions of the equation (6.14) have zero of infinite order, i.e. these solutions w satisfy the condition

$$\lim_{z \to 0} \frac{w(z)}{(z - z_0)^k} = 0, \quad k = 0, 1, 2, \ldots$$

In these cases $\operatorname{Re}\lambda < 0$ the equation (6.14) also has regular solutions w (their class is sufficiently wide) also such that $z = 0$ is limit point of their zeroes.

The equation (6.14) is the particular case of the equation (1.6) with the coefficients

$$\begin{aligned} A(z) &= \frac{\lambda \cdot \exp(i\,\varphi)}{r^\nu} + \frac{A_0(z)}{r^{\nu_1}} + \overline{h(z)}, \\ B(z) &= \frac{B_0(z)}{r^\mu}, \end{aligned} \tag{6.18}$$

where the real numbers ν, ν_1, μ satisfy the condition

$$\mu \geqslant 0, \quad \nu \geqslant \max\big\{[\mu]+2, [\nu_1]+2\big\}, \tag{6.19}$$

and the functions

$$h \in \mathfrak{A}_0^*(G), \quad A_0, B_0 \in L_p(G), \quad p > 2 \tag{6.20}$$

(G is a bounded domain containing the origin).

Most statements formulated above for the model equation (6.14) can be proved for the equation (1.6) with the coefficients (6.18) also.

The picture, described above for the class $\mathfrak{A}^*(A, B, G)$, sharply changes, if we carry the apparently insignificant change in the coefficient $A(z)$ from (6.18), namely we get a very interesting picture if we consider the coefficients

$$\begin{gathered} A(z) = \frac{\lambda \cdot \exp(i\,m\,\varphi)}{|z|^{\nu}} + \frac{A_0(z)}{|z|^{\nu_1}} + \overline{h(z)}, \\ B(z) = \frac{B_0(z)}{|z|^{\mu}}\,, \end{gathered} \tag{6.21}$$

where m is a natural number and with respect to other parameters of the functions A, B the above assumptions (6.19) and (6.20) are fulfilled.

It is clear, that the equation (1.6) with the coefficients (6.21) is quasiregular. Using the relation (6.2) for this equation first and then applying a modification of the well known principle of Phragmen-Lindelöf from the function theory we get the following theorem

Theorem 6.2 *Let the generating pair of the class $\mathfrak{A}^*(A, B, G)$ be of the form (6.21), let the conditions (6.19) and (6.20) be fulfilled and*

$$\lambda \neq 0, \quad m > 1, \quad m \neq \nu, \tag{6.22}$$

then every solution $w \in \mathfrak{A}^*(A, B, G)$ satisfying the condition

$$w(z) = O\big(\Psi(z)\big), \quad z \to 0, \tag{6.23}$$

for some function $\Psi \in \mathfrak{A}_0^*(G)$ is identically zero.

The essential extension of the Theorem 6.2 is proved; the existence of the real number $\delta_0 > 0$, such that every solution w of $\mathfrak{A}^*(A, B, G)$, (the generating pair should satisfy the conditions of the Theorem 6.2) satisfying the following condition

$$w(z) = O\left(\Psi(z) \cdot \exp\left\{\frac{\delta}{|z|^{\nu-1}}\right\}\right), \quad z \to 0, \tag{6.24}$$

for some $\delta < \delta_0$, $\Psi \in \mathfrak{A}_0^*(G)$ is identically zero is also proved.

From Theorem 6.2 (taking as the analytic function $\Psi \equiv 1$) we get immediately the triviality of solution of the class $\mathfrak{A}^*(A, B, G)$ bounded in the neighborhood of the singular for the equation point $z = 0$. Further, let the solution $w \in \mathfrak{A}^*(A, B, G)$ has the power of the growth

$$w(z) = O(\frac{1}{|z|^\sigma}), z \to 0 \tag{6.25}$$

for some $\sigma > 0$. Taking as $\Psi(z)$ the function

$$\Psi(z) = \frac{1}{z^{[\sigma]+1}}$$

we conclude that $w \equiv 0$.

As the next application of the Theorem 6.2 consider arbitrary solution $w \in \mathfrak{A}^*(A, B, G)$ (the generating pair should satisfy the conditions of the Theorem 6.2) and let the analytic function $\Psi \in \mathfrak{A}_0^*(G)$ satisfy the condition

$$\Psi(z) = O\big(w(z)\big), \quad z \to 0. \tag{6.26}$$

Applying the functions w, Ψ we construct the function

$$W = \frac{\Psi}{w},$$

which is bounded in the neighborhood of $z = 0$.

Direct checking gives, that

$$\frac{\partial W}{\partial \overline{z}} - AW - \frac{B\,\overline{\Psi}}{\Psi}\left(\frac{W}{\overline{\overline{W}}}\right)^2 \overline{W} = 0, \tag{6.27}$$

i.e. W is a quasiregular solution of the quasiregular equation (6.27). It is evident, that for the coefficients of this equation all conditions of the Theorem 6.2 are fulfilled and therefore the solution $W \equiv 0$, i.e. $\Psi \equiv 0$.

Summarizing the above we conclude that the following theorem is valid.

Theorem 6.3 *Let the generating pair of the class $\mathfrak{A}^*(A, B, G)$ be of the form (6.21) and let the conditions (6.19),(6.20),(6.22) be fulfilled. Then every function $\Psi \in \mathfrak{A}_0^*(G)$ satisfying the condition (6.26) for some solution $w \in \mathfrak{A}^*(A, B, G)$ is identically zero.*

It follows from the Theorems 6.2, 6.3 that the quasiregular solutions of the equation (1.6) of sufficiently wide class in the neighborhood of singular point of the equation doesn't admit the estimation (neither from above nor from below) by the module of the analytic function and therefore the behavior of the solution is non-analytic. But these solutions have one common property: they are remaining the behavior of the analytic functions in the neighborhood of the essentially singular point. Namely, these solutions have no limit in singular (for the equation) point. Indeed, the unboundedness of every nontrivial function $w \in \mathfrak{A}^*(A, B, G)$ imply that no finite limit exists in the point $z = 0$.

Let us prove the impossibility of the equality

$$\lim_{z\to 0} w(z) = \infty. \tag{6.28}$$

In fact, if the equality (6.28) takes place then there exists a real number $\rho > 0$ such, that

$$|w(z)| \geqslant 1, \quad 0 < |z| < \rho,$$

however, this is impossible by virtue of the Theorem 6.3. Summarizing what has been told we conclude that the following theorem is valid.

Theorem 6.4 *Let the generating pair of the class $\mathfrak{A}^*(A, B, G)$ be of the form 6.21) and let the conditions(6.19), (6.20), (6.22) be fulfilled. Then every nontrivial function $w \in \mathfrak{A}^*(A, B, G)$ has no limit (neither finite nor infinite) in singular for the equation point $z = 0$.*

In conclusion of this section we note, that we have investigated the boundary problems for a sufficiently wide class of quasiregular equations (1.6) and their complete analysis in some sense.

6.3 Correct boundary value problems for some classes of singular elliptic differential equations on a plane

The investigation of differential equations of the type

$$\frac{\partial^n \omega}{\partial \overline{z}^n} + a_{n-1}\frac{\partial^{n-1}\omega}{\partial \overline{z}^{n-1}} + a_{n-2}\frac{\partial^{n-2}\omega}{\partial \overline{z}^{n-2}} + \cdots + a_0\omega = 0$$

with sufficiently smooth coefficients $a_0, a_1, \ldots, a_{n-1}$ (the theory of meta-analytic functions) traces back to the work of G. Kolosov [72]. Subsequently, numerous papers in this direction were published by many authors. The present section deals with some singular cases of the above-given equation. Correct boundary value problems are pointed out, and their in some sense complete analysis are given.

In the domain G containing the origin of the plane of a complex variable $z = x+iy$ we consider a differential equation of the type

$$E_\nu\omega \equiv z^{2\nu}\frac{\partial^2\omega}{\partial \overline{z}^2} + Az^\nu\frac{\partial\omega}{\partial \overline{z}} + B\omega = 0, \tag{6.29}$$

where A and B are given complex numbers, $\nu \geq 2$ is a given natural number and as usual $\frac{\partial}{\partial \overline{z}} \equiv \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)$. To avoid more simple case we assume that

$$B \neq 0. \tag{6.30}$$

The function $\omega(z)$ is said to be a solution of the equation (6.29), if it belongs to the class $C^2(G \setminus \{0\})$ and satisfies (6.29) at every point of the domain $G \setminus \{0\}$. We denote by $\mathbb{K}$ the set of such functions; it should be noted that it is wide enough.

Every non-trivial (not identically equal to zero) function from the set $\mathbb{K}$, being a classical solution of an elliptic differential equation in the neighborhood of any non-zero point of the domain G, has an isolated singularity at the point $z = 0$. The analysis of the structure of the functions $\omega \in \mathbb{K}$ shows highly complicated nature of their behaviour (in the vicinity of the singular point $z = 0$) and, undoubtedly, is of independent interest because it allows one to obtain a priori estimates of solutions and of their derivatives which in turn are necessary for the correct statement and for the investigation of boundary value problems. A highly complicated nature of behaviour of solutions in the vicinity of the origin can be explained first by the fact that the equation (6.29) at the point $z = 0$ degenerates up to the zero order.

For every function $\omega(z) \in \mathbb{K}$ we introduce the following natural characteristic, i.e., the function of the real argument $\rho > 0$,

$$T_\omega(\rho) \equiv \max_{0\le\varphi\le 2\pi} \Big\{ \Big| \omega(\rho e^{i\varphi}) \Big| + \Big| \frac{\partial \omega}{\partial \overline{z}}(\rho e^{i\varphi}) \Big| \Big\}. \tag{6.31}$$

According to Theorem 6.2 proved below, we in particular conclude that for every non-trivial solution $\omega(z)$ the function (6.31) increases more rapidly not only than an arbitrary power of $\frac{1}{\rho}$ as $\rho \to 0$, but more rapidly than the function $\exp\{\frac{\delta}{\rho^{\nu-1}}\}$ for certain positive numbers δ.

With the equation (6.29) is closely connected the characteristic equation

$$\lambda^2 + A\lambda + B = 0,$$

where λ is an unknown complex number, which, by (6.30), has two non-zero, possibly coinciding, roots; we denote them by λ_1 and λ_2, and in what follows it will be assumed that

$$|\lambda_1| \le |\lambda_2|. \tag{6.32}$$

Having the roots λ_1 and λ_2, we can factorize the operator E_ν in the form

$$E_\nu = \Big(z^\nu \frac{\partial}{\partial \overline{z}} - \lambda_1 I \Big) \circ \Big(z^\nu \frac{\partial}{\partial \overline{z}} - \lambda_2 I \Big),$$

and immediately obtain that every function $\omega(z) \in \mathbb{K}$ under the condition

$$\lambda_1 \ne \lambda_2$$

is representable as

$$\omega(z) = \phi(z) \exp\Big\{ \frac{\lambda_1 \overline{z}}{z^\nu} \Big\} + \psi(z) \exp\Big\{ \frac{\lambda_2 \overline{z}}{z^\nu} \Big\}, \tag{6.33}$$

and under the condition

$$\lambda_0 \equiv \lambda_1 = \lambda_2$$

as

$$\omega(z) = [\phi(z)\overline{z} + \psi(z)] \exp\Big\{ \frac{\lambda_0 \overline{z}}{z^\nu} \Big\}, \tag{6.34}$$

where $\phi(z)$ and $\psi(z)$ are arbitrary holomorphic functions in the domain $G \setminus \{0\}$; $z = 0$ is an isolated singular point for $\phi(z)$ and $\psi(z)$.

2^0. We will need the following two statements whose proof is based on the well-known Fragman–Lindelöf principle (see, e.g., [120] and also [85])in the sequel.

Lemma 6.5 *Let $\phi(z)$ be a function holomorphic in the deleted neighborhood of the point $z = 0$ and such that*

$$\phi(z) = 0\,(\exp\{g(z)\}), \quad z \to 0, \tag{6.35}$$

where

$$g(z) = \frac{1}{|z|^{k-2}}\{\delta + a\cos(k\arg z) + b\sin(k\arg z)\},$$

$k \geq 3$ is natural, δ, a, b are real numbers, and

$$\delta = \sqrt{a^2 + b^2}\cos\pi\beta, \quad \beta = \max\Big\{0, \frac{k-4}{2k-4}\Big\}.$$

Then the function $\phi(z)$ is identically equal to zero.

Lemma 6.6 *Let ϕ be a function holomorphic in the deleted neighborhood of the point $z = 0$ and such that the condition (6.35) is fulfilled with*

$$g(z) = \frac{1}{|z|}\{\sqrt{a^2 + b^2} + a\cos(3\arg z) + b\sin(3\arg z)\}$$

and a, b real numbers. Then the function $\phi(z)$ has the removable singularity at the point $z = 0$.

3^0. The following theorem holds

Theorem 6.7 *Let δ be a real number such that $\delta < |\lambda_1|\cos\pi\beta$, where*

$$\beta = \max\Big\{0, \frac{\nu - 3}{2\nu - 2}\Big\}. \tag{6.36}$$

Then for every non-trivial solution $\omega(z) \in \mathbb{K}$

$$\varlimsup_{\rho \to 0+} \frac{T_\omega(\rho)}{\exp\Big\{\frac{\delta}{\rho^{\nu-1}}\Big\}} = +\infty. \tag{6.37}$$

Proof. First, let $\lambda_1 \neq \lambda_2$. Then differentiating the general solution (6.33) with respect to $\overline{z}$, we have

$$\frac{\partial\omega}{\partial\overline{z}} = \frac{\lambda_1}{z^\nu}\phi(z)\exp\Big\{\frac{\lambda_1\overline{z}}{z^\nu}\Big\} + \frac{\lambda_2}{z^\nu}\psi(z)\exp\Big\{\frac{\lambda_2\overline{z}}{z^\nu}\Big\},$$

which together with (6.33)gives

$$\begin{aligned}\phi(z)\exp\Big\{\frac{\lambda_1\overline{z}}{z^\nu}\Big\} &= \frac{1}{\lambda_1-\lambda_2}\Big(\lambda_1\omega - z^\nu\frac{\partial\omega}{\partial\overline{z}}\Big),\\ \psi(z)\exp\Big\{\frac{\lambda_2\overline{z}}{z^\nu}\Big\} &= \frac{1}{\lambda_1-\lambda_2}\Big(\lambda_2\omega - z^\nu\frac{\partial\omega}{\partial\overline{z}}\Big).\end{aligned} \tag{6.38}$$

Let for some solution $\omega(z)\in\mathbb{K}$ the condition (6.37) be violated; this means that there exist positive numbers M and ρ_0 such that

$$T_\omega(\rho)\le M\exp\Big\{\frac{\delta}{\rho^{\nu-1}}\Big\},\quad 0<\rho<\rho_0,$$

whence, with considering (6.31), we obtain

$$\begin{gathered}|w(\rho e^{i\varphi})|\le M\cdot\exp\{\frac{\delta}{\rho^{\nu-1}}\},\\ \Big\|\frac{\partial\omega}{\partial\overline{z}}(\rho e^{i\varphi})\Big\|\le M\cdot\exp\Big\{\frac{\delta}{\rho^{\nu-1}}\Big\},\quad 0<\rho<\rho_0,\quad 0\le\varphi\le 2\pi.\end{gathered} \tag{6.39}$$

In its turn, from (6.39) and (6.38) the existence of a positive number M_0 follows such that

$$\begin{gathered}|\phi(z)|\le M_0\exp\Big\{\frac{1}{|z|^{\nu-1}}[\delta-|\lambda_1|\cos(\psi_1-(\nu+1)\varphi)]\Big\},\\ |\psi(z)|\le M_0\exp\Big\{\frac{1}{|z|^{\nu-1}}[\delta-|\lambda_2|\cos(\psi_2-(\nu+1)\varphi)]\Big\},\\ 0<|z|<\rho_0,\quad 0\le\varphi\le 2\pi,\end{gathered} \tag{6.40}$$

where $\varphi=\arg z$, $\psi_k=\arg\lambda_k$, $k=1,2$.

From the inequalities (6.40) by virtue of Lemma 6.5 we get that $\phi(z)\equiv\psi(z)\equiv 0$, i.e., the solution $\omega(z)$ is trivial.

Let now $\lambda_0\equiv\lambda_1=\lambda_2$. Then differentiating the general solution (6.34) with respect to $\overline{z}$, we have

$$\frac{\partial\omega}{\partial\overline{z}}=\Big[\phi(z)\Big(1+\frac{\lambda_0\overline{z}}{z^\nu}\Big)+\frac{\lambda_0}{z^\nu}\psi(z)\Big]\exp\Big\{\frac{\lambda_0\overline{z}}{z^\nu}\Big\},$$

which together with (6.34) gives

$$\begin{aligned}z^\nu\phi(z)\exp\Big\{\frac{\lambda_0\overline{z}}{z^\nu}\Big\} &= z^\nu\frac{\partial\omega}{\partial\overline{z}}-\lambda_0\omega,\\ z^\nu\psi(z)\exp\Big\{\frac{\lambda_0\overline{z}}{z^\nu}\Big\} &= (z^\nu+\lambda_0\overline{z})\omega-\overline{z}z^\nu\frac{\partial\omega}{\partial\overline{z}}.\end{aligned} \tag{6.41}$$

The formulas (6.41) obtained above are analogous to the formulas (6.38) which allows to repeat our reasoning and conclude that the non-trivial solutions $\omega(z)\in\mathbb{K}$ are unable to violate the condition (6.37).

From the above-proved theorem it follows immediately that for every non-trivial solution $\omega(z) \in \mathbb{K}$

$$\overline{\lim_{\rho\to 0+}} \frac{T_\omega(\rho)}{\exp\left\{\frac{\delta}{\rho\sigma}\right\}} = +\infty,$$

where δ is some real number, and the real number $\sigma < \nu - 1$.

4^0. Theorem 6.7 admits generalizations to more general systems of differential equations of the type

$$\sum_{k=0}^{m} z^{\nu k} A_k \frac{\partial^k \omega}{\partial \overline{z}^k} = 0, \tag{6.42}$$

where $\nu \geq 2$, $m \geq 1$ are given natural numbers A_k, $k = 0, 1, \ldots, m$, are given complex square matrices of dimension $n \times n$, and

$$\det A_0 \neq 0, \quad \det A_m \neq 0, \tag{6.43}$$

$$A_k A_j = A_j A_k, \quad j, k = 0, 1, \ldots, m. \tag{6.44}$$

Under a solution of the system (6.42) we mean the vector function $\omega(z) = (\omega_1(z), \omega_2(z), \ldots, \omega_n(z))$ belonging to the class $C^m(G \setminus \{0\})$ and satisfying (6.42) at every non-zero point of the domain G.

By Λ we denote the set of all possible complex roots of the polynomial

$$\sum_{k=0}^{m} \tau_k \lambda^k = 0,$$

where the coefficient τ_k is some eigenvalue of the matrix A_k, $k = 0, 1, \ldots, m$. Introduce the number

$$\delta_0 \equiv \min_{\lambda\in\Lambda} |\lambda|,$$

which by (6.43) satisfies the inequality $\delta_0 > 0$.

The following theorem holds.

Theorem 6.8 *Let $\psi(z)$ be a function analytic in some deleted neighborhood of the point $z = 0$ and having possibly arbitrary isolated singularities (concentration of singularities of the function $\psi(z)$ at the point $z = 0$ is not excluded). Further, let δ, σ be real numbers such that either $\sigma < \nu - 1$ (σ is arbitrary) or $\sigma = \nu - 1$, $\delta < \delta_0 \cos \pi\beta$ where the number β is given by the formula (6.36). Then there are no non-trivial solutions of the system (6.42) satisfying the asymptotic condition*

$$\widetilde{T}_\omega(|z|) = 0\Big(|\psi(z)| \exp\left\{\frac{\delta}{|z|^\sigma}\right\}\Big), \quad z \to 0,$$

where

$$\widetilde{T}_\omega(\rho) \equiv \max_{0\leq\varphi\leq 2\pi} \sum_{k=1}^{n} \sum_{p=0}^{m-1} \Big|\frac{\partial^p \omega_k}{\partial \overline{z}^p}(\rho e^{i\varphi})\Big|, \quad \rho > 0.$$

5^0. Everywhere below G will denote a finite domain (containing the origin of coordinates of the complex plane) with the boundary Γ consisting of a finite number of simple, closed, non-intersecting Lyapunov contours. In the sequel, we will consider the special case of the equation (6.29), when $\nu = 2$, i.e., we consider the equation

$$z^4 \frac{\partial^2 \omega}{\partial \bar{z}^2} + Az^2 \frac{\partial \omega}{\partial \bar{z}} + B\omega = 0, \tag{6.45}$$

and study the following two boundary value problems.

Problem $R(\delta, \sigma)$. On the contour Γ there are prescribed, Hölder continuous functions $a(t)$, $\gamma(t)$ where the function $\gamma(t)$ is real and $a(t) \neq 0$, $t \in \Gamma$. Real positive numbers δ, σ are also given. It is required to find a continuously extendable to $\overline{G} \setminus \{0\}$ solution of the equation (6.45) satisfying both the asymptotic condition

$$\overline{\lim_{\rho \to 0}} \frac{T_\omega(\rho)}{\exp\left\{\frac{\delta}{\rho^\sigma}\right\}} < +\infty \tag{6.46}$$

and the boundary condition

$$\mathrm{Re}\{a(t)\omega(t)\} = \gamma(t), \quad t \in \Gamma. \tag{6.47}$$

Problem $Q(\delta, \sigma)$**.** On the contour Γ there are prescribed Hölder continuous functions $\gamma_k(t)$, $a_{k,m}(t)$, $k, m = 1, 2$, where $\gamma_1(t)$, $\gamma_2(t)$ are real and

$$\det \|a_{k,m}(t)\| \neq 0, \quad t \in \Gamma.$$

Real positive numbers δ, σ are also given. It is required to find a continuously extendable (together with its derivative $\frac{\partial \omega}{\partial \bar{z}}$) to $\overline{G} \setminus \{0\}$ solution of the equation (6.45) satisfying both the condition (6.46) and the boundary condition

$$\mathrm{Re}\{a_{k,1}(t)\omega(t) + a_{k,2}(t)\frac{\partial \omega}{\partial \bar{z}}(t)\} = \gamma_k(t), \quad t \in \Gamma, \quad k = 1, 2. \tag{6.48}$$

Along with the problems formulated above, let us consider the following boundary value problems.

Problem $R_0(\overline{p})$. Given an integer p, it is required to find a function $\phi_0(z)$ holomorphic in the domain G, continuously extendable to $\overline{G}$ and satisfying the boundary condition

$$\mathrm{Re}\{\alpha(t)\phi_0(t)\} = \gamma(t), \quad t \in \Gamma, \tag{6.49}$$

where $\alpha(t) = a(t)t^{2-p}\exp\{\frac{\lambda_1 \bar{t}}{t^2}\}$.

Problem $Q_0'(p)$. Given an integer p, it is required to find a vector function $(\phi_0(z), \psi_0(z))$ holomorphic in the domain G, continuously extendable to $\overline{G}$ and satisfying the boundary condition

$$\mathrm{Re}\{\alpha_{k,1}(t)\phi_0(t) + \alpha_{k,2}(t)\psi_0(t)\} = \gamma_k(t), \quad t \in \Gamma, \quad k = 1, 2, \tag{6.50}$$

where

$$\alpha_{k,m}(t)=\Big[a_{k,1}(t)t^{2-p}+\frac{\lambda_m a_{k,2}(t)}{t^p}\Big]\exp\Big\{\frac{\lambda_m\bar t}{t^2}\Big\},\quad k,m=1,2.$$

Problem $Q_0''(p)$**.** Given an integer p, it is required to find a vector function $(\phi_0(z)),\ \psi_0(z)$ holomorphic in the domain G, continuously extendable to $\overline{G}$ and satisfying the boundary condition

$$\operatorname{Re}\{\beta_{k,1}(t)\phi_0(t)+\beta_{k,2}(t)\psi_0(t)\}=\gamma_k(t),\quad t\in\Gamma,\quad k=1,2,\tag{6.51}$$

where

$$\beta_{k,1}(t)=\Big[\frac{a_{k,1}(t)}{t^p}|t^2|+a_{k,2}(t)\Big(t^{1-p}+\frac{\lambda_0\bar t}{t^{2+p}}\Big)\Big]\exp\Big\{\frac{\lambda_0\bar t}{t^2}\Big\}$$
$$\beta_{k,2}(t)=\Big[a_{k,1}(t)t^{2-p}+\frac{\lambda_0}{t^p}a_{k,2}(t)\Big]\exp\Big\{\frac{\lambda_0\bar t}{t^2}\Big\}.$$

On the basis of the following obvious relations

$$\alpha(t)\neq 0,\quad t\in\Gamma,$$
$$\det\|\beta_{k,m}(t)\|=-t^{3-2p}\det\|a_{k,m}(t)\|e^{\frac{2\lambda_0\bar t}{t^2}}=0,\quad t\in\Gamma,$$
$$\det\|\alpha_{k,m}(t)\|=(\lambda_2-\lambda_1)t^{2-2p}\det\|a_{k,m}(t)\|e^{\frac{\lambda_1+\lambda_2}{t^2}\bar t}\neq 0,\quad t\in\Gamma,$$

if only $\lambda_1\neq\lambda_2$, we conclude that for every integer p the problems $R_0(p)$, $Q_0'(p)$, $Q_0''(p)$ refer to those boundary value problems which are well-studied (see, e.g., [23], [99]). In particular, it is known that the corresponding homogeneous problems $(\gamma(t)\equiv\gamma_1(t)\equiv\gamma_2(t)\equiv 0)$ have finite numbers of linearly independent solutions [1] (and as it is easy to see, these numbers become arbitrarily large as $p\to+\infty$). Also formulas for the index and the solvability criteria of the problems are available.

6^0. We have the following

Theorem 6.9 *Let* $|\lambda_1|<|\lambda_2|$. *Then the boundary value problems* $R(|\lambda_1|,1)$ *and* $R_0(0)$ *are simultaneously solvable (unsolvable), and in case of their solvability the relation*

$$\omega(z)=z^2\phi_0(z)\exp\Big\{\frac{\lambda_1\bar z}{z^2}\Big\},\quad z\in G\setminus\{0\},\tag{6.52}$$

establishes a bijective relation between the solutions of these problems.

Proof. First we have to find a general representation of solutions of the equation (6.45) which are continuously extendable to $\overline{G}\setminus\{0\}$ and satisfy the condition (6.46), where $\delta=|\lambda_1|$, $\sigma=1$. Towards this end, we use the equalities (6.38) and find that the functions $\phi(z)$ and $\psi(z)$, holomorphic in the domain $G\setminus\{0\}$, satisfy the conditions

$$\phi(z)=0\Big(\exp\Big\{\frac{|\lambda_1|}{|z|}\Big[1-\cos(\psi_1-3\arg z)\Big]\Big\}\Big),\quad z\to 0,$$

[1] Here and everywhere, the linear independence is understood over the field of real numbers.

$$\psi(z) = 0\Big(\exp\Big\{\frac{|\lambda_1|}{|z|}\Big[1 - \frac{|\lambda_2|}{|\lambda_1|}\cos(\psi_2 - 3\arg z)\Big]\Big\}\Big), \quad z \to 0,$$
$$\psi_k = \arg\lambda_k, \quad k = 1, 2.$$

The first of the above conditions on the basis of Lemma 6.6 shows that $z = 0$ is a removable singular point for the function $\phi(z)$. Next, if we take into account the inequality $\left|\frac{\lambda_2}{\lambda_1}\right| > 1$, then by virtue of Lemma 6.5 the second condition shows that the function $\psi(z) \equiv 0$. This immediately implies that the relation

$$\frac{\partial\omega}{\partial\overline{z}} = \frac{\lambda_1}{z^2}\phi(z)\exp\Big\{\frac{\lambda_1\overline{z}}{z^2}\Big\}$$

is valid. Consequently,

$$\left|\frac{\lambda_1}{z^2}\right| |\phi(z)| = 0\Big(\exp\Big\{\frac{|\lambda_1|}{|z|}(1 - \cos(\psi_1 - 3\arg z))\Big\}\Big), \quad z \to 0. \tag{6.53}$$

In turn, (6.53) yields

$$\left|\frac{\lambda_1}{z^2}\right| |\phi(z)| = 0(1), \quad z \to 0, \quad \arg z = \frac{\psi_1}{3}. \tag{6.54}$$

Considering the Taylor series expansion of the holomorphic function $\lambda_1\phi(z)$

$$\lambda_1\phi(z) = a_0 + a_1 z + a_2 z^2 + \cdots,$$

and substituting this expansion in (6.54), we obtain

$$\left|\frac{a_0 + a_1 z}{z^2}\right| = 0(1), \quad z \to 0, \quad \arg z = \frac{\psi_1}{3},$$

and hence $a_0 = a_1 = 0$. From the above–said it follows that

$$\omega(z) = z^2\phi_0(z)\exp\Big\{\frac{\lambda_1\overline{z}}{z^2}\Big\}, \quad z \in G\setminus\{0\}, \tag{6.55}$$

where $\phi_0(z)$ is a function holomorphic in the domain G. Further, if the solution $\omega(z)$ is continuously extendable on $\overline{G}\setminus\{0\}$, then the function $\phi_0(z)$ is likewise continuously extendable on $\overline{G}$.

Conversely, it is obvious that any function of the type (6.55) provides us with a solution of the equation (6.45), which is continuously extendable to $\overline{G}\setminus\{0\}$ and satisfies the condition (6.46), where $\delta = |\lambda_1|$, $\sigma = 1$.

It remains to take into account the boundary conditions (6.47) and (6.49) (where $p = 0$) which immediately leads us to the validity of the theorem.

Since any linearly independent system of functions $\phi_0(z)$ by means of the relation (6.52) transforms into the functions $\omega(z)$ (and conversely), on the basis of the above proved Theorem 6.9 it is possible to carry out the complete investigation of the boundary value problem $R(|\lambda_1|, 1)$ under the assumption $|\lambda_1| < |\lambda_2|$.

We have the following

Theorem 6.10 *Let at least one of the relations*

$$\delta = |\lambda_1|, \quad \sigma = 1, \quad |\lambda_1| < |\lambda_2|, \tag{6.56}$$

be violated. Then either the homogeneous problem $R(\delta, \sigma)$ has an infinite set of linearly independent solutions or the inhomogeneous problem is unsolvable for any right-hand side $\gamma(t) \not\equiv 0$.

Proof. By the inequality (6.32), violation at least of one of the relations (6.56) means the fulfilment of one of the following conditions

$$\delta \neq |\lambda_1|, \quad \sigma = 1, \quad |\lambda_1| < |\lambda_2|, \tag{6.57}$$

or

$$\sigma \neq 1 \quad (\delta \quad \text{is arbitrary}), \quad |\lambda_1| < |\lambda_2|, \tag{6.58}$$

or

$$\delta = |\lambda_1|, \quad \sigma = 1, \quad |\lambda_1| = |\lambda_2|, \tag{6.59}$$

or

$$\delta \neq |\lambda_1|, \quad \sigma = 1, \quad |\lambda_1| = |\lambda_2|, \tag{6.60}$$

or

$$\sigma \neq 1 \quad (\delta \quad \text{is arbitrary}) \quad |\lambda_1| = |\lambda_2|. \tag{6.61}$$

We consider these cases separately. Let (6.57) be fulfilled. In its turn, this case splits into the following two cases: either

$$\delta < |\lambda_1|, \quad \sigma = 1, \quad |\lambda_1| < |\lambda_2|, \tag{6.62}$$

or

$$\delta > |\lambda_1|, \quad \sigma = 1, \quad |\lambda_1| < |\lambda_2|. \tag{6.63}$$

Let the case (6.62) be fulfilled, and let $\omega(z)$ be a solution of the equation (6.45) satisfying the condition (6.46). Since $\nu = 2$, the number β given by the formula (6.36) is equal to zero. On the basis of Theorem 1, this implies that the solution $\omega(z) \equiv 0$, and hence the inhomogeneous boundary value problem $R(\delta, 1)$ is unsolvable for any right-hand side $\gamma(t) \not\equiv 0$.

Let now the condition (6.63) be fulfilled. We call an arbitrary real number N and prove that the number of linearly independent solutions of the homogeneous boundary value problem $R(\delta, 1)$ is greater than N. Indeed, we select a natural number p so large that the number of linearly independent solutions of the homogeneous boundary value problem $R_0(p)$ be greater than N. Denote these solutions by $\phi_0^{(1)}(z), \phi_0^{(2)}(z) \cdots, \phi_0^{(m)}(z)$, $(m > N)$ and introduce the functions

$$\omega_k(z) = z^{2-p}\phi_0^{(k)} \exp\left\{\frac{\lambda_1 \overline{z}}{z^2}\right\}, \quad k = 1, 2, \ldots, m. \tag{6.64}$$

It is clear that the system of functions (6.64) is likewise independent.

By the representation (6.33), every function from (6.64) is a continuously extendable to $\overline{G}\setminus\{0\}$ solution of the equation (6.45) which by virtue of (6.49) satisfies the homogeneous boundary condition (6.47). Further, since the condition (6.63) is fulfilled, on the basis of the obvious relation

$$\frac{\partial \omega_k}{\partial \overline{z}} = \frac{\lambda_1}{z^p}\phi_0^{(k)}(z)\exp\left\{\frac{\lambda_1 \overline{z}}{z^2}\right\} = 0\Big(\exp\left\{\frac{\delta}{|z|}\right\}\Big), \quad z \to 0,$$

we can conclude immediately that every function of the system (6.64) satisfies the asymptotic condition (6.46), and hence the homogeneous boundary value problem $R(\delta, 1)$ has infinitely many linearly independent solutions.

Let now the condition (6.58) be fulfilled. This case in its turn splits into two cases: either

$$\sigma < 1 \quad (\delta \quad \text{is arbitrary}), \quad |\lambda_1| < |\lambda_2|, \tag{6.65}$$

or

$$\sigma > 1 \quad (\delta \quad \text{is arbitrary}), \quad |\lambda_1| < |\lambda_2|. \tag{6.66}$$

It is evident that in the case (6.65) (analogously to the case (6.62)) the inhomogeneous boundary value problem $R(\delta, \sigma)$ is unsolvable for any right-hand side $\gamma(t) \not\equiv 0$, and in the case (6.66) (analogously to the case (6.63) the homogeneous boundary value problem $R(\delta, \sigma)$ has infinitely many linearly independent solutions.

Let now the condition (6.59) be fulfilled. This case in its turn splits into two cases: either

$$\delta = |\lambda_1|, \quad \sigma = 1, \quad |\lambda_1| = |\lambda_2|, \quad \lambda_1 \neq \lambda_2, \tag{6.67}$$

or

$$\delta = |\lambda_1|, \quad \sigma = 1, \quad \lambda_1 = \lambda_2. \tag{6.68}$$

Let us prove that in both cases (6.67) and (6.68) the homogeneous boundary value problem $R(\delta, 1)$ has infinitely many linearly independent solutions. We start with the case (6.67). Evidently, every function of the type

$$\omega(z) = z^2\phi_0(z)e^{\frac{\lambda_1 \overline{z}}{z^2}} + z^2\psi_0(z)e^{\frac{\lambda_2 \overline{z}}{z^2}}, \quad z \in G \setminus \{0\} \tag{6.69}$$

(where $\phi_0(z)$ and $\psi_0(z)$ are holomorphic in the domain G) is a solution of the equation (6.45) satisfying the condition (6.46), where $\sigma = |\lambda_1|$, $\sigma = 1$ (in proving Theorem 6.12 below we will show that the converse statement is valid, i.e., every solution of the equation (6.45) satisfying the condition (6.46) with $\delta = |\lambda_1|$, $\sigma = 1$ has the form (6.69)). Next, if the holomorphic functions $\phi_0(z)$ and $\psi_0(z)$ are continuously extendable to $\overline{G}$, then the solution $\omega(z)$ is likewise continuously extendable on $\overline{G} \setminus \{0\}$. Consider the following problem: find two functions $\phi_0(z)$ and $\psi_0(z)$, holomorphic in the domain G and continuously extendable to G by the boundary condition

$$\operatorname{Re}\left\{a(t)t^2\phi_0(t)e^{\frac{\lambda_1 \overline{t}}{t^2}} + a(t)t^2\psi_0(t)e^{\frac{\lambda_2 \overline{t}}{t^2}}\right\} = 0, \quad t \in \Gamma. \tag{6.70}$$

Therefore every solution of the problem (6.70) provides us by the formula (6.69) with solution of the homogeneous boundary value problem $R(|\lambda_1|, 1)$.

On the other hand, the problem (6.70) has infinitely many linearly independent solutions. Indeed, let

$$\phi_1^*(z), \phi_2^*(z), \dots, \phi_l^*(z)$$

be a complete system of solutions of the conjugate boundary value problem: given a real Hölder continuous function $\beta(t)$, find the function $\phi_0(z)$ holomorphic in the domain G and continuously extendable on $\overline{G}$ by the boundary condition

$$\operatorname{Re}[\alpha(t)\phi_0(t)] = \beta(t), \quad t \in \Gamma, \tag{6.71}$$

where

$$\alpha(t) = a(t)t^2 \exp\Big\{\frac{\lambda_1 \bar{t}}{t^2}\Big\}.$$

Take arbitrary natural number N_0 and consider natural number N such that

$$N + 1 - 2l > N_0.$$

Introduce now the polynomial

$$\psi_0(z) = C_0 + C_1 z + \cdots + C_n z^N, \tag{6.72}$$

where C_j, $j = 0, 1, \dots, N$, are yet undefined real coefficients. Further, taking the right-hand side of the problem (6.71) in the form

$$\beta(t) = -\operatorname{Re}\Big[a(t)t^2 \exp\Big\{\frac{\lambda_2 \bar{t}}{t^2}\Big\}\psi_0(t)\Big], \quad t \in \Gamma,$$

we obtain a boundary value problem which will certainly be solvable if

$$\int_\Gamma \alpha(t)\beta(t)\phi_k^*(t)dt = 0, \quad 1 \le k \le l.$$

Thus if real constants C_j are chosen such that

$$\sum_{j=0}^{N} D_{kj} C_j = 0, \quad k = 1, 2, \dots, l, \tag{6.73}$$

where

$$D_{kj} = \int_\Gamma \alpha(t)\phi_k^*(t) \operatorname{Re}\Big[a(t)t^{2+j} e^{\frac{\lambda_2 \bar{t}}{t^2}}\Big]dt,$$

then the problem (6.71) is solvable. In turn, the conditions (6.73) form a system consisting of (6.40) linear algebraic homogeneous equations with $N+1$ real unknowns, of which at least $N + 1 - 2l$ can be taken arbitrarily. This means that in the decomposition (6.72) we can take $N + 1 - 2l$ real coefficients. Substituting this decomposition in the boundary condition (6.70), we can find the function $\phi_0(z)$. It is obvious that the problem (6.70) has an infinite number of linearly independent solutions.

If the condition (6.68) is fulfilled, then any function of the type

$$\omega(z) = (z\overline{z}\phi_0(z) + z^2\psi_0(z))e^{\frac{\lambda_1 \overline{z}}{z^2}}, \quad z \in G \setminus \{0\} \tag{6.74}$$

(where $\phi_0(z)$ and $\psi_0(z)$ are functions holomorphic in G), is a solution of the equation (6.45) satisfying the condition (6.46), where $\delta = |\lambda_1|$, $\sigma = 1$ (in proving Theorem 6.13 below, we will establish the validity of the converse statement, i.e., any solution of the equation (6.45) satisfying the condition (6.46), where $\delta = |\lambda_1|$, $\sigma = 1$, has the form (6.74)). Moreover, if the holomorphic functions $\phi_0(z)$ and $\psi_0(z)$ are continuously extendable on $\overline{G}$ then the solution $\omega(z)$ is likewise continuously extendable on $\overline{G}\setminus\{0\}$.

Let us consider the following boundary value problem. Find two functions $\phi_0(z)$ and $\psi_0(z)$, holomorphic in the domain G and continuously extendable on $\overline{G}$ by the boundary condition

$$\operatorname{Re}\left[a(t)(t\overline{t}\phi_0(t) + t^2\psi_0(t))e^{\frac{\lambda_1 \overline{t}}{t^2}}\right] = 0, \quad t \in \Gamma. \tag{6.75}$$

Any solution of the problem (6.75) provides us by the formula (6.74) with a solution of the boundary value problem $R(|\lambda_1|, 1)$. But the problem (6.75), just as the problem (6.70), has an infinite number of linearly independent solutions. Hence the homogeneous problem $R(|\lambda_1|, 1)$ has an infinite number of linearly independent solutions.

The case (6.60) splits into the following two cases: either

$$\delta < |\lambda_1|, \quad \sigma = 1, \quad |\lambda_1| = |\lambda_2|, \tag{6.76}$$

or

$$\delta > |\lambda_1|, \quad \sigma = 1, \quad |\lambda_1| = |\lambda_2|. \tag{6.77}$$

In the case (6.76), just as in the case (6.62), on the basis of Theorem 6.9 we immediately find that the equation (6.45) has no non-trivial solution satisfying the condition (6.46), and hence the inhomogeneous boundary value problem $R(\delta, 1)$ is unsolvable for any right-hand side $\gamma(t) \not\equiv 0$.

In the case (6.76) it is obvious that every solution of the boundary value problem $R(|\lambda_1|, 1)$ is also a solution of the problem $R(\delta, 1)$. But the homogeneous boundary value problem $R(|\lambda_1|, 1)$ has an infinite number of linearly independent solutions (see the case (6.59) above), consequently the homogeneous problem $R(\delta, 1)$ has an infinite number of linearly independent solutions as well.

The case (6.61) splits into the following two cases: either

$$\sigma < 1 \quad (\delta \quad \text{is arbitrary}), \quad |\lambda_1| = |\lambda_2| \tag{6.78}$$

or

$$\sigma > 1 \quad (\delta \quad \text{is arbitrary}), \quad |\lambda_1| = |\lambda_2|. \tag{6.79}$$

In the case (6.78), just as in the case (6.76), on the basis of Theorem 6.8 we immediately find that the inhomogeneous boundary value problem $R(\delta, \sigma)$ is unsolvable for any right-hand side $\gamma(t) \not\equiv 0$, $t \in \Gamma$, and in the case (6.79) (just as in the case (6.77)) the homogeneous boundary value problem $R(\delta,\sigma)$ has an infinite number of linearly independent solutions.

On the basis of the above proved Theorems 6.9 and 6.10 we have

Theorem 6.11 *The boundary value problem $R(\delta,\sigma)$ is Noetherian if and only if the relations* (6.56) *are fulfilled.*

7^0. Next we have investigated the boundary value problem $R(\delta,\sigma)$. As we have found out, this problem is correct only under the condition (6.56). The last of these relations allows one to exclude from the consideration wide a class of equations of the type (6.45).

If not mentioning it specially, we assume that $|\lambda_1| = |\lambda_2|$, and for equations of the type (6.45) we give the correct statement and investigation of the boundary value problems.

Everywhere below by δ_0 we denote the number $\delta_0 = |\lambda_1|$. We have the following

Theorem 6.12 *If*

$$\arg\lambda_1 \neq \arg\lambda_2,$$

then the boundary value problems $Q(\delta_0,1)$ and $Q_0'(0)$ are simultaneously solvable (*unsolvable*) *and in case they are solvable, the relation* (6.69) *allows us to establish bijective correspondence between the solutions of these problems.*

Proof. First we have to find general representation of those solutions of the equation (6.45) which (together with its derivative with respect on $\overline{z}$) are continuously extendable to $G\setminus\{0\}$ and satisfy the condition (6.46), where $\delta=\delta_0$, $\sigma=1$. To this end we again use the equalities (6.38) and find that the functions $\phi(z)$ and $\psi(z)$, holomorphic in the domain $G\setminus\{0\}$, satisfy the conditions

$$\phi(z) = 0\Big(\exp\Big\{\frac{\delta_0}{|z|}[1-\cos(\psi_1 - 3\arg z)]\Big\}\Big), \quad z\to 0,$$
$$\psi(z) = 0\Big(\exp\Big\{\frac{\delta_0}{|z|}[1-\cos(\psi_2 - 3\arg z)]\Big\}\Big), \quad z\to 0,$$
$$\psi_k = \arg\lambda_k, \quad k=1,2.$$

Thus on the basis of Lemma 6.6 we conclude that $z=0$ is a removable singular point for the functions $\phi(z)$ and $\psi(z)$. Further, it is obvious that

$$\frac{\partial\omega}{\partial\overline{z}} = \frac{\lambda_1\phi(z)}{z^2}\exp\Big\{\frac{\lambda_1\overline{z}}{z^2}\Big\} + \frac{\lambda_2\psi(z)}{z^2}\exp\Big\{\frac{\lambda_2\overline{z}}{z^2}\Big\} =$$
$$= 0\Big(\exp\Big\{\frac{\delta_0}{|z|}\Big\}\Big), \quad z\to 0.$$

Hence we obtain the following two relations:

$$\frac{\delta_0}{r^2}\Big|\phi\Big(r\exp\Big\{\frac{i\psi_1}{3}\Big\}\Big)\Big| \leq \text{const} +$$

$$+\frac{\delta_0}{r^2}\Big|\psi\Big(r\exp\Big\{\frac{i\psi_1}{3}\Big\}\Big)\Big|\exp\Big\{\frac{\delta_0}{r}[\cos(\psi_2-\psi_1)-1]\Big\},$$
$$\frac{\delta_0}{r^2}\Big|\psi\Big(r\exp\Big\{\frac{i\psi_2}{3}\Big\}\Big)\Big|\le \text{const}+$$
$$+\frac{\delta_0}{r^2}\Big|\phi\Big(r\exp\Big\{\frac{i\psi_1}{3}\Big\}\Big)\Big|\exp\Big\{\frac{\delta_0}{r}[\cos(\psi_2-\psi_1)-1]\Big\},$$

whence it respectively follow

$$\Big|\frac{\phi(z)}{z^2}\Big|=0(1),\quad z\to 0,\quad \arg z=\frac{\psi_1}{3},$$

and

$$\Big|\frac{\psi(z)}{z^2}\Big|=0(1),\quad z\to 0,\quad \arg z=\frac{\psi_2}{3}.$$

This implies that the functions $\phi(z)$ and $\psi(z)$ admit the representations

$$\phi(z)=z^2\phi_0(z),\quad \psi(z)=z^2\psi_0(z),$$

where $\phi_0(z)$ and $\psi_0(z)$ are functions holomorphic in the domain G.

Consequently, every solution of the equation (6.45) satisfying the condition (6.46) ($\delta=\delta_0$, $\sigma=1$) is representable in the form

$$\omega(z)=z^2\phi_0(z)\exp\Big\{\frac{\lambda_1\overline{z}}{z^2}\Big\}+z^2\psi_0(z)\exp\Big\{\frac{\lambda_2\overline{z}}{z^2}\Big\},\tag{6.80}$$

and hence

$$\frac{\partial\omega}{\partial\overline{z}}=\lambda_1\phi_0(z)\exp\Big\{\frac{\lambda_1\overline{z}}{z^2}\Big\}+\lambda_2\psi_0(z)\exp\Big\{\frac{\lambda_2\overline{z}}{z^2}\Big\}.\tag{6.81}$$

Next, if the solution (6.80) (together with its derivative (6.81)) is continuously extendable on $\overline{G}\setminus\{0\}$, then we find that the functions $\phi_0(z)$ and $\psi_0(z)$ are likewise continuously extendable on $\overline{G}$.

Conversely, it is evident that any function of the type (6.80) provides us with continuously extendable (together with its derivative $\frac{\partial\omega}{\partial\overline{z}}$) solution of the equation (6.45), satisfying the condition (6.46), where $\delta=\delta_0$, $\sigma=1$. It remains to take into account the boundary conditions (6.48) and (6.50) (where $p=0$) which directly leads to the conclusion of our theorem.

On the basis of the above proved Theorem 6.12 in particular it follows that the number of linearly independent solutions of the homogeneous boundary value problem $Q(\sigma_0,1)$ is finite. This number coincides with that of the linearly independent solutions of the homogeneous boundary value problem $Q_0'(0)$, because any linearly independent system of holomorphic vector functions

$$(\phi_k(z),\psi_k(z)),\quad 1\le k\le m,\tag{6.82}$$

transforms by the relation

$$\omega_k(z)=\phi_k(z)\exp\Big\{\frac{\lambda_1\overline{z}}{z^2}\Big\}+\psi_k(z)\exp\Big\{\frac{\lambda_2\overline{z}}{z^2}\Big\},\quad k=1,2,\dots,m,\tag{6.83}$$

into linearly independent system of functions $\omega_k(z)$, $k = 1, 2, \dots, m$, and vice versa. Indeed, let the system of holomorphic vector functions (6.82) be independent, and

$$\sum_{k=1}^{m} C_k \omega_k(z) \equiv 0,$$

where C_k are complex (in particular, real) coefficients. Then

$$\sum_{k=1}^{m} C_k \phi_k(z) \equiv -e^{\frac{\lambda_2-\lambda_1}{z^2}\overline{z}} \sum_{k=1}^{m} C_k \psi_k(z). \tag{6.84}$$

Differentiating both parts of the equality (6.84) with respect to $\overline{z}$, we obtain

$$\frac{\lambda_2-\lambda_1}{z^2}\, e^{\frac{\lambda_2-\lambda_1}{z^2}\overline{z}} \sum_{k=1}^{m} C_k \psi_k(z) \equiv 0.$$

Hence (since $\lambda_2 \neq \lambda_1$)

$$\sum_{k=1}^{m} C_k \psi_k(z) \equiv 0. \tag{6.85}$$

It follows from (6.84) and (6.85) that

$$\sum_{k=1}^{m} C_k \phi_k(z) \equiv 0, \tag{6.86}$$

while (6.86) and (6.85), by virtue of the fact that the system (6.82) is linearly independent, yield $C_k = 0$, $k = 1, 2, \dots, m$.

The converse statement is obvious because the linear dependence of the system of vector functions (6.82) immediately implies that of the system of functions (6.83).

We have the following

Theorem 6.13 *If*

$$\psi_1 \equiv \arg \lambda_1 = \arg \lambda_2,$$

then the boundary value problems $Q(\delta_0, 1)$ and $Q''_0(0)$ are simultaneously solvable (unsolvable), and if they are solvable, then the relation (6.74) *allows us to establish bijective correspondence between the solutions of these problems.*

Proof. First of all, just as in the proof of Theorems 6.9 and 6.12, we have to find general representation of those solutions of the equation (6.45) which (together with the derivative $\frac{\partial \omega}{\partial \overline{z}}$) are continuously extendable on $\overline{G} \setminus \{0\}$ and satisfy the condition (6.46), where $\delta = \delta_0$, $\sigma = 1$. Towards this end, we use the equalities (6.41) and find that the functions $\phi(z)$ and $\psi(z)$, holomorphic in $G \setminus \{0\}$, satisfy the conditions

$$z^2\phi(z) = 0(g(z)), \quad z^2\psi(z) = 0(g(z)), \quad z \to 0, \tag{6.87}$$

where

$$g(z) = \exp\left\{\frac{\delta_0}{|z|}(1 - \cos(\psi_1 - 3\arg z))\right\}.$$

By virtue of the relations (6.87) and Lemma 6.6 we obtain that $z = 0$ is a removable singular point for the functions $z^2\phi$ and $z^2\psi$, i.e. the solution ω is representable in the form

$$\omega(z) = H(z)\exp\left\{\frac{\lambda_1\overline{z}}{z^2}\right\}, \quad z \in G \setminus \{0\}, \tag{6.88}$$

where

$$H(z) = \overline{z}\frac{\widetilde{\phi}(z)}{z^2} + \frac{\widetilde{\psi}(z)}{z^2},$$

and $\widetilde{\phi}$ and $\widetilde{\psi}$ are functions holomorphic in G. In its turn from the representation (6.88) it follows

$$\frac{\partial\omega}{\partial\overline{z}} = H_1(z)\exp\left\{\frac{\lambda_1\overline{z}}{z^2}\right\}, \quad z \in G \setminus \{0\},$$

where

$$H_1(z) = \frac{\widetilde{\phi}(z)}{z^2}\left(1 + \frac{\lambda_1\overline{z}}{z^2}\right) + \frac{\lambda_1}{z^4}\widetilde{\psi}(z).$$

Further, taking into account the condition (6.46), we get

$$H(z) = 0(1), \quad z \to 0, \arg z = \frac{1}{3}(\psi_1 + 2\pi k), \tag{6.89}$$

$$H_1(z) = 0(1), \quad z \to 0, \arg z = \frac{1}{3}(\psi_1 + 2\pi k), \tag{6.90}$$

$$k = 0, 1, 2, \ldots .$$

Expanding the holomorphic functions $\widetilde{\phi}$ and $\widetilde{\psi}$ into their Taylor series

$$\begin{aligned}\widetilde{\phi}(z) &= a_0 + a_1 z + a_2 z^2 + \cdots, \\ \widetilde{\psi}(z) &= b_0 + b_1 z + b_2 z^2 + \cdots,\end{aligned} \tag{6.91}$$

and substituting them in (6.89), we have

$$\frac{a_0\overline{z} + b_1 z + b_0}{z^2} = 0(1), \quad \arg z = \frac{\psi_1 + 2\pi k}{3}, \tag{6.92}$$

where the coefficient $b_0 = 0$. Taking this into account and using the relation (6.92) for the coefficients a_0 and b_1, we obtain the following equalities

$$a_0 e^{-2i\varphi_0} + b_1 = 0, \quad \varphi_0 = \frac{\psi_1}{3},$$

$$a_0 e^{-2i\varphi_0} + b_1 = 0, \quad \varphi_1 = \frac{\psi_1 + 2\pi}{3},$$

which (with regard for $e^{-2i\varphi_0} - e^{-2i\varphi_1} \neq 0$) show that the coefficients $a_0 = b_1 = 0$.

Substituting now the expansions (6.91) and (6.90), we have

$$\frac{1}{r^3}[\lambda_1 a_1 e^{-4i\varphi_k} + \lambda_1 b_2 r e^{-i\varphi_k} + r^2(a_1 + \lambda_1 a_2 e^{-2i\varphi_k} +$$
$$+\lambda_1 b_3)] = 0(1), \quad r \to 0, \quad \varphi_k = \frac{\psi_1 + 2\pi k}{3}, \quad k = 0, 1, 2, \dots,$$

which immediately gives $a_1 = 0$. Taking this fact into account, we obtain

$$\frac{1}{r^2}[\lambda_1 b_2 e^{-i\varphi_k} + \lambda_1 r(a_2 e^{-2i\varphi_k} + b_3)] = O(1), \quad r \to 0,$$

and therefore $b_2 = 0$. In its turn we have

$$a_2 e^{-2i\varphi_0} + b_3 = 0, \quad \varphi_0 = \frac{\psi_1}{3},$$
$$a_2 e^{-2i\varphi_1} + b_3 = 0, \quad \varphi_1 = \frac{\psi_1 + 2\pi}{3},$$

by virtue of which $a_2 = b_3 = 0$.

Thus the holomorphic functions $\widetilde{\phi}$ and $\widetilde{\psi}$ have the form

$$\widetilde{\phi}(z) = z^3 \phi_0(z), \quad \widetilde{\psi}(z) = z^4 \psi_0(z), \tag{6.93}$$

where the functions ϕ_0 and ψ_0 are holomorphic in the domain G. Substituting (6.93) and (6.88), we obtain the representation (6.74). Next, if the solution (6.74) together with its derivative

$$\frac{\partial \omega}{\partial \overline{z}} = \Big[\phi_0(z)\Big(z + \frac{\lambda_1 \overline{z}}{z^2}\Big) + \lambda_1 \psi_0(z)\Big] e^{\frac{\lambda_1 \overline{z}}{z^2}} \tag{6.94}$$

is continuously extendable on $\overline{G} \setminus \{0\}$, we will find that the holomorphic functions ϕ_0 and ψ_0 are continuously extendable on $\overline{G}$.

Conversely, every function of the type (6.74) provides us with continuously extendable (together with its derivative (6.94)) on $\overline{G} \setminus \{0\}$ solution of the equation (6.45), satisfying the condition (6.46) with $\delta = \delta_0$, $\sigma = 1$. It remains to take into account the boundary conditions (6.48) and (6.51) (with $p = 0$) which immediately leads us to the conclusion of our theorem.

It is not difficult to see that any linearly independent system of holomorphic vector functions (6.82) transforms by the relation

$$\omega_k(z) = \Big(z\overline{z}\phi_k(z) + z^2 \psi_k(z) \exp\Big\{\frac{\lambda_1 \overline{z}}{z^2}\Big\}\Big), \quad z \in G \setminus \{0\}$$

(analogously to the relation (6.83)) into linearly independent system of functions $\omega_k(z)$, $k = 1, 2, \dots, m$, and vice versa. Therefore the numbers of linearly independent solutions of homogeneous boundary problems $Q(\delta_0, 1)$ and $Q_0''(0)$ coincide.

We have the following

Theorem 6.14 *Let at least one of the equalities*

$$\delta = \delta_0, \quad \sigma = 1, \tag{6.95}$$

be violated. Then either the homogeneous boundary value problem $Q(\delta, \sigma)$ has an infinite number of linearly independent solutions, or the inhomogeneous problem is unsolvable for any right-hand side $(\gamma_1(t), \gamma_2(t)) \not\equiv 0$.

Proof. The violation of at least of one of the equalities (6.95) implies that one of the following conditions is fulfilled:

$$\delta < \delta_0, \quad \sigma = 1, \tag{6.96}$$

or

$$\delta > \delta_0, \quad \sigma = 1, \tag{6.97}$$

or

$$\sigma < 1 \quad (\sigma \quad \text{is arbitrary}), \tag{6.98}$$

or

$$\sigma > 1 \quad (\sigma \quad \text{is arbitrary}). \tag{6.99}$$

Under the condition (6.96) (and under the condition (6.98)), on the basis of Theorem 6.14 it immediately follows that the equation (6.45) has no non-trivial solution satisfying the condition (6.46) and hence the inhomogeneous boundary value problem $Q(\delta, \sigma)$ is unsolvable for any right-hand side $(\gamma_1(t), \gamma_2(t)) \not\equiv 0$.

Let us prove that under the condition (6.97) the homogeneous boundary value problem $Q(\delta, 1)$ has an infinite number of linearly independent solutions. Indeed, let the condition (6.97) be fulfilled and moreover, $\arg\lambda_1 \neq \arg\lambda_2$. We take an arbitrary natural number N and choose natural number p so that the number of linearly independent solutions of the homogeneous boundary value problem $Q_0'(p)$ be greater than N. We denote these solutions by

$$(\phi_0^{(k)}(z), \psi_0^{(k)}(z)), \quad k = 1, 2, \ldots, m, \quad m > N. \tag{6.100}$$

It is easy to see that the system of functions (6.100) transforms by the relation

$$\omega_k(z) = z^2\phi_0^{(k)}(z) \exp\left\{\frac{\lambda_1\overline{z}}{z^2}\right\} + \\ + z^2\psi_0^{(k)}(z) \exp\left\{\frac{\lambda_2\overline{z}}{z^2}\right\}, \quad z \in G \setminus \{0\},$$

into a linearly independent system of solutions of the homogeneous boundary value problem $Q(\delta, \sigma)$. Therefore this problem has an infinite number of linearly independent solutions.

Let now the condition (6.97) be fulfilled, and $\arg\lambda_1 = \arg\lambda_2$. We take an arbitrary natural number N and choose a natural number p so large that the number of linearly independent solutions of the homogeneous boundary value problem $Q_0''(p)$

be greater than N. We denote again these solutions by (6.100). It is not difficult to see that the system of functions (6.100) transforms by the relation

$$\omega_k(z) = (z\overline{z}\phi_0^{(k)}(z) + z^2\psi_0^{(k)}(z))\exp\left\{\frac{\lambda_1\overline{z}}{z^2}\right\},$$
$$z \in G \setminus \{0\}, \quad k = 1, 2, \dots, m,$$

into linearly independent system of solutions of the homogeneous boundary value problem $Q(\delta, \sigma)$. Therefore this problem has an infinite number of linearly independent solutions.

It remains to consider the case (6.99). But any solution of the homogeneous boundary value problem $Q(\delta, 1)$ (for $\delta > \delta_0$) is likewise the solution of the homogeneous boundary value problem $Q(\delta, \sigma)$ (for $\sigma > 1$). Therefore the latter problem has an infinite number of linearly independent solutions.

On the basis of the above-proved Theorems 6.13 and 6.14 we have the following theorem.

Theorem 6.15 *The boundary value problem $Q(\delta, \sigma)$ is Noetherian if and only if the condition* (6.95) *is fulfilled.*

6.4 One class of two-dimensional third kind Fredholm integral equation

Consider the following integral equation

$$z^\nu w + p_n w = f, \tag{6.101}$$

where $\nu > 0$ is a given integer, n is a natural number, f is a given function on the bounded domain G containing the origin,

$$\begin{aligned} p_1 w(z) &= -\frac{1}{\pi}\iint_G \frac{a_1(\zeta)\,w(\zeta)}{\zeta - z}\,dG(\zeta), \\ &\dots\dots\dots\dots\dots\dots \\ p_n w(z) &= -\frac{1}{\pi}\iint_G \frac{a_n(\zeta)\,p_{n-1}(w(\zeta))}{\zeta - z}\,dG(\zeta), \end{aligned} \tag{6.102}$$

is the sequence of singular integral operators, w is a desired function; by $a_k = a_k(z)$ $(1 \le k \le n)$ are denoted the holomorphic in G and continuous in $\overline{G}$ functions. Here we mean that

$$m = \frac{\nu}{n} \geqslant 2. \tag{6.103}$$

Consider now the homogeneous integral equation

$$z^\nu w + p_n w = 0. \tag{6.104}$$

Under the solution of this equation we mean the function $w \in L_p(G)$, $p > 2$, satisfying (6.104) almost everywhere in $\overline{G}$. From the definition and on the basis of known properties of the operator $p_1 w$ [124] it follows that every solution of the equation (6.104) is continuous in every point of $\overline{G}$, except may be the origin. Moreover, the function

$$w_0(z) \equiv z^{\nu}\, w(z), \quad z \in G, \tag{6.105}$$

has a continuous derivatives of order $n-1$ in G and the function

$$w_1(z) \equiv z^{\nu}\, \frac{\partial^{n-1} w}{\partial \overline{z}^{\,n-1}}\,, \quad z \in G, \tag{6.106}$$

has a generalized derivative by $\overline{z}$ and

$$w_{\overline{z}} + R(z)\, w = 0, \tag{6.107}$$

where $R(z) = \prod\limits_{k=1}^{n} a_k(z)$, $z \in G$.

Assume that there exists a holomorphic in the domain G function $b(z)$, such that

$$\big[b(z)\big]^n = -R(z), \quad z \in G \quad \text{and} \quad b(0) \neq 0. \tag{6.108}$$

Denote by

$$\alpha_k = \cos\frac{2\pi(k-1)}{n} + i\sin\frac{2\pi(k-1)}{n}\,, \quad 1 \le k \le n$$

all possible roots from 1 of order n. Taking into account (6.103), (6.106), (6.107) and (6.108) we obtain that every solution of the homogeneous equation (6.104) is representable in the form

$$w(z) = \sum_{k=1}^{n} Q_k(z), \quad z \in G \setminus \{0\}, \tag{6.109}$$

where $Q_k(z) = \Phi_k(z)\exp\{\alpha_k\, c(z)\,\overline{z}\}$, $c(z) = \frac{b(z)}{z^m}$ and $\Phi_k(z)$ $(k = \overline{1,n})$ are holomorphic in $G \setminus \{0\}$ functions.

From (6.109) follows the validity of the following representation

$$\frac{\partial^q w}{\partial \overline{z}^{\,q}} = \sum_{k=1}^{n} \alpha_k^q\, c(z)^q\, Q_k(z) \quad (1 \le q \le n-1), \tag{6.110}$$

which simultaneously with (6.109) admits the matrix form

$$A \cdot \Psi(z) = \Omega(z), \tag{6.111}$$

$$\Psi(z) = \text{column}\,\big(Q_1(z), Q_2(z), \ldots, Q_n(z)\big),$$

$$\Omega(z) = \text{column}\left(w, \frac{1}{c(z)}\frac{\partial w}{\partial \overline{z}}\,, \ldots, \frac{1}{c(z)^{n-1}}\frac{\partial^{n-1} w}{\partial \overline{z}^{\,n-1}}\right), \tag{6.112}$$

$$A = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ \alpha_1 & \alpha_2 & \cdots & \alpha_n \\ \alpha_1^2 & \alpha_2^2 & \cdots & \alpha_n^2 \\ \cdots & \cdots & \cdots & \cdots \\ \alpha_1^{n-1} & \alpha_2^{n-1} & \cdots & \alpha_n^{n-1} \end{pmatrix}.$$

From (6.111) we have that every function Q_k $(k = \overline{1,n})$ is a linear combination of the components of the vector (6.111). Therefore, in the neighborhood of $z = 0$, the following estimation holds

$$Q(z) = O\Big(\frac{1}{\overline{z}^{\,\nu}}\Big), \quad z \to 0. \tag{6.113}$$

Consequently, we get the result:

Theorem 6.16 *The homogeneous integral equation (6.104) has no nontrivial solutions in $L_p(G)$, $p > 2$.*

Let us consider the inhomogeneous case (6.101), when the right hand side function is s-analytic in G

$$f(z) = F_0(z) + \overline{z}\,F_1(z) + \cdots + \overline{z}^{\,s-1}\,F_{s-1}(z), \tag{6.114}$$

here s is a natural number, F_k $(0 \le k \le s-1)$ are arbitrary holomorphic functions in G.

The following theorem is valid.

Theorem 6.17 *The inhomogeneous integral equation (6.101) is unsolvable in the class $L_p(G)$, $p > 2$, if the right hand side function has the form (6.114), where at least one function F_k $(0 \le k \le s-1)$ isn't identically zero and the following inequality holds*

$$s \le n. \tag{6.115}$$

Remark. We have established that the homogeneous integral equation (6.104) and the inhomogeneous equation (6.101) with sufficiently general right-hand side have no solutions. Note, that the exact description of the image of our integral operator for which the equation (6.101) will have solutions is very complicated.

7 Some problems of $2n$-elliptic systems on the plane

7.1 Maximum modulus theorem

The first order system of partial differential equations

$$\frac{\partial u}{\partial x} + A(x,y)\frac{\partial u}{\partial y} + B(x,y)u = 0, \tag{7.1}$$

where $u = (u_1, u_2, \ldots, u_{2n})$ is $2n$-component desired vector, A, B are given real $2n \times 2n$-matrices depending on two real variables x, y is called elliptic in some domain $G \subset R^2_{(x,y)}$, if

$$\det(A - \lambda I) \neq 0, \tag{7.2}$$

for every real λ and $(x,y) \in G$; I is an identity matrix. In other words the system (7.1) is elliptic if the matrix A has no real characteristic numbers in G.

In this subsection we study the problem of *validity of the maximum modulus theorem.* To this end let us mention some auxiliary explanations. Under the solution of the system (7.1) we mean the classical solution of the class $C^1(G) \cap C(\overline{G})$.

Denote by

$$\Lambda(A,B) \tag{7.3}$$

the class of all possible solutions of the system (7.1); the matrices A and B are called the generating pair of the class (7.3).

Introduce

$$\rho_u(x,y) = \left[\sum u_k^2(x,y)\right]^{\frac{1}{2}}, \quad (x,y) \in \overline{G} \tag{7.4}$$

for every u of the class (7.3).The following question arises (cf. Bojarski [23]):

Is the inequality

$$\rho_u(x_0,y_0) \leq \max_{(x,y)\in\Gamma} \rho_u(x,y) \tag{7.5}$$

valid for arbitrary u from (7.3) *and* $(x_0,y_0) \in \overline{G}$? Here Γ is a boundary of the domain G.

Of course, in case $n = 1$ and

$$A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \qquad B = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

the condition (7.5) if fulfilled.

Consider now $G = \{x^2 + y^2 < 1\}$, $n = 1$, A is the same matrix, $B = \begin{pmatrix} 2x & 0 \\ 2y & 0 \end{pmatrix}$ and $u = \text{column}(e^{-x^2-y^2}, 0) \in \Lambda(A,B)$.

It is evident, that $\rho_u(0,0) = 1$ and $\rho_u(x,y) = \frac{1}{e}$, i.e. the condition (7.5) is not fulfilled. In this example the matrix B is not a constant matrix. This example shows, that the maximum modulus theorem for minimal dimensional elliptic system is not always true. It is easy to construct the example of higher dimensional system when the condition (7.5) is disturbed in case A and B are constant. In fact, consider G is the same domain $G = \{x^2 + y^2 < 1\}$,

$$A = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \qquad B = \begin{pmatrix} -2 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ -6 & 0 & -2 & 0 \\ 0 & -6 & 0 & -2 \end{pmatrix}$$

and $u = \text{column}(u_1, u_2, u_3, u_4) \in \Lambda(A, B)$, where

$$u_1 = e^x(x\cos y + y\sin y), \qquad u_2 = e^x(y\cos y - x\sin y),$$
$$u_3 = 3(x^2 + y^2 - 1)e^x\cos x, \qquad u_4 = -3(x^2 + y^2 - 1)e^x\sin y.$$

It is clear, that

$$\rho_u(0,0) = 3, \quad \max_{(x,y)\in\Gamma} \rho_u(x,y) = e$$

and therefore the condition (7.5) is not fulfilled.

In case the dimension of the system (7.1)–(7.3) is minimal, i.e. when $n = 1$ and moreover, when

$$A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}, \quad b_{kq} \in L_p(G), \quad p > 2.$$

We have with the great effort of very famous mathematicians, in some sense complete theory which is in very close connection with the theory of analytic functions of complex variable. In particular, it is well-known that there exists the number $M \geqslant 1$ (depending only on the matrix B) such, that

$$\rho_u(x_0, y_0) \leq M \max_{(x,y)\in\Gamma} \rho_u(x,y) \tag{7.6}$$

for every $u \in \Lambda(A, B)$ and $(x_0, y_0) \in G$.

The inequality (7.6) is weaker than (7.5), but it is also a very interesting problem, as was noted by Bojarski in [30].

Now we describe the sufficiently wide class of the elliptic systems (7.1)–(7.3), for which the inequality (7.6) as well as stronger inequality (7.5) holds. Consider the case of constant coefficients.

Theorem 7.1 *Let for the matrices A and B there exists the orthogonal matrix D such, that*

$$D^{-1}AD = \begin{pmatrix} 0 & -1 & 0 & 0 & \cdots & 0 & 0 \\ 1 & 0 & 0 & -1 & \cdots & 0 & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & -1 \\ 0 & 0 & 0 & 0 & \cdots & 1 & 0 \end{pmatrix} \tag{7.7}$$

$$D^{-1}BD = \begin{pmatrix} d_{11} & -h_{11} & d_{12} & -h_{12} & \cdots & d_{1n} & -h_{1n} \\ h_{11} & d_{11} & h_{12} & d_{12} & \cdots & h_{1n} & d_{1n} \\ d_{21} & -h_{21} & d_{22} & -h_{22} & \cdots & d_{2n} & -h_{2n} \\ h_{21} & d_{21} & h_{22} & d_{22} & \cdots & h_{2n} & d_{2n} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ d_{n1} & -h_{n1} & \cdots & \cdots & \cdots & d_{nn} & -h_{nn} \\ h_{n1} & d_{n1} & \cdots & \cdots & \cdots & h_{nn} & d_{nn} \end{pmatrix} \tag{7.8}$$

where d_{kp}, h_{kp}, $1 \le k \le n$, $1 \le p \le n$ are arbitrary real numbers and the constructed complex matrix

$$B_0 = \begin{pmatrix} d_{11} + ih_{11} & d_{12} + ih_{12} & \cdots & d_{1n} + ih_{1n} \\ d_{21} + ih_{21} & d_{22} + ih_{22} & \cdots & d_{2n} + ih_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ d_{n1} + ih_{n1} & d_{n2} + ih_{n2} & \cdots & d_{nn} + ih_{nn} \end{pmatrix} \tag{7.9}$$

is a normal matrix, i.e. $B_0 \overline{B}_0^T = \overline{B}_0^T B_0$. Then the inequality (7.5) holds for any $u \in L(A, B)$, $(x_0, y_0) \in \overline{G}$. Moreover, if the equality holds in some inner point of the domain G then the function ρ_u (but not necessarily vector-function u) is constant.

In the above mentioned example, for the case $n = 2$, the conditions (7.7) are fulfilled, but the constructed complex B_0 is not normal and therefore (7.5) is violated.

7.2 Generalized Beltrami equation

The first order linear system of partial differential equations

$$\frac{\partial}{\partial x} u(x, y) = A(x, y) \frac{\partial u}{\partial y} + B(x, y)\, u(x, y) + F(x, y), \tag{7.10}$$

where $u = u(u_1, u_2, \dots, u_n)$ is $2n$-desired vector, A, B are given real $2n \times 2n$-matrices, depending on two variables x, y F is given real $2n$-vector, is said to be elliptic in the domain D, is

$$\det(A - \lambda I) \neq 0, \tag{7.11}$$

for all real λ and for all points $(x, y) \in D$; I is a unit matrix. In other words the system is elliptic in some plane domain D if and only if the matrix A has no real characteristic numbers in D.

As it is well-known, when $n = 1$ in case of sufficient smoothness of the coefficients of (7.10), after corresponding changing of variables we can reduce the system to one complex equation

$$\partial_{\overline{z}} w + A_1 w + B_1 \overline{w} + F = 0 \quad \left(\partial_{\overline{z}} = \frac{1}{2} \Big(\frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \Big) \right). \tag{7.12}$$

The complete theory of functions, satisfying this equation, the theory of generalized analytic functions was constructed by I. Vekua [124]. Later on B. Bojarski has shown, that the methods of the theory of generalized analytic functions admit far-going generalizations on case of an elliptic system of first order in the complex form, which has the following form

$$\partial_{\bar z} w - Q(z)\partial_z w + Aw + B\overline{w} = 0, \quad \Big(\partial_z = \frac{1}{2}(\partial x - i\,\partial y)\Big), \tag{7.13}$$

where $Q(z), A(z), B(z)$ are given square matrices of order u, $Q(z)$ is a matrix of special quasi-diagonal form [21], $Q(z) \in W_p^1(\mathbb{C})$, $p > 2$, $|q_{ii}| \le q_0 < 1$, $Q(z) \equiv 0$ outside some circle, A, B are bounded measurable matrices. (The notation $A \in K$, where A is a matric and K is some class of functions, means that every element $A_{\alpha\beta}$ of A belongs to K).

The regular solutions of the equation (7.13) are called the generalized analytic vectors. In case $A = B = 0$ such solutions are called the Q-holomorphic vectors.

In the works of B. Bojarski by the full analogy with the theory of generalized analytic functions are given the formulae of general representations. On this basis the boundary value problems of linear conjugation and Riemann-Hilbert boundary value problem with Hölder-continuous coefficients are considered. These results of B. Bojarski and some further development of the theory of generalized analytic vectors are presented in the monograph [124].

The section 7.4 deals with discontinuous problems of the theory of analytic vectors. By analogy with the case of analytic functions under these problems we mean the problems, where desired vectors in considered domains have angular boundary continuous are to be fulfilled only almost everywhere on Γ, in addition given coefficients of the boundary conditions are to be piecewise continuous matrices.

7.3 The solvability of the problem (V)

Differential boundary value problem is such boundary value problem for which the boundary condition contains the boundary values of derivatives of the desired functions. In the theory of differential boundary value problems for holomorphic functions an integral representation formula constructed by I. Vekua [130] plays an important role.

Let D be a finite domain bounded by a simple smooth curve Γ, $0 \in D$, let $\Phi(z)$ be holomorphic in D. Suppose the derivative of order m $(m \geqslant 1)$ of $\Phi(z)$ has boundary values on Γ satisfying Hölder-condition. Then $\Phi(z)$ can be represented by the formula

$$\Phi(z) = \int_\Gamma \mu(t)\Big(1 - \frac{z}{t}\Big)^{m-1} \ln\Big(1 - \frac{z}{t}\Big) ds + \int_\Gamma \mu(t) ds + ic, \tag{7.14}$$

where $\mu(t)$ is a real-valued function, $\mu(t) \in H(\Gamma)$ and c is a real constant; $\mu(t)$ and c are uniquely determined by $\Phi(z)$.

This representation gave I. Vekua the possibility to study the differential boundary value problem for holomorphic functions in Hölder-classes.

We introduce the suitable classes of generalized analytic vectors and for the elements of these classes the analog of I. Vekua representations, which allow us to investigate discontinuous differential boundary value problems in these classes.

Denote by $E_p(D,Q)$, $p \geqslant 1$, $Q(z) \in W^1_{p_0}(\mathbb{C})$, $p_0 > 2$, the class of Q-holomorphic vectors in D satisfying the conditions

$$\int_{\delta_{k\Gamma}} |w_k(z)|^p \, |dz| \leq c, \quad k = 1,2,\dots,n, \tag{7.15}$$

where c is a constant, $\delta_{k\Gamma}$ is the image of the circumference $|\zeta| = r$, $r < 1$, under quasi-conformal mapping

$$\zeta = \omega_k(s_k(z)) \tag{7.16}$$

of unit circle $|\zeta| < 1$ onto D, ω_k is a schlicht analytic function in the domain $s_k(D)$, $s_k(z)$ is a fundamental homeomorphism of the Beltrami equation

$$\partial_{\overline{z}} S - q_{kk}(z)\, \partial_z S = 0, \quad k = 1,2,\dots,n, \tag{7.17}$$

q_{kk} are the main diagonal elements of the matrix Q.

By $E_{m,p}(d,Q)$ denote the class of Q-holomorphic vectors satisfying the inequalities

$$\int_{\delta_{kr}} \left| \frac{\partial^m w_k(z)}{\partial z^m} \right|^p |dz| \leq c, \quad k = 1,2,\dots,n, \tag{7.18}$$

where c is a constant and δ_{kr} denotes the same.

By $E_{m,p}(D,Q,\rho)$ denote the class of the vectors $w(z)$ belonging to the class $E_{m,\lambda}(D,Q)$ for some $\lambda > 1$ such that the boundary values of the vector $\partial^m w / \partial z^m$ belong to the class $L_p(\Gamma,\rho)$.

If $w(z)$ is a Q-holomorphic vector from $E_{m,p}(D,Q,\rho)$, $Q(z) \in W^m_{p_0}(\mathbb{C})$, $p_0 > 2$. Then the analogous formula of (7.14) holds.

$$\begin{aligned} w(z) = &\int_\Gamma \left[I - \zeta^{-1}(t)\right]^{m-1} \ln\left[I - \zeta(z)\,\zeta^{-1}(t)\right]\left[I + Q(t)\overline{t'}^{\,2}\right]\mu(t)\,ds \\ &+ \int_\Gamma M(t)\,\mu(t) ds + iC, \end{aligned} \tag{7.19}$$

where $C = \operatorname{Im} w(0)$, $M(t) = \operatorname{diag}[M_1(t),\dots,M_n(t)]$ is a definite real continuous diagonal matrix depending only on Q and Γ; the real vector $\mu(t) \in L_p(\Gamma,\rho)$ is defined uniquely by the vector $w(z)$. By $\ln[I - \zeta(z)\,\zeta'(t)]$ we mean the branch on the plane, cut along the curve l_t (l_t connects the point t on Γ with the point $z = \infty$ and lies outside of D) which is zero-matrix at the point $z = 0$.

$E_{m,p}(D,Q,A,B,\rho)$ is the subclass of the class $E_{m,\lambda}(D,Q,A,B)$ for some $\lambda > 1$ containing vectors whose angular boundary values $\partial^m w / \partial z^m$ belong to $L_p(\Gamma,\rho)$.

The following formula holds [124]:

$$w(z) = \Phi(z) + \int_D \left[\Gamma_1(z,t)\,\Phi(t) + \Gamma_2(z)\,\overline{\Phi(t)}\,\right]dt + \sum_{k=1}^N c_k W_k(z), \tag{7.20}$$

where $\Phi(z)$ is a Q-holomorphic vector, c_k are real constants, $\{W_k(z)\}$ $(k = 1, \ldots, N)$ is a complete system of linearly independent solutions of the Fredholm equation

$$Kw \equiv w(z) - \frac{1}{\pi} \int_D V(t, z) \big[A(t)\, w(t) + B(t)\, \overline{w(t)} \,\big] d\sigma_t. \tag{7.21}$$

$W_k(z)$ turn out to be continuous vectors in the whole plane vanishing at infinity; the kernels $\Gamma_1(z, t)$ and $\Gamma_2(z, t)$ satisfy the system of integral equations

$$\begin{aligned} &\Gamma_1(z,t) + \frac{1}{\pi} V(t,z)\, A(t) + \frac{1}{\pi} \int_D V(\tau,z) \big[A(\tau)\, \Gamma_1(\tau,t) + B(\tau) \overline{\Gamma_2(\tau,t)} \big] d\sigma_\tau \\ &\qquad = -\frac{1}{2} \sum_{k=1}^{N} \big\{ v_k(z), \overline{v}_k(t) \big\}, \\ &\Gamma_2(z,t) + \frac{1}{\pi} V(t,z)\, A(t) + \frac{1}{\pi} \int_D V(\tau,z) \big[A(\tau)\, \Gamma_2(\tau,t) + B(\tau) \overline{\Gamma_1(\tau,t)} \big] d\sigma_\tau \\ &\qquad = -\frac{1}{2} \sum_{k=1}^{N} \big\{ v_k(z), \overline{v}_k(t) \big\}, \end{aligned} \tag{7.22}$$

where $v_k(z) \in L_p(\overline{D})$ $(k = 1, \ldots, N)$ form a system of linearly independent solutions of the Fredholm integral equation

$$v(z) + \frac{\overline{A'(z)}}{\pi} \int_D \overline{V'(z,t)}\, v(t)\, d\sigma_t + \frac{\overline{B'(z)}}{\pi} \int_D V'(z,t)\, \overline{v(t)}\, d\sigma_t = 0. \tag{7.23}$$

In (7.22) the curly bracket $\{v, w\}$ mains a diagonal product of the vectors v and w, the matrix $V(t, z)$ is a generalized Cauchy kernel for the equation (4) in case $A(z) \equiv B(z) \equiv 0$. $\Phi(z)$ in (7.20) has to satisfy the following conditions

$$\mathrm{Re} \int_D \Phi(z)\, v_k(z)\, d\sigma_z = 0, \quad k = 1, \ldots, N. \tag{7.24}$$

Note that generally speaking, the Liouville theorem in not true for solutions of the equation (7.13). This explains the appearence of the constants c_k in the representation formula (7.20) and the conditions (7.24).

From (7.24) we have

$$w(z) = \Phi(z) + h(z), \tag{7.25}$$

where $\Phi(z) \in E_{m,p}(D, Q, \rho)$ and $h(z) \in H^m(D)$, $W_k(z) \in H^m(D)$.

Next we consider differential boundary value problem of linear conjugation type for generalized analytic vectors, i.e. the boundary condition contains the boundary values of the desired vector and its derivatives on both sides of jump line.

Let Γ be a smooth simple curve. Denote by $D^+ (D^-)$ the finite (infinite) domain which is bounded by Γ. Suppose $0 \in D^+$. Consider the pair of equations

$$\frac{\partial w}{\partial \overline{z}} - Q_+(z) \frac{\partial}{\partial z} + A_+(z)\, w(z) + B_+(z)\, \overline{w(z)} = 0 \quad \text{in} \quad D^+ \tag{7.26}$$

and

$$\frac{\partial w}{\partial \overline{z}} - Q_-(z)\frac{\partial}{\partial z} + A_-(z)\,w(z) + B_-(z)\,\overline{w(z)} = 0 \quad \text{in} \quad D^-, \tag{7.27}$$

where $Q_+ \in W_p^l(\mathbb{C})$, $Q_- \in W_p^m(\mathbb{C})$, $A_+, B_+ \in H^{l-1}(D^+)$, $A_-, B_- \in H^{m-1}(D)$, $A_- = B_- = 0$ in certain neighborhood of $z = \infty$. By $E^{\pm}_{l,m,p}(\Gamma, Q_\pm, A_\pm, B_\pm, \rho)$ we denote the class of solutions of equations (7.26) and (7.27) respectively,belonging to the class $E_{l,p}(D^+, Q_+, A_+, B_+, \rho)$ $[E_{m,p}(D^-, Q_-, A_-, B_-, \rho)]$ in the domain D^+ $[D^-]$. The classes $E^{\pm}_{l,m,p}(\Gamma, Q_\pm, 0, 0, \rho)$ will be denoted by $E^{\pm}_{l,m,p}(\Gamma, Q_\pm, \rho)$.

Problem (V). Find a vector $w(z)$ of the class $E^{\pm}_{l,m,p}(\Gamma, Q_\pm, A_\pm, B_\pm)$ satisfying the boundary condition

$$\sum_{k=0}^{l}\left[a_k(t)\left(\frac{\partial^k w}{\partial t^k}\right)^+ + b_k(t)\overline{\left(\frac{\partial^k w}{\partial t^k}\right)^+}\right] + \sum_{k=o}^{m}\left[c_k(t)\left(\frac{\partial^k w}{\partial t^k}\right)^- + d_k(t)\overline{\left(\frac{\partial^k w}{\partial t^k}\right)^-}\right] = f(t), \tag{7.28}$$

almost everywhere on Γ, where $a_k(t), b_k(t), c_k(t), d_k(t)$ are given piecewise continuous square matrices of order k, and $f(t)$ is a given vector of the class $L_p(\Gamma, \rho)$.

Boundary condition can also contain integral term, which we omit for the sake of simplicity.

First we consider this problem in case $A_\pm = B_\pm = 0$, i.e. the class $E^{\pm}_{l,m,p}(\Gamma, Q_\pm, \rho)$. For vectors of this class the following representation formula

$$w(t) = \begin{cases} \dfrac{1}{2\pi i}\displaystyle\int_\Gamma S_+(z,\tau,l)\,d\zeta_+(\tau)\,\mu(\tau), & z \in D^+ \\ -\dfrac{1}{2\pi i}\displaystyle\int_\Gamma S_-(z,\tau,m)\,d\zeta_-(\tau)\,\mu(\tau), & z \in D^- \end{cases} \tag{7.29}$$

holds, the kernels $S_+(z,t,l)$ and $S_-(z,t,m)$ are represented by the matrices ζ_+ $[\zeta_-]$ respectively. Thee are fundamental matrices for $Q_+(z)$ $[Q_-(z)]$, $\mu(t)$ is the solution of the equation

$$N_\mu = (\dot{D})^l\big(\zeta_+^l(t)\,\Phi_+(t)\big) - \zeta_-^m\,\dot{D}^m\,\Phi_-(t) \quad \text{in} \quad L_p(\Gamma, \rho), \tag{7.30}$$

where

$$\begin{aligned} \dot{D}\,f(z) &= \alpha(z)\,f_{\overline{z}}(z) + \beta(z)\,f_z(z), \\ \alpha(z) &= -\overline{\zeta_{\overline{z}}(z)}\big[\zeta_z(z)\,\overline{\zeta_z(z)} - [\zeta_{\overline{z}}(z)\,\overline{\zeta_{\overline{z}}(z)}\big]^{-1}, \\ \beta(z) &= -\zeta_z(z)\big[\zeta_z(z)\,\overline{\zeta_z(z)} - [\zeta_{\overline{z}}(z)\,\overline{\zeta_{\overline{z}}(z)}\big]^{-1}. \end{aligned} \tag{7.31}$$

Substituting the representation (7.29) into the boundary condition for the desired vector $\mu(t)$ we obtain the following system of singular integral equations

$$K_\mu = K_1\mu + \overline{K_2}\mu = 2f(t), \tag{7.32}$$

where

$$K_s\mu = A_s(t)\,\mu(t) + \frac{B_s(t)}{\pi i}\int_\Gamma \frac{\mu(\tau)d\tau}{\tau - t} + \int_\Gamma k_s(t,\tau)\,\mu(\tau)\,d\tau \qquad (7.33)$$
$$(s = 1, 2),$$

and

$$\begin{aligned}
A_1(t) &= a_l(t)\Big(\frac{\partial}{\partial t}\,\zeta_+(t)\Big)^l \zeta_+^{-l}(t) - c_m(t)\Big(\frac{\partial}{\partial t}\,\zeta_-(t)\Big)^m \zeta_-^{-m}(t),\\
A_2(t) &= \overline{b_l(t)\Big(\frac{\partial}{\partial t}\,\zeta_+(t)\Big)^l}\,\zeta_+^{-l}(t) - \overline{d_m(t)\Big(\frac{\partial}{\partial t}\,\zeta_-(t)\Big)^m}\,\zeta_-^{-m}(t),\\
B_1(t) &= a_l(t)\Big(\frac{\partial}{\partial t}\,\zeta_+(t)\Big)^l \zeta_+^{-l}(t) + c_m(t)\Big(\frac{\partial}{\partial t}\,\zeta_-(t)\Big)^m \zeta_-^{-m}(t),\\
B_2(t) &= \overline{b_l(t)\Big(\frac{\partial}{\partial t}\,\zeta_+(t)\Big)^l}\,\zeta_+^{-l}(t) + \overline{d_m(t)\Big(\frac{\partial}{\partial t}\,\zeta_-(t)\Big)^m}\,\zeta_-^{-m}(t),
\end{aligned} \qquad (7.34)$$

$k_s(\tau, t)$ are certain matrices with weak singularities.

In the general case the problem (7.28) is to be considered in the class $E^{\pm}_{l,m,p}(\Gamma, Q_\pm, A_\pm, B_\pm, \rho)$, and we use the integral formula

$$\begin{aligned}
w_\pm(z) = \Phi_\pm(z) &+ \int_\Gamma \big[\Gamma^1_\pm(z,\tau)\,\Phi_\pm(\tau) + \Gamma^2_\pm(z,\tau)\,\overline{\Phi(\tau)}\big]d\sigma_\tau\\
&+ \sum_{k=1}^{N^\pm} c^k_\pm\, W^k_\pm(z), \quad z \in D^\pm,
\end{aligned} \qquad (7.35)$$

where the resolvents Γ^1, Γ^2 and the vector $W_k(z)$ are as introduced above. $c^k_\pm$ $(k = 1, \ldots, N^\pm)$ unknown real constants, $\Phi_\pm(z)$ are unknown vectors of the class $E^{\pm}_{l,m,p}(\Gamma, Q_\pm, \rho)$, satisfying additional conditions

$$\operatorname{Im}\int_\Gamma \Phi_\pm(t)\,d_{Q^\pm}t\,\Psi^j_\pm(t) = 0, \quad j = 1, \ldots, N^\pm, \qquad (7.36)$$

where $\{\Psi^j_\pm\}$ form a complete system of linearly independent solutions of conjugate equations, they are continuous in the whole plane and vanish at infinity.

The formula (7.25) allows us to reduce the problem (7.28) to the case of Q-holomorphic vectors. Note that the vectors $W^k_\pm(z)$, $k = 1, \ldots, N^\pm$ have continuous derivatives up to the required order because of smoothness of the coefficients of the equations (7.26) and (7.27).

Finally we obtain the following result

Theorem 7.2

$$\inf_{t\in\Gamma}\big|\det\Omega(t)\big| > 0 \qquad (7.37)$$

holds, then the problem (7.28) is Noetherian in the class $E^{\pm}_{l,m,p}(\Gamma, Q_\pm, A_\pm, B_\pm, \rho)$ *if and only if*

$$\frac{1+\nu_k}{p} \neq \mu_{jk}, \qquad (7.38)$$

where $\mu_{jk} = 1/2\pi \arg \lambda_{jk}$, $0 \leq \arg \lambda_{jk} < 2\pi$, $k = 1, \ldots, r$, $j = 1, \ldots, n$, λ_{jk} *are the roots of the equation*

$$\det \left[\Omega^{-1}(t_k + 0)\, \Omega(t_k - 0) - \lambda I\right] = 0 \tag{7.39}$$

and $\Omega(t)$ *is the block-matrix*

$$\Omega(t) = \begin{pmatrix} \dfrac{c_m(t)}{d_l(t)} & \dfrac{b_m(t)}{a_l(t)} \end{pmatrix}. \tag{7.40}$$

Using I. Vekua representations we obtain necessary and sufficient solvability conditions and index formulae for Problem (V) in case when the plane is cut along several regular arcs for analytic functions so-called cut plane in various functional classes. These problems are important in applications. We have considered the general differential boundary value problems for analytic vectors as well as boundary value problems with shift complex conjugation on a cut plane [89]-[94].

7.4 Boundary value problems for elliptic system on the plane with angular points

In this subsection by the full analogy with the theory of generalized analytic functions the formulas of general representation of regular solutions of the system (7.13) are given, the so-called generalized analytic vectors. On this basis the boundary value problems of Riemann-Hilbert and linear conjugation in case of Holder-continuous coefficients are considered. These results and some further development of the theory of generalized analytic vectors are presented in the monograph [90].

In the work of Bojarski [31] it was shown that the methods of generalized analytic functions admits further generalization on the case of the first order elliptic systems the complex form of which is the following

$$\partial_{\overline{z}}\, \omega - Q(z)\, \partial_z\, \omega + A\, \omega + B\, \overline{\omega} = 0, \tag{7.41}$$

$\partial_z \equiv \frac{1}{2}(\partial_x - i\partial_y)$, $Q(z)$, $A(z)$, $B(z)$ are given square matrices of order n, $Q(z)$ is a given square matrix of order n of the special quasi-diagonal form: every block $Q^r = (q_{ik})^r$ is a lower (upper) triangular matrix satisfying the conditions

$$\begin{gathered} q^r_{11} = \cdots = q^r_{m_s,m_s} = q^r, \quad |q^r| \leq q_0 < 1, \\ q^r_{ik} = q^r_{i+s,k+s} \quad (i + s \leq n, \quad k + s \leq n), \end{gathered}$$

moreover $Q(z) \in W^1_p(\mathbb{C})$, $p > 2$ and $Q(z) \equiv 0$ outside some circle of the complex plane $\mathbb{C}$.

A vector $w(z) = (w_1, \ldots, w_n)$ is called a *generalized analytic vector* in some domain D of the complex plane $\mathbb{C}$, if it is a solution of an elliptic system (7.41).

Under the solution of the system (7.41) we mean the so-called regular solution, i.e. $w(z) \in L_2(\overline{D})$, $w_{\overline{z}}$, $w_z \in L_\lambda(D')$, $\lambda > 2$, $\overline{D}\,' \subset D$. The system (7.41) is to be fulfilled almost everywhere on D.

The following equation

$$\partial_{\overline{z}}\,\Psi - \partial_z(Q'\,\Psi) - A'(z)\,\Psi - \overline{B'(z)\,\Psi} = 0 \tag{7.42}$$

is called the conjugate equation of the equation (7.41) (the prime' denotes the transposition of the matrices).

When $A = B = 0$, system (7.41) takes form

$$\partial_{\overline{z}}\,w - Q(z)\,\omega(z) = 0. \tag{7.43}$$

The solutions of the equation (7.43) are called Q-holomorphic vectors.

We introduce the suitable classes of generalized analytic vectors. Let Γ be a piecewise smooth curve. Denote by $E_p^{\pm}(\Gamma, Q, A, B, \rho)$ the class of solutions of the system (7.42) representable by generalized Cauchy type integrals

$$\begin{aligned} w(z) =& \frac{1}{2\pi i}\int_\Gamma \Big\{\Omega_1(z,t)\,d_Q\,t\,\varphi(t) - \Omega_2(r,t)\,\overline{d_Q\,t\,\varphi(t)}\Big\} + \\ &+ \sum_{k=1}^{N} c_k\,W_k(z), \end{aligned} \tag{7.44}$$

where c_k are arbitrary real constants, $W_k(z)$ are continuous vectors on the whole plane vanishing at infinity, $\{W_k\}$ $(k = \overline{1,N})$ forms a complete system of linearly independent solutions of the Fredholm equation

$$Kw \equiv w(z) - \frac{1}{\pi}\int_D v(t,z)\big[A(t)\,w(t) + B(t)\,\overline{W(t)}\,\big]\,d\sigma_t = 0, \tag{7.45}$$

the matrix $v(t,z)$ is generalized Cauchy kernel for the equation (7.43), the weight function $\rho(t) = \prod\limits_{k=1}^{m} |t - t_k|^{\nu_k}$, $-1 < \nu_k < p-1$, $p > 1$, $\varphi(t) \in L_p(\Gamma,\rho)$ and satisfies the condition

$$\operatorname{Im}\int_\Gamma \big(d_Q\,t\,\varphi(t), \Psi_j\big) = 0 \quad (j = \overline{1,N}\,), \quad \big(d_Q\,t = I\,dt + Q\,\overline{dt}\,\big), \tag{7.46}$$

here Ψ_j form a similar system for the conjugate equation (7.42), Ω_1 and Ω_2 are the fundamental kernels of (7.41) representable by the resolvent Γ_1 and Γ_2

$$\begin{aligned} \Omega_1(z,t) &= v(t,z) + \int_D \Gamma_1(z,\tau)\,v(t,\tau)\,d\sigma_\tau\,, \\ \Omega_2(z,t) &= \int_D \Gamma_2(z,t)\,\overline{v(t,\tau)}\,d\sigma_\tau\,. \end{aligned} \tag{7.47}$$

Denote by $E_q^{\pm}(\Gamma, Q', -A', -B', \rho^{1-q})$, $q = p/p-1$ the class of solutions of the equation (7.42) representable in the form

$$\Psi(z) = \frac{1}{2\pi i}\int_\Gamma \Big\{\Omega_1'(t,z)\,d_{Q'}\,t\,h(t) - \overline{\Omega_r'(t,z)}\;\overline{d_{Q'}\,t\,h(t)}\Big\} +$$

$$+\sum_{k=1}^{N} c_k\,\Psi_k(z), \tag{7.48}$$

where the density $h(t) \in L_q(\Gamma, \rho^{1-q})$ satisfies the conditions

$$\operatorname{Im}\int_\Gamma \big(d_{Q'}\,t\,h(t), W_j(t)\big) = 0 \quad (j = \overline{1,N}). \tag{7.49}$$

Our model problem is the following:

Find a vector $w(z) = (w_1, \ldots, w_n) \in E_p^{\pm}(\Gamma, Q, A, B, \rho)$ satisfying the boundary condition

$$w^+(t) = a(t)\,w^-(t) + b(t)\,\overline{w^-(t)} + c(t), \tag{7.50}$$

almost everywhere on Γ. Γ is piecewise smooth closed curve, the knot points of Γ (where Γ loosed the smoothness) are included in the set of $\{t_k\}$ points; $a(t)$ and $b(t)$ are given piecewise continuous $n \times n$-matrices on Γ, $\inf|\det a(t)| > 0$ and $c(t)$ is a given vector of the class $L_p(\Gamma, \rho)$.

The boundary value problem (7.50) for holomorphic vectors was studied by A. Markushevich and is called the generalized Hilbert problem [95]. The boundary condition (7.50) contains the conjugate value of the desired vector. Therefore, the Noetherity condition and the index formula of this problem depend on the values of the angles in the knot points of the boundary curve Γ.

Substituting the integral representation formula (7.44) into the boundary condition (7.50) for the unknown vector $\varphi(t)$, we obtain the following singular integral equation

$$(M\,\varphi)(t) = f(t), \tag{7.51}$$

$$f(t) = 2c(t) + 2\sum_{k=1}^{N} c_k\,[a(t) - I]\,W_k(t) + b(t)\,\overline{W_k(t)}$$

and the solution is subjected to the conditions (5).

It is easy to see that the problem (7.50) is Noetherian in the class $E_p^{\pm}(\Gamma, Q, A, B, \rho)$ if and only if the singular integral operator $(M\,\varphi)(t)$ is Noetherian in $L_p(\Gamma, \rho)$.

The necessary and sufficient Noetherity condition for the integral operator $M\,\varphi$ is [99]

$$\inf_{t\in\Gamma,\,\xi\in R^1} \big|\det M(t_k, \xi)\big| > 0 \quad (k = \overline{1,m}), \tag{7.52}$$

where the matrix $M(t_k, \xi)$ is defined in the following way: in the points different from the knot points

$$M(t_k, \xi) = (1 + \operatorname{sgn}\xi)\,I + (1 - \operatorname{sgn}\xi)\,a(t), \tag{7.53}$$

and in the knot points

$$M(t_k,\xi) = \begin{pmatrix} (1-s_0)I+(1+s_0)a(t-0) & s_1(I-a(t-0)) \\ s_2(I-a(t+0)) & (1+s_0)I+(1+s_0)a(t+0) \end{pmatrix} \tag{7.54}$$

$s_0 = \cos th\omega$, $s_1 = e^{(\alpha-1)\omega}/\sin h\omega$, $s_2 = e^{(1-\alpha)\omega}/\sin h\omega$, $\alpha = \eta/\pi$, where η is the angle between the tangents at the knot points, $0 < \eta < 2\pi$, $\omega_j = \pi(i\beta_j + \xi)$, $\beta_j = \frac{1+\nu_j}{p}$.

If the Noetherity condition (7.52) is fulfilled, then the necessary and sufficient solvability conditions are the following

$$\operatorname{Im} \int_\Gamma \big(f(t), d_Q\, f\, \Psi_k(t)\big) = 0, \tag{7.55}$$

where $\Psi_k(z)$ form a complete system of linearly independent solutions of the homogeneous problem

$$\Psi^+(t) = a'(t)\,\Psi^-(t) + b(t)[t'] + Q'(t)\,[\bar{t}\,']\,\Psi^-(t) \tag{7.56}$$

for the equation (7.42) in the class $E_q^\pm(\Gamma, Q', -A', -B', \rho^{1-q})$.

Starting from the properties of Mellin transform and the hyperbolic trigonometric functions we define the index of singular integral operator and therefore the index of our problem (7.50) in $E_q^\pm(\Gamma, Q, A, B, \rho)$.

Consider the first order system of partial differential equations in the complex plane C

$$w_{\overline{z}} = Q(z)\,w_z\,, \tag{7.57}$$

where Q is abovementioned matrix.

Following G. Hile [60] if Q is self-commuting in C, which means

$$Q(z_1)\,Q(z_2) = Q(z_2)\,Q(z_1),$$

for any $z_1, z_2 \in C$ and $Q(z)$ has eigenvalues with the modulus less than one, then the system (7.57) is called *generalized Beltrami system.* Solutions of this equation are called Q-holomorphic vectors. Under the solution in some domain D we understand so-called regular solution [124], [31]. Equation (7.57) is to be satisfied almost everywhere in D.

The matrix valued function $\Phi(z)$ is a generating solution of the system (7.57) if it satisfies the following properties [31]:

(i) $\Phi(z)$ is a C^1 solution of (7.57) in C;

(ii) $\Phi(z)$ is self-commuting and commutes with Q in C;

(iii) $\Phi(t) - \Phi(z)$ is invertible for all z, t in C, $z \neq t$;

(iv) $\Phi(z)$ is invertible for all z in C.

We call the matrix

$$V(t,z) = \partial_t \Phi(t)\big[\Phi(t) - \Phi(z)\big]^{-1}$$

the generalized Cauchy kernel for the system (7.57).

Let now Γ be a union of simple closed non-intersecting Liapunov-smooth curves, bounding finite or infinite domain. If Γ is one closed curve then D^+ denotes the finite domain; if Γ consists of several curves then by D^+ we denote the connected domain with the boundary Γ, on these curves the positive direction is chosen such, that when moving to this direction D^+ remains left; the complement of open set $D^+ \cup \Gamma$ in the whole plane will be denoted by D^-.

Consider the following integral

$$\Phi(z) = \frac{1}{2\pi i} \int_\Gamma V(t,z)\, d_Q\, t\, \varphi(t), \tag{7.58}$$

where $\varphi(t) \in L(\Gamma)$, $d_Q(t) = Idt + Qd\bar{t}$, I is an identity matrix. It is evident, that (3) is a Q-holomorphic vector everywhere outside of Γ, $\Phi(\infty) = 0$. We call the integral (7.58) the generalized Cauchy-Lebesgue type integral for the system (7.57) with the jump line Γ.

The boundary values of $\Phi(z)$ on Γ are given by the formulas:

$$\Phi^\pm = \pm\frac{1}{2}\,\varphi(t) + \frac{1}{2\pi i} \int_\Gamma V(\tau,t)\, d_Q\, \tau\, \mu(\tau). \tag{7.59}$$

The formulas (7.59) are to be fulfilled almost everywhere on Γ, provided that $\Phi^\pm$ are angular boundary values of the vector $\Phi(z)$ and the integral in (7.59) is to be understood in the sense of Cauchy principal value.

Theorem 7.3 *Let $\Phi(z)$ be a Q-holomorphic vector on the plane cut along Γ, $\Phi(\infty) = 0$. Let $\Phi(z)$ have the finite angular boundary values $\Phi^\pm$. The vector $\Phi(z)$ is represented by the Cauchy-Lebesgue type integral (7.58) if and only if the following equality*

$$\frac{1}{\pi i} \int_\Gamma V(t,t_0)\, d\big[\Phi^+(t) - \Phi^-(t)\big] = \Phi^+(t_0) + \Phi^-(t_0) \tag{7.60}$$

is fulfilled almost everywhere on Γ.

Introduce some classes of Q-holomorphic vectors. Let

$$\rho(t) = \prod_{k=1}^{r} |t-t_k|^{\rho_k}, \quad -\frac{1}{p} < \rho_k < \frac{1}{p^*} \quad p^* = \frac{p}{p-1}\,, \tag{7.61}$$
$$k = 1, \ldots, r.$$

t_k are some fixed points on Γ.

We say that the Q-holomorphic vector $\Phi(z)$ belongs to the class

$$E_p(D^+,\rho,Q)|E_p(D^-,\rho,Q)|, \quad p > 1,$$

if $\Phi(z)$ is represented by generalized Cauchy-Lebesgue type integral in the domain D^+ (D^-) with the density from the class $L_p(\Gamma,\rho) = \langle \varphi | \rho\varphi \in L_p(\Gamma) \rangle$. It follows from (7.61) that $E_p(D^\pm, \rho, Q \subseteq E_{1+\varepsilon}(D^\pm, Q))$ for sufficiently small positive ε.

The following theorems are valid [90], [91], [100],[101], [7],[8], [5]:

Theorem 7.4 *If $Q \in E_p(D^\pm, \rho, Q)$ then it can be represented by generalized Cauchy-Lebesgue integral with respect to its angular boundary values.*

Theorem 7.5 *Let Q-holomorphic vector $\Phi(z)$ be represented by generalized Cauchy-Lebesgue type integral in the domain D^+ (D^-) with the summable density. If the angular boundary values Φ^+ (Φ^-) belong to the class $L_p^n(\Gamma,\rho,Q)$ for some weight function (7.61) then $\Phi(z)\in E_p(D^+,\rho,Q)$ ($\Phi(z)\in E_p(D^-,\rho,Q)$).*

Theorem 7.6 *Let D be a domain of the complex plane bounded by the union of simple closed non-intersecting Liapunov curves $\Gamma=\Gamma_0\cup\Gamma_1\cup\cdots\cup\Gamma_m$, $\Gamma_1,\dots,\Gamma_m$ are situated outside of each other but inside of Γ_0. If $Q\in E_p(D,\rho,Q)$ then it admits the following representation*

$$\Phi(z)=\frac{1}{\pi i}\int_\Gamma V(t,z)\,d_Q t\,\mu(t)+i\,C, \tag{7.62}$$

where $\mu(t)\in L_p(\Gamma,\rho)$ is a real vector, is a real constant vector. The vector $\mu(t)$ is defined on Γ_j, $j\geqslant 1$, uniquely within the constant vector, $\mu(t)$ on Γ_0 and the constant vector C are defined by $\Phi(z)$ uniquely.

Theorem 7.7 *Let D be the domain defined as in the above theorem. If $\Phi(z)\in E_{1+\varepsilon}(D,Q)$ and $\operatorname{Re}\Phi^+(t)(\operatorname{Im}\Phi^+(t))$ belongs to the class $L_p(\Gamma,\rho)$, $p>1$, ρ has the form* (6)*, then $\operatorname{Im}\Phi^+(t)(\operatorname{Re}\Phi^+(t))$ also belongs to the class $L_p(\Gamma,\rho)$.*

On the basis of introduced and investigated weight Cauchy-Lebesgue classes for the generalized analytic vectors can be considered the discontinuous boundary value problems of generalized analytic vectors since they are natural classes for such problems. Similarly, as in case of analytic functions [124], we mean the problems when the desired vectors in the considered case have the angular boundary values almost everywhere on boundary Γ and the boundary conditions are fulfilled almost everywhere on Γ. In this connection given coefficients of the boundary conditions are piecewise-continuous non-singular matrices.

For example in our view the Riemann-Hilbert type discontinuous boundary value problems can be solved by means of these classes. Reducing these problems to the corresponding singular integral systems one can establish the solvability criterions and index formulas of corresponding functional classes. While investigating such problems some difficulties appear, connected with the fact that the Liouwille theorem is not valid in general as well as the unique-ness theorem. In most cases these difficulties may be successfully avoided.

Recently in paper [61] detailed analysis of Q-holomorphic functions is given in the case when the matrix Q is *self-commuting*. The authors investigate generalized Q-holomorphic functions under the additional assumptions that A and B commute with Q, that is, $Q(z_1)A(z_2)=A(z_2)Q(z_1)$ and $Q(z_1)B(z_2)=B(z_2)Q(z_1)$, respectively.

The main tool of the investigations is generalized Pompeiu operator, that is, a domain integral operator J_Ω in the domain Ω which generalizes the well known (weakly singular) T_Ω-operator from I. N. Vekua's theory. One has $D(J_\Omega\nu)=\nu$, where D is the operator $\partial_{\overline{z}}-Q\partial_z$. Also a *Cauchy-Pompeiu integral formula* and the *compactness* of $\widetilde{J}w = J(Aw + B\overline{w})$ in the space of continuous and bounded functions in $\mathbb{C}$ are proved.

Special assumptions about the commutability of Q lead to (complicated) relations between the entries of the matrix Q. Using them, the authors prove the following Liouville type theorem generalized Q-holomorphic functions:

Theorem 7.8 *[61] Suppose w is a generalized Q-holomorphic continuous function in the whole plane which commutes with Q, then w can be represented in the form $w(z) = C\,\boldsymbol{exp}\,\omega(z)$, where C is a constant lower diagonal matrix and ω is matrix-function satisfies the condition: $\omega(z) = O(|z|^{-\alpha})$ as $|z| \to \infty$.*

The given proof makes use of rewriting $w_{\overline{z}} - Qw_z = Aw + B\overline{w}$ of the differential equation for generalized Q-holomorphic functions.

Theorem 7.9 *[61] If h commutes with Q, then the uniquely determined solution w of*

$$w + \widetilde{J}w = h$$

also commutes with Q.

This theorem permits us to give an analog to the generating pair of Bers. Following the paper [61] we give the definition. The $m \times m$ square matrix functions F and G are called a generating pair in some bounded domain U if they satisfies the condition:

a) F, G bounded and continuous in U;

b) $F_x, F_y, G_x, G_y \in L^p(U)$;

c)there exists a positive number ε such that $Im \sqcap_{j=1} m\overline{f_{jj}}g_{jj} \geq \varepsilon$, where f_{jj} and g_{jj} main diagonal elements of F and G, respectively;

d) F and G are commuting with Q.

The generating pair for matrix function introducing this way has all properties similar to Bers generating pair in scalar case.

The proof is obtained by rewriting the differential equation for generalized Q-holomorphic function $\omega(z)$.

8 Monodromy of generalized analytic functions

The global theory of generalized analytic functions [124] both in one-dimensional and multi-dimensional case [31], involves studying the space of horizontal sections of a holomorphic line bundle with connection on a complex manifold with singular divisor. In this context one should require that a connection is complex analytic. An interesting class of such connections is given by L_p-connections, and their moduli spaces have many applications. Such connections and their moduli spaces are the object of intensive study [122], [48], [41], [58].

8.1 System of elliptic equations on the Riemann surfaces

We study holomorphic vector bundles with L_p-connections from the viewpoint of the theory of generalized analytic vector [31]. To this end we consider a matrix elliptic system of the form

$$\partial_{\bar{z}}\Phi(z) = A(z)\Phi(z). \tag{8.1}$$

The system (8.1) is a particular case of the *Carleman-Bers-Vekua system*

$$\partial_{\bar{z}} f(z) = A(z)f(z) + B(z)\overline{f(z)}, \tag{8.2}$$

where $A(z)$, $B(z)$ are bounded matrix functions on a domain $U \subset \mathbb{C}$ and $f(z) = (f^1(z), ..., f^n(z))$ is an unknown vector function. The solutions of the system (8.2) is called *generalized analytic vectors*, by analogy with the one-dimensional case [124], [31].

Along with similarities between the one-dimensional and multi-dimensional cases, there also exist essential differences. One of them, as noticed by B.Bojarski [31], is that there may exist solutions of system (8.1) for which there is *no analogue of the Liouville theorem* on the constancy of bounded entire functions.

We present first some necessary fundamental results of the theory of generalized analytic functions and vectors [124],[31],[90] in the form convenient for our purposes. A modern consistent exposition of this theory was gives by A.Soldatov [115], [116].

Let $f \in \mathrm{L}^p(U)$, where U is a domain in $\mathbb{C}$. As above (see section 2.1) we write $f \in \mathrm{W}_p(U)$, if there exist functions θ_1 and θ_2 of class $\mathrm{L}^p(U)$ such that the equalities

$$\iint_U f\frac{\partial\varphi}{\partial\bar{z}}dU = -\iint_U \theta_1\varphi dU, \iint_U f\frac{\partial\varphi}{\partial z}dU = -\iint_U \theta_2\varphi dU$$

hold for any function $\varphi \in \mathrm{C}^1(U)$.

Let us define two differential operators on $\mathrm{W}_p(U)$

$$\partial_{\bar{z}} : \mathrm{W}_p(U) \to \mathrm{L}_p(U), \quad \partial_z : \mathrm{W}_p(U) \to \mathrm{L}_p(U),$$

by setting $\partial_{\bar{z}} f = \theta_1$, $\partial_z f = \theta_2$. The functions θ_1 and θ_2 are called the generalized partial derivatives of f with respect to $\bar{z}$ and z respectively. Sometimes we will use the shorthand notation $f_{\bar{z}} = \theta_1$ and $f_z = \theta_2$. It is clear that ∂_z and $\partial_{\bar{z}}$ are linear operators satisfying the Leibnitz equality.

Define the following singular integral operator on the Banach space $\mathrm{L}_p(U)$:

$$T : \mathrm{L}_p(U) \to W_p(U),$$

$$T(\omega) = -\frac{1}{\pi} \iint_U \frac{\omega(t)}{t-z} dU, \omega \in L_p(U). \tag{8.3}$$

The integral (8.3) makes sense for all $\omega \in \mathrm{L}_p(U)$, almost all $z \in U$, and all $z \notin \bar{U}$ and (8.3) determines a function $\varphi(z) = T(\omega)$ on the whole $\mathbb{C}$. For $\omega \in \mathrm{L}_p(U)$ with $p > 2$, the function φ is continuous.

Any element of $\mathrm{W}_p(U)$ can be represented by the integral (8.3). In particular, if $f_{\bar{z}} = \omega$, then $f(z)$ can be represented in the form

$$f(z) = h(z) - \frac{1}{\pi} \iint_U \frac{\omega(t)}{t-z} dU,$$

where $h(z)$ is holomorphic in U. The converse is also true, i.e., if $h(z)$ is holomorphic in U and $\omega \in \mathrm{L}_p(U)$, then $h(z) - \frac{1}{\pi} \iint_U \frac{\omega(t)}{t-z} dU$ determines an element $f(z)$ of $\mathrm{W}_p(U)$ satisfying the equality $f_{\bar{z}} = \omega$.

As we have seen, the generalized derivative with respect to $\bar{z}$ of the integral (8.3) is ω. Similarly, there exists a generalized derivative of this integral with respect to z. It equals

$$-\frac{1}{\pi} \iint_U \frac{\omega(t)}{(t-z)^2} dU. \tag{8.4}$$

The integral (8.4) is understood in the sense of Cauchy principal value and by definition equals

$$S(\omega) = \lim_{\varepsilon \to 0} f_\varepsilon(z) \equiv -\frac{1}{\pi} \iint_{U_\varepsilon} \frac{\omega(t)}{(t-z)^2} dU, \tag{8.5}$$

where $U_\varepsilon = \mathbb{C} \setminus \overline{D_\varepsilon(z)}$, with $D_\varepsilon(z)$ being the disk of radius ε centered at z. In the equality (8.4), the limit converges to function $f(z)$ in L_p-metric, $p > 1$.

It is known [124] that in one dimensional case a solution of (8.1) can be represented as

$$\Phi(z) = F(z) \exp(\omega(z)), \tag{8.6}$$

where F is a holomorphic function in U, and $\omega = -\frac{1}{\pi} \int\int_U \frac{A(z)}{\xi - z} dU$. In multi-dimensional case an analogue of factorization (8.6) is given by the following theorem

Theorem 8.1 *[31] Each solution of the matrix equation (8.1) in U can be represented as*

$$\Phi(z) = F(z) V(z), \tag{8.7}$$

where $F(z)$ is an invertible holomorphic matrix function in U, and $V(z)$ is a single-valued matrix function invertible outside $\overline{U}$.

The representation of the solution of the system (8.1) of the form (8.7) we use for the construction holomorphic vector bundle on the Riemann sphere and for computation of monodrmy matrices of the elliptic system of the form (8.1).

We recall some properties of solutions of (8.1). The product of two solutions is again a solution. From Theorem 8.1 it follows (see also [45]) that the solutions constitute an algebra and the invertible solutions are a subfield of this algebra.

Proposition 8.2 *Let $C(z)$ be a holomorphic matrix function, then $[C(z), \partial_{\overline{z}}] = 0$.*
Indeed,

$$[C(z), \partial_{\overline{z}}]\Phi(z) = C(z)\partial_{\overline{z}}\Phi(z) - \partial_{\overline{z}}C(z)\Phi(z) = C(z)\partial_{\overline{z}}\Phi(z) - C(z)\partial_{\overline{z}}\Phi(z) = 0.$$

Here we have used that $\partial_{\overline{z}}C(z) = 0$.

Definition 8.3 *Two systems $\partial_{\overline{z}}\Phi(z) = A(z)\Phi(z)$ and $\partial_{\overline{z}}\Phi(z) = B(z)\Phi(z)$ called gauge equivalent if there exists a non-degenerate holomorphic matrix function $C(z)$, such that $B(z) = C(z)A(z)C(z)^{-1}$.*

Proposition 8.4 *Let the matrix function $\Psi(z)$ be a solution of the system $\partial_{\overline{z}}\Phi(z) = A(z)\Phi(z)$ and let $\Phi_1(z) = C(z)\Phi(z)$, where $C(z)$ is a nonsingular holomorphic matrix function. Then $\Phi(z)$ and $\Phi_1(z)$ are solutions of the gauge equivalent systems.*

The converse is also true: if $\Phi(z)$ and $\Phi_1(z)$ satisfy systems of equations

$$\partial_{\overline{z}}\Phi(z) = A(z)\Phi_(z),$$

$$\partial_{\overline{z}}\Phi_1(z) = B(z)\Phi_1(z)$$

and $A(z) = C^{-1}(z)B(z)C(z)$, then $\Phi_1 = D(z)\Phi(z)$ for any holomorphic matrix function $D(z)$.

Proof. By Proposition 8.2 we have $C(z)\partial_{\overline{z}}\Phi_1(z) = A(z)C(z)\Phi_1(z)$, and therefore $\Phi_1(z)$ satisfies the equation $\partial_{\overline{z}}\Phi_1(z) = C^{-1}(z)A(z)C(z)\Phi_1(z)$. To prove the converse let us substitute in $\partial_{\overline{z}}\Phi(z) = A(z)\Phi_(z)$, instead of $A(z)$ the expression of the form $C^{-1}B(z)C(z)$ and consider

$$\partial_{\overline{z}}\Phi_1(z) = C^{-1}B(z)C(z)\Phi(z) \Longrightarrow$$

$$C(z)\partial_{\overline{z}}\Phi(z) = B(z)C(z)\Phi(z)$$

but for the left hand side of the last equation we have $C(z)\partial_{\overline{z}}\Phi(z) = \partial_{\overline{z}}C(z)\Phi(z)$, therefore

$$\partial_{\overline{z}}(C(z)\Phi(z)) = B(z)(C(z)\Phi(z)).$$

From this it follows that Φ and $C\Phi$ are the solutions of equivalent systems, which means that $\Phi_1 = D\Phi$.

The above arguments for solutions of (8.1) are of a local nature, so they are applicable for an arbitrary compact Riemann surface X, which enables us to construct a holomorphic vector bundle on X (see [56], [58]). Moreover, using the solutions of system (8.1) one can construct the matrix 1-form $\Omega = D_{\overline{z}} F F^{-1}$ on X which is analogous to holomorphic 1-forms on Riemann surfaces.

Let X be a Riemann surface. Denote by $L_p^{\alpha,\beta}(X)$ the space of L_p-forms of the type $(\alpha,\beta), \alpha,\beta = 0,1$, with the norm $\|\omega\|_{L_p^{\alpha,\beta}(X)} = \sum_j \|\omega\|_{L_p^{\alpha,\beta}(U_j)}$, where $\{U_j\}$ is an open covering of X and denote by $W_p(U) \subset L_p(U)$ the subspace of functions which have generalized derivatives.

We define the operators

$$D_z = \frac{\partial}{\partial z} : W_p(U) \to L_p^{1,0}(U), f \mapsto \omega_1 dz = \partial_z f dz,$$

$$D_{\bar{z}} = \frac{\partial}{\partial \bar{z}} : W_p(U) \to L_p^{0,1}(U), f \mapsto \omega_2 d\,\bar{z} = \partial_{\overline{z}} f d\overline{z}.$$

It is clear that $D_{\bar{z}}^2 = 0$ and hence the operator $D_{\bar{z}}$ can be used to construct the *de Rham cohomology*.

Let us denote by $\mathbb{C}L_p^1(X)$ the *complexification* of $L_p^1(X)$, i.e. $\mathbb{C}L_p^1(X) = L_p^1(X) \otimes \mathbb{C}$. Then we have the natural decomposition

$$\mathbb{C}L_p^1(X) = L_p^{1,0}(X) \oplus L_p^{0,1}(X) \tag{8.8}$$

according to the eigenspaces of the Hodge operator $* : L_p^1(X) \to L_p^1(X)$, $* = -\imath$ on $L_p^{1,0}(X)$ and $* = \imath$ on $L_p^{0,1}(X)$.

The decomposition (8.8) splits the operator $d : L_p^0(X) \to L_p^0(X)$ in the sum $d = D_z + D_{\bar{z}}$.

Next, let, as above, $\mathcal{E} \to X$ be C^∞-vector bundle on X, let $L_p(X, \mathcal{E})$ be the sheaf of the L_p-sections of $\mathcal{E}$ and let $\Omega \in L_p^1(X, \mathcal{E}) \otimes GL_n(\mathbb{C})$ be a matrix valued 1-form on X. If the above arguments are applied to the complex $L_p^*(X, \mathcal{E})$ with covariant derivative $\bigtriangledown_\Omega$, we obtain again the decompositions of the space $\mathbb{C}L_p^1(X, \mathcal{E})$ and the operator $\bigtriangledown_\Omega$

$$\mathbb{C}L_p^1(X, \mathcal{E}) = L_p^{1,0}(X, \mathcal{E}) \oplus L_p^{0,1}(X, \mathcal{E}),$$

$$\bigtriangledown_\Omega = \bigtriangledown'_\Omega + \bigtriangledown''_\Omega.$$

Locally, on the domain U, we have $\bigtriangledown_\Omega^U = d_U + \Omega$, where $\Omega \in L_p^1(X, U) \otimes GL_n(\mathbb{C})$ is a 1-form. Therefore $\bigtriangledown_\Omega^U = (D_z + \Omega_1) + (D_{\bar{z}} + \Omega_2)$, where Ω_1 and Ω_2 are, respectively, holomorphic and anti-holomorphic part of the matrix value 1-form on U. We say that a W_p-section f of the bundle $\mathcal{E}$ with L_p-connection is holomorphic if it satisfies the system of equations

$$\partial_{\bar{z}} f(z) = A(z) f(z), \tag{8.9}$$

where $A(z)$ is a $n \times n$ matrix-function with entries in $L_p^0(X) \otimes GL_n(\mathbb{C})$ and $f(z)$ is a vector function $f(z) = (f_1(z), f_2(z), ..., f_n(z))$, or in equivalent form (8.9) may be written as:

$$D_{\overline{z}} f = \Omega f,$$

where $\Omega \in L_p^1(X) \otimes GL_n(\mathbb{C})$.

8.2 Associated bundles with regular connections

We now use the above arguments for constructing a holomorphic vector bundle over Riemann sphere $\mathbb{CP}^1$ by system (8.1). Let $\{U_j\}$, j=1,2, be an open covering of the $\mathbb{CP}^1$. Then in any domain U_j, a solution $\Phi(z)$ can be represented as $\Phi(z) = V_j(z)F(z)$, where $V_j(z)$ is a holomorphic non-degenerate matrix function on $U_j^c - S_j$ where S_j being a finite set of points. Restrict $\Phi(z)$ on the $(U_1^{\mathbb{C}} \cap U_2^{\mathbb{C}}) - S = (U_1 \cup U_2)^{\mathbb{C}} - S$, $S = S_1 \cup S_2$ and consider the holomorphic matrix-function $\varphi_{12} = V_1(z)V_2(z)^{-1}$ on $(U_1 \cup U_2)^{\mathbb{C}} - S$. It is a cocycle and therefore defines a holomorphic vector bundle $\mathcal{E}'$ on $\mathbb{CP}^1 - S$. From the proposition 8.1 it follows, that $\mathcal{E}' \to \mathbb{CP}^1 - S$ is independent of the choice of solutions in the same gauge equivalence class. The extension of this bundle to a holomorphic vector bundle $\mathcal{E} \to \mathbb{CP}^1$ can be done by a well-known construction (see [56], [58]) and the obtained bundle is holomorphically nontrivial.

It is now possible to verify that the operator $\frac{\partial}{\partial \bar{z}} + \Omega(z, \overline{z})$ is a L_p-connection of this bundle. It turns out that its index coincides with the index of Cauchy-Riemann operator on X. This follows since the index of Cauchy-Riemann operator is equal to the Euler characteristic of the sheaf of holomorphic sections of the holomorphic vector bundle $\mathcal{E}$.

For the given loop $G : \Gamma \to GL_n(\mathbb{C})$, find the piecewise continuous generalized analytic vector $f(z)$ with the jump on contour Γ such that on Γ it satisfies the conditions

$$\text{a) } f^+(t) = G(t)f^-(t), t \in \Gamma, \ \text{ b) } |f(t)| \leq c|z|^{-1}, |z| \to \infty.$$

It is known, that for G there exists a Birkhoff factorization (see [59], i.e.

$$G(t) = G_+(t)d_K(t)G_-(t),$$

setting this equality in a) we obtain the following boundary problem

$$G_+^{-1}(t)f^+(t) = d_K(t)G_-(t)f^-(t).$$

Since $G_+^{-1}(t)f^+(t)$, $f^+(t)$ and $G_-(t)f^-(t)$, $f^-(t)$ are solutions of the gauge equivalent systems, the holomorphic type of the corresponding vector bundles on Riemann sphere is defined by $K = (k_1, ..., k_n)$.

Proposition 8.5 *The cohomology groups $H^i(\mathbb{CP}^1, \mathcal{O}(\mathcal{E}))$ and $H^i(\mathbb{CP}^1, \mathcal{G}(\mathcal{E}))$ are isomorphic for $i = 0, 1$, where $\mathcal{O}(\mathcal{E}))$ and $\mathcal{G}(\mathcal{E})$, respectively, are the sheaves of holomorphic and generalized analytic sections of $\mathcal{E}$.*

From this proposition follows that the number of linear independent solutions of the Riemann-Hilbert boundary problem is equal to $\sum_{k_j<0} k_j$. Its holomorphic type is determined by an integer vector. In terms of cohomology groups $H^i(\mathbb{CP}^1, \mathcal{O}(\mathcal{E}))$ and $H^i(\mathbb{CP}^1, \mathcal{G}(\mathcal{E}))$ one can describe the number of solutions and stability of the Riemann-Hilbert boundary value problem [25], [23] the topological constructions related with the sheaf $\mathcal{O}(\mathcal{E})$ can be extended to the sheaf $\mathcal{G}(\mathcal{E})$ [56].

Theorem 8.6 *There exists one-to-one correspondence between the space of gauge equivalent Carleman-Bers-Vekua systems and the space of holomorphic structures on the bundle* $E \to X$.

For the investigation of the monodromy problem for Phaff system an important role is played by a representation of solution of the system in exponential form, which in one-dimensional case was studied by W.Magnus in [82]. We use iterated path integrals and the theory of formal connections (as a paralel transport operator) developed by K.-T.Chen [39].

Let $\omega_1, ..., \omega_r \in L^1_p(X)$ and $\gamma : [0,1] \to X$ be a piecewise continuous path. Let a_j be functions defined on $[0,1]$ and satisfying the identity $\gamma^*\omega_j = a_j(t)dt, j = 1, ..., r$.

Definition 8.7 *The* r*-iterated integral from 1-form* $\omega_1, ..., \omega_r$ *is defined as the function on the space of piecewise continuous paths whose value on a path* γ *is the number* $\gamma \to \int_\gamma \omega_1...\omega_r$*, where* $\int_\gamma \omega_1...\omega_r$ *is*

$$\int_\gamma \omega_1...\omega_r = \int_{\triangle_r} a_1(t_1)a_2(t_2)...a_r(t_r)dt_1dt_2...dt_r.$$

Here on the right hand side is an ordinary Lebesque integral on the simplex

$$\triangle_r = \{(t_1, ..., t_r) : 0 \leq t_1 \leq t_2 \leq ... \leq t_r \leq 1\}.$$

By passing to a multiple integral the value of r-iterated integral can be expressed by the formula:

$$\int_\gamma \omega_1...\omega_r = \int_0^1 a_r(t_r)... \int_0^{t_3} a_2(t_2)(\int_0^{t_2} a_1(t_1)dt_1)dt_2...dt_r.$$

If $r = 1$, then we obtain ordinary path integral.

Let $P_{z_0}X$ be the space of piecewise continuous loops. It is known that it is a differentiable space [39] and the operator of exterior differentiation d is defined $P_{z_0}X$. Let d $\int_\gamma \omega_1...\omega_r = 0$, then $\int_\gamma \omega_1...\omega_r = 0$ depends only on homotopy class of γ and therefore we obtain a function on $\pi_1(X, z_0)$.

Let $\Omega_1, ..., \Omega_r$ be $m \times m$ matrix forms with entries of $L^1_p(X)$. The iterated integral from $\Omega_1, ..., \Omega_r$ is defined as follows: consider the form product of matrix form $\Omega = \Omega_1, ..., \Omega_r$ and iterated integral of Ω is defined elementwise.

Proposition 8.8 *[50] The parallel transport corresponding to the elliptic system (8.1), has an exponential representation.*

Since the elliptic system (8.1) defines a connection the proof of proposition follows from the general theory of formal connections.

From the identity $\partial_{\bar{z}}\Phi\Phi^{-1} = \Omega$ it follows that singular points of Ω are zeros of the matrix function Φ, in particular ∞. This means that it makes sense to speak of singular and apparent singular points of the system (8.1).

In case $n = 1$ from (8.6) it follows, that given analytic function $F(z)$, one can define by (8.6) generalized analytic function $\Phi(z)$ uniquely. Besides that, if $z_1, ..., z_m$ are poles (or branching points) for $F(z)$, then they are poles (correspondingly, branching points) for $\Phi(z)$ too. For $n > 1$ this does not hold. Thus we want to emphasize once more the difference between the one-dimensional and multi-dimensional theory of generalized analytic functions. In general the correspondence between holomorphic vectors and generalized analytic vectors is not one-to-one.

Despite that, properties of the generalized analytic functions allow one to construct a generalized analytic function with given monodromy.

From the integrability of (8.1) it follows that for the iterated integral $\int \Omega\Omega...\Omega$ *we have* $d\int \Omega\Omega...\Omega = 0$ *and therefore we have a representation of the fundamental group* $\pi_1(X - S, z_0)$.

We can say that $z_i \in \{z_1, ..., z_m\}$ is a regular singular point of (8.1), if any element of $F(z)$ has at most polynomial growth as $z \to z_i$. If the solution $\Phi(z)$ at any singular point $z_i, i = 1, ..., m$ has a regular singularity, then we call the system (8.1) a regular system.

In case $n = 1$ the singular integral (8.6) is well studied. In particular, it is known that $\omega(z)$ is holomorphic in $\mathbb{C}_m \setminus \bar{U}_{z_0}$ and equal to zero at infinity. Here $\mathbb{C}_m = \mathbb{CP}^1 \setminus \{z_1, ..., z_m\}$.

Let $\tilde{z} \in U_{z_0}$ be any point and let $\gamma_1, \gamma_2, ..., \gamma_m$ be loops at $\tilde{z}$ such that γ_i goes around z_i without going around any $z_j \neq z_i$. Consider the holomorphic continuation of the function $F(z)$ around γ_i. Then we obtain the analytical element $\tilde{F}(z)$ of the holomorphic function $F(z)$, which are related by the equality $\tilde{F}_i(z) = m_i F(z)$, where $m_i \in \mathbb{C}^*$. It is independent of the choice of the homotopy type of loop γ_i. Therefore, we obtain a representation of the fundamental group $\pi_1(\mathbb{CP}^1 \setminus \{z_1, ..., z_m\}, \tilde{z}) \to \mathbb{C}^*$, which is defined by the correspondence $\gamma_i \to m_i$.

Let us summarize the above arguments.

Proposition 8.9 *Let the system (8.1) have regular singularities at points* $z_1, ..., z_m$. *Then it defines a monodromy representation of the fundamental group*

$$\rho : \pi_1(\mathbb{C} \setminus \{z_1, ..., z_m\}, \tilde{z}) \to GL_n(\mathbb{C}).$$

In this situation the monodromy matrices are given by Chen iterated integrals

$$\rho(\gamma_j) = 1 + \int_{\gamma_j} \Omega + \int_{\gamma_j} \Omega\Omega + \int_{\gamma_j} \Omega\Omega\Omega + ... + ... \tag{8.10}$$

The convergence properties of series (8.10) can be described as follows. Let a 1-form Ω be smooth except the points $s_1, ..., s_m \in X$. Let, as above, $S = \{s_1, s_2, ..., s_m\}$ and $X_m = X - S$. Thus, for every $\gamma \in PX_m$, there exists a constant $C > 0$ such that

$$|\int_{\gamma_j} \overbrace{\Omega...\Omega}^{r}| = 0(\frac{C^r}{r!})$$

and the series (8.10) converges absolutely [52].

In addition to system (8.1) consider the system of ordinary differential equations with regular singularities at points $z_1, z_2, ..., z_m$:

$$\frac{df}{dz} = A(z)f(z) \tag{8.11}$$

Let

$$\rho : \pi_1(\mathbb{CP}^1 \backslash \{z_1, ..., z_m\}, \tilde{z}) \to GL_n(\mathbb{C}) \tag{8.12}$$

be the monodromy representation induced from (8.11). Denote by $F(z)$ the fundamental matrix of solutions of (8.11). Then $F(z)$ satisfies the system of equations

$$dF = \widehat{\Omega} F(z),$$

where $\widehat{\Omega} = A(z)dz$ denotes now the corresponding matrix valued $(1,0)$-form on the surface $\mathbb{CP}^1 \backslash \{z_1, ..., z_m\}$.

Let $\Omega(z) = *\widehat{\Omega}$, where $*$ is the Hodge star operator. Then $\Omega(z)$ is a (0,1)-form and it makes sense to consider the system of type (8.1)

$$\partial_{\bar{z}} \Phi = \Phi \Omega(z), \tag{8.13}$$

which has regular singularities at the points $z_1, z_2, ..., z_m$.

For the regular system (8.11), the Poincaré theorem is valid, which gives that the fundamental matrix of solutions (8.11) has the form

$$F(z) = (z - z_i)^{E_i} Z(z), \tag{8.14}$$

where $E_i = \frac{1}{2\pi i} \ln M_i$ and M_i is the monodromy matrix corresponding to the singular point z_i. It follows that any solution of the system (8.13) in the neighborhood U_{z_i} has the form

$$\Phi(z) = (z - z_i)^{E_i} Z(z) V(z, \overline{z}). \tag{8.15}$$

All what was said above remains true for arbitrary compact Riemann surface of genus g. It is clear, that in this case the singular integral (8.3) should contain the Cauchy kernel for the given Riemann surface.

As was noted, a system (8.13) without singularities induces flat vector bundle $\mathcal{E} \to X$ on the Riemann surface X. Analogously, the system (8.13) with regular singularity gives flat vector bundle on the surface $X_m = X \backslash \{z_1, z_2, ..., z_m\}$ which we denote by $\mathcal{E}' \to X_m$. The representation of solutions in the form (8.14), (8.15) gives possibility to extend $\mathcal{E}'$ to the whole of X and if we choose the canonical extension then we obtain the uniquely defined (possibly topologically nontrivial) vector bundle $\mathcal{E} \to X$ which is induced from (8.13). The Chern number of the bundle $\mathcal{E} \to X$ can be calculated from monodromy matrices in the following way: $c_1(\mathcal{E}) = \sum_{i=1}^{m} tr(E_i)$, here matrices E_j are chosen in such a way that eigenvalues $\lambda_i, i = 1, ..., n$ satisfy inequalities $0 \leq \Re \lambda_i^j < 1, j = 1, ..., n$.

The matrix-function Φ is a holomorphic section of the bundle $End\,\mathcal{E} \to X$. Assume that $\mathcal{E} \to X$ is stable in the sense of Mumford. Since stability implies

$H^0(X, \mathcal{O}(End\,\mathcal{E})) \cong \mathbf{C}$, from the Riemann-Roch theorem for the bundle $End\,\mathcal{E} \to X$ we obtain

$$\dim H^1(X, \mathcal{O}(End\,\mathcal{E})) = n^2(g-1) + 1. \tag{8.16}$$

Since there exists the one-to-one correspondence between the gauge equivalent systems (8.13) and the holomorphic structures on the bundle $\mathcal{E} \to X$, we obtain that if the system (8.13) induces a stable bundle, then the dimension d of the gauge equivalent solutions of the system (8.13) is calculated by formula (8.16).

9 Carleman-Bers-Vekua equation on Riemann surfaces

9.1 Cauchy-type integral on Riemann surfaces

Let X be a compact Riemann surface. Boundary-value problems on such surfaces were considered by many authors (see [109] and the references therein). We describe below some constructions and problems of a global nature in the spirit of the geometric approach to (a) *Riemann-Hilbert boundary value problem* and (b) *Riemann-Hilbert monodromy problem* (see [58], [56], [59]). In view of what was said above, it is clear that the representation of functions of the considered class in the form of Cauchy-type integral is one of the key tools for solving such boundary-value problems. Recall that it is also possible to construct a Cauchy-type integral on the Riemann surface X [11], [16]-[19], [50], [107]. Denote the corresponding kernel (it is described, e.g., in [109]) by $K_X(\tau, z)d\tau$. If we cut the surface X along the cycles of a canonical homology basis, we obtain polygon $\widehat{X}$ and $K_X(\tau, z)d\tau$ is single valued in $\widehat{X}$. If γ is some sufficiently smooth curve on $\widehat{X}$ and $\psi(\tau)$ is a Hölder-continuous function on Γ, we can consider the Cauchy-type integral

$$f(z) = \frac{1}{2\pi i}\int_\gamma \psi(\tau)K_X(\tau, z)dz.$$

Boundary-value problems on Riemann surfaces are closely connected with differential forms. Recall that a differential 1-form on a compact Riemann surface which is analytic everywhere except for a finite number of poles is called the *Abelian differential* and the corresponding indefinite integral is called the *Abelian integral.* Analytic Abelian differentials on surfaces without singular points are called *Abelian differentials of the first kind.* Abelian differentials having a finite number of poles with zero residues are called *Abelian differentials of the second kind*, and Abelian differentials having poles with nonzero residues called *Abelian differentials of the third kind.*

If M is a compact Riemann surface of genus g, then the real dimension of the linear space of Abelian differentials of the first kind is equal to $2g$ and the sum of the residues of Abelian differential is equal to zero. Let $\gamma_1, ..., \gamma_{2g}$ be a *canonical homology basis* of M and let z_0 and z be arbitrary different points on M. The normalized Abelian differentials of the third kind $d\omega_{z_0 z}(\tau)$ and $d\Omega_{z_0 z}(\tau)$ are defined as differentials which have poles with residues -1 and +1 at the points z_0 and z, respectively, and periods satisfying the conditions

$$\int_{\gamma_{2j-1}} d\omega_{z_0 z}(\tau) = 0, j = 1, ..., g,$$

$$\int_{\gamma_j} d\Omega_{z_0 z}(\tau) = 0, j = 1, ..., 2g.$$

The Cauchy type integral is the main tool for the investigation of boundary problems on the Riemann sphere. The existence of Cauchy-type integrals, in particular, yields the Cauchy integral representation and the Sokhotsky formulas. The Cauchy-type integral can be described rather explicitly by analogy with the case of a complex plane, where its properties can be formulated as follows.

Theorem 9.1 *The kernel $K(\tau, z)d\tau$ of Cauchy-type integral on the complex plane is Abelian differential of the third kind with respect to the variable τ having first-order poles at the points $\tau = z, \infty$ with residues ± 1, respectively, and it is analytic function with respect to z having a pole at $z = t$ and zero at $z = \infty$. In explicit form $K_{\mathbb{CP}^1}(\tau, z)d\tau = \frac{d\tau}{\tau - z}$.*

Starting from the Cauchy-type integral on the Riemann sphere, we can construct the Cauchy-type integral on arbitrary compact Riemann surface. A direct generalization is not possible but there exist natural analogs. First of all, let us require that $K_X(\tau, z)d\tau$ be a function with respect to z and a differential form with respect to τ. Moreover, every analog of Cauchy-type integral should satisfy the condition

$$K_X(\tau, z)d\tau = \frac{d\tau}{\tau - z} + \text{regular terms}, \tag{9.1}$$

which implies the Cauchy formula and the Sokhotski-Plemelj formula. Following the paper [50], denote by $d\omega_{qq_0}(p)$ the Abelian differential of the third kind on X which (a) has vanishing periods and (b) has two simple poles at the points $p = q$ and $p = q_0$ with residues $+1$ and -1, respectively. Denote by

$$d\omega_1(p), d\omega_2(p), ..., d\omega_g(p)$$

a canonical basis of Abelian differentials of the first kind (i.e., periods of this basis form the identity $(g \times g)$-matrix). Let $d\widetilde{\omega}_{qq_0}(p)$ be a single-valued branch of multi-valued differential 1-form $d\omega_{qq_0}(p)$ on $\widetilde{X}$, where $\widetilde{X}$ is obtained from X by cutting along $a_1, ..., a_g$ and $d\widetilde{\omega}_{qq_0}(p)$ satisfies the condition $d\widetilde{\omega}_{qq_0}(p) = 0$. This branch can be changed by imposing conditions

$$d\widetilde{\omega}_{qq_0}(p) = \int_{q_0}^{q} d_t[d\omega_{tq_0}(p)],$$

where the integration path does not intersect $a_1, ..., a_g$. This differential is easily seen to satisfy condition (9.1). For this reason, $d\widetilde{\omega}_{qq_0}(p)$ can be considered as a Cauchy kernel and is called the *discontinuous Cauchy kernel* since it is discontinuous on curves a_j. In some cases it possesses further nice properties similar to Theorem 9.1.

Let $\Delta = p_1^{n_1}...p_k^{n_k}$ be a divisor on the surface X. A meromorphic analog of the Cauchy kernel with minimal characteristic divisor Δ is an expression of the form $A(p, q)dp$, which satisfies the following conditions:

(1) $A(p, q)dp = \frac{dp}{p-q} +$ regular terms and it is Abelian differential with respect to the variable p with divisor $q^{-1}\Delta$;

(2) with respect to the variable q, $A(p,q)dp$ is a meromorphic function on X with divisor $p^{-1}\Delta^{-1}$.

Actually, there are several other ways of constructing an analog of the Cauchy kernel (see [1], [16], [111]). We described only the most natural one, and in the sequel we always use the Cauchy kernel constructed

$$f(z) = \frac{1}{2\pi i}\int_{\gamma} \psi(\tau)K_X(\tau, z)d\tau$$

for the surface X and obtain the Plemelj-Sokhotski formulas [109]:

$$f^{\pm}(z) = \pm\psi(z) + \frac{1}{2\pi i}\int_{\gamma} \psi(\tau)K_X(\tau, z)d\tau$$

the boundary values to the right and left of the contour. Recall that, on each Riemann surface X, we have an analog of the usual $\partial_{\overline{z}}$ operator in the plane which is called the *Dolbeault operator* and is again denoted by $\partial_{\overline{z}}$ [109]. Using this operator, one can obtain elliptic systems of differential equations on X and develop an analog of the classical theory.

Namely, consider a homogeneous first-order elliptic system on the Riemann surface X of the form

$$\partial_{\overline{z}}w(z) + z(z)w(z) + b(z)\overline{w(z)} = 0, a, b \in L_{p,2}, p > 2. \tag{9.2}$$

Solutions of this system can be called generalized analytic functions on X. Let $s_1, ..., s_m$ be singular points of forms a and b. We say that the point s_j is a regular singular point of system (9.2) if each of its solutions has no more than polynomial growth in each small sector with vertex at s_j. The system is called regular if all of its singular points are regular. In the investigation of such systems, an important role is played by the operator

$$Pv = -\frac{1}{\pi}\iint_{\widetilde{X}} (a(\tau)v(\tau) + b(\tau)\overline{v(\tau)}K_X(\tau, z)d\widetilde{X}, a, b \in L_{p,2}, p > 2. \tag{9.3}$$

where the integral is taken with respect to surface area.

Theorem 9.2 *(see [110]). Operator (9.3) defines a completely continuous mapping*

$$P : L^0_m(X) \to L^0_l(X)$$

where $\frac{1}{2} \le \frac{1}{m} + \frac{1}{p} < 1, \frac{1}{m} + \frac{1}{p} - \frac{1}{2} < \frac{1}{l} < 1$. *The function* Pf *has a generalized derivative and*

$$\partial_{\overline{z}}f(z) = a(z)f(z) + b(z)\overline{f(z)}.$$

For $l > \frac{2p}{p-2}$, *the function* $(Pf)(z)$ *is Hölder continuous.*

Using this operator, one can relate the above system to integral equations.

Theorem 9.3 *(see [111]). If $v(z)$ is a generalized analytic solution of system (9.3) with the orders of poles not exceeding one then $v(z)$ satisfies the following integral equation:*

$$v(z) + \frac{1}{\pi}\iint_{\widetilde{X}}(a(\tau)v(\tau) + b(\tau)\overline{v(\tau)}K_X(\tau, z)d\widetilde{X} = F(z),$$

where $\widehat{F}(z)$ is an Abelian integral.

It is known that the real dimension of the space of generalized constants c and the dimension of the space of generalized analytic differentials of the first kind h are finite and

$$c - h = \chi(X),$$

where χ denotes the Euler characteristic.

Denote by $L(D)$ and $H(D)$ the space of generalized analytic functions which are multiples of the divisor $-D$, and by $H(D)$ denote the space of generalized analytic differentials which are multiples of D. Then one has the following analog of the *Riemann-Roch* formula [109]:

$$dimL(D) - dimH(D) = 2degD - \chi(X).$$

9.2 Hypercomplex function

In many practically important situations, Q has in matrix Beltrami equation some special form and then this equation can be written in the canonical form, which simplifies its investigation. For example, if elements of Q are lower triangular matrices whose shape is the same throughout the domain of the system then matrix Beltrami equation is called the *Bojarski normal form* (see section 7). If Q is a constant quasi-triangular matrix, then this equation is said to be the *Douglis normal form* of the elliptic system in the plane. Let us illustrate the above discussion by considering the latter case in some detail. Denote by $\mathbb{A}$ the commutative associative algebra generated by two elements i and e and relations

$$i^2 = -1, e^r = 0, r \in \mathbb{N}.$$

Elements of this algebra are called hypercomplex numbers and have the form $\sum_{k=1}^{r-1}(a_k + ib_k)e^l$. Now any function $V : \mathbb{R}^2 \to \mathbb{A}$ can be written as

$$V(x, y) = \sum_{k=0}^{r-1}(u_k(x, y) + iv_k(x, y))e^k$$

where $v_k, u_k : \mathbb{R}^2 \to \mathbb{R}$ are real functions. Continuous functions of such form are called *hypercomplex* functions and their totality is denoted by $\mathcal{H}$. Define on $\mathcal{H}$ a differential operator

$$D : \mathcal{H} \to \mathcal{H}$$

by the formula

$$D = D_x + (a + ib + e)D_y, \text{ where } D_x = \frac{\partial}{\partial x}, D_y = \frac{\partial}{\partial x}.$$

This operator satisfies the Leibnitz rule, i.e., for any pair of hypercomplex functions $U, V \in \mathbb{H}$ we have

$$D(UV) = DU \cdot V + U \cdot DV.$$

The operator D is called the *generalized Beltrami operator* and the equation

$$DV = 0$$

is called the *generalized Beltrami equation.* Now we consider the following differential equation

$$DV + \sum_{k=0}^{r-1} e^k \sum_{l=0}^{k} (a_{kl} W_l + b_{kl} \overline{W}_l) = 0, \tag{9.4}$$

where W is a hypercomplex function and a_{kl} and $b_k l$ are complex functions.

Theorem 9.4 *(see [49]). Let W be a continuous and bounded solution of (9.4) in the whole plane and $a_{kl}, b_{kl} \in L^{p,2}(\mathbb{C})$. Then W admits the exponential representation*

$$W(z) = Cexp\omega(z),$$

where C is a hypercomplex constant and ω is a hypercomplex Hölder-continuous function with exponent $\frac{p-2}{p}$.

This theorem is generalization of the Vekua representation for generalized analytic functions [124]. At the same time, it can be deduced from Theorem 9.4 using the algebraic properties of the algebra $\mathbb{A}$. Note that if a and b are hypercomplex numbers, then

$$\exp(ab) = \exp a \exp b, ln(\exp(a)) = \ exp(ln(a)) = a.$$

Moreover, each hypercomplex function f admits a canonical factorization [49], i.e.,

$$f(t) = f^+(t)\tau(t)^\kappa f^-(t), \tag{9.5}$$

where $f^+(z)$ is hyperanalytic and invertible in U^+, $f^-(z)$ is hyperanalytic in $U \cup \{\infty\}$, κ is an integer, and $\tau(t)$ is a generating solution for the system

$$DW = 0, \tag{9.6}$$

where

$$DW = \sum_{k=0}^{r} e^k W_{k\overline{z}} + \sum_{k=1}^{r} \sum_{j=1}^{r} e^{j+k} q_k W_{j\overline{z}}.$$

A generating solution is special solution of (9.6), and it is expressed in the following form by the Pompeiu operator T (see section 2)

$$\tau(t) = z + T(z). \tag{9.7}$$

For more detailed analysis, see [49].

Consider Riemann-Hilbert boundary value problem for hypercomplex functions which consists in finding a piecewise hyperanalytic function satisfying the boundary condition

$$\Phi^{+}(t) = f(t)\Phi^{-}(t), t \in \Gamma,$$

where $f(t)$ is a quasi-diagonal matrix with $det f(t) \neq 0$. It follows from [107] that $f(t)$ defines a holomorphic vector bundle on $\mathbb{CP}^1$, which is denoted by $\mathcal{E} \to \mathbb{CP}^1$ as above. The number of linearly independent solutions of the above problem can be calculated as the rank of the 0-th cohomology group of $\mathbb{CP}^1$ with coefficients in the sheaf of germs of hyperanalytic sections of the bundle $\mathcal{E}$.

In conclusion, we mention that the Riemann-Hilbert problem for generalized analytic functions gives rise to some more subtle phenomena which cannot be described in topological terms. For example, as is well known, the solvability of Riemann-Hilbert monodromy problem for Fuchsian system depends on the location of singular points of Fuchsian system [55], [54]. It turned out that an analog of this phenomenon takes place for boundary-value problems for generalized analytic functions.

To be more precise, Begehr and Dai in [14] considered the following Hilbert problem for a singular elliptic system

$$\partial_{\overline{z}} w = \frac{\mu}{\overline{z} - \overline{z}_0} w + aw + b\overline{w}, \tag{9.8}$$

where Γ is the unit circle considered as the boundary of the unit disk U, μ is constant and $z_0 \in U \cup S^1$. The Hilbert problem in question was to find a regular solution w of (9.8) in U which is continuous in $U \cup S^1, w_{\overline{z}}, w_z \in L^p(U \cup S^1)$ for some $p > 2$, and satisfies on S^1 the following boundary condition:

$$Re[G(z)w(z)] = g(z), z \in \Gamma,$$

where G and g are given functions on S^1. Begehr has shown that the number of linearly independent continuous solutions depends not only on the index but also on the location of the singularity z_0. It is easy to show that, in general, the number of linearly independent continuous solutions depends on the location and types of all singularities.

Moreover, in [109] Rodin and Turakulow consider the system

$$\overline{\partial} W = AW + B\overline{W}$$

with singular coefficients on Riemann surface and show that the index of this system depends not only on the genus of the surface but also on the type of singularities of the coefficients. These results show that Riemann-Hilbert boundary value problem and Rieamnn-Hilbert monodromy problem exhibit interesting features in the setting of hyperanalytic and generalized analytic vectors (see [59]).

10 Appendix

10.1 On the analytic solutions of the system degenerated in the point

The results of the present section deal with the analytic solutions of sufficiently wide class of the system of the equations degenerated in the point. It has the following complex form

$$z^{\nu}\,\overline{z}^{\,\mu}\,\frac{\partial^{n+q}w}{\partial\overline{z}^{\,n}\,\partial z^{q}} = B\,\overline{w}, \tag{10.1}$$

where ν, μ, n, q are given non-negative integers, B is a given analytic function of the variables x, y on the domain G

We call the *solution of the equation* (10.1) the analytic function w satisfying (10.1) everywhere in the domain G, which contains the origin $z = 0$.

10.1.1 Some auxiliary statements

This subsection contains some auxiliary statements concerning analytic function of two real variables x and y needed in sequel.

Let G be a given domain of the complex plane. If the function $\varphi(x, y)$ of real variables x and y is representable in the form of the following convergent double power series with complex coefficients

$$\varphi(x, y) = \sum_{k=0}^{\infty}\sum_{m=0}^{\infty} a_{km}(x - x_0)^k\,(y - y_0)^m.$$

in every $z = x + iy$ point of the domain G^*.

Denote by $\zeta \in G^*$ the domain defined as follows: $\zeta \in G$ if $\overline{\zeta} \in G$. i.e. G^* is the symmetric domain of the domain G according to real number axis. Let G_4 be a domain of four-dimensional space defined as: $G_4 = G \times G^*$.

As is known, every analytic function $\varphi(x, y)$ of real variables x, y in the domain G is analytically extendable in the domain G_4 where $\Phi(z, \zeta)$. Denote by $\Phi(z, \zeta)$ the analytic extension of the function z, ζ in the domain G_4. Therefore $\Phi(z, \zeta)$ is an analytic function of the complex variables z, ζ in the domain G_4. Moreover, the following equality is fulfilled

$$\Phi(z, \overline{z}) = \varphi(z), \quad z \in G. \tag{10.2}$$

The following theorem is valid.

Theorem 10.1 *Let $\Phi(z, \zeta)$ be an analytic extension in the domain $G_4$of the analytic function $\varphi(x, y)$ of the real variables x, y in the domain G. Then the analytic extension of the function $\overline{\varphi(x, y)}$ in the domain G_4 will be the function $\Phi^*(z, \zeta)$ given by the formula:*

$$\Phi^*(z, \zeta) = \overline{\Phi(\overline{\zeta}, \overline{z})}, \quad z \in G, \quad \zeta \in G^*. \tag{10.3}$$

Proof. First prove, that the function $\Phi^*(z,\zeta)$ is the analytic function of the complex variables z,ζ in the domain G_4. Consider arbitrary point $(z_0,\zeta_0)\in G_4$ of the domain G_4, $z_0\in G$, $\zeta_0\in G^*$. Since $\Phi(z,\zeta)$ is the analytic function of the complex variables z,ζ in the domain G_4, then it may be expanded in the form of double power series in the neighborhood of the point $(\overline{\zeta}_0,\overline{z}_0)$ as follows

$$\Phi(z,\zeta)=\sum_{k=0}^{\infty}\sum_{m=0}^{\infty}C_{km}(z-\overline{\zeta}_0)^k(\zeta-\overline{z}_0)^m. \tag{10.4}$$

Then in the neighborhood of the point (z_0,ζ_0) the following equalities are valid

$$\begin{aligned}\Phi^*(z,\zeta)=\overline{\Phi(\overline{\zeta},\overline{z})}&=\overline{\sum_{k=0}^{\infty}\sum_{m=0}^{\infty}C_{km}(\overline{\zeta}-\overline{\zeta}_0)^k(\overline{z}-\overline{z}_0)^m}=\\&=\sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\overline{C}_{km}(\zeta-\zeta_0)^k(z-z_0)^m=\\&=\sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\overline{C}_{km}(z-z_0)^k(\zeta-\zeta_0)^m.\end{aligned} \tag{10.5}$$

i.e. $\Phi^*(z,\zeta)$ expands in the form of double power series in the neighborhood of the point (z_0,ζ_0). Therefore, the function $\Phi^*(z,\zeta)$ is the analytic function of the complex variables z,ζ in the domain G_4.

On the other hand, from (10.3) we have

$$\Phi^*(z,\overline{z})=\overline{\Phi(z,\overline{z})}=\overline{\varphi(z)},\quad z\in G. \tag{10.6}$$

It follows from (10.5) and (10.6), that the function $\Phi^*(z,\zeta)$ is the analytic extension of the function $\overline{\varphi(z)}$ in the domain G_4. The theorem is proved.

Theorem 10.2 *Let $\Phi(z,\zeta)$ be an analytic extension in the domain G of the analytic function $\varphi(x,y)$ of the variables x,y in the domain G. Then the analytic extension of the function $\dfrac{\partial\varphi}{\partial\overline{z}}$ in the domain G_4 is the function $\dfrac{\partial\Phi(z,\zeta)}{\partial\zeta}$.*

Proof. At first prove, that the function $\dfrac{\partial\Phi(z,\zeta)}{\partial\zeta}$ is the analytic function of the complex variables z,ζ in the domain G_4. Consider arbitrary point $(z_0,\zeta_0)\in G_4$, $z_0\in G$, $\zeta_0\in G^*$ of the domain G_4. Since $\Phi(z,\zeta)$ is the analytic function of the complex variables z,ζ in the domain G_4 then it is expanding in the form of double power series

$$\Phi(z,\zeta)=\sum_{k=0}^{\infty}\sum_{m=0}^{\infty}C_{km}(z-z_0)^k(\zeta-\zeta_0)^m. \tag{10.7}$$

Differentiating by terms the double series (10.7) with respect to ζ in the neighborhood of the point (z_0,ζ_0) we get the following equality

$$\frac{\partial\Phi(z,\zeta)}{\partial\zeta}=\sum_{k=0}^{\infty}\sum_{m=1}^{\infty}C_{km}(z-z_0)^k\,m(\zeta-\zeta_0)^{m-1}=$$

$$= \sum_{k=0}^{\infty}\sum_{m=0}^{\infty} C_{k,m+1}(m+1)(z-z_0)^k(\zeta-\zeta_0)^m. \qquad (10.8)$$

It follows from the $\dfrac{\partial\Phi(z,\zeta)}{\partial\zeta}$ that the function (z_0,ζ_0) is expanding in the form of double power series in the neighborhood of the point z,ζ and therefore this function is the analytic function of the complex variables in the domain G_4.

On the other hand the following equality is valid

$$\frac{\partial\Phi(z,\overline{z})}{\partial\zeta} = \frac{\partial\varphi(z)}{\partial\overline{z}}, \quad z\in G. \qquad (10.9)$$

In fact, since the function $\Phi(z,\zeta)$ is the analytic extension of the function $\varphi(z)$ in the domain G_4, then

$$\varphi(z) = \varphi(x,y) = \Phi(z,\overline{z}) = \Phi(x+iy, x-iy), \quad z\in G. \qquad (10.10)$$

Differentiating both sides of the equality (10.10) with respect to x and y, we obtain

$$\begin{aligned} \frac{\partial\varphi}{\partial x} &= \frac{\partial\Phi(z,\overline{z})}{\partial z} + \frac{\partial\Phi(z,\overline{z})}{\partial\zeta}, \\ \frac{\partial\varphi}{\partial y} &= i\,\frac{\partial\Phi(z,\overline{z})}{\partial z} - i\,\frac{\partial\Phi(z,\overline{z})}{\partial\zeta}. \end{aligned} \qquad (10.11)$$

If we recall the definition of $\dfrac{\partial\varphi}{\partial\overline{z}}$ from (10.11) we conclude, that

$$\begin{aligned} \frac{\partial\varphi(z)}{\partial\overline{z}} &= \frac{1}{2}\Big(\frac{\partial\varphi}{\partial x} + i\,\frac{\partial\varphi}{\partial y}\Big) = \\ &= \frac{1}{2}\Big(\frac{\partial\Phi(z,\overline{z})}{\partial z} + \frac{\partial\Phi(z,\overline{z})}{\partial\zeta} - \frac{\partial\Phi(z,\overline{z})}{\partial z} + \frac{\partial\Phi(z,\overline{z})}{\partial\zeta}\Big) = \frac{\partial\Phi(z,\overline{z})}{\partial\zeta}. \end{aligned}$$

Therefore the equality (10.9) is proved. It follows from (10.8) and (10.9), that the analytic extension of analytic function $\dfrac{\partial\varphi}{\partial\overline{z}}$ in the domain G_4 is $\dfrac{\partial\Phi(z,\zeta)}{\partial\zeta}$. The theorem is completely proved.

10.1.2 The existence of the analytic solutions

Theorem 10.3 *Let $B(0)\neq 0$, $\nu+\mu > n+q$. Then the equation (10.1) has the only trivial solution.*

Proof. Recall, that B and w are the analytic functions of the variables x,y in the domain G. Extend the functions $B(z)$ and $w(z)$ analytically in the domain G_4. Denote correspondingly by $B(z)$ and $w(z)$ the analytical extensions of the functions $B(z)$ and $w(z)$ in domain G_4. Therefore, the functions $B(z,\zeta)$ and $w(z,\zeta)$ are the analytic functions of the complex variables z,ζ in the domain G_4. It follows from the theorem 3.1, that the function $\overline{w(z)}$ will be the analytic extension of the function $w^*(z,\zeta)$ in the domain G_4 which is defined by the following formula

$$w^*(z,\zeta) = \overline{w(\overline{\zeta},\overline{z})}\,. \tag{10.12}$$

It follows from the Theorem 10.3, that $\dfrac{\partial w}{\partial \overline{z}}$ is the analytic extension of the function $\dfrac{\partial w(z,\zeta)}{\partial \zeta}$ in the domain G_4. From here, we get, that the analytic extension of the function $\dfrac{\partial^{n+q} w}{\partial \overline{z}^{\,n}\,\partial z^q}$ is $\dfrac{\partial^{n+q} w}{\partial \zeta^n\,\partial z^q}$ in the domain G_4. Then the equation (10.1) has the following form

$$z^\nu\,\zeta^\mu\,\frac{\partial^{n+q} w}{\partial \zeta^n\,\partial z^q} = B(z,\zeta)\,w^*(z,\zeta). \tag{10.13}$$

Since $B(z,\zeta)$ and $w(z,\zeta)$ are the analytic functions of the complex variables z,ζ in the domain G_4 then in the neighborhood of the point $(0,0)$ these functions are representable in the form of double power series

$$B(z,\zeta) = \sum_{k=0}^{\infty}\sum_{m=0}^{\infty} b_{km}\,z^k\,\zeta^m, \tag{10.14}$$

$$w(z,\zeta) = \sum_{k=0}^{\infty}\sum_{m=0}^{\infty} w_{km}\,z^k\,\zeta^m. \tag{10.15}$$

It follows from the equalities (10.14) and (10.12), that the function $w^*(z,\zeta)$ in the neighborhood of the point (0, 0) will expand in the form of the following double power series

$$w^*(z,\zeta) = \sum_{k=0}^{\infty}\sum_{m=0}^{\infty} \overline{w_{mk}}\,z^k\,\zeta^m. \tag{10.16}$$

Consider the left-hand side of the equality (10.13). Differentiating by terms the double power series (10.15) with respect to z and ζ- we get the following equalities

$$\begin{aligned}
z^\nu\,\zeta^\mu\,\frac{\partial^{n+q} w}{\partial \zeta^n\,\partial z^q} &= z^\nu\,\zeta^\mu \sum_{k=q}^{\infty}\sum_{m=n}^{\infty} m(m-1)(m-2)\cdots(m-n+1)\cdot \\
&\cdot k(k-1)(k-2)\cdots(k-q+1)w_{km}\,z^{k-q}\,\zeta^{m-n} = \\
&= \sum_{k=q}^{\infty}\sum_{m=n}^{\infty} \frac{m!}{(m-n)!}\,\frac{k!}{(k-q)!}\,w_{km}\,z^{k+\nu-q}\,\zeta^{m+\mu-n} = \\
&= \sum_{k=\nu}^{\infty}\sum_{m=\mu}^{\infty} \frac{(m+n-\mu)!\,(k+q-\nu)!}{(m-\mu)!\,(k-\nu)!}\,w_{k+q-\nu,m+n-\mu}\,z^k\,\zeta^m.
\end{aligned} \tag{10.17}$$

Consider now the right-hand side of the equation (10.15). Multiplying the double power series (10.14) and (10.16) we obtain

$$B(z,\zeta)\,w^*(z.,\zeta) = \sum_{k=0}^{\infty}\sum_{p=0}^{\infty} b_{kp}\,z^k\,\zeta^p \sum_{k=0}^{\infty}\sum_{p=0}^{\infty} \overline{w}_{pk}\,z^k\,\zeta^p =$$

$$= \sum_{k=0}^{\infty} \sum_{p=0}^{\infty} \sum_{k_1=0}^{\infty} \sum_{p_1=0}^{\infty} b_{k_1 p_1} \overline{w}_{pk} \, z^{k+k_1} \, \zeta^{p+p_1} =$$

$$k + k_1 = l, \quad p + p_1 = m, \quad k_1 = l - k, \quad p_1 = m - p$$

$$= \sum_{l=0}^{\infty} \sum_{m=0}^{\infty} \left(\sum_{k=0}^{l} \sum_{p=0}^{m} b_{l-k,m-p} \, \overline{w}_{pk} \right) z^l \, \zeta^m =$$

$$= \sum_{k=0}^{\infty} \sum_{m=0}^{\infty} \left(\sum_{l=0}^{l} \sum_{p=0}^{m} b_{k-l,m-p} \, \overline{w}_{pl} \right) z^k \, \zeta^m. \tag{10.18}$$

Inserting the equalities (10.17) and (10.18) in the equation (10.13), we get the following equation

$$\sum_{k=\nu}^{\infty} \sum_{m=\mu}^{\infty} \frac{(m+n-\mu)! \, (k+q-\nu)!}{(m-\mu)! \, (k-\nu)!} \, w_{k+q-\nu,m+n-\mu} \, z^k \, \zeta^m =$$

$$= \sum_{k=0}^{\infty} \sum_{m=0}^{\infty} \left(\sum_{l=0}^{k} \sum_{p=0}^{m} b_{k-l,m-p} \, \overline{w}_{pl} \right) z^k \, \zeta^m. \tag{10.19}$$

Equating in the equation (10.19) the coefficients of the same powers of the variables z and ζ we obtain the algebraic system with respect to w_{mk}

$$\sum_{l=0}^{k} \sum_{p=0}^{m} b_{k-l,m-p} \, \overline{w}_{pl} = 0, \tag{10.20}$$

In case $0 \le m \le \mu - 1$, $k = 0, 1, 2, 3, \ldots$ or $m = 0, 1, 2, 3, \ldots$, $0 \le k \le \nu - 1$.

$$\sum_{l=0}^{k} \sum_{p=0}^{m} b_{k-l,m-p} \, \overline{w}_{pl} = \frac{(m+n-\mu)! \, (k+q-\nu)!}{(m-\mu)! \, (k-\nu)!} \, w_{k+q-\nu,m+n-\mu}, \tag{10.21}$$

when $m \geqslant \mu$, $k \geqslant \nu$.

Prove, that $w_{mk} = 0$, $m = 0, 1, 2, 3, \ldots$, $k = 0, 1, 2, 3, \ldots$ w_{mk}, we call h the height (altitude) of the coefficient w_{mk} the non-negative integer $h = m + k$.

Proof this by means of the method of mathematical induction with respect to h. Let $h = 0$. Then $m = k = 0$. From the equation (10.20) we have

$$b_{00} \, \overline{w}_{00} = 0.$$

Since $b_{00} = B(0,0) \neq 0$, when $w_{00} = 0$.

Assume that by the assumption of mathematical induction all coefficients w_{mk} the height of which is not greater than h, are equal to zeroes. Consider the coefficient w_{mk} with the height $h+1$, $m+k = h+1$. Rewrite the left-hand sides of the equalities (10.20) and (10.21) as follows

$$\sum_{l=0}^{k} \sum_{p=0}^{m} b_{k-l,m-p} \, \overline{w}_{pl} = \sum_{l=0}^{k} \left(\sum_{p=0}^{m-1} b_{k-l,m-p} \, \overline{w}_{pl} + b_{k-l,0} \, \overline{w}_{ml} \right) =$$

$$= \sum_{l=0}^{k-1} \left(\sum_{p=0}^{m-1} b_{k-l,m-p}\, \overline{w}_{pl} + b_{k-l,0}\, \overline{w}_{ml} \right) +$$

$$+ \sum_{p=0}^{m-1} b_{0,m-p}\, \overline{w}_{pk} + b_{00}\, \overline{w}_{mk} = \sum_{l=0}^{k-1} \sum_{p=0}^{m-1} b_{k-l,m-p}\, \overline{w}_{pl} +$$

$$+ \sum_{l=0}^{k-1} b_{k-l,0}\, \overline{w}_{ml} + \sum_{p=0}^{m-1} b_{0,m-p}\, \overline{w}_{pk} + b_{0,0}\, \overline{w}_{mk}.$$

Let $0 \leq m \leq \mu - 1$, $k = 0, 1, 2, 3, \ldots$, or $0 \leq k \leq \nu - 1$, $m = 0, 1, 2, 3, \ldots$ then the equation (10.20) takes the form

$$\overline{w}_{mk} = -\sum_{l=0}^{k-1} \sum_{p=0}^{m-1} \frac{b_{k-l,m-p}}{b_{00}}\, \overline{w}_{pl} - \sum_{l=0}^{k-1} \frac{b_{k-l,0}}{b_{00}}\, \overline{w}_{ml} - \sum_{p=0}^{m-1} \frac{b_{0,m-p}}{b_{00}}\, \overline{w}_{pk}\,. \tag{10.22}$$

We conclude from the equality (10.22), that w_{mk} is the linear combination of the coefficients w_{pl}, w_{ml}, w_{pk} the height of them is not greater than h. But by the assumption of mathematical induction such w_{pl} coefficients are equal to zero. Therefore, $w_{mk} = 0$.

Let $m \geqslant \mu$, $k \geqslant \nu$. Then the equation will take the following form

$$\overline{w}_{mk} = -\sum_{l=0}^{k-1} \sum_{p=0}^{m-1} \frac{b_{k-l,m-p}}{b_{00}}\, \overline{w}_{pl} - \sum_{l=0}^{k-1} \frac{b_{k-l,0}}{b_{00}}\, \overline{w}_{ml} -$$

$$- \sum_{p=0}^{m-1} \frac{b_{0,m-p}}{b_{00}}\, \overline{w}_{pk} + \frac{(m+n-\mu)!\,(k+q-\nu)!}{(m-\mu)!\,(k-\nu)!\, b_{00}}\, w_{k+q-\nu,m+n-\mu}\,. \tag{10.23}$$

Since $\nu + \mu > n + q$, Then

$$(k + q - \nu) + (m + n - \mu) = (k + m) + (n + q) - (\nu + \mu) < k + m. \tag{10.24}$$

It follows from the equality (10.23) and the inequality (10.24), that w_{mk} is the linear combination of the coefficients w_{pl}, w_{ml}, w_{pk}, $w_{k+q-\nu,m+n-\mu}$ with the heights not greater than h. But by means of the assumption of the method of mathematical induction such w_{pl} coefficients are equal to zero. Therefore $w_{mk} = 0$.

At last we prove, that

$$w_{mk} = 0, \quad m = 0, 1, 2, 3, \ldots, \quad k = 0, 1, 2, 3, \ldots$$

Hence $w(z, \zeta)$is identically equal to zero in the neighborhood of the point (0,0). From the uniqueness theorem we have, that $w(z, \zeta)$ is identically equal to zero in the domain G_4. Since $w(z, \zeta)$ is the analytic extension of the analytic function $w(z)$ in the domain G_4, then $w(z)$ is identically zero on the whole domain G. The theorem is proved.

Examples. 1. Let $\nu = 1$, $\mu = 0$, $n = 1$, $q = 0$, $B(z) \equiv 1$. Then we get the equation

$$z\,\frac{\partial w}{\partial \overline{z}} = \overline{w}$$

This equation has the non-zero analytic solution $w(z) = \overline{z}$.

2. Let ν, μ, n, q be the non-negative integers satisfying the condition $\nu+\mu = n+q$ Let m and k be non-negative integers satisfying the conditions

$$m - k = \nu - q = n - \mu, \quad m \geqslant n, \quad k \geqslant q.$$

Let $B(z) = \dfrac{m!\,k!}{(m-n)!\,(k-q)!}$. Then the equation (10.1) takes the form

$$z^{\nu}\,\overline{z}^{\,\mu}\,\frac{\partial^{n+q} w}{\partial \overline{z}^{\,n}\,\partial z^{q}} = \frac{m!\,k!}{(m-n)!\,(k-q)!}\,\overline{w}. \tag{10.25}$$

The equation (10.25) has the non-zero analytic solution $w(z) = z^k\,\overline{z}^{\,m}$. In fact,

$$z^{\nu}\,\overline{z}^{\,\mu}\,\frac{\partial^{n+q} z^k\,\overline{z}^{\,m}}{\partial \overline{z}^{\,n}\,\partial z^{q}} = z^{\nu}\,\overline{z}^{\,\mu}\,m(m-1)\cdots(m-n+1)\,\overline{z}^{\,m-n}\,k(k-1)\cdots$$
$$\cdots(k-q+1)\,z^{k-q} = \frac{m!}{(m-n)!}\,\frac{k!}{(k-q)!}\,z^{\nu+k-q}\,\overline{z}^{\,\mu+m-n} =$$
$$= \frac{m!\,k!}{(m-n)!\,(k-q)!}\,z^{m}\,\overline{z}^{\,k} = \frac{m!\,k!}{(m-n)!\,(k-q)!}\,\overline{z^{k}\,\overline{z}^{\,m}}.$$

This examples shows, that the condition $\nu + \mu > n + q$ from the theorem is important.

10.2 Darboux transformation and Carleman-Bers-Vekua system

Recently in [73] new application of the theory of pseudoanalytic functions to differential equations of mathematical physics is presented. The author's applications of pseudoanalytic functions to differential equations of mathematical physics are based on the factorization of second order differential operator in product of two first order differential operators whose one of these two factors leads to so called *main Vekua equation*

$$w_{\overline{z}} = \frac{f_{\overline{z}}}{f}\overline{w}, \tag{10.26}$$

where f is a real valued function.

In particular it is show, that if f, h, ψ are real-valued functions, $f, \psi \in C_2(\Omega)$, $\Omega \subset \mathbb{C}$ and besides f is a positive particular solution of the two dimensional stationary Schrodinger equation

$$(-\Delta + h)f = 0 \tag{10.27}$$

in the domain $\Omega \subset \mathbb{C}$, where $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ is a two dimensional Laplace operator, then

$$(\Delta - h)\psi = 4(\partial_{\overline{z}} + \frac{f_z}{f}C)(\partial_{\overline{z}} - \frac{f_z}{f}C)\psi = 4(\partial_z + \frac{f_{\overline{z}}}{f}C)(\partial_z - \frac{f_{\overline{z}}}{f}C)\psi, \qquad (10.28)$$

where C denotes the operator of complex conjugation.

Let $w = w_1 + iw_2$ be a solution of the equation (10.26). Then the functions $u = f^{-1}w_1$ and $v = fw_2$ are the solutions of the following conductivity and associated conductivity equations

$$div(f^2\nabla u) = 0, \text{ and } div(f^2\nabla u) = 0, \qquad (10.29)$$

respectively. The real and imaginary part of the solution of the equation (10.26) w_1 and w_2 are solutions of the stationary Schrodinger and associated stationary Schrodinger equations

$$-\Delta w_1 + r_1 w_1 = 0 \quad \text{and} \quad -\Delta w_2 + r_2 w_2 = 0, \qquad (10.30)$$

respectively, where $r_1 = \frac{\Delta f}{f}$, $r_2 = \frac{2(\nabla f)^2}{f^2} - r_1$, $\nabla f = f_x + f_y$ and $(\nabla f)^2 = f_x^2 + f_y^2$.

On the other hand it is known that the elliptic equation

$$\partial_z \partial_{\overline{z}} \psi + h\psi = 0 \qquad (10.31)$$

is covariant with respect to the Darboux transformation [96]

$$\psi \to \psi[1] = \theta(\psi, \psi_1)\psi_1^{-1}, \quad \theta(\psi, \psi_1) = \int_{(z_0,\overline{z}_0)}^{(z,\overline{z})} \Omega, \qquad (10.32)$$

$$h[1] = h + 2\partial_z\partial_{\overline{z}} ln\psi_1, \qquad (10.33)$$

where ψ_1 is a fixed solution of equation (10.30), Ω is closed 1-differential form

$$\Omega = (\psi\partial_z\psi_1 - \psi_1\partial_z\psi)dz - (\psi\partial_{\overline{z}}\psi_1 - \psi_1\partial_{\overline{z}}\psi)d\overline{z}.$$

Here covariant properties means, that $\psi[1]$ satisfies the following equation

$$\partial_z\partial_{\overline{z}}\psi[1] + h[1]\psi[1] = 0.$$

From the equality $d\Omega = 0$ it follows, that the function $\theta(\psi_1, \psi)$ in (10.32) doest not depend on the path of integration.

Theorem 10.4 *[47] Let $w = w_1 + iw_2$ be the solution of the main Vekua equation*

$$w_{\overline{z}} = \frac{\psi_{1\overline{z}}}{\psi_1}\overline{w}. \qquad (10.34)$$

Then $w_1 = \psi_1$ and $w_2 = -\frac{1}{2}\psi[1]$, where ψ_1 is the real positive solution of the equation

$$-\Delta\psi + h\psi = 0 \qquad (10.35)$$

$h = \frac{\triangle\psi_1}{\psi_1}$ and $\psi[1]$ its Darboux transformation defined by (10.32), (10.33).

Conversely, if ψ_1 is the real positive solution of the equation (10.35) and $\psi[1]$ its Darboux transformation then the solution of main Vekua equation (10.34) equal to $w = \psi_1 - \frac{1}{2}\psi[1]$.

First part of the theorem follows from (10.29), (10.30). Here we prove the second part of theorem. Let ψ be a real solution of (10.35), then in this case the Darboux transformation (10.32),(10.33) has the form

$$h[1] = h - 2\Delta ln\psi_1 \quad \text{and} \quad \psi[1] = 2i\psi_1^{-1} Im \int (\psi\psi_{1\overline{z}} - \psi_{\overline{z}}\psi_1)d\overline{z}.$$

We seek the solution of the equation (10.34) in the form $w = \psi + iw_2$. Then

$$\psi_{1\overline{z}} + iw_{2\overline{z}} = \frac{\psi_{1\overline{z}}}{\psi_1}\psi - i\frac{\psi_{1\overline{z}}}{\psi_1}w_2,$$

from this the solution of the corresponding homogenous equation is $w_2 = \frac{C(z)}{\psi_1}$, where $C(z)$ is arbitrary holomorphic function. Let $w_2 = \frac{C(z,\overline{z}}{\psi_1})$ be the solution of the above equation. Then

$$\psi_{\overline{z}} + i\frac{C_{\overline{z}}}{\psi_1} - i\frac{\psi_{1\overline{z}}}{(\psi_1)^2}C(z,\overline{z}) = \frac{\psi_{1\overline{z}}}{\psi_1}\psi - i\frac{\psi_{1\overline{z}}}{(\psi_1)^2}C(z,\overline{z}) \Rightarrow$$

$$\Rightarrow C_{\overline{z}} = -i(\psi\psi_{1\overline{z}} - \psi_{\overline{z}}\psi_1) \Rightarrow C(z,\overline{z}) = -i\int(\psi\psi_{1\overline{z}} - \psi_{\overline{z}}\psi_1)d\overline{z} + \widetilde{C}(z).$$

From this we obtain

$$w_2 = \psi_1^{-1}(b(z) - i\int(\psi\psi_{1\overline{z}} - \psi_{\overline{z}}\psi_1)d\overline{z}).$$

We choose $b(z)$ in last expression such that w_2 was real. Then $w_2 = \psi_1^{-1} Im \int(\psi\psi_{1\overline{z}} - \psi_{\overline{z}}\psi_1)d\overline{z}$, from this it follows, that $-2iw_2 = \psi[1]$, therefore $w = \psi_1 - \frac{1}{2}\psi[1]$ is the solution of (10.35).

Here we give new formulation and proof of Theorem 10.4.

Theorem 10.5 *1)Let $W = W_1 + iW_2$ be the solution of the equation $W_{\overline{z}} = \frac{f_{\overline{z}}}{f}\overline{W}$, then W_1 and W_2 are related to by Darboux transformation $W_2 = iW_1[1]$ and $W_1 = -iW_1[1]$.*

2) If W_1 is a solution of the equation $(\triangle - \frac{\triangle f}{f})\psi = 0$, then $W_1 - W_1[1]$ is the solution of the equation $W_{\overline{z}} = \frac{f_{\overline{z}}}{f}\overline{W}$.

3) If W_2 is the solution of the equation $(\triangle + \frac{\triangle f}{f} - 2(\frac{\nabla f}{f})^2)\psi = 0$, then $-iW_2[1] + iW_2$ is a solution of the equation $W_{\overline{z}} = \frac{f_{\overline{z}}}{f}\overline{W}$.

From the theorem 33 [73] it follows, that $W_1 + iW_2 = W$ is the solution of the equation $W_{\overline{z}} = \frac{f_{\overline{z}}}{f}\overline{W}$, then

$$W_2 = f^{-1}\overline{A}[if^2\partial_{\overline{z}}(f^{-1}W_1)] \text{ and } W_1 = -f\overline{A}[i\overline{f}^2\partial_{\overline{z}}(fW_2)],$$

where $\overline{A}[\phi] = 2Re\int\phi d\overline{z} = 2Im\int i\phi d\overline{z}$. Therefore,

$$W_2 = -f^{-1}2Im\int f^2\partial_{\overline{z}}(f^{-1}W_1)d\overline{z} \text{ and } W_1 = f2Im\int f^{-2}\partial_{\overline{z}}(fW_2)d\overline{z}.$$

Consider the equation $(\triangle - \frac{\triangle f}{f})\psi = 0$ and take the function f as particular solution of this equation, then by theorem 33 [73] the function W_1 is the solution of this equation. Consider the Darboux transformation W_1 :

$$W_1 \to W_1[1] = f^{-1} \int \Omega(W_1, f),$$

$$\Omega(W_1, f) = (W_1 f_z - W_{1z} f)dz - (W_1 f_{\overline{z}} - W_{1\overline{z}} f)d\overline{z} = 2iIm[f^2 \partial_{\overline{z}}(f^{-1}W_1)],$$

$$W_1[1] = f^{-1} 2iIm \int f^2 \partial_{\overline{z}}(f^{-1}W_1)d\overline{z}.$$

Therefore $W_2 = iW_1[1]$.

Now consider the function $\frac{1}{f}$ as particular solution of the equation $(\triangle - \frac{\triangle f}{f})\psi = 0$, then from theorem 33 [73] it follows, that W_2 is a solution of this equation. Consider the Darboux transformation of W_2:

$$W_2 \to W_2[1] = (\frac{1}{f})^{-1} \int \Omega(W_2, f^{-1}) = f \int \Omega(W_2, f^{-1}),$$

$$\Omega(W_2, f^{-1}) = (W_2 \partial_z(\frac{1}{f}) - W_{2z}\frac{1}{f})dz - (W_2 \partial_{\overline{z}}(\frac{1}{f}) - W_{2\overline{z}}\frac{1}{f})d\overline{z} =$$

$$= (-W_2 \frac{f_{\overline{z}}}{f^2} - W_{2z}\frac{1}{f})dz + \frac{1}{f^2}(W_2 f_{\overline{z}} + W_{2\overline{z}} f)d\overline{z} = 2iIm[f^{-2}\partial_{\overline{z}}(fW_2)].$$

Therefore, $W_1 = -iW_2[1]$.

Remark. In [50] the authors studied intertwining relations, supersymmetry and Darboux transformations for time-dependent generalized Schrodinger equations and obtained these relations in an explicit form, it means that it is possible to construct arbitrary-order Darboux transformations for some class of equations. The authors develop corresponding supersymmetric formulation and prove equivalence of the Darboux transformations with the supersymmetry formalism. In our opinion the method given in this subsection may be applied also in this direction.

10.3 Topological properties of generalized analytic functions

The Carleman-Bers-Vekua equation is invariant with respect to conformal transformations [124], [11], therefore it is possible to consider generalized analytic functions on Riemann surfaces. The global theory of generalized analytic functions demands consideration of many-valued generalized analytic functions. One of the first results in this direction was the Riemann-Hurvitz theorem (formula) given in [11]. In particular, Bers computed the genus g of covering Riemann surface $w : X \to \mathbb{CP}^1$ and showed that

$$g = \frac{B}{2} - 1 + N,$$

where w is a pseudoanalytic function from the compact Riemann surface X to the Riemann sphere $\mathbb{CP}^1$, B is the sum of the orders of all branching points of w and

N is the number of sheets (topological degree of w). In section 9 we consider the monodromy theorem for generalized analytic functions. *Different branches of the generalized analytic functions,* in general, *satisfy different Carleman-Bers-Vekua equations* [109]. The results given in this paper is a certain tool for investigation of many-valued generalized analytic functions on the Riemann sphere with finitely many points removed so that it is possible to consider these points as singular points of Carleman-Bers-Vekua equations.

Let f be a map of topological space X into topological space Y. If whenever U is open in X, $f(U)$ is open in Y, then f is interior. By introducing the interior transformations, S. Stoilow to solve the Brouwer problem, the topological characterization of the analytic functions (see [118]). Stoilow proved two basic properties of the interior transformation $f : X \to Y$, where X and Y are domains (open, connected sets) in the plane:

a) The interior transformation are discrete;

b) they are locally topological equivalent to non-constant analytic functions;

Properties b) is contained in the well-known Stoilow inverse theorem, which shows that in the neighborhood of a point in X an interior transformation behaves as z^n, $n \in \mathbb{N}$ in the neighborhood of $z = 0$.

Theorem 10.6 *(see [42]) If $A, B \in L_{p,2}(\mathbb{C}), p > 2$, then for every given point z_0 in the small neighborhood U_0 of this point, there exists single valued generalized analytic functions.*

Theorem 10.7 *(see [42]) Let the function $A(x, y), B(x, y)$ be analytic with respect to arguments x and y in closed circle $\overline{K}$. Suppose there exists the extension of $A(x, y), B(x, y)$ with respect to x, y as complex analytic functions and let $A(z, \zeta), B(z\zeta)$ be analytic functions on $\overline{K_z \times K_{zeta}}$, where $z = x + iy$ and $\zeta = x - iy$, i.e.$z = \overline{\zeta}$. Then there exists $f = u + iv$ solution of Carlemann-Vekua equation, which is an interior mapping of the unit disc on some Riemann surface.*

Topological properties of the generalized analytic functions differ from the topological properties of the analytic functions. B.Shabat was first to consider the generalized analytic functions from the topological point of view in [112]. He proved that some of the solutions of Carleman-Bers-Vekua equations are topologicaly equivalent to analytic functions, while the others don't have this property. This is true, e.g. for the regular equations with constant coefficients, see [112]). Generally the Riemann mapping theorem doesn't hold for the system (1.9) (see [124], p.278, [42]), but it holds in particular cases (see [105]).

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