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CLASSIFYING SPACES OF CATEGORIES AND TERM REWRITING

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ABSTRACT. In this paper we show how collapsing schemes can give us information on the homotopy type of the classifying space of a small category, when this category is presented by a complete rewrite system.

1. Introduction

Given a small category $\mathbf{C}$, the classifying space $B\mathbf{C}$ of $\mathbf{C}$ is constructed by assigning to $\mathbf{C}$ its nerve $N\mathbf{C}$ and then taking the geometric realization $B\mathbf{C} = |N\mathbf{C}|$ of the simplicial set $N\mathbf{C}$. The definition of the geometric realization functor $|\_|$ is relatively plain and it is well-known that the space $|X|$ associated to a simplicial set X is a CW-complex whose n-cells correspond to non degenerate n-simplices of X ([Mil57]). But simplicial sets tend to be very large objects and the number of non degenerate simplices is usually much greater than the number of cells which are necessary to describe the homotopy type of realization. Can we reduce this number? Kenneth S. Brown and Ross Geoghegan ([BG84], [Bro92]) introduced the notion of collapsing scheme. This scheme distinguishes a class of non degenerate simplices of a simplicial set X, which are called the essential simplices of X. The main theorem on collapsing schemes states that the geometric realization $|X|$ of the simplicial set X has the homotopy type of a CW-complex $\mathcal{E}(X)$ whose n-cells correspond to essential n-simplices of X. When we consider a "complete presentation" of a monoid M (i.e. a presentation given by a complete rewrite system), then the classifying space of M can be obtained as the geometric realization of a simplicial set X. The paper [Bro92] shows that this simplicial set X is canonically endowed with a collapsing scheme.

We recall the definition of collapsing scheme in Section 2. Then we show that simplicial sets endowed with collapsing schemes form a category $\mathcal{C}\mathbf{Set}^{\mathbf{\Delta}^{op}}$ and that the construction $\mathcal{E}$ can be viewed as a functor $\mathcal{E} : \mathcal{C}\mathbf{Set}^{\mathbf{\Delta}^{op}} \to \mathbf{Top}$ into the category of topological spaces and maps. Moreover, the canonical homotopy equivalence $|X| \to \mathcal{E}(X)$ is a natural transformation $q : |\_| \Rightarrow \mathcal{E}$ from the geometric realization to the functor $\mathcal{E}$. In Section 3, we recall basic definitions and facts on rewrite systems (main references are [ES87], [Str96]). We reach the main result in Section 4: a functorial extension to categories of the construction given by K. S. Brown for monoids. The power of this technique is emphasized by an example, where we give a one-line proof of the fact that the classifying space of a free category is (up to homotopy) the geometric realization of its generating graph.

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This paper is only a small part of a wider work (whose starting point is [Cit00]) on collapsing schemes and their applications to bisimplicial sets and classifying spaces of 2-categories.

2. Collapsing schemes

Let X be a simplicial set whose face operators are denoted by d, and degenerate operators by s. We shall write D for the graded set of degenerate simplices (i.e. the simplices which are in the image of s), and $X^{nd} = X - D$ for the graded set of non-degenerate simplices. Assume that non-degenerate simplices have been partitioned into three classes: $E = \{\text{essential simplices}\}$, $R = \{\text{redundant simplices}\}$, $C = \{\text{collapsible simplices}\}$. We shall say that (E, R, C) is a partition of X^{nd} and write $X^{nd} = E \sqcup R \sqcup C$ (equivalently (E, R, C, D) is a partition of X and $X = E \sqcup R \sqcup C \sqcup D$); this means that for every $n \geq 0$ we have $X^{nd}_n = E_n \sqcup R_n \sqcup C_n$.

Writing $[m] = \{0, 1, \dots, m\}$ for the standard finite set with $m+1$ elements, assume that there exist functions $c_n : R_n \to C_{n+1}$ and $\iota_n : R_n \to [n+1]$ such that, for every redundant n-simplex τ we have $d_{\iota_n(\tau)} c_n(\tau) = \tau$. Functions c_n and ι_n define the following relation $>_{R_n}$ on R_n:

$$\tau >_{R_n} \tau' \quad (\tau, \tau' \in R_n) \quad \text{if and only if} \quad \tau' = d_j c_n(\tau) \quad (j \neq \iota_n(\tau))$$

and we say that τ' is an *immediate predecessor* of τ. The relation $>_{R_n}$ is said to be *noetherian* (or *terminating* or *well-founded*) when there is no infinite descending chain $\tau > \tau' > \tau'' > \dots$ of redundant n-simplices.

2.1. Definition. A *collapsing scheme* $\mathcal{C} = (E, R, C; c, \iota)$ on a simplicial set X consists of

1. a partition (E, R, C, D) of X with C_0 empty
2. a function $c_n : R_n \to C_{n+1}$ for every $n \geq 0$
3. a function $\iota_n : R_n \to [n+1]$ for every $n \geq 0$

and axioms

CS1. $\forall n \geq 0$, the function $c_n : R_n \to C_{n+1}$ is a bijection and $d_{\iota_n} c_n = \mathrm{id}_{R_n}$

CS2. $\forall n \geq 0$, the immediate predecessor relation $>_{R_n}$ is noetherian.

