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THE HURWITZ ACTION AND BRAID GROUP ORDERINGS

JONATHON FUNK

ABSTRACT. In connection with the so-called Hurwitz action of homeomorphisms in ramified covers we define a groupoid, which we call a ramification groupoid of the 2-sphere, constructed as a certain path groupoid of the universal ramified cover of the 2-sphere with finitely many marked-points. Our approach to ramified covers is based on cosheaf spaces, which are closely related to Fox's complete spreads. A feature of a ramification groupoid is that it carries a certain order structure. The Artin group of braids of n strands has an order-invariant action in the ramification groupoid of the sphere with $n + 1$ marked-points. Left-invariant linear orderings of the braid group such as the Dehornoy ordering may be retrieved. Our work extends naturally to the braid group on countably many generators. In particular, we show that the underlying set of a free group on countably many generators (minus the identity element) can be linearly ordered in such a way that the classical Artin representation of a braid as an automorphism of the free group is an order-preserving action.

Introduction

Patrick Dehornoy [8, 9] has discovered through a connection with self-distributive operations that an Artin braid group carries a left-invariant linear ordering. Geometric explanations of the ordering have subsequently appeared. Fenn, Greene, Rolfsen, Rourke and Wiest [10] have presented such an explanation, and Thurston has suggested another approach based on Nielsen [25, 28], which has been described and analyzed in Short and Wiest [27]. Christian Kassel has provided a description of Dehornoy's discovery and a survey of all these achievements in a recent article [17].

The author's previous interest [13] in *ramified*, or *branched covers* (in the sense of Fox [11]) has led to the present investigation of a connection between the so-called Hurwitz action and linear orderings of braid groups. Our explanations focus on a certain path groupoid of a ramified cover, which we shall call a ramification groupoid. The objects of this groupoid are taken to be the elements of the branch-point set of the ramified cover. One of our goals is to show how these groupoids are involved in braid group orderings. In particular, we describe an order-structure that a ramification groupoid carries.

Our approach involves cosheaf spaces [12], which we review in §2. Cosheaf spaces are defined by an adjointness with cosheaves, but they have a topological characterization that is almost equivalent to Fox's notion of a complete spread. Fox had introduced complete spreads as a framework by which to study topologically ramified covers. It may interest the reader that a cosheaf can be equivalently regarded as a *topos distribution* in the sense of Lawvere [20, 21]. We shall not pursue this connection with topos distributions, but the reader can consult [6] (and references cited therein) for information and recent

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developments concerning topos distributions. Remark 11.4 explains another connection with topos theory through the classifying topos for the theory of dense linear orders [26].

The role played by the Poincaré disk and its circle at infinity in Nielsen, and in Short and Wiest, is in this investigation played by a ramified cover. Remark 6.3 explains how the ramified cover and the Poincaré disk are related. From Nielsen [28]:

> It now proves useful to *close* the universal covering surface by addition of its "points at infinity"; in order to make these conveniently accessible, a conformal mapping of the hyperbolic plane onto the interior of a disk is used. Just as proper elements of the fundamental group are given by finite products of the generators, the attempt to characterize individual boundary points leads us to consider improper group elements, given by *infinite sequences of generators*.

The points at infinity that we require are represented by the objects of the ramification groupoid. We can thus avoid words with infinitely many 'edge'-symbols by introducing a second type of symbol, a 'vertex'-symbol, which is used to encode the groupoid objects. But we can pass from the groupoid to the collection of infinite words by a 'spiral' map (Remark 13.12).

If $Y \xrightarrow{\psi} X$ is a ramified cover, then composition with a homeomorphism $X \xrightarrow{h} X$ produces another ramified cover $h \cdot \psi$. This action of homeomorphisms in ramified covers has been called the Hurwitz action (surveyed in [4], but also see [3, 18]). If we regard two ramified covers ψ and ψ' to be equivalent just when there is a homeomorphism $Y \to Y'$ over X, then a ramified cover ψ is fixed under this action by a homeomorphism h precisely when h can be lifted to a homeomorphism $Y \xrightarrow{\tilde{h}} Y$, meaning that $\psi \cdot \tilde{h} = h \cdot \psi$. We are particularly interested in the question: what ψ are fixed by a given collection of homeomorphisms h? Or can we even find one such non-trivial ψ? It is desirable to be able to do this functorially in the sense that for a given collection of homeomorphisms $X \xrightarrow{h} X$ can we find a ramified cover ψ, and a section of the monoidal functor

$$\Sigma(\psi) = \{(h, \tilde{h}) \mid \tilde{h} \text{ is a lifting of } h \text{ over } \psi\} \longrightarrow Aut(X) \; ; \; (h, \tilde{h}) \mapsto h \, ?$$

The morphisms in the monoidal[1] categories $\Sigma(\psi)$ and $Aut(X)$ may vary depending on the particular purpose one has in mind. $Aut(X)$ acts in X (or in the fundamental groupoid $\pi_1(X)$), but notice that a section of the above functor yields an action of $Aut(X)$ in the map ψ (or in the homomorphism ψ_*). For example, let S denote a 2-sphere, and suppose that in $Aut(S)$ we take for morphisms just those isotopies in S that permute n marked-points and pointwise fix an open neighbourhood of an $n + 1^{\text{st}}$ one. The group of isomorphism classes of objects $S \to S$ (homeomorphisms) of this monoidal category is isomorphic to the Artin group of braids of n strands, denoted $\mathbf{B}_n$. In effect, a section of the above functor provides a representation of $\mathbf{B}_n$.

[1] These categories are only 'almost monoidal' because composition of isotopies is not associative, and the constant isotopies are not identities for vertical composition. This technicality does not interest us here.

In §3 we construct a certain universal ramified cover ψ_m of a 2-sphere with m marked-points as a cosheaf space. (For the reader interested in 'exponential' matters, this ψ_m is an instance of a cosheaf space that is not exponentiable in spaces over the 2-sphere - see Remark 3.6.) The ramification groupoid is derived from ψ_m. We then produce an action of $\mathbf{B}_n$ in the ramification groupoid of ψ_{n+1} in the manner we have tried to describe in the previous paragraph. It follows that this action respects the order structure of the ramification groupoid, so that a free element of the action provides a (left)-invariant linear ordering of the braid group (§12).

In §13 we show that these methods extend naturally to the countably generated braid group $\mathbf{B}_\infty$. We obtain the following result (Corollary 13.9): the free group on countably many generators (except the identity) can be linearly ordered in such a way that the classical Artin action [1, 15] of $\mathbf{B}_\infty$ in the free group is order-preserving. This ordering of the free group is not a group ordering in the usual sense of being invariant under multiplication on the left (or right). Although this action has no free elements, for any n it has elements that are free for the subgroup $\mathbf{B}_n$. Free elements for the whole action may be obtained by admitting infinite words. We arrive at Corollary 13.9 by considering another action of $\mathbf{B}_\infty$ in the free group, which is isomorphic to Artin's. Larue [19] has considered this other action in connection with the "Dehornoy bracket." Remark 13.11 explains another connection with Dehornoy [9].

An appropriate setting in which to examine paths in a ramified cover is the Sierpinski fibration, as we shall call it. We shall begin in §1 by providing some details about homotopy theory in the Sierpinski fibration.

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1. Homotopy theory over Sierpinski space

Let *Tsp* denote the category of topological spaces and continuous functions (henceforth called maps). Let *Sier* (for Sierpinski) denote the space consisting of two elements $\{0, 1\}$, such that $\{1\}$ is open and $\{0\}$ is not. An object of *Tsp*//*Sier* is a map $X \xrightarrow{U} \mathit{Sier}$, which we also denote as X_U, and a morphism $X_U \to Y_V$ is a map $X \xrightarrow{f} Y$ such that $U \le V \cdot f$ in the specialization order for maps in *Tsp*. Equivalently, an object X_U is a topological space X together with a designated open set U, and a morphism $X_U \to Y_V$ is a map $X \xrightarrow{f} Y$ such that $U \subseteq f^{-1}V$. Let $[X_U, Y_V]$ denote the set of such morphisms.

We have a functor

$$\mathcal{X} : Tsp//Sier \longrightarrow Tsp\ ,$$

which associates with an object X_U the space $\mathcal{X}(X_U) = X$. This functor has left and right adjoints given by $\mathcal{L}(Z) = Z_\phi$, where ϕ denotes the empty space, and respectively $\mathcal{R}(Z) = Z_Z$. These adjoints are full and faithful. Furthermore, $\mathcal{R}$ has a right adjoint $\mathcal{U}$ given by $\mathcal{U}(X_U) = U$. Thus, we have adjoints

$$\mathcal{L} \dashv \mathcal{X} \dashv \mathcal{R} \dashv \mathcal{U}\ .$$

The functors $\mathcal{U}$ and $\mathcal{X}$ are related by a natural transformation $t : \mathcal{U} \Rightarrow \mathcal{X}$, where the component morphism t_{X_U} is equal to the inclusion $U \subseteq X$. The natural transformation $\mathcal{R}t$ is equal to the composite of the counit $\mathcal{R}\mathcal{U} \Rightarrow id$ with the unit $id \Rightarrow \mathcal{R}\mathcal{X}$.

1.1. Definition. *We shall refer to $\mathcal{X}$ (with all the adjoints, and the base and total categories) as the* Sierpinski fibration. *A morphism $X_U \xrightarrow{f} Y_V$ for which $U = f^{-1}V$ is said to be* cartesian.

Every map $X \xrightarrow{f} \mathcal{X}(Y_V) = Y$ has a *cartesian lifting*: $f^\star(Y_V) \xrightarrow{f} Y_V$, where $f^\star(Y_V)$ denotes $X_{f^{-1}V}$. Furthermore, every morphism $X_U \xrightarrow{f} Y_V$ factors uniquely as a *vertical morphism* $X_U \to f^\star(Y_V)$ followed by a cartesian morphism.

The functors $\mathcal{L}$, $\mathcal{X}$, and $\mathcal{R}$ preserve colimits, and $\mathcal{X}$, $\mathcal{R}$, and $\mathcal{U}$ preserve limits. We use these facts to examine limits and colimits in *Tsp//Sier*. For instance, the product $X_U \times Y_V$ must be isomorphic to $(X \times Y)_{U \times V}$, but an infinite product may not exist.

We next investigate exponentials in the Sierpinski fibration. For the following proposition, recall that if an exponential Y^X exists in *Tsp*, then its underlying set must be equal to the collection of maps $X \to Y$.

1.2. Proposition. *If ${Y_V}^{X_U}$ exists, then Y^X exists, given by $\mathcal{X}({Y_V}^{X_U})$, and the designated open set $\mathcal{U}({Y_V}^{X_U})$ is equal to $[X_U, Y_V]$. Conversely, if Y^X exists in Tsp and if $[X_U, Y_V]$ is an open set of Y^X, then ${Y_V}^{X_U}$ exists, given by Y^X with designated open set $[X_U, Y_V]$.*

Proof. The adjoint pair $\mathcal{L} \dashv \mathcal{X}$ satisfies

$$\mathcal{L}(Z \times \mathcal{X}(Y_V)) \cong \mathcal{L}(Z) \times Y_V\ . \tag{1}$$

Therefore, we have

$$\mathcal{X}({Y_V}^{X_U}) \cong \mathcal{X}(Y_V)^{\mathcal{X}(X_U)} = Y^X\ .$$

Of course, the elements of the open set $\mathcal{U}({Y_V}^{X_U})$ are in bijection with maps $1 \to \mathcal{U}({Y_V}^{X_U})$, which are in bijection with morphisms $\mathcal{R}(1) \to {Y_V}^{X_U}$. In turn, these correspond to morphisms $X_U \to Y_V$. The converse is just as easily established. ∎

1.3. Remark. $\mathcal{R} \dashv \mathcal{U}$ does not satisfy condition (1), and $\mathcal{U}$ does not preserve exponentials.

We are going to consider homotopy theory in the Sierpinski fibration. Let I denote the real unit interval $[0, 1]$.

1.4. Definition. *A* homotopy *between two morphisms in the Sierpinski fibration is a morphism*

$$H : X_U \times \mathcal{R}(I) \;\to\; Y_V \;,$$

where $X_U \times \mathcal{R}(I) = (X \times I)_{U \times I}$. *An* isotopy *is a homotopy H such the pairing*

$$X_U \times \mathcal{R}(I) \to Y_V \times \mathcal{R}(I)$$

of H with the projection $X_U \times \mathcal{R}(I) \to \mathcal{R}(I)$ is an isomorphism in the Sierpinski fibration.

If ${Y_V}^{X_U}$ exists, then by transposing, a homotopy may be equivalently given as a map

$$\hat{H} : I \longrightarrow \mathcal{U}({Y_V}^{X_U}) \;.$$

Let (I) denote the open unit interval, and consider the object $I_{(I)}$. By a *path p in X_U* we shall mean a morphism

$$p : I_{(I)} \to X_U$$

in the Sierpinski fibration.

1.5. Example. Let S denote a 2-sphere, and let U denote an open subset of S. Then S_U is an object of the Sierpinski fibration. When $S - U$ is a finite set we refer to S_U as a finitely marked 2-sphere. A path in S_U is a path in S that maps (I) into U.

Since the inverse operation in $Aut(S)$ (= homeomorphisms $S \to S$) is continuous for the compact-open topology, it follows that an isotopy in S_U is equivalently given as a map $H : S \times I \to S$ such that each map $H(\_\,, t)$ is a homeomorphism of S that permutes the marked-points.

Let $Path(X_U)$ denote the set of paths in X_U, which is $[I_{(I)}, X_U]$ in our previous notation. It is well-known that the exponential X^I always exists in *Tsp*: it is the set of paths in X, endowed with the compact-open topology. Thus we may always consider $Path(X_U)$ as a topological space by regarding it as a subspace of X^I. By Proposition 1.2, if $Path(X_U)$ is an open set of X^I, then ${X_U}^{I_{(I)}}$ exists and we have $Path(X_U) = \mathcal{U}({X_U}^{I_{(I)}})$. Then by adjointness, the bijection expressed in the following proposition must hold. This bijection holds even when ${X_U}^{I_{(I)}}$ does not exist. We omit the straightforward proof.

1.6. Proposition. *We have a functor*

$$Path : Tsp//Sier \longrightarrow Tsp \;.$$

There are natural bijections

$$\frac{Z \;\to\; Path(X_U)}{I_{(I)} \times \mathcal{R}(Z) \;\to\; X_U}$$

between maps in Tsp and morphisms in the Sierpinski fibration. Thus, Path has a left adjoint. In particular, Path preserves limits.

1.7. Remark. Our concept of path in the Sierpinski fibration is apparently flawed because there may not be a way to compose paths (as in S_U). For our purposes this apparent flaw is really a valuable feature because for a particular object X_U a natural composition for (homotopy classes of) paths in X_U may be available. The universal ramified cover of S_U provides an illustration of this (§3).

1.8. Definition. *A* homotopy of paths in X_U *is a morphism*

$$H : I_{(I)} \times \mathcal{R}(I) \rightarrow X_U$$

in the Sierpinski fibration.

By Proposition 1.6, a homotopy may be equivalently regarded as a map

$$\hat{H} : I \rightarrow Path(X_U) \ .$$

We may speak of paths in the Sierpinski fibration with common endpoints by using the domain and codomain *natural transformations*: $Path \Rightarrow \mathcal{X}$. Then we may consider homotopy equivalence classes of paths with common endpoints, in the sense of Definition 1.8.

1.9. Definition. *Let us denote the collection of homotopy classes of paths in X_U with common endpoints by $\pi_1(X_U)$. Its objects (or vertices) are the points of X, while U supplies the morphisms (or edges).*

1.10. Remark. The collection $\pi_1(X_U)$ is a directed graph with an involution, but as we have said it may not have a law of composition.

When we use $\pi_1(X)$ with no subscript on X, we shall mean the ordinary fundamental groupoid.

2. Review of cosheaf spaces and complete spreads

We review the notion of cosheaf space [12] and the slightly more general notion of complete spread, due to R. Fox. Our terminology is a mixture coming from [2, 11, 12]. We first review complete spreads. Following [11], a *spread* is a continuous map $\varphi : Y \rightarrow X$, where Y is locally connected, such that the components of sets $\varphi^{-1}(U)$, for U open in X, are a base for the topology on Y. We shall assume throughout that the domain space of a spread is locally connected, even though the notion can be sensibly generalized to the case of an arbitrary domain space by using quasi-components [11, 22]. In order to formulate completeness we recall that a *cogerm of a map* $\varphi : Y \rightarrow X$ *at* $x \in X$ (with Y locally connected) is a consistent choice of components $\alpha = \{\alpha_U \subseteq \varphi^{-1}(U)\}$, where U ranges over all neighbourhoods of x. By consistent, we mean that $U \subseteq V \Rightarrow \alpha_U \subseteq \alpha_V$. Then a spread over a space X is *complete* if for every $x \in X$, and every cogerm α at x, the intersection $\bigcap_{x \in U} \alpha_U$ is non-empty.

We next turn to cosheaf spaces. For a given map $\varphi : Y \to X$ (with Y locally connected) consider the collection

$$\widetilde{Y} = \{(x, \alpha) \mid \alpha \text{ is a cogerm of } \varphi \text{ at a point } x \in X\} .$$

$\widetilde{Y}$ is topologized by the basic sets

$$(U, \beta) = \{(x, \alpha) \mid x \in U,\ \alpha_U = \beta\} ,$$

where U is an open set of X, and β is a connected component of $\varphi^{-1}(U)$. We refer to the topological space $\widetilde{Y}$ as the *display space* associated with the *cosheaf* $U \mapsto \pi_0(\varphi^{-1}(U))$, where π_0 denotes connected components. $\widetilde{Y}$ is continuously fibered over X in the obvious way. A fiber over a given $x_0 \in X$ is sometimes called a *costalk*; it consists of the collection of pairs $\{(x_0, \alpha)\}$. A costalk may also be regarded as the limit

$$\varprojlim_{x_0 \in U} \pi_0(\varphi^{-1}(U)) ,$$

taken over the filter of open neighbourhoods of x_0. $\widetilde{Y}$ is locally connected. Every element $y \in Y$ determines a cogerm at $\varphi(y)$: take α_U to be the unique component of $\varphi^{-1}(U)$ that contains y. This defines a continuous map from Y into $\widetilde{Y}$ (over X), denoted

$$\eta : Y \to \widetilde{Y} \ ; \ \eta(y) = (\varphi(y), \alpha) .$$

The inverse image set $\eta^{-1}(U, \beta)$ is equal to β. A *cosheaf space* is then a map $\varphi : Y \to X$ for which Y is locally connected and η is a homeomorphism. If Y is locally connected and $Y \xrightarrow{\varphi} X$ is any map, then the canonical projection $\widetilde{Y} \xrightarrow{\tilde{\varphi}} X$ is a cosheaf space. Furthermore, there are adjoint functors connecting cosheaves and cosheaf spaces, which induces an equivalence between the category of cosheaf spaces over X and a full subcategory of cosheaves on X (called the *spatial cosheaves* in [12]). The above map η is the unit of this adjointness. The inclusion of spatial cosheaves in cosheaves has a right adjoint.

Cosheaf spaces and complete spreads are nearly the same. The space $\widetilde{Y}$ is precisely Fox's construction of the completion of a spread. A map with locally connected domain over a T_1 space is a cosheaf space if and only if it is a complete spread with T_1 domain ([12], 5.17). The following result from [14] may help clarify the notion of cosheaf space and its connection with complete spreads.

2.1. Proposition. (Topological characterization of cosheaf spaces [14].) *For any map $\varphi : Y \to X$ with locally connected domain, the following are equivalent:*

1. *φ is a cosheaf space.*
2. *φ is a spread and η is a bijection, in which case the inverse of η is given by:*
$$\eta^{-1}(x, \alpha) \ = \ (\bigcap_{x \in U} \alpha_U) \cap \varphi^{-1}(x) .$$
3. *φ is a spread, and for every $x \in X$ and every cogerm α of φ at x, the set*
$$(\bigcap_{x \in U} \alpha_U) \cap \varphi^{-1}(x)$$
is equal to a singleton.

2.2. Remark. By Proposition 2.1.3, a cosheaf space is a complete spread with locally connected domain.

The display space construction applies to any map $Y \to X$ for which Y is locally connected. Thus, we may always factor such a map as

$$Y \xrightarrow{\eta} \widetilde{Y} \xrightarrow{\psi} X\,,$$

where ψ is a cosheaf space. The map η is *pure* in the sense that for every non-empty, connected open set U, $\eta^{-1}(U)$ is again non-empty and connected. (In particular, η is dense.) This factorization of a map with locally connected domain into its pure and cosheaf space parts (such that the middle space is also locally connected) is unique up to unique homeomorphism.

3. The universal ramified cover of a marked sphere

Consider the finitely marked 2-sphere regarded as an object S_U of the Sierpinski fibration (Eg. 1.5). We refer to the open set U as a punctured sphere. Let us denote the universal covering space of U by $P : Y \to U$. If $m = |S - U|$, let $\psi_m : \widetilde{Y} \to S$ denote the cosheaf space factor of the map $Y \xrightarrow{P} U \to S$ (as in §2). We have

$$\begin{tikzcd} Y \arrow[r, "\eta"] \arrow[d, "P"'] & \widetilde{Y} \arrow[d, "\psi_m"] \\ U \arrow[r, tail] & S \end{tikzcd}$$

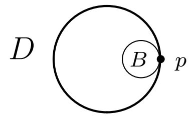
where the first factor η is pure. The open inclusion $U \to S$ is also pure. The cosheaf space ψ_m is a ramified, or branched covering space in the sense of [11]. We shall call ψ_m *the universal ramified cover of the m-marked 2-sphere.*

The following is an instance of a general fact about the pure, cosheaf space factorization of a map such as $Y \xrightarrow{P} U \to S$ ([13], Lemma 4.2).

3.1. Proposition. *The above square is a pullback, so that $\eta : Y \to \widetilde{Y}$ is an open inclusion, and Y is homeomorphic to $\psi_m^{-1}(U)$.*

3.2. Proposition. *The space $\widetilde{Y}$ is connected, simply connected, locally path-connected, and locally simply connected.*

Proof. By using Proposition 3.1 and well-known properties of Y, the stated connectedness properties of $\widetilde{Y}$ can be argued directly from the definition of the cosheaf space topology that $\widetilde{Y}$ carries. Another way to see that $\widetilde{Y}$ has these properties is to use the Poincaré disk with its boundary at infinity. Let D denote this space. Topologically, D is a closed disk in the plane, but with the following additional open sets: in the following diagram, $B \cup \{p\}$ is also a basic open set in D, where B is an open disk with tangent point p.



Then (for $m \geq 3$) $\widetilde{Y}$ is homeomorphic to a subspace of D in such a way that (only) the branch-points of $\widetilde{Y}$ correspond to points on the boundary of D. ∎

A fact that may be of independent interest is that ψ_m is an open map. We have the following explanation of this fact. For any given element c of a Heyting algebra $(H, \leq, \Rightarrow)$ one may define a 'thinner' partial order on the elements of H:

$$a \leq_c b \text{ if } a \leq b \text{ and } a \Rightarrow c \leq b \Rightarrow c\ .$$

Consider our situation where H is the open set lattice of S, and c is taken to be U. Since $S - U$ is a closed set that lies discretely in S, for any two open sets V and V' of S we have

$$V \subseteq_U V' \text{ iff } V \subseteq V' \text{ and } V \cap (S - U) = V' \cap (S - U)\ .$$

Let C_{ψ_m} denote the cosheaf associated with ψ_m. By definition, for any open set $W \subseteq S$ we have

$$C_{\psi_m}(W) = \pi_0(P^{-1}(U \cap W))\ .$$

If $W \subseteq W'$, then there is an obvious transition map $C_{\psi_m}(W) \rightarrow C_{\psi_m}(W')$. The proof of Proposition 3.4 depends on the following lemma. By convention, a connected set is always non-empty.

3.3. Lemma. *The cosheaf C_{ψ_m} has the following property: for any two connected, simply connected open subsets V, V' of S, we have*

$$V \subseteq_U V' \Rightarrow C_{\psi_m}(V) \rightarrow C_{\psi_m}(V') \text{ is an isomorphism}\ .$$

3.4. Proposition. *The universal ramified cover ψ_m is an open surjection.*

Proof. The collection of (non-empty) connected, simply connected open subsets of S is a base for S. Let (V, β) be a basic open subset of $\widetilde{Y}$, where V is connected and simply connected, and β is a component of $P^{-1}(U \cap V)$. By definition, we have

$$\psi_m(V, \beta) = \{y \in V \mid \text{ there is a cogerm } \alpha \text{ at } y \text{ for which } \alpha_V = \beta\} \subseteq V\ .$$

We shall show that $\psi_m(V, \beta)$ is equal to V, hence open. Let $y \in V$. If $y \in U$, then clearly there is a cogerm α at y for which $\alpha_V = \beta$ because we have only to work with the cover P, and not with ψ_m. On the other hand, if y is a marked-point, it suffices to define a cogerm α at y on the connected, simply connected open subsets $B \subseteq S$ that satisfy $B \cap (S - U) = \{y\}$. For any two such B, B' obviously we have $B \subseteq B'$ iff $B \subseteq_U B'$. Choose one such B and any member of $C_{\psi_m}(B)$, which we denote α_B, such that α_B goes to β under $C_{\psi_m}(B) \rightarrow C_{\psi_m}(V)$. Then we are able to define an α at y because if $B \subseteq B'$, then by Lemma 3.3 the transition map $C_{\psi_m}(B) \rightarrow C_{\psi_m}(B')$ is an isomorphism. ∎

We have the object $\widetilde{Y}_Y = \psi_m^\star(S_U)$ and the morphism $\psi_m : \widetilde{Y}_Y \to S_U$ in the Sierpinski fibration, which is a cartesian morphism in the sense of fibrations (Definition 1.1).

3.5. Remark. The closed complement $\widetilde{Y} - Y$ is discrete in $\widetilde{Y}$, and countable. Furthermore, for $m \geq 2$, the space $\widetilde{Y}$ is not locally compact ($m = 1$ is an exception because ψ_1 is a homeomorphism). Indeed, local compactness fails at every point of $\widetilde{Y} - Y$. In particular, $\widetilde{Y}$ is not a topological manifold, $m \geq 2$.

3.6. Remark. The cosheaf space ψ_m cannot be exponentiable [23, 24, 7] in the category of topological spaces over S (in contrast with sheaf spaces, which are always exponentiable). Indeed, if ψ_m were exponentiable, then $\widetilde{Y}$ would be exponentiable in topological spaces because S is exponentiable and such things compose. Hence, $\widetilde{Y}$ would be locally compact. All assertions made here about exponentiability are explained in [23]. In this connection, Susan Niefield's [24] recent example of a poset morphism that is a discrete opfibration, and whose induced essential geometric morphism is not exponentiable, provides another example of a cosheaf space that is not exponentiable. Indeed, a discrete opfibration of posets is a cosheaf space when regarded as a continuous map with respect to the down-closed topology (use Prop. 2.1 to see this).

4. The ramification groupoids $\pi_1^r(S, m)$

For each natural number m we now define a groupoid, which we shall call *a ramification groupoid of the m-marked 2-sphere*, denoted $\pi_1^r(S, m)$. Here $m = |S - U|$ denotes the number of marked-points on S. By definition, $\pi_1^r(S, m)$ is the full subgraph of $\pi_1(\widetilde{Y}_Y)$ whose vertices are the elements of $\widetilde{Y} - Y$ (Definition 1.9 and §3).

We make $\pi_1^r(S, m)$ into a groupoid as follows. Suppose that we have paths

$$u \xrightarrow{p} x \xrightarrow{q} v$$

in $\widetilde{Y}_Y$, where $u, x, v \in \widetilde{Y} - Y$. It is possible to find an open set $V \subseteq \widetilde{Y}$ that contains x, such that $V - \{x\} \subseteq Y$, and such that $V - \{x\}$ is connected and simply connected. There is a t such that $p(t, 1] \subseteq V$ and $q[0, 1 - t) \subseteq V$. Choose s such that $t < s < 1$. There is a path γ in $V - \{x\}$, which is unique up to homotopy in $V - \{x\}$, joining the points $p(s)$ and $q(1 - s)$. We use γ to connect p and q, producing a path $u \to v$ in $\widetilde{Y}_Y$. This is a well-defined associative composition for homotopy classes of paths in $\widetilde{Y}_Y$. Furthermore, it is readily verified that this composition has identities and inverses.

4.1. Example. $\pi_1^r(S, 1)$ has a single object, and is equal to the trivial group.

4.2. Remark. The groupoid $\pi_1^r(S, m)$ is connected and simply connected in the sense that between any two objects there is exactly one isomorphism. It may seem that we could have saved ourselves the trouble of doing homotopy theory in the Sierpinski fibration because $\pi_1^r(S, m)$ is isomorphic to the full subgroupoid of the ordinary fundamental groupoid $\pi_1(\widetilde{Y})$ determined by the elements of $\widetilde{Y} - Y$. However, we wish to emphasize that one of our aims is to describe an order structure that $\pi_1^r(S, m)$ carries. We shall do this not in terms of Y or even in terms of $\widetilde{Y}$, but rather in terms of the object $\widetilde{Y}_Y$ of the Sierpinski fibration.

5. A combinatorial presentation of $\pi_1^r(S, n+1)$

Our later description of an order structure carried by the groupoid $\pi_1^r(S, n+1)$ is in terms of the following combinatorial presentation of $\pi_1^r(S, n+1)$. (Our change in notation in this section from m to $n+1$ is convenient for our later purposes.) First consider the following presentation of the ordinary fundamental groupoid $\pi_1(U)$ of the $n+1$-punctured sphere U, as we have been denoting it. Let

$$v_0 \xrightarrow{e_1} v_1 \xrightarrow{e_2} v_2 \;\cdots\; v_{n-1} \xrightarrow{e_n} v_n$$

denote pairwise disjoint line segments connecting $n+1$ marked-points v_i on the sphere, which when deleted gives U. Up to homotopy, we may encode any path in U (whose endpoints do not lie on a line segment) as a word in the names e_i of the line segments, and their 'inverses' e_i^{-1}. We also call e_i an edge-symbol, or just an edge. We build such a word from a path in U by including e_i just when the path crosses the line segment e_i going up the page, and e_i^{-1} when the path crosses the line segment e_i going down the page. Thus, we have the well-known fact that we may present any homotopy class of paths in U as an element of the free group on the symbols $e_1, \dots, e_n$. (Perhaps it should be emphasized that we are describing the fundamental group*oid* of U, so it makes sense to describe paths in U using line segments without equal endpoints as generators.)

For our later purposes we need a presentation of just the isomorphisms with common domain $x \in \widetilde{Y} - Y$. We denote this collection of isomorphisms by $x/\pi_1^r(S, n+1)$. Let us fix such an x. As above, we label the $n+1$ marked-points on the sphere by $v_0, \dots, v_n$, such that

$$\psi_{n+1}(x) = v_n \; . \tag{2}$$

We introduce vertex-symbols v_i^+ and v_i^-, which are not to be confused with the marked-points v_i on the sphere. Consider all words of the form

$$v_i^{\pm} w v_n \; ; \; i \in \{0, \dots, n\} \; ,$$

where w is an element of the free group on the edge-symbols $e_1, \dots, e_n$, and where $v_i^{\pm}$ means either v_i^- or v_i^+, subject to the following 'codomain winding relations:'

$$\left.\begin{array}{l} v_{i-1}^+ e_i^{-1} \sim v_{i-1}^- \\ v_i^+ e_i^{-1} \sim v_i^- \\ v_{i-1}^- e_i \sim v_{i-1}^+ \\ v_i^- e_i \sim v_i^+ \end{array}\right\} \; i = 1, \dots, n \; .$$

The meaning of these relations is clarified in the proof of Proposition 5.5. We also identify the 'ends' by including the relations:

$$\begin{array}{l} v_0^+ \sim v_0^- \\ v_n^+ \sim v_n^- \; . \end{array}$$

5.1. Remark. We shall write $v_0 = v_0^+ = v_0^-$, and $v_n = v_n^+ = v_n^-$, which should not lead to any confusion.

5.2. Proposition. *Every such equivalence class of words $v_i^{\pm} w v_n$ has a unique member of least length* (up to exchanging v_n^+ with v_n^-, or v_0^+ with v_0^-).

Thus, there is no need to make a distinction in the notation between an equivalence class and a representing word, since we can assume it is the word of least length. Let us call these words of least length *reduced words.*

5.3. Example. For instance, we have $v_n e_n v_n \sim v_n v_n$. But $v_n e_{n-1} e_n v_n$ is a reduced word.

In the proof of the next proposition we shall consider leaves of the universal cover $P : Y \twoheadrightarrow U$. A *leaf* of P is a connected component of $P^{-1}(U - \bigcup e_i)$. Thus, each leaf of P is canonically identified with $U - \bigcup e_i$. An *edge* of a leaf of P is the part of the closure of the leaf in Y that is mapped by P to a line segment e_i. The edges and branch-points of the closure in $\widetilde{Y}$ of a leaf of P may be labelled by edge and vertex-symbols in a circular fashion illustrated by Figure 1 (§6). A *leaf of a point $y \in \widetilde{Y}$* is a leaf of P whose closure in $\widetilde{Y}$ contains y.

5.4. Remark. If a path exits a leaf across an edge e_i^{ν}, $\nu = \pm 1$, then the adjacent leaf it enters is labelled so that the edge just crossed is now named $e_i^{-\nu}$. The endpoints of e_i^{ν} also change sign.

5.5. Proposition. *With x as in* (2)*, every isomorphism $x \cong x'$ in $\pi_1^r(S, n+1)$ may be uniquely presented by a reduced word $v_i^{\pm} w v_n$, where $\psi_{n+1}(x') = v_i$* (without $\pm$) *for some $i \in \{0, \ldots, n\}$. The identity on x is presented by the reduced word $v_n v_n$. Conversely, every such reduced word presents such an isomorphism.*

Proof. We shall establish a bijection between reduced words and homotopy classes of paths by first arbitrarily fixing a leaf of x. Suppose we have a path $p : x \twoheadrightarrow x'$ in $\widetilde{Y}_Y$. There is a connected and simply connected open neighbourhood $B \subseteq \widetilde{Y}$ of x, such that $B - \{x\} \subseteq Y$ is open, connected and simply connected, and contained in the union of the leaves of x. By suitably winding finitely many times around x in $B - \{x\}$, the homotopy class of p can be represented by a path that exits x via the fixed leaf. The winding around x is recorded in the word $v_i^{\pm} w v_n$ by the power of e_n occurring as the rightmost symbols of w. For instance, if $w = z e_n^2$, then the 'path' $v_i^{\pm} w v_n$ exits x via the fixed leaf and cycles around x crossing e_n twice. But we cannot identify $e_n^2 v_n$ with v_n in $v_i^{\pm} w v_n$, for then we would identify two possibly different homotopy classes of paths in $\widetilde{Y}_Y$.

Now consider the codomain x' of the path p. Again we may find a neighbourhood B' as above such that $B' - \{x'\}$ is contained in the union of the leaves of x'. There is a $0 < t_0 < 1$ such that $p[t_0, 1]$ lies in B', and furthermore p crosses only finitely many leaf edges between leaving B and reaching the point $p(t_0)$. The word we are building to represent the homotopy class of p may be ended with a vertex-symbol $v_i^{\pm}$ that labels x'. This vertex-symbol is determined by the labelling of the leaf of x' in which $p(t_0)$ lies. The codomain winding relations arise because any path from any point in B' into x' is presented by the vertex-symbol that labels x'. ∎

6. The order structure carried by $\pi_1^r(S, n+1)$

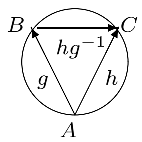
From the point of view of analytic topology a circle is a continuum that is disconnected by the omission of any two of its points ([29], Chap. 3, §7). A *linear order* is a reflexive, transitive, and anti-symmetric relation, which is total. Consider the following property of a connected groupoid $\mathbf{G}$: for every object $A \in \mathbf{G}$, the collection

$$(A/\mathbf{G})^\star = \{ \text{ isomorphisms } g : A \cong B \mid g \neq id_A \}$$

forms a linear order. If for any object A we denote the linear ordering in $(A/\mathbf{G})^\star$ by $\leq_A$, then we furthermore require that the composition of the groupoid cooperates with the linear orderings as follows: for any two *distinct* isomorphisms $g : A \cong B$ and $h : A \cong C$ other than id_A, we have

$$g \leq_A h \Rightarrow hg^{-1} \leq_B g^{-1} . \tag{3}$$

The picture below may help explain this rule. In this picture regard $\leq_A$ and $\leq_B$ as increasing in the clockwise direction (so that $g \leq_A h$ starting from A).



$\mathbf{G}$ is thus a 'circle.'

The groupoid $\pi_1^r(S, m)$ carries an order structure such as defined above. (Since $\pi_1^r(S, m)$ is connected and simply connected, $x/\pi_1^r(S, m)$ is in bijection with the objects of $\pi_1^r(S, m)$ (= elements of $\widetilde{Y} - Y$), but really it is the collection of isomorphisms with domain x, excluding id_x, that are linearly ordered.) In Definition 6.1 we define a linear ordering of $(x/\pi_1^r(S, n+1))^\star$ in terms of the reduced words $v_i^{\pm} w v_n$ (except $v_n v_n$, although w may be empty in which case $v_i^{\pm}$ may not be v_n, for this would give $v_n v_n$). A word is a *right subword* of another word if the former appears in the latter regarded from the right.

6.1. Definition. *Let $v_i^{\pm} w v_n$ and $v_j^{\pm} w' v_n$ be two reduced words as above. Let e_k^ν, $\nu = \pm 1$, be the leftmost symbol of the greatest common right subword of w and w'* (the right-hand v_n in the word has an inert role in this ordering criteria). *Let a and b be the next symbols to the left of e_k^ν in $v_i^{\pm} w$, respectively $v_j^{\pm} w'$. Starting from the edge-symbol that immediately follows $e_k^{-\nu}$* (see Remark 5.4) *clockwise in the* circular *list in Figure 1, we declare $v_i^{\pm} w v_n < v_j^{\pm} w' v_n$ if a occurs before b (clockwise).* (See Remark 5.1 about v_0 and v_n.)

6.2. Remark. In Definition 6.1 it may happen that w is a right subword of w' so that $a = v_i^{\pm}$ (or that w' is a right subword of w so that $b = v_j^{\pm}$), and hence we must compare the vertex $v_i^{\pm}$ with an edge of w', but the list in Figure 1 accounts for this. (If $w = w'$ and $a = b$, then of course the two words $v_i^{\pm} w v_n$ and $v_j^{\pm} w' v_n$ are identical.) It may happen that the greatest common right subword of w and w' is empty, in which case by definition we start with e_n^{-1} when finding the positions of a and b in the circular list.

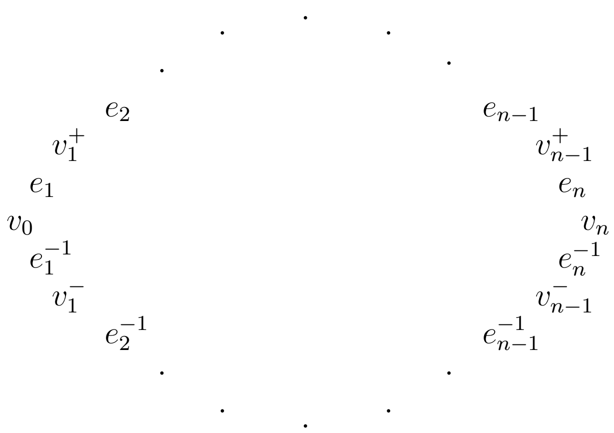


Figure 1: The circular list of vertex and edge-symbols.

6.3. Remark. From a geometric point of view, the term "hyperbolic" for the above ordering seems appropriate because the space $\widetilde{Y}$ is a subspace of the Poincaré disk and its circle at infinity (as described in the proof of Prop. 3.2) such that *certain* geodesics map to the 'combinatorial geodesics' $v_i^{\pm} w v_n$. An analysis of this connection with hyperbolic geometry is given in [27]. The ordering imposed by Definition 6.1 might also be called warped, or twisted-lexicographical (Franz-Victor Kuhlmann's suggestion) because although the ordering has a lexicographical aspect, the order of the alphabet varies (by a rotation) depending on the last common letter encountered when comparing two words.

The following is not difficult to establish.

6.4. Proposition. *The ordering of* $(x/\pi_1^r(S, n+1))^\star$ *given by Definition 6.1 is a dense linear order* ("dense" means that between any two isomorphisms $p < q$ with domain x there is another one $p < r < q$).

6.5. Proposition. *The order structure in* $\pi_1^r(S, n+1)$ *has property* (3).

We omit a proof of Proposition 6.5. In any case, we do not use (3) to obtain linear orderings of a braid group.

6.6. Remark. There is a one-to-one function into $x/\pi_1^r(S, n+1)$ from the subset of the free group on $e_1, \ldots, e_n$ consisting of all reduced words *not* beginning on the left with e_n or with e_n^{-1}, which sends such a word w to $v_n w v_n$. The word w should not begin on the left with e_n or e_n^{-1} because of the codomain winding relations $v_n e_n \sim v_n$ and $v_n e_n^{-1} \sim v_n$. Therefore, we may linearly order this particular *subset* of non-empty reduced words in $e_1, \ldots, e_n$ The action of a braid group restricts to this subset, and it is order-preserving. By admitting countably many generators e_i it is possible to avoid the codomain winding relations. In effect we may work without the vertex-symbols. The details are provided in §13.

7. Lifting a homeomorphism of S_U to one of $\widetilde{Y}_Y$

In this section we show how to lift a homeomorphism of S_U to one of $\widetilde{Y}_Y = \psi_m^\star(S_U)$ in the Sierpinski fibration, where $m = |S - U|$. (As we were saying in the Introduction, such a lifting exists just when ψ_m is fixed under the Hurwitz action.) Much of what we do holds in greater generality, but we shall work only with the universal ramified cover ψ_m. Later we shall derive, by lifting homeomorphisms, an action of the braid group on n strands in the ramification groupoid $\pi_1^r(S, n+1)$.

Recall that we denote the universal covering space of U by $P : Y \twoheadrightarrow U$. We shall use P_* to denote the induced functor of fundamental groupoids. By definition, for any $y \in Y$, the object $P_*(y)$ is equal to the point $P(y)$.

7.1. Proposition. *Let h be a homeomorphism of S_U. Then a point b on Y, and an isomorphism $\gamma_b : P(b) \cong h(P(b))$ of the groupoid $\pi_1(U)$ define a homeomorphism $\tilde{h}_{\gamma_b}$ of $\widetilde{Y}_Y$, which furthermore is a lifting of h. Moreover, if $\gamma_c : P(c) \cong h(P(c))$ is another isomorphism such that $h_*(P_*(\rho)) \cdot \gamma_b = \gamma_c \cdot P_*(\rho)$ in $\pi_1(U)$, where $\rho : b \cong c$ is the unique isomorphism of $\pi_1(Y)$, then $\tilde{h}_{\gamma_b} = \tilde{h}_{\gamma_c}$* (so we may denote the lifting as $\tilde{h}_\gamma$).

$$\begin{tikzcd} \widetilde{Y}_Y \arrow[r, "\tilde{h}_\gamma"] \arrow[d, "\psi_m"'] & \widetilde{Y}_Y \arrow[d, "\psi_m"] \\ S_U \arrow[r, "h"'] & S_U \end{tikzcd}$$

If there is isotopy between h and h' in S_U (Example 1.5) *with induced natural isomorphism $i : h_* \cong h'_*$, and if for some $b \in Y$, we have isomorphisms γ_b and γ'_b such that $\gamma'_b = i_{P(b)} \cdot \gamma_b$, then the given isotopy lifts to one of $\tilde{h}_\gamma$ and $\tilde{h}_{\gamma'}$.*

Proof. By the well-known procedure of path-lifting, there is a homeomorphism $\bar{h}_\gamma$ such that the following diagram commutes.

$$\begin{tikzcd} Y \arrow[r, "\bar{h}_\gamma"] \arrow[d, "P"'] & Y \arrow[d, "P"] \\ U \arrow[r, "h"'] & U \end{tikzcd}$$

(Here we mean the restriction of h to U, or $\mathcal{U}(h)$ in the notation of §1.) Indeed, given $y \in Y$, we define $\bar{h}_\gamma(y)$ by lifting the path $P(b) \xrightarrow{\gamma_b} h(P(b)) \xrightarrow{h(P(!))} h(P(y))$ in U to a path $b \to \bar{h}_\gamma(y)$ in Y such that $P(\bar{h}_\gamma(y)) = h(P(y))$, where $b \xrightarrow{!} y$ denotes the unique path in Y (up to homotopy in Y). The stated uniqueness property is straightforward to establish. We may lift isotopies in U along P in essentially the same manner, but working over I.

Any homeomorphism $\bar{h}$ of Y over a homeomorphism $h : S_U \to S_U$ may be uniquely completed to one of $\widetilde{Y}_Y$ over h, as depicted in the following diagram.

$$\begin{tikzcd}
Y \arrow[r, "\bar{h}"] \arrow[d, tail, "\eta"'] & Y \arrow[d, tail, "\eta"] \\
\widetilde{Y} \arrow[r, "\tilde{h}"'] \arrow[d, two heads, "\psi_m"'] & \widetilde{Y} \arrow[d, two heads, "\psi_m"] \\
S \arrow[r, "h"'] & S
\end{tikzcd}$$

The reason that there is such an $\tilde{h}$ is explained by the following simple fact about factorization systems, applied in our case to the pure, cosheaf space factorization. Suppose we have a commutative diagram

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "e"'] & A \arrow[d, "e"] \\
B \arrow[d, "m"'] & B \arrow[d, "m"] \\
C \arrow[r, "g"'] & C
\end{tikzcd}$$

where e and m are orthogonal, and where f and g are isomorphisms. Then because $g \cdot m$ and $e \cdot f$ are orthogonal, there is a unique isomorphism $B \xrightarrow{\tilde{g}} B$ making both squares commute.

Finally suppose we have an isotopy $\bar{H} : Y \times I \to Y$ over an isotopy $H : S_U \times I_I \to S_U$. The previously mentioned fact about factorization systems also applies for completing $\bar{H}$ to an isotopy $\widetilde{H} : \widetilde{Y}_Y \times I_I \to \widetilde{Y}_Y$ over H. This fact applies because the pure, cosheaf space factorization is stable under pullback along a *locally* 0-*acyclic map*. A locally 0-acyclic map $f : A \to B$ is an open map with the property that A has a base of open sets each of which meet any fiber of f in a connected or empty set. This pullback stability property has a topos theory proof: a locally 0-acyclic map induces what is usually called a *locally connected* geometric morphism of sheaf toposes [16]. The pullback stability of the pure, cosheaf factorization along a locally connected geometric morphism has been established in [5]. The projection map $S \times I \to S$ is locally 0-acyclic, so that $\psi_m \times I$ is a cosheaf space and $\eta \times I$ is pure. ∎

Lifting a homeomorphism and lifting a braid are not exactly the same matter because a braid is really an isotopy-equivalence class of homeomorphisms. We review this in the next section.

8. Artin braids as homeomorphisms of S_U

Let $\mathbf{B}_n$ denote the Artin group of braids of n strands. Let D_n denote the closed disk with n interior marked-points. Consider the collection of homeomorphisms and isotopies of D_n that pointwise fix the boundary and permute the marked-points. The group of isotopy-equivalence classes of this collection is called a *mapping class group*[2] $\mathit{Mcg}(D_n)$ [3]. It is well-known that $\mathbf{B}_n$ is isomorphic to $\mathit{Mcg}(D_n)$. Generators of $\mathit{Mcg}(D_n)$, which we denote by $\sigma_1, \ldots, \sigma_{n-1}$, may be described as follows. The well-known *Dehn twist about a circle* is defined by choosing a cylinder that contains the circle so that the cylinder is parameterized by (y, θ), for $-1 \leq y \leq 1$ with the circle at $y = 0$. Then the Dehn twist about this circle is the function $(\, y, \theta + \pi(1 - y)\,)$ on the cylinder. To describe σ_i, let $v_0, \ldots, v_{n-1}$ denote the n marked-points of the disk. Consider the homeomorphism defined to be the Dehn twist about the circle on which v_{i-1} and v_i lie diametrically opposed (with v_i at $(0, 0)$) in a cylinder that contains no other marked-point, and defined to be the identity elsewhere on the disk. This homeomorphism exchanges v_{i-1} with v_i and leaves the other marked-points fixed. Let σ_i denote the isotopy class of this homeomorphism.

We shall want to regard a braid $\beta \in \mathbf{B}_n$ as an isotopy-equivalence class $\beta = [h]$ of homeomorphisms $h : S_U \to S_U$, where $n + 1 = |S - U|$. This introduces a complication because isotopies on the sphere are not the same as on the disk, but we feel that it is worth the trouble because working with ramified covers seems better suited to the sphere. We shall say that an isotopy $H : S \times I \to S$ *fixes an open neighbourhood* $V \subseteq S$ if

$$\begin{tikzcd}
V \times I \arrow[r] \arrow[d, tail] & V \arrow[d, tail] \\
S \times I \arrow[r, "H"'] & S
\end{tikzcd}$$

commutes, where the top horizontal arrow denotes the projection map. (This also makes sense for isotopies in the Sierpinski fibration.) In other words, an isotopy fixes an open set V if every homeomorphism of the isotopy fixes V. (Note: by fix we shall always mean fix pointwise.) Now consider S_U, where $S - U = \{v_0, \ldots, v_{n-1}, v_n\}$. There are n 'Dehn twist' generators $\sigma_1, \ldots, \sigma_n$ of the mapping class group of S_U. An Artin braid may always be represented by a homeomorphism of S_U that fixes an open neighbourhood of v_n. (Such a homeomorphism necessarily preserves orientation.) Represented this way, the Artin braids are generated by the Dehn twists $\sigma_1, \ldots, \sigma_{n-1}$, just as on the disk. Furthermore, two such homeomorphisms represent the same Artin braid if and only if they are isotopic in S_U under an isotopy that fixes an open neighbourhood of v_n.

8.1. Remark. The notions of isotopy in S_U and isotopy in S_U fixing an open neighbourhood are not equivalent. For instance, if $S - U = \{v_0, v_1, v_2\}$, with Dehn twist generators σ_1, σ_2, then the square of the Dehn twist that represents σ_1 is isotopic to the identity

[2]A member of a mapping class group is required to *preserve orientation*, but in the case of D_n, a homeomorphism that pointwise fixes the boundary necessarily preserves orientation.

homeomorphism of S_U;[3] however, σ_1^2 is not equal to the identity Artin braid. I.e., the square of this Dehn twist is homotopic to the identity through orientation-preserving homeomorphisms that permute (in this case, fix) v_0, v_1, v_2, but this homeomorphism is not isotopic to the identity under an isotopy that fixes an open neighbourhood of v_2.

9. The action of $\mathbf{B}_n$ in $\pi_1^r(S, n+1)$

9.1. Proposition. *Depending on a chosen isomorphism* $\gamma_b : P(b) \cong h(P(b))$ *of* $\pi_1(U)$ (as in Prop. 7.1)*, a homeomorphism* h *of* S_U *induces a groupoid automorphism of* $\pi_1^r(S, m)$*,* $m = |S - U|$. (How this automorphism depends on the isotopy-equivalence class of h and on the isomorphism γ_b is explained in Prop. 7.1.)

Proof. From Proposition 7.1, two isotopic homeomorphisms of S_U have liftings to isotopic homeomorphisms of $\widetilde{Y}_Y$. Since the branch-point set $\widetilde{Y} - Y$ (= objects of $\pi_1^r(S, m)$) is a discrete subspace of $\widetilde{Y}$, it follows that two isotopic homeomorphisms of $\widetilde{Y}_Y$ must induce the same automorphism of the branch-point set. Further, two homotopic paths (in the Sierpinski fibration) are homotopic under isotopic images. We omit further details. ∎

By Proposition 9.1, a braid of n strands induces an automorphism of $\pi_1^r(S, n+1)$, but when working with braid homeomorphisms and the restricted notion of 'fixed open set' isotopy in S_U we can do better: for any given object x of $\pi_1^r(S, n+1)$ we may define a covariant (or left) action of $\mathbf{B}_n$ in $\pi_1^r(S, n+1)$ by groupoid automorphisms, such that every braid fixes x. We now explain how to do this. Let x be any object of $\pi_1^r(S, n+1)$. We label $S - U = \{v_0, \ldots, v_{n-1}, v_n\}$ such that

$$\psi_{n+1}(x) = v_n \ .$$

Once (pairwise disjoint) line segments e_i joining the v_i have been chosen, we may define leaves of the cover $Y \xrightarrow{P} U$. (A leaf of P is a connected component of $P^{-1}(U - \bigcup e_i)$.) Let $A \subseteq Y$ be a leaf of x, i.e., a leaf of P whose closure in $\tilde{Y}$ contains x. Let $\beta = [h]$ denote an arbitrary member of $\mathbf{B}_n$, where h is a homeomorphism of S_U that fixes an open neighbourhood $V \subseteq S$ of v_n. Let $b \in A$ be any point such that $P(b) \in V$, so that h fixes $P(b)$. Then, as in Prop. 9.1, we may define an induced automorphism of $\pi_1^r(S, n+1)$ taking for γ_b the identity on $P(b)$. The definition of this automorphism does not depend on the point $b \in A$ chosen such that $P(b) \in V$. If h' is another homeomorphism that is isotopic to h under an isotopy that fixes an open neighbourhood of v_n, then the same automorphism of $\pi_1^r(S, n+1)$ is induced. Thus, we may denote this automorphism as β_*. It is clear that $\beta_*(x) = x$. We denote the action by

$$\beta \cdot g = \beta_*(g) \ ,$$

[3] This can be seen using the system of generators and relations for the mapping class group of the marked sphere given in [3]. It can also be seen directly: rotate one revolution, and in opposite directions, the two closed disks that are disjoint from the open cylinder in which the Dehn twist for σ_1 (squared) is defined, whilst keeping v_0 and v_1 fixed.

where g is an isomorphism of the groupoid $\pi_1^r(S, n+1)$. It may be verified that the automorphism $(\beta\beta')_*$ is equal to the composite automorphism $\beta_* \cdot \beta'_*$. The definition of this action depends (only) on x and on the leaf of x chosen.

9.2. Remark. Since x is fixed under this action, we may pass to an action of $\mathbf{B}_n$ in $x/\pi_1^r(S, n+1)$ that fixes the identity isomorphism $id_x : x \cong x$.

10. A presentation of the action of $\mathbf{B}_n$ in $\pi_1^r(S, n+1)$

The order structure of $\pi_1^r(S, n)$ has been defined combinatorially, so in order to show (in §11) that the action of $\mathbf{B}_n$ in $\pi_1^r(S, n+1)$ respects this order structure we shall provide a combinatorial presentation of the action (which is (4) below). This action is isomorphic to the Artin action (Proposition 13.8). Larue [19] has considered (4) in his study of the 'Dehornoy bracket.'

In §5 we have described a geometric representation of a free group(oid) using line segments and edge-symbols. In order to prescribe the action of $\mathbf{B}_n$ in $\pi_1(U)$ precisely, let σ_i, $1 \leq i \leq n-1$, denote the braid group generators. We have previously explained in §8 that σ_i may be represented by a Dehn twist (on the sphere), which permutes v_{i-1} and v_i, where $S - U = \{v_0, \ldots, v_n\}$. From this geometric representation of σ_i we are led to define the action of the σ_i on the generators e_j of $\pi_1(U)$ as follows:

$$\left.\begin{array}{l} \sigma_i \cdot e_i = e_{i-1} e_i^{-1} e_{i+1} \\ \sigma_i \cdot e_j = e_j \ \ ; \ j \neq i \end{array}\right\} \ i = 1, \ldots, n-1 \ . \tag{4}$$

The first identity in (4) arises by observing what the Dehn twist σ_i does to a simple path that crosses the line segment e_{i+1} going up the page (which we regard as the positive direction). When $i = 1$, the first identity does not make sense because there is no edge e_0. In this case, we define

$$\sigma_1 \cdot e_1 = e_1^{-1} e_2 \ .$$

The braid relations are easily seen to hold for the above definitions, so that we have a well-defined action of $\mathbf{B}_n$ in $\pi_1(U)$. This action is faithful in the sense that for any braid β, if for every member w of the free group on the e_i, we have $\beta \cdot w = w$, then $\beta = id$.

We complete the presentation of the action of $\mathbf{B}_n$ in $\pi_1^r(S, n+1)$ by defining the effect on the vertex-symbols $v_0, v_1^{\pm}, \ldots, v_{n-1}^{\pm}, v_n$.

$$\left.\begin{array}{l} \sigma_i \cdot v_{i-1}^+ = v_i^+ \\ \sigma_i \cdot v_i^+ = v_{i-1}^- e_{i+1} \\ \sigma_i \cdot v_{i-1}^- = v_i^+ e_{i-1}^{-1} \\ \sigma_i \cdot v_i^- = v_{i-1}^- \end{array}\right\} \ i = 1, \ldots, n-1 \tag{5}$$

The third line above does not make sense when $i = 1$ again because there is no edge e_0. We define

$$\sigma_1 \cdot v_0^- = v_1^+ \ .$$

This is consistent with the other definitions. For instance, since $v_0^+ \sim v_0^-$, we have

$$\sigma_1 \cdot v_0^- = \sigma_1 \cdot v_0^+ = v_1^+ \ ,$$

where the second equality is by the first identity in (5). In all other cases, we set $\sigma_i \cdot v_k^\pm = v_k^\pm$. The definition extends to arbitrary words $v_i^\pm w v_n$, and braids β in the obvious way. For instance, we have $\sigma_i \cdot v_n = v_n$ for $i = 1, \ldots, n-1$, so that every braid fixes the word $v_n v_n$. Also, the action of σ_i^{-1} on the vertex-symbols may be inferred from the above definitions and the codomain winding relations. For instance, we have

$$\sigma_i^{-1} \cdot v_i^- = v_{i-1}^+ e_{i+1}^{-1} \ .$$

This completes the combinatorial presentation of the action of $\mathbf{B}_n$ in $\pi_1^r(S, n+1)$.

11. The action of $\mathbf{B}_n$ in $\pi_1^r(S, n+1)$ respects order

We shall prove the following.

11.1. Theorem. *The action of $\mathbf{B}_n$ in $\pi_1^r(S, n+1)$ respects the order structure carried by $\pi_1^r(S, n+1)$: for every object x of $\pi_1^r(S, n+1)$, isomorphisms $g, h \in (x/\pi_1^r(S, n+1))^\star$, and braid $\beta \in \mathbf{B}_n$, we have*

$$g \leq_x h \ \Rightarrow \ \beta \cdot g \leq_x \beta \cdot h \ ,$$

where the action is defined in §9 (depending on x and a choice of leaf of x)*, and where $\leq_x$ denotes the linear ordering in $(x/\pi_1^r(S, n+1))^\star$* (given by Definition 6.1)*.*

Our argument for Theorem 11.1 is combinatorial in nature since that is how we have defined the order structure of $\pi_1^r(S, n+1)$. *For this argument only, we shall write the words $v_i^\pm w v_n$ just as $v_i^\pm w$ since the right-hand v_n is inert anyway.* We begin with some terminology.

11.2. Definition. *Let $1 \leq k \leq n$.*

1. *We shall say that a reduced word b in the e_i is k-*positive definite *if only positive exponents of e_k appear in b, and if the first and last symbol of b is e_k. Similarly, we define k-*negative definite*.*

2. *Let $\mathrm{sgn}_k(w)$ denote the sum of the exponents of e_k in w. Let $|w|$ denote the positive word obtained from w by replacing every exponent by its absolute value.*

3. *Obviously any reduced word w in the e_i can be written uniquely as*

$$w_0 b_1 w_1 \cdots b_m w_m \ ,$$

where each b_j is either k-positive definite or k-negative definite, $\mathrm{sgn}_k(b_j)$ alternates between positive and negative, and $\mathrm{sgn}_k(|w_j|) = 0$. We shall call this the k-definite form of w*.*

An examination of the definitions (4) and (5) reveals the following. As always, σ_i denotes a braid group generator.

11.3. LEMMA. *Fix $h \in \{1, \ldots, n-1\}$, and let $v_k^{\pm} z$ denote $\sigma_h \cdot v_i^{\pm} w$.*

1. *If $\mathrm{sgn}_h(|w|) = 0$, then either $z = e_{h-1}^{-1} w$, $z = e_{h+1} w$, or z is a (possibly empty) right subword of w.*

2. *Assume that $\mathrm{sgn}_h(|w|) \neq 0$, and write w in its h-definite form $w_0 b_1 w_1 \cdots b_m w_m$* (each b_j is either h-positive definite or h-negative definite, $sgn_h(b_j)$ alternates between positive and negative, and $sgn_h(|w_j|) = 0$). *Then $v_k^{\pm} z$ has a similar form $v_k^{\pm} z_0 c_1 z_1 \cdots c_m z_m$, where $\mathrm{sgn}_h(|z_j|) = 0$, and $\mathrm{sgn}_h(c_j) = -\mathrm{sgn}_h(b_j)$. Furthermore, if w_m is empty, then $z_m = e_{h+1}^{\nu}$, where ν is equal to the sign of $\mathrm{sgn}_h(b_m)$.*

We proceed with the proof of Theorem 11.1. Assume that $v_k^{\pm} z < v_i^{\pm} w$. We may assume that the rightmost symbols of these two words are different, so that the start-symbol in the circular list in Figure 1 is e_n^{-1}. There must be vertex-symbols $v_j^{\pm}, v_{j'}^{\pm}$ such that

$$v_k^{\pm} z \leq v_j^{\pm} \leq v_{j'}^{\pm} \leq v_i^{\pm} w \ ,$$

where in the circular list $v_j^{\pm}$ is clockwise adjacent to the rightmost symbol of $v_k^{\pm} z$, and $v_{j'}^{\pm}$ is clockwise adjacent to the rightmost symbol of $v_i^{\pm} w$. We thus have three cases to argue: when w is empty and z is not, when both w and z are empty, and when z is empty and w is not. We shall argue only the case when z is empty and w is not: this is the case $v_k^{\pm} < v_i^{\pm} w$, where $v_k^{\pm}$ is clockwise adjacent to the rightmost symbol of $v_k^{\pm} w$. This symbol is an edge-symbol since we are assuming that w is non-empty. Furthermore, we shall assume that $v_k^{\pm} = v_{n-1}^{-}$ (the general case is similarly argued), so that the rightmost symbol of $v_k^{\pm} w$ is therefore e_{n-1}^{-1}. To summarize: we are assuming that $v_{n-1}^{-} < v_i^{\pm} w$ (starting from e_n^{-1}), and that the rightmost symbol of w is e_{n-1}^{-1}. We must show that $\sigma_h \cdot v_{n-1}^{-} < \sigma_h \cdot v_i^{\pm} w$ and $\sigma_h^{-1} \cdot v_{n-1}^{-} < \sigma_h^{-1} \cdot v_i^{\pm} w$, for $h = 1, \ldots, n-1$.

First we consider the case $h \neq n-1$, so that $\sigma_h^{\pm 1} \cdot v_{n-1}^{-} = v_{n-1}^{-}$. The word $v_i^{\pm} w$ may be uniquely expressed as

$$v_i^{\pm} w_0 e_n^{p_1} w_1 \ldots e_n^{p_m} w_m \ ; \ m \geq 0 \ , \tag{6}$$

where every subword w_j does not contain e_n^{-1}, and the p_j's are negative integers. For $j \geq 1$, w_j is non-empty, but w_0 may be empty, in which case we must have $m \geq 1$. For $h \neq n-1$, the action of $\sigma_h^{\pm 1}$ (on vertex or edge-symbols) does not introduce or remove $e_n^{\pm 1}$, and we have $\sigma_h^{\pm 1} \cdot e_n^{\pm 1} = e_n^{\pm 1}$. Hence, for $j \geq 1$, $\sigma_h^{\pm 1} \cdot w_j$ is non-empty and contains no e_n^{-1}, so that no change in m or in the p_j in (6) can occur under the action of $\sigma_h^{\pm 1}$. In particular, the rightmost symbol of $\sigma_h^{\pm 1} \cdot v_i^{\pm} w$ (= rightmost symbol of $\sigma_h^{\pm 1} \cdot w_m$) cannot be e_n^{-1}. Hence, $v_{n-1}^{-} < \sigma_h^{\pm 1} \cdot v_i^{\pm} w$ (starting from e_n^{-1}).

Next assume that $h = n-1$. We shall show that $\sigma_{n-1}^{-1} \cdot v_{n-1}^{-} < \sigma_{n-1}^{-1} \cdot v_i^{\pm} w$, leaving $\sigma_{n-1} \cdot v_{n-1}^{-} < \sigma_{n-1} \cdot v_i^{\pm} w$ for the reader. We have $\sigma_{n-1}^{-1} \cdot v_{n-1}^{-} = v_{n-2}^{+} e_n^{-1}$. Let $v_k^{\pm} z$ denote $\sigma_{n-1}^{-1} \cdot v_i^{\pm} w$. If the rightmost symbol of $v_k^{\pm} z$ is not e_n^{-1}, then the largest common right subword of $v_{n-2}^{+} e_n^{-1}$ and $v_k^{\pm} z$ is empty, which means that we begin from e_n^{-1} in Figure 1. But then $v_{n-2}^{+} e_n^{-1} < v_k^{\pm} z$. Hence, we can assume that the rightmost symbol of $v_k^{\pm} z$ is

e_n^{-1}. Then the greatest common right (edge) subword of $v_k^{\pm}z$ and $\sigma_{n-1}^{-1} \cdot v_{n-1}^{-} = v_{n-2}^{+}e_n^{-1}$ is e_n^{-1}. Thus, we determine the order of $v_{n-2}^{+}e_n^{-1}$ and $v_k^{\pm}z$ by comparing v_{n-2}^{+} with the right-penultimate symbol of $v_k^{\pm}z$, starting from the edge-symbol that immediately follows e_n clockwise in the circular list, which is again e_n^{-1}. By assumption, the rightmost symbol of $v_i^{\pm}w$ is e_{n-1}^{-1}. Hence $sgn_{n-1}(|w|) \neq 0$, so that when we consider $v_i^{\pm}w$ written in its $n-1$-definite form, we have $m \geq 1$. Moreover, in this case w_m is empty. So according to Lemma 11.3.2 (for σ_{n-1}^{-1}), we have $z_m = e_n^{-1}$, and that the right-penultimate symbol of $v_k^{\pm}z$ is e_{n-1} (= rightmost symbol of c_m). This concludes the argument because v_{n-2}^{+} comes before e_{n-1} starting from e_n^{-1}.

11.4. Remark. Kassel [17] notes that Nielsen [25, 28] gives an action of $\mathbf{B}_n$ in the unit circle that preserves circular order and has a fixed point. Furthermore, this action has free elements. We have the following topos theory version of Nielsen's result. Let $\mathcal{B}_n$ denote the topos of (left) $\mathbf{B}_n$-sets. Let A denote the topological group of order preserving bijections of the rationals in $(0, 1)$. It is well-known that the topos $\mathcal{A}$ of continuous (left) A-sets classifies dense linear orders without endpoints [26]. Thus, corresponding to the dense linear order that we have constructed in $\mathcal{B}_n$ there is a geometric morphism $\mathcal{B}_n \longrightarrow \mathcal{A}$.

12. Linear orderings of $\mathbf{B}_n$

A *preorder* is a reflexive and transitive relation. A *partial order* is a preorder that is also anti-symmetric. If a preorder $(A, \leq)$ is equipped with an order-preserving (left) action by a group $\mathbf{G}$, then an element $a \in A$ induces a left-invariant preorder $\leq_a$ in $\mathbf{G}$ by:

$$g \leq_a h \text{ if } g \cdot a \leq h \cdot a ,$$

for any $g, h \in \mathbf{G}$. Moreover, if $(A, \leq)$ is a partial order, then $\leq_a$ is anti-symmetric if and only if a is *free* (meaning that the stabilizer of a is trivial). We shall also use the symbol $<$, which means $\leq$ and not equal. For our purposes, we are only interested in *total* or *linear* orders. Everything we have said about preorders and partial orders applies equally well to linear preorders and orders.

Thus, we may produce left-invariant linear orderings of $\mathbf{B}_n$ by producing free elements of the left $\mathbf{B}_n$-set $(x/\pi_1^r(S, n+1))^{\star}$. We have the following.

12.1. Proposition. *For any n, the isomorphism $x \cong x'$ of $\pi_1^r(S, n+1)$ presented by the word*

$$d_n = v_n e_{n-1}^{-1} e_n \cdots e_2^{-1} e_3 e_1^{-1} e_2 e_1 v_n$$

is free under the action of $\mathbf{B}_n$. (Note: we have $\psi_{n+1}(x) = \psi_{n+1}(x') = v_n$.)

Proof. We sketch a topological argument. Suppose that $\beta \in \mathbf{B}_n$ fixes d_n. Then we may find a path p in $\widetilde{Y}_Y$ that represents d_n (with $\psi_{n+1}(p(0)) = \psi_{n+1}(p(1)) = v_n$), and a homeomorphism h of S_U representing β (that fixes an open neighbourhood of v_n), such that h pointwise fixes the path $\psi_{n+1} \cdot p$ in S_U. Now consider the components of the open set $S - \psi_{n+1} \cdot p$, of which there are just finitely many. Each marked-point except v_n is an interior point of some component, while some components may contain no marked point. The components must be permuted under h; however, further straightforward argumentation shows that this permutation must be the identity. In particular, h must fix the points v_i (hence, the permutation of β is the identity). To each component we may now apply a classical result due to Alexander (called the Alexander trick - proof provided in [15]): a homeomorphism of a closed disk that pointwise fixes the boundary is isotopic to the identity under an isotopy that also pointwise fixes the boundary. Furthermore, if the homeomorphism fixes an interior point, then the isotopy may be chosen to fix that point. Note: we can find a closed disk C that contains v_n as an interior point, that is fixed by h, and which meets *only* those components of $S - \psi_{n+1} \cdot p$ whose closure contains v_n (there is at least one such component). We now just delete the interior of C, and apply the Alexander trick (without interior point) to the components of $S - \psi_{n+1} \cdot p$ that meet C, using the boundary of C wherever necessary. The result is that we are able to build an isotopy of h with the identity that fixes the interior of C. After all this, we finally conclude that the h with which we began is isotopic to the identity, under an isotopy that fixes an open neighbourhood of v_n. Thus, $\beta = id$. ■

Let $\leq_{d_n}$ denote the linear ordering induced in $\mathbf{B}_n$ by the free element d_n.

12.2. Remark. By the results of Short and Wiest [27], it may be deduced through the connection with their work explained in Remark 6.3 that $(\mathbf{B}_n, \leq_{d_n})$ is the Dehornoy ordering. A more direct link with the Dehornoy ordering is explained in Remark 13.11.

12.3. Example. We calculate the order of the braid generators σ_1, σ_2, σ_3 in $(\mathbf{B}_4, \leq_{d_4})$, where $d_4 = v_4 e_3^{-1} e_4 e_2^{-1} e_3 e_1^{-1} e_2 e_1 v_4$. Direct calculation yields

$$\begin{aligned} \sigma_1 \cdot d_4 &= v_4 e_3^{-1} e_4 e_2^{-1} e_3 e_2^{-1} e_1 e_2 e_1^{-1} e_2 v_4 \\ \sigma_2 \cdot d_4 &= v_4 e_3^{-1} e_4 e_3^{-1} e_2 e_1^{-1} e_3 e_2^{-1} e_3 e_1 v_4 \\ \sigma_3 \cdot d_4 &= v_4 e_3 e_2^{-1} e_4 e_3^{-1} e_4 e_1^{-1} e_2 e_1 v_4 \ , \end{aligned}$$

so that $id \leq_{d_4} \sigma_3 \leq_{d_4} \sigma_2 \leq_{d_4} \sigma_1$.

13. Ordering $\mathbf{B}_\infty$

Our methods extend naturally to the braid group $\mathbf{B}_\infty$. This group is the group colimit of the $\mathbf{B}_n$ organized by the natural inclusions $\mathbf{B}_n \rightarrowtail \mathbf{B}_m$, $n < m$. $\mathbf{B}_\infty$ is also isomorphic to the quotient of the free group in countably many generators, given by the same relations as in the finite case (but now there are countably many). A member of $\mathbf{B}_\infty$ may be thought of as a braid of countably many strands for which all but finitely many strands are unbraided.

Consider a collection of countably many marked points v_i on the sphere S that has just one limit point v_ω. We suppose that these marked points lie on a closed line segment with endpoints v_0 and v_ω.

$$v_0 \xrightarrow{e_1} v_1 \xrightarrow{e_2} v_2 \ \cdots \ v_{n-1} \xrightarrow{e_n} v_n \ \cdots \ v_\omega$$

Let U denote the open set $S - \{v_0, \ldots, v_n, \ldots, v_\omega\}$. U is connected, locally path-connected and locally simply connected with a universal covering space $P : Y \twoheadrightarrow U$. Consider the pure, cosheaf space factorization of $Y \overset{P}{\twoheadrightarrow} U \rightarrowtail S$.

$$\begin{tikzcd} Y \arrow[r, "\eta"] \arrow[d, two heads, "P"'] & \widetilde{Y} \arrow[d, two heads, "\psi_{\omega+1}"] \\ U \arrow[r, tail] & S \end{tikzcd}$$

The open subset $U \rightarrowtail S$ is pure, the square is a pullback, and $\psi_{\omega+1}$ is a ramified covering space. As in the case of finitely many marked-points, $\widetilde{Y}$ is connected, locally path-connected, semi-locally simply connected, and simply connected.

An interesting feature of $\widetilde{Y}$ is that no branching occurs at the points in the fiber of v_ω, even though these points are members of the branch-point set $\widetilde{Y} - Y$. A precise expression of this fact is as follows.

13.1. Proposition. *If $x \in \widetilde{Y}$ lies over v_ω, then x lies in the closure of exactly one leaf of P.* (Whereas a point over v_i, $i < \omega$, lies in the closures of countably many leaves of P.)

The objects of the groupoid $\pi_1^r(S, \omega + 1)$ are the elements of $\widetilde{Y} - Y$ as in the finite case. Morphisms and composition in $\pi_1^r(S, \omega + 1)$ are also defined as in the finite case. Composition at points of the fiber $\psi_{\omega+1}^{-1}(v_\omega)$ is simpler than over a v_i, $i < \omega$, because there can be no cycling around a point over v_ω. On S_U, we compose two paths (that can be composed) at v_ω by moving their end-points appropriately to the 'right' of v_ω.

Let $F(X)$ denote the free group on a set X. Let E denote a set of countably many 'edge'-symbols $e_1, e_2, \ldots$. Fix a point $x \in \widetilde{Y} - Y$ lying over v_ω. The paths in $\widetilde{Y}_Y$ with domain x and codomain in $\widetilde{Y} - Y$ may be encoded as words

$$v_i^{\pm} w v_\omega \ ; \ w \in F(E) \ , \ i \in \{0, 1, 2, \ldots, \omega\} = \omega + 1 \ , \tag{7}$$

where we identify $v_0^+ \sim v_0^-$ and $v_\omega^+ \sim v_\omega^-$. There are codomain winding relations for the $v_i^{\pm}$, $i < \omega$, so if $w \in F(E)$ is a reduced word, then $v_i^{\pm} w v_\omega$ may not be reduced. But there are no codomain winding relations for v_ω, so a word $v_\omega w v_\omega$ is reduced if $w \in F(E)$ is. We have the following.

13.2. Proposition. *The following collections are in bijective correspondence* (in an obvious way).

1. *The fiber $\psi_{\omega+1}^{-1}(v_\omega) = \{y \in \widetilde{Y} \mid \psi_{\omega+1}(y) = v_\omega\}$.*

2. *For any x such that $\psi_{\omega+1}(x) = v_\omega$, the subset of $x/\pi_1^r(S, \omega + 1)$ consisting of those isomorphisms whose codomain also lies over v_ω.*

3. *The collection of reduced words $v_\omega w v_\omega$.*

4. *$F(E)$.*

If $\mathbf{G}$ is a group, let $\mathbf{G}^\star$ denote $\mathbf{G}-\{id\}$. For instance, $F(E)^\star$ consists of all non-empty reduced words in E. The collection of reduced words (7) (except $v_\omega v_\omega$) may be linearly ordered just as in the finite case. Hence, the subset of the collection (7) consisting of the reduced words $v_\omega w v_\omega$ (except $v_\omega v_\omega$) can also be linearly ordered. This subset corresponds to $F(E)^\star$. We shall use the following terminology.

13.3. Definition. *We shall refer to the ordering of $F(E)^\star$ given by Definition 6.1 (ignoring the vertex-symbols and admitting countably many edge-symbols) as the* hyperbolic *ordering of $F(E)^\star$.*

13.4. Remark. We can work without the vertex-symbols $v_i^\pm$, but we cannot ignore the vertex-symbol v_ω in Definition 13.3. For instance, if we wish to compare e_2e_3 and e_3, we compare e_2 with v_ω starting from the edge that is clockwise adjacent to e_3^{-1}, which is e_2^{-1}. Then $e_2e_3 \leq e_3$ since e_2 occurs before v_ω clockwise from e_2^{-1}. If the largest common right subword of two words is empty, then we start from v_ω. (In the finite case we start from the 'least' edge e_n^{-1}.) How to compare the empty word with another one is not defined because in this case we must compare an $e_i^{\pm 1}$ with v_ω starting from v_ω; however, v_ω is neither the top nor the bottom of the order.

13.5. Proposition. *The hyperbolic ordering of $F(E)^\star$ is a dense linear ordering (without endpoints).*

13.6. Remark. The hyperbolic ordering of $F(E)^\star$ is neither invariant under left multiplication nor under right multiplication. For instance, we have $e_1 \leq e_3$, but $e_3e_2^{-1} \leq e_1e_2^{-1}$. If we multiply $e_1 \leq e_3$ on the left by $e_1e_3^{-1}$, then we have $e_1 \leq e_1e_3^{-1}e_1$. Thus, the hyperbolic ordering is not a group ordering in the usual sense. (In anycase, the identity is left out of the ordering.) But the hyperbolic ordering does satisfy property (3), which in the present context reads as follows:

$$\forall\, w, z \in F(E)^\star,\ w < z \Rightarrow zw^{-1} < w^{-1}\ . \tag{8}$$

For instance, we have $e_3e_2^{-1} < e_4^{-1}e_2^{-1}$. Then $e_4^{-1}e_3^{-1} < e_2e_3^{-1}$ as predicted by (8).

A member of $\mathbf{B}_\infty$ may be regarded as a homeomorphism class $\beta = [h]$ of S_U, where $U = S-\{v_0, \ldots, v_n, \ldots, v_\omega\}$, such that the homeomorphism h fixes an open neighbourhood of v_ω. If x lies over v_ω, then we may lift such a homemorphism to a homeomorphism of $\widetilde{Y}_Y$ that fixes x. In this way we obtain an action of $\mathbf{B}_\infty$ in $\pi_1^r(S, \omega+1)$ such that every braid fixes the chosen x. Then we may pass to an action of $\mathbf{B}_\infty$ in $(x/\pi_1^r(S, \omega+1))^\star$. (Unlike the finite case, by Proposition 13.1 we do not need to choose a leaf of x to define the action because x has just one leaf.) This action is presented by (4) and (5) just as in the finite case.

The action of $\mathbf{B}_\infty$ in $(x/\pi_1^r(S, \omega+1))^\star$ is order-preserving. Moreover we may restrict the action to those isomorphisms whose codomain also lies over v_ω, which we have encoded as precisely the members of $F(E)^\star$. We have the following.

13.7. Proposition. *The following action of* $\mathbf{B}_\infty$ *in* $F(E)$ (as in (4)) *preserves the hyperbolic ordering of* $F(E)^\star$.

$$\left.\begin{array}{l}\sigma_i \cdot e_i = e_{i-1}e_i^{-1}e_{i+1}\\ (\sigma_1 \cdot e_1 = e_1^{-1}e_2)\\ \sigma_i \cdot e_j = e_j \ \ ; \ j \neq i\end{array}\right\} \ i = 1, 2, \ldots \ . \tag{9}$$

For any n*, the only member of the subgroup* $\mathbf{B}_n \rightarrow \mathbf{B}_\infty$ *that stabilizes the word*

$$d_n = e_{n-1}^{-1}e_n \ \cdots \ e_2^{-1}e_3e_1^{-1}e_2e_1$$

is the identity braid. (We shall say that d_n is n-free.) *Thus, the linear preorder* $\leq_{d_n}$ *that* d_n *induces in* $\mathbf{B}_\infty$ *is a linear order in* $\mathbf{B}_n$ (meaning anti-symmetric). *The natural inclusions* $\mathbf{B}_n \rightarrow \mathbf{B}_m$ *are compatible with the linear orderings induced by* d_n *and* d_m*, so that all the* d_n *taken together induce a linear order* $\leq_d$ *of* $\mathbf{B}_\infty$*:* define $id \leq_d \beta$ if for some N for which $\beta \in \mathbf{B}_N$, we have $id \leq_{d_N} \beta$ in $\mathbf{B}_N$.

Let V denote a set of countably many 'vertex'-symbols $v_0, v_1, \ldots$. The classical Artin action of $\mathbf{B}_\infty$ in $F(V)$ is given by:

$$\left.\begin{array}{l}\sigma_i \cdot v_{i-1} = v_i\\ \sigma_i \cdot v_i = v_i^{-1}v_{i-1}v_i\end{array}\right\} \ i = 1, 2, \ldots \ . \tag{10}$$

Of course, the map $v_i \mapsto e_{i+1}$ provides a group isomorphism of $F(V)$ with $F(E)$, but we wish to know how $F(V)$ and $F(E)$ are related as $\mathbf{B}_\infty$-sets. Proposition 13.8 has been observed by Larue [19].

13.8. Proposition. (Larue) *The group homomorphism* $a : F(V) \rightarrow F(E)$ *given by* $a(v_0) = e_1$ *and* $a(v_i) = e_i^{-1}e_{i+1}$*,* $i = 1, 2, \ldots$*, is an isomorphism of* $\mathbf{B}_\infty$*-sets.* (The inverse of a is given by $a^{-1}(e_i) = v_0v_1 \cdots v_{i-1}$.)

By Proposition 13.8, we may rephrase Proposition 13.7 as follows.

13.9. Corollary. $F(V)^\star$ *may be given the structure of a dense linear order such that the Artin representation* (10) *is an order-preserving action. In this ordering* (which is not a group ordering) *we have* $v_i^{-1} < v_i < v_{i+1}^{-1}$*. For any* n*, the word* $v_{n-1} \cdots v_1v_0$ *is* n*-free, so that the linear preorder it induces in* $\mathbf{B}_\infty$ *is a linear ordering of the subgroup* $\mathbf{B}_n$*. All the words* $v_{n-1} \cdots v_1v_0$ *taken together induce a linear order in* $\mathbf{B}_\infty$ *as explained in Proposition 13.7.*

Proof. The ordering of $F(V)^\star$ is given by $w \leq w'$ just when $a(w) \leq a(w')$ in the hyperbolic ordering of $F(E)^\star$. For instance, we have $v_i < v_{i+1}^{-1}$ because $e_i^{-1}e_{i+1} < e_{i+2}^{-1}e_{i+1}$. To see this we must first determine the starting symbol: it is the symbol clockwise adjacent to e_{i+1}^{-1}, which is e_i^{-1}. But then e_i^{-1} occurs before e_{i+2}^{-1} (clockwise) starting from e_i^{-1}. The word $v_{n-1} \cdots v_1v_0 = a^{-1}(d_n)$ is n-free by Proposition 12.1. ∎

13.10. Remark. It ought to be possible to explicitly describe the ordering in $F(V)^\star$ given by Corollary 13.9. We leave this as an exercise.

13.11. Remark. Dehornoy [9] has provided an analysis of his ordering in terms of the Artin action of a braid group in a free group. Using Corollary 13.9 as a link to Dehornoy's analysis should provide another way to show the the ordering $\leq_{d_n}$ in $\mathbf{B}_n$ coincides with Dehornoy's. We shall not carry out the details of this.

The action (9) of $\mathbf{B}_\infty$ in $F(E)$ has no free elements. We can obtain free elements by admitting (reduced) infinite words

$$w = \cdots e_{i_3}^{\nu_3} e_{i_2}^{\nu_2} e_{i_1}^{\nu_1} \ ; \ \nu_j \in \{-1, 1\} \ , \ i_j \in \{1, 2, \ldots\} \ . \tag{11}$$

Let $\widehat{F(E)}$ denote the collection of infinite reduced words. By convention, $\widehat{F(E)}$ includes all finite words as well. Geometrically, such an infinite word w can be thought of as an 'open-ended' path

$$p : [0,1)_{(0,1)} \twoheadrightarrow \widetilde{Y}_Y \ ,$$

where $\psi_{\omega+1}(p(0)) = v_\omega$.

13.12. Remark. If $\psi_{\omega+1}(x) = v_\omega$, then there are two 'spiral' maps (clockwise and counter-clockwise):

$$x/\pi_1^r(S, \omega+1) \longrightarrow \widehat{F(E)} \ .$$

For instance, the counter-clockwise spiral map is given by

$$\left.\begin{array}{l} v_i^+ w v_\omega \mapsto \cdots e_{i+1} e_i^{-1} e_{i+1} w \\ v_i^- w v_\omega \mapsto \cdots e_i^{-1} e_{i+1} e_i^{-1} w \end{array}\right\} \ i < \omega \ ,$$
$$v_\omega w v_\omega \mapsto w \ \text{ (these are the finite words).}$$

On the sphere, all three paths above begin at v_ω, but whereas the first two appear to spiral into v_i, the third one makes finitely many loops around the marked-points and then returns to v_ω.

Consider the infinite word

$$d = \cdots e_2^{-1} e_3 e_1^{-1} e_2 e_1 = a(\cdots v_2 v_1 v_0) \ .$$

On S_U we think of d as a presentation of a 'wild' path p_d that begins and ends at v_ω, and loops once around every v_i in a loop whose diameter converges to 0. Such a path $p_d : [0,1] \twoheadrightarrow S_U$ may be lifted (uniquely) to an open-ended one

$$\overline{p_d} : [0,1)_{(0,1)} \twoheadrightarrow \widetilde{Y}_Y \ ,$$

provided that $\overline{p_d}(0)$ is given lying over v_ω. However, it is not possible to continuously define $\overline{p_d}(1)$ such that $\psi_{\omega+1}(\overline{p_d}(1)) = v_\omega$.

The hyperbolic ordering of $F(E)^\star$ may be extended to the collection $\widehat{F(E)}^\star$ of non-empty infinite words (11). Likewise, the action (9) of $\mathbf{B}_\infty$ in $F(E)^\star$ may be extended to $\widehat{F(E)}^\star$. Although we cannot necessarily calculate this action effectively, we can effectively compare d and $\beta \cdot d$ because d has the property that for any braid $\beta \in \mathbf{B}_\infty$, $\beta \cdot d$ differs from d only by an effectively computable finite right subword. We have the following.

13.13. Proposition. *The infinite word d is free for the action of $\mathbf{B}_\infty$ in $\widehat{F(E)}^\star$. The linear order it induces in $\mathbf{B}_\infty$ is precisely the one $\leq_d$ obtained in Proposition 13.7. It is the Dehornoy ordering of $\mathbf{B}_\infty$.*

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Dept. of Mathematics and Statistics
University of Saskatchewan
Saskatoon, SK., S7N 5E6 CANADA
Email: funk@snoopy.usask.ca

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