The axiom CS2 implies that a chain of redundant simplices cannot include twice the same redundant simplex. In particular, for any $\tau \in R_n$, we cannot have $\tau > \tau$ and there exists no index $i \neq \iota(\tau)$ such that $\tau = d_i c(\tau)$. We say that the redundant τ is *the free face* of the collapsible $\sigma = c_n(\tau)$. The axiom CS1 implies that every redundant simplex is the free face of a unique collapsible simplex.

Since every redundant simplex has only finitely many immediate predecessors, there cannot exist arbitrarily long descending chains. The maximum length of a chain $\tau >_{R_n} \tau' >_{R_n} \cdots$ starting from τ will be called the *height* of τ and written $h(\tau)$.

2.2. Theorem. [K. S. Brown] *Let X be a simplicial set with a collapsing scheme $\mathcal{C}$. The geometric realization $|X|$ of X admits a canonical quotient CW-complex $\mathcal{E}$, whose cells are in 1–1 correspondence with essential simplices of X. The quotient map $q_X : |X| \to \mathcal{E}$ is a homotopy equivalence; it maps each open essential cell of $|X|$ homeomorphically onto the corresponding open cell of $\mathcal{E}$ and it maps each collapsible $(n+1)$-cell into the n-skeleton of $\mathcal{E}$.*

We present here the proof of [Bro92] mainly because we shall need the constructions and notations. It also describes the geometry of collapsing schemes. We shall use the same notation for simplices of the simplicial set and corresponding cells of the geometric realization; as usual we shall identify degenerate cells with cells of lower dimension.

Proof. The geometric realization of a simplicial set X endowed with a collapsing scheme can be viewed as the colimit of the sequence of topological spaces and cofibrations

$$X_0^e \overset{j_0}{\hookrightarrow} X_0^+ \hookrightarrow X_1^e \overset{j_1}{\hookrightarrow} \cdots \hookrightarrow X_n^e \overset{j_n}{\hookrightarrow} X_n^+ \hookrightarrow X_{n+1}^e \overset{j_{n+1}}{\hookrightarrow} \cdots$$

where X_0^e consists of essential 0-cells, X_n^+ is obtained from X_n^e by adjoining redundant n-cells and collapsible $(n+1)$-cells, and X_{n+1}^e is obtained from X_n^+ by adjoining essential $(n+1)$-cells. The description of this filtration is the key of the proof.
After CS2, if R_n is not empty, there exist redundant n-simplices whose height is 1. Let τ be one of them. Clearly the faces of τ are $(n-1)$-cells and so we can adjoin it to X_n^e obtaining the space $X_n^e(\tau)$. Let $c_n(\tau)$ be the collapsible of τ. Observe that the faces of $c_n(\tau)$ are $\tau \in X_n^e(\tau)$ and either essential or collapsible or degenerate n-cells which are in X_n^e. This means that X_n^e is a strong deformation retract of the space $X_n^e(\tau, c_n(\tau))$ obtained from $X_n^e(\tau)$ by adjoining $c_n(\tau)$. Chosen a retraction (collapsing map) from the geometric simplex Δ^{n+1} onto its boundary deprived of the open $\iota(\tau)$-face $\Lambda^{n+1}_{\iota(\tau)}$, this induces an elementary retraction $X_n^e(\tau, c_n(\tau)) \to X_n^e$ which collapses the $(n+1)$-cell $c_n(\tau)$ onto its boundary deprived of the open free face τ°. Since $c_n : R_n \to C_{n+1}$ is a bijection, X_n^e is a deformation retract of the space $X_n^{e,1}$ obtained from X_n^e by adjoining all the redundant n-cells with height one and the corresponding collapsible $(n+1)$-cells. Gluing the elementary retraction, we get a homotopy inverse of the inclusion $X_n^e \hookrightarrow X_n^{e,1}$.
Let $X_n^{e,k}$ be the space obtained from X_n^e by attaching all the redundant n-cells with height less or equal to k and the corresponding collapsible cells. Observe that, given a redundant τ' with height $k+1$, every face of its collapsible $c_n(\tau')$ other than its free face τ' is either an immediate predecessor of τ' (hence with height less or equal to k) or else is essential, collapsible or degenerate. These faces are therefore already present in $X_n^{e,k}$ and we can proceed as above.
We have factored the inclusion $j_n : X_n^e \hookrightarrow X_n^+$ as a sequence of strong deformation retracts

$$X_n^e \subseteq X_n^{e,1} \subseteq X_n^{e,2} \subseteq \cdots \subseteq X_n^{e,k} \subseteq X_n^{e,k+1} \subseteq \cdots$$

Observe that, fixed for any $i \in [n+1]$ a collapsing map $\Delta^{n+1} \to \Lambda_i^{n+1}$, retractions $X_n^{e,k+1} \to X_n^{e,k}$ are canonically given. This proves that the cofibration j_n is a homotopy equivalence with a *canonical* homotopy inverse r_n.

$$\begin{tikzcd} X_n^e \arrow[r, hook, shift left, "j_n"] & X_n^+ \arrow[l, shift left, "r_n"] \end{tikzcd}$$

We finally consider the diagram

$$\begin{tikzcd} X_0^e \arrow[r, hook, "j_0"] \arrow[d, equal] & X_0^+ \arrow[r, hook] \arrow[d, "r_0"'] & X_1^e \arrow[r, hook, "j_1"] \arrow[d, "\overline{r_0}"'] & X_1^+ \arrow[r, hook] \arrow[d, "\overline{r_0}r_1"'] & X_2^e \arrow[r, hook, "j_2"] \arrow[d, "\overline{\overline{r_0}r_1}"'] & X_2^e \arrow[r, hook] \arrow[d, "\overline{\overline{r_0}r_1}r_2"'] & \cdots \\ \mathcal{E}_0 \arrow[r, equal] & \mathcal{E}_0 \arrow[r, hook] & \mathcal{E}_1 \arrow[r, equal] & \mathcal{E}_1 \arrow[r, hook] & \mathcal{E}_2 \arrow[r, equal] & \mathcal{E}_2 \arrow[r, hook] & \cdots \end{tikzcd}$$

where $\mathcal{E}_0 = X_0^e$, the space $\mathcal{E}_{n+1}$ is the push-out $\mathcal{E}_n +_{X_n^+} X_{n+1}^e$ and overlined maps are induced by push-outs; horizontal maps are cofibration, while vertical maps are homotopy equivalences. The colimit of this diagram gives us the required canonical projection $q_X : |X| \to \mathcal{E}(X)$. ∎

2.3. Example. Consider the category $\mathbf{C}$ that we picture below without drawing composites and identities

$$\begin{tikzcd} x & y \arrow[l, bend right, "a"'] \arrow[l, bend left, "c"] & z \arrow[l, bend right, "b"'] \arrow[l, bend left, "d"] \end{tikzcd}$$

The classifying space $B\mathbf{C}$ of $\mathbf{C}$ is a CW-complex whose cells are in 1-1 correspondence with non-degenerate simplices of the nerve $N\mathbf{C}$:

$$(N\mathbf{C})_0^{nd} = \{x, y, z\}\,, \qquad (N\mathbf{C})_1^{nd} = \{a, b, c, d, ab, ad, cb, cd\}\,,$$
$$(N\mathbf{C})_2^{nd} = \{(a, b), (a, d), (c, b), (c, d)\}\,, \qquad (N\mathbf{C})_n^{nd} = \emptyset \quad (n \geq 3),$$

then $B\mathbf{C}$ is a space which has surely the same homotopy type as $S^1 \vee S^1$. The following collapsing scheme makes this intuition rigorous:

	E_n	R_n	$\xrightarrow{c_n}$	C_{n+1}
$n = 0$	z	y	$\mapsto$	d
		x	$\mapsto$	c
$n = 1$	$a,\ b$	fg	$\mapsto$	(f, g)
$n \geq 2$	-	-	-	-

where $f \in \{a, c\}$, $g \in \{b, d\}$ and $\iota_n = 1$ (constant function) for every n. Now we can state that $B\mathbf{C} \sim S^1 \vee S^1$, since it has the same homotopy type as a CW-complex with one 0-cell and two 1-cells.

Observe that identifications need some care when we want to construct the geometric realization of a simplicial set starting from a collapsing scheme; even in this simple example (with the obvious meaning of symbols) $d_1 a = x$ is not essential, but it is identified with z collapsing first c over y and then d over x.

2.4. Example. *Any (small) category with terminal object is contractible.*

Let $\mathbf{C}$ be a category with a terminal object $*$ (i.e. for any object $x \in \mathbf{C}$, there exists a unique arrow $x \to *$). A collapsing scheme $(E, R, C; c, \iota)$ on the nerve of $\mathbf{C}$ is defined by $E_0 = \{*\}$, $R_0 = \mathrm{Obj}\mathbf{C} - \{*\}$ and, for $n \geq 1$, $E_n = \emptyset$, $R_n = \{x_0 \xleftarrow{f_1} \cdots \xleftarrow{f_n} x_n : x_0 \neq *\}$, $C_n = \{* \leftarrow x_1 \xleftarrow{f_2} \cdots \xleftarrow{f_n} x_n\}$ with $c_n(x_0 \xleftarrow{f_1} \cdots \xleftarrow{f_n} x_n) = * \leftarrow x_0 \xleftarrow{f_1} \cdots \xleftarrow{f_n} x_n$ and $\iota_n = 0$ (constant function). Clearly c_n is a bijection and any redundant n-simplex τ has height $h(\tau) = 1$ since for $i \neq 0$, $d_i c(\tau)$ is either collapsible or degenerate. CS1 and CS2 are verified and $B\mathbf{C} \sim \mathcal{E}(N\mathbf{C})$ is contractible.

Simplicial sets endowed with a collapsing scheme are the objects of the category $\mathcal{C}\mathbf{Set}^{\mathbf{\Delta}^{op}}$ whose arrows are $\mathcal{C}$-morphisms defined as follows.

Writing $\mathcal{C}^X = (E^X, R^X, C^X; c^X, \iota^X)$ for a collapsing scheme on the simplicial sets X, and writing $f|_S$ for the graded function obtained from a simplicial function $f : X \to Y$ by restricting it to the graded subset S of X, we say that

2.5. Definition. A *$\mathcal{C}$-morphism* $f : \mathcal{C}^X \to \mathcal{C}^Y$ is a simplicial function $f : X \to Y$ such that

1. f respects the partitions, i.e. $\mathrm{Im} f|_{E^X} \subseteq E^Y$, $\mathrm{Im} f|_{R^X} \subseteq R^Y$, $\mathrm{Im} f|_{C^X} \subseteq C^Y$;
2. f commutes with c and ι , i.e. $\forall x \in R^X$, $f c^X(x) = c^Y f(x)$, $\iota^Y f(x) = \iota^X(x)$.

We are going to show the functoriality of the construction $\mathcal{E} : \mathcal{C}\mathbf{Set}^{\mathbf{\Delta}^{op}} \to \mathbf{Top}$ and the naturality of the canonical projection $q : |\ \_\ | \Rightarrow \mathcal{E}$ (where the geometric realization functor is defined from $\mathcal{C}\mathbf{Set}^{\mathbf{\Delta}^{op}}$ by composing it with the functor which forgets collapsing schemes). We first need to state some lemmas.

2.6. Lemma. *Any $\mathcal{C}$-morphism $f : \mathcal{C}^X \to \mathcal{C}^Y$ preserves the immediate predecessor relation and the height of redundant simplices.* ∎

Going back to the construction introduced in the proof of Theorem 2.2, we emphasize the following observation.

2.7. Lemma. *Let $\mathcal{C}^X$ and $\mathcal{C}^Y$ be collapsing schemes for the simplicial sets X and Y. If $f : X \to Y$ is a simplicial function such that $\mathrm{Im} f|_{E^X} \subseteq E^Y \cup D^Y$ and $\mathrm{Im} f|_{C^X} \subseteq C^Y \cup D^Y$, then f gives rise to the following commutative diagram of spaces and maps, where f_n^e and f_n^+ are restrictions of the geometric realization $|f|$ of f.*

$$\begin{tikzcd}
X_0^e \arrow[r, hook, "j_0^X"] \arrow[d, "f_0^e"'] & X_0^+ \arrow[r, hook] \arrow[d, "f_0^+"'] & X_1^e \arrow[r, hook, "j_1^X"] \arrow[d, "f_1^e"'] & \cdots \arrow[r, hook] & X_n^e \arrow[r, hook, "j_n^X"] \arrow[d, "f_n^e"'] & X_n^+ \arrow[r, hook] \arrow[d, "f_n^+"'] & X_{n+1}^e \arrow[r, hook, "j_{n+1}^X"] \arrow[d, "f_{n+1}^e"'] & \cdots \\
Y_0^e \arrow[r, hook, "j_0^Y"'] & Y_0^+ \arrow[r, hook] & Y_1^e \arrow[r, hook, "j_1^Y"'] & \cdots \arrow[r, hook] & Y_n^e \arrow[r, hook, "j_n^Y"'] & Y_n^+ \arrow[r, hook] & Y_{n+1}^e \arrow[r, hook, "j_{n+1}^Y"'] & \cdots
\end{tikzcd}$$

The colimit of the diagram is $|f| : |X| \to |Y|$. ∎

For any cofibration $j_n^X : X_{n+1}^e \hookrightarrow X_n^+$, we have introduced a *canonical* retract $r_n^X : X_n^+ \to X_{n+1}^e$. A $\mathcal{C}$-morphism verifies the hypotheses of Lemma 2.7 and

2.8. Lemma. *Any $\mathcal{C}$-morphism $f : \mathcal{C}^X \to \mathcal{C}^Y$ commutes with r, i.e. $f_n^e r_n^X = r_n^Y f_n^+$ for any n.* ∎

We are now ready to draw the following diagram which arises from a $\mathcal{C}$-morphism $f : \mathcal{C}^X \to \mathcal{C}^Y$.

$$
\begin{tikzcd}
 & \mathcal{E}_0^X \arrow[rr, equal] \arrow[dd, "e_0"] & & \mathcal{E}_0^X \arrow[rr, hook] \arrow[dd, "e_0"] & & \mathcal{E}_1^X \arrow[rr, equal] \arrow[dd, "e_1"] & & \mathcal{E}_1^X \arrow[rr, hook] \arrow[dd, "e_1"] & & \mathcal{E}_2^X \arrow[dd, "e_2"] & \cdots \\
X_0^e \arrow[ur, equal] \arrow[rr, hook, "j_0^X"] \arrow[dd, "f_0^e"] & & X_0^+ \arrow[ur, "r_0^X"] \arrow[rr, hook] \arrow[dd, "f_0^+"] & & X_1^e \arrow[ur, "\overline{r_0^X}"] \arrow[rr, hook, "j_1^X"] \arrow[dd, "f_1^e"] & & X_1^+ \arrow[ur, "\overline{r_0^X} r_1^X"] \arrow[rr, hook] \arrow[dd, "f_1^+"] & & X_2^e \arrow[ur, "\overline{\overline{r_0^X} r_1^X}"] \arrow[dd, "f_2^e"] & \cdots & \\
 & \mathcal{E}_0^Y \arrow[rr, equal] & & \mathcal{E}_0^Y \arrow[rr, hook] & & \mathcal{E}_1^Y \arrow[rr, equal] & & \mathcal{E}_1^Y \arrow[rr, hook] & & \mathcal{E}_2^Y & \cdots \\
Y_0^e \arrow[ur, equal] \arrow[rr, hook, "j_0^Y"'] & & Y_0^+ \arrow[ur, "r_0^Y"] \arrow[rr, hook] & & Y_1^e \arrow[ur, "\overline{r_0^Y}"] \arrow[rr, hook, "j_1^Y"'] & & Y_1^+ \arrow[ur, "\overline{r_0^Y} r_1^Y"] \arrow[rr, hook] & & Y_2^e \arrow[ur, "\overline{\overline{r_0^Y} r_1^Y}"] & \cdots &
\end{tikzcd}
\tag{1}
$$

The upper and the lower parts are the constructions for $\mathcal{C}^X$ and $\mathcal{C}^Y$ described in the proof of Theorem 2.2. Defined the map $e_0 : \mathcal{E}_0^X \to \mathcal{E}_0^Y$ as $e_0 := f_0^e : X_0^e \to Y_0^e$, the first cube (the one with index zero) commutes. The map $e_1 : \mathcal{E}_1^X \to \mathcal{E}_1^Y$ is induced by the push-outs which are the upper and the lower squares of the second cube. So the third cube (with index one) commutes and we can iterate. Thus we obtain the commutative diagram

$$
\begin{tikzcd}
\mathcal{E}_0^X \arrow[r, hook] \arrow[d, "e_0"'] & \mathcal{E}_1^X \arrow[r, hook] \arrow[d, "e_1"'] & \mathcal{E}_2^X \arrow[r, hook] \arrow[d, "e_2"'] & \mathcal{E}_3^X \arrow[r, hook] \arrow[d, "e_3"'] & \cdots \\
\mathcal{E}_0^Y \arrow[r, hook] & \mathcal{E}_1^Y \arrow[r, hook] & \mathcal{E}_2^Y \arrow[r, hook] & \mathcal{E}_3^Y \arrow[r, hook] & \cdots
\end{tikzcd}
$$

whose colimit is a map $\mathcal{E}f : \mathcal{E}^X \to \mathcal{E}^Y$.

2.9. Proposition. *The assignment* $f \rightsquigarrow \mathcal{E}f$ *defines a functor*

$$\mathcal{E} : \mathcal{C}\mathbf{Set}^{\mathbf{\Delta}^{op}} \to \mathbf{Top}$$

and the canonical projection $q_X : |X| \to \mathcal{E}^X$ *is a natural transformation*

$$q : |\_| \Rightarrow \mathcal{E} : \mathcal{C}\mathbf{Set}^{\mathbf{\Delta}^{op}} \to \mathbf{Top} \, .$$

Proof. The functoriality of $\mathcal{E}$ comes from standard properties of colimits (i.e. the functoriality of coLim). The naturality of the canonical projection q arises from the diagram

$$
\begin{tikzcd}
|X| \arrow[r, "q_X"] \arrow[d, "|f|"'] & \mathcal{E}^X \arrow[d, "\mathcal{E}f"] \\
|Y| \arrow[r, "q_Y"'] & \mathcal{E}^Y
\end{tikzcd}
$$

obtained as colimit of diagram (1). ∎

3. Term rewriting for categories

In this section we recall definitions and basic facts on term rewriting for categories.

3.1. Graphs, words and free categories. An *n-graph* G consists of $n+1$ sets G_i and n pairs of functions $s_i, t_i : G_{i+1} \to G_i$ such that $s_{i-1}s_i = s_{i-1}t_i$ and $t_{i-1}s_i = t_{i-1}t_i$ for every $i < n$.

$$G \;:\; G_0 \overset{t_0}{\underset{s_0}{\leftleftarrows}} G_1 \overset{t_1}{\underset{s_1}{\leftleftarrows}} G_2 \overset{t_2}{\underset{s_2}{\leftleftarrows}} \cdots \overset{t_{n-1}}{\underset{s_{n-1}}{\leftleftarrows}} G_n$$

Elements of G_0 are called *objects* (or *vertices* or *points*); elements of G_1 are called *arrows* (or *edges*); elements of G_i are called *i-arrows*. The functions s and t are called *source* and *target*. We can picture an arrow $f \in G_1$ as $t_0(f) \xleftarrow{f} s_0(f)$. A typical picture of a 2-arrow $\alpha \in G_2$ will be

$$\begin{tikzcd} x & y \arrow[l, bend right, "f"'] \arrow[l, bend left, "g"] \arrow[l, phantom, "\Downarrow\alpha"] \end{tikzcd}$$

where $f, g \in G_1$ and $x, y \in G_0$ with $f = s_1\alpha,\ g = t_1\alpha$ and $x = t_0f = t_0g = t_0s_1(\alpha) = t_0t_1(\alpha),\ y = s_0f = s_1g = s_0s_1(\alpha) = s_0t_1(\alpha)$.

A morphism $\varphi : G \to G'$ of n-graphs is an $(n+1)$-tuple $(\varphi_i)_0^n$ of functions $\varphi_i : G_i \to G'_i$ such that $\varphi_i t_i = t_i\varphi_{i+1}$ and $\varphi_i s_i = s_i\varphi_{i+1}$ for every $i < n$. The category **n-Graph** consists of n-graphs and morphisms of n-graphs. Observe that **0-Graph** is the category of sets, while **1-Graph** is the category **Graph** of graphs.

We can get a graph by truncating an n-graph G; in particular, we shall write $\mathrm{sk}_{i,i+1}G$ for the graph

$$\mathrm{sk}_{i,i+1}G \;:\; G_i \overset{t_i}{\underset{s_i}{\leftleftarrows}} G_{i+1} \qquad 0 \le i < n$$

A pair of adjoint functors links the category of graphs and the category of small categories. The forgetful functor $U : \mathbf{Cat} \to \mathbf{Graph}$ assigns the *underlying graph* $\mathbf{C} : C_0 \leftleftarrows C_1$ to the small category $\mathbf{C}$, where C_0 is the set of objects, C_1 is the set of arrows of $\mathbf{C}$, source and target are domain and codomain. The left adjoint $F : \mathbf{Graph} \to \mathbf{Cat}$ of U assigns the *free category* FG to a graph G. Objects of FG are the objects of G, while arrows of FG are words in G, described as follows. Given a graph $G \;:\; G_0 \overset{t}{\underset{s}{\leftleftarrows}} G_1$, a *word* (or *path*) w of length $l(w) = m$ is an alignment $w = (x_0, f_1, x_1, f_2, \cdots, f_m, x_m) = (f_1, f_2, \cdots, f_m)$ where $x_* \in G_0,\ f_* \in G_1,\ x_0 = t(f_1),\ s(f_i) = x_i = t(f_{i+1})\ (1 \le i < m)$, $x_m = s(f_m)$.

$$w \;=\; x_0 \xleftarrow{f_1} x_1 \xleftarrow{f_2} \cdots \xleftarrow{f_m} x_m \;=\; \xleftarrow{f_1}\ \xleftarrow{f_2}\ \cdots \xleftarrow{f_m}$$

An *empty word* w is a word of length zero; this means that $w = (x_0)$ and there exits exactly one empty word for every object of G_0. A word of length one (that is an element of G_1) may also be called *letter* and G_1 may be called the *alphabet*. We thus obtain a new graph (the underlying graph of the category FG)

$$FG \;:\; G_0 \overset{t}{\underset{s}{\leftleftarrows}} FG_1$$

where FG_1 is the set of words of finite length and s and t are defined by $s(f_1, f_2, \ldots, f_m) = s(f_m)$ and $t(f_1, f_2, \ldots, f_m) = t(f_1)$. Two words w and w' are composable when $s(w) = t(w')$ and their composition $w \circ_{FG} w'$ is the concatenation ww'

$$w \circ_{FG} w' = (f_1, \ldots, f_m) \circ_{FG} (g_1, \ldots, g_k) = (f_1, \ldots, f_m, g_1, \ldots, g_k) = ww'.$$

Empty words are the identities. Observe that a letter $f \in G_1$ is a word of length one in FG_1, thus a word w can be written as

$$w = (f_1, f_2, \ldots, f_m) = f_1 f_2 \cdots f_m = f_1 \circ_{FG} f_2 \circ_{FG} \cdots \circ_{FG} f_m .$$

A word w' is a *sub-word* of a word w ($w' \subseteq w$) when $w = uw'v$ (u and v words). A non-empty word w' is a *proper sub-word* of w ($w' \subset w$) when either u or v is not empty. A non-empty word w' is an *initial sub-word* of w when $w = w'v$. Observe that if $w = (x_0, f_1, x_1, f_2, \cdots, f_m, x_m)$, then any empty word (x_i) with $0 \leq i \leq m$ is a sub-word of w.

3.2. Rewrite systems. A *rewrite system* $\mathcal{R} = (G, R)$ is a 2-graph such that $\mathrm{sk}_{01}\mathcal{R}$ is the free category FG over the graph G

$$\mathcal{R} \;:\; G_0 \underset{s_0}{\overset{t_0}{\leftleftarrows}} FG_1 \underset{s_1}{\overset{t_1}{\leftleftarrows}} R$$

R is called the set of *rewrite rules.* A morphism of rewrite systems $\varphi : \mathcal{R} \to \mathcal{R}'$ is a pair $\varphi = (\varphi_G, \varphi_R)$ where $\varphi_G = (\varphi_0, \varphi_1) : G \to G'$ is a morphism of graphs and $\varphi_R : R \to R'$ is a function such that $(F\varphi_G, \varphi_R)$ is a morphism of 2-graphs. We write **RS** for the category of rewrite systems.

The *whiskering* $w\mathcal{R}$ of a rewrite system $\mathcal{R} = (G, R)$ is the 2-graph

$$w\mathcal{R} \;:\; G_0 \underset{s_0}{\overset{t_0}{\leftleftarrows}} FG_1 \underset{s_1'}{\overset{t_1'}{\leftleftarrows}} wR$$

where $wR = \{(u, \alpha, v) \in FG_1 \times R \times FG_1 : s_0(u) = t_0 t_1(\alpha),\ s_0 s_1(\alpha) = t_0(v)\}$ is the set of *reductions* (or *applications of rewrite rules*, or *elementary derivations*, or *whiskered* 2-*arrows*). Source and target are defined by $s_1'(u, \alpha, v) = u(s_1\alpha)v$ and $t_1'(u, \alpha, v) = u(t_1\alpha)v$. A whiskered 2-arrow (u, α, v) may be viewed as the "horizontal composition" of α with u (on left) and v (on right).

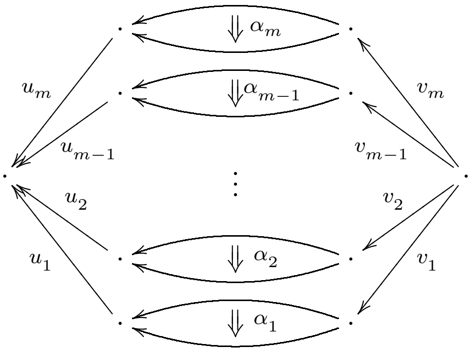
$$\begin{tikzcd} x & y \arrow[l, bend right, "u(s_1\alpha)v"'] \arrow[l, bend left, "u(t_1\alpha)v"] \arrow[l, phantom, "\Downarrow(u,\alpha,v)"] \end{tikzcd} \quad = \quad \begin{tikzcd} x & y \arrow[l, "u"'] & z \arrow[l, bend right, "s_1\alpha"'] \arrow[l, bend left, "t_1\alpha"] \arrow[l, phantom, "\Downarrow\alpha"] & t \arrow[l, "v"'] \end{tikzcd}$$

When no ambiguity occurs, we write $u\alpha v$ (or also $u \circ_{FG} \alpha \circ_{FG} v$) instead of (u, α, v), while, given a reduction $p = (u, \alpha, v)$, we write $u'pv'$ (or also $u' \circ_{FG} p \circ_{FG} v'$) for $(u'u, \alpha, vv')$.

The *derivation scheme* $d\mathcal{R}$ of a rewrite system $\mathcal{R} = (G, R)$ is the 2-graph

$$d\mathcal{R} \;:\; G_0 \underset{s_0}{\overset{t_0}{\leftleftarrows}} FG_1 \underset{s_1''}{\overset{t_1''}{\leftleftarrows}} dR$$

where dR is the set of paths of elementary derivations (called *chains of reductions* or simply *derivations*).



A derivation is then a sequence $\big((u_1, \alpha_1, v_1), (u_2, \alpha_2, v_2), \cdots, (u_m, \alpha_m, v_m)\big)$ of elementary derivations $(u, \alpha, v) \in wR$ such that

$$u_i(s_1\alpha_i)v_i = s_1'(u_i, \alpha_i, v_i) = t_1'(u_{i+1}, \alpha_{i+1}, v_{i+1}) = u_{i+1}(t_1\alpha_{i+1})v_{i+1}.$$

Source and target of a derivation are defined by

$$\begin{aligned} s_1''\big((u_1, \alpha_1, v_1), \cdots, (u_m, \alpha_m, v_m)\big) &= s_1'(u_m, \alpha_m, v_m) = u_m(s_1\alpha_m)v_m \\ t_1''\big((u_1, \alpha_1, v_1), \cdots, (u_m, \alpha_m, v_m)\big) &= t_1'(u_1, \alpha_1, v_1) = u_1(t_1\alpha_1)v_1. \end{aligned}$$

Observe that $\mathrm{sk}_{12}(d\mathcal{R})$ is the underlying graph of the category $F\mathrm{sk}_{12}(w\mathcal{R})$, when we admit that a derivation can be empty.

A rewrite system $\mathcal{R}$ is said to be

- *confluent* when, for any pair of derivations $p : w \Rightarrow u$, $p' : w \Rightarrow u'$ in dR, there exist chains of reductions $q : u \Rightarrow w'$, $q' : u' \Rightarrow w'$ in dR;
- *terminating* when there is no infinite chain of reductions;
- *complete* when it is both confluent and terminating.

Before listing some basic facts on rewrite systems, we recall that a word is *irreducible* when it is not the source of any reduction.

- Sub-words of irreducible words are irreducible.
- Empty sub-words of an irreducible word are irreducible.
- In a complete rewrite system $\mathcal{R} = (G, R)$, any word $w \in FG_1$ can be reduced to a unique irreducible $\overline{w} \in FG_1$ by a (not necessarily unique) derivation.
- In a complete rewrite system, any empty word is irreducible.

Complete rewrite systems are the objects of the category **CRS**, whose arrows are those morphisms of rewrite systems which send irreducible words to irreducible words. Observe that **CRS** is not a full subcategory of **RS**.

3.3. Categories of normal forms and presentations of categories. Given a rewrite system $\mathcal{R} = (G, R)$, and its derivation scheme $d\mathcal{R}$, one usually defines a relation P on the set FG_1 of words by saying that $(w, w') \in P$ when there exists a derivation form w to w'. The quotient $FG_1/\widetilde{P}$ of FG_1 modulo the equivalence relation $\widetilde{P}$ generated by P is the set $\pi_0(\mathrm{sk}_{12}d\mathcal{R})$ of connected components of $\mathrm{sk}_{12}d\mathcal{R}$. An element of the set $\pi_0(\mathrm{sk}_{12}(d\mathcal{R}))$ is therefore a class $[w]$ of words belonging to FG_1, and $\tilde{w} \in [w]$ if and only if then there exist reductions (not necessarily in the same direction) connecting $\tilde{w}$ and w (e.g. $\ w \Rightarrow\Leftarrow \cdots \Rightarrow\Rightarrow\Leftarrow \tilde{w}\ $). Since a reduction has been defined as a 2-arrow of a 2-graph, all the words in an equivalence class have the same source and target. Writing $s_0[w] = s_0(w)$ and $t_0[w] = t_0(w)$, we obtain the graph

$$Q\mathcal{R} \ : \ G_0 \overset{t_0}{\underset{s_0}{\leftleftarrows}} \pi_0\big(\mathrm{sk}_{12}(d\mathcal{R})\big).$$

If $p : w \Rightarrow \tilde{w}$ is a reduction and a word w' is composable on right (or on left) with w, then we have a reduction $pw' : ww' \Rightarrow \tilde{w}w'$ (or $w'p : w'w \Rightarrow w'\tilde{w}$). This means that a composition law is well defined by $[w] \circ_{Q\mathcal{R}} [w'] = [w \circ_{FG} w'] = [ww']$ and the graph $Q\mathcal{R}$ is indeed a category: the *quotient category* of the rewrite system $\mathcal{R} = (G, R)$. It is well-known that Q is a functor $Q : \mathbf{RS} \to \mathbf{Cat}$.

A presentation of a (small) category $\mathbf{C}$ is a rewrite system $\mathcal{R}$ such that the quotient category $Q\mathcal{R}$ is isomorphic to $\mathbf{C}$.

From $Q\mathcal{R}$ we obtain the graph of normal forms by choosing a representative for any class $[w]$ of equivalent words. When the rewrite system $\mathcal{R}$ is complete, there exists a canonical choice. Let IFG_1 be the subset of FG_1 consisting of irreducible words of the complete rewrite system $\mathcal{R} = (G, R)$ and let $\bar{q} : IFG_1 \to \pi_0\,(\mathrm{sk}_{12}(d\mathcal{R}))$ be the restriction to IFG_1 of the quotient function $q : FG_1 \to \pi_0(\mathrm{sk}_{12}(d\mathcal{R}))$, $q(w) = [w]$. Remember now that the completeness of $\mathcal{R}$ gives us a function $r : FG_1 \to IFG_1$ which assigns to a word $w \in FG_1$ the corresponding irreducible word $r(w) = \overline{w}$. It is easy to show that $\bar{q}$ is a bijection whose inverse is (well-defined by) $\bar{r}[w] = r(w)$. The *graph of normal forms* $I\mathcal{R}$ of a complete rewrite system $\mathcal{R}$

$$I\mathcal{R} \ : \ G_0 \overset{t_0}{\underset{s_0}{\leftleftarrows}} IFG_1$$

is therefore isomorphic to the graph $Q\mathcal{R}$. The concatenation of irreducible words is not, a priori, an irreducible word, but we can introduce a composition in $I\mathcal{R}$ by defining $w \circ_{I\mathcal{R}} w' = r(ww') = \overline{ww'}$. This makes $\bar{q}$ and $\bar{r}$ isomorphisms between the quotient category $Q\mathcal{R}$ and the *category of normal forms* $I\mathcal{R}$. Therefore the category $I\mathcal{R}$ comes from $Q\mathcal{R}$ by choosing, for each class $[w]$, the unique irreducible word $\overline{w} \in [w]$ as normal form and defining a composition such that $I\mathcal{R}$ results isomorphic to the category $Q\mathcal{R}$. In this way we have defined a functor $I : \mathbf{CRS} \to \mathbf{Cat}$.

A complete presentation of a (small) category $\mathbf{C}$ is a rewrite system $\mathcal{R}$ such that the category of normal forms $I\mathcal{R}$ is isomorphic to $\mathbf{C}$.

We observed that in a complete rewrite system any object is irreducible, when we think of it as a word of length zero. A priori it is not true that any (1-)arrow is irreducible, but one can easily show that "reducible letters can always be deleted from the alphabet". In fact, given a complete rewrite system $\mathcal{R}$, there exists a complete rewrite system $\mathcal{R}^* = (G^*, R^*)$ such that $I\mathcal{R} = I\mathcal{R}^*$ and $G_1^* \subseteq IFG_1^*$. For this reason one can consider only complete rewrite systems where every letter is irreducible.

4. Collapsing schemes and rewrite systems

We are going to define a canonical collapsing scheme on the nerve $NI\mathcal{R}$ of the category of normal forms $I\mathcal{R}$ of a complete rewrite system $\mathcal{R}$. When $\mathcal{R}$ is a complete presentation of a small category $\mathbf{C}$, then the collapsed space $\mathcal{E}(NI\mathcal{R})$ is (up to homotopy) the classifying space $B\mathbf{C}$ of $\mathbf{C}$. It is useful to remark now that, given two irreducible words w and w', we will use both the notations ww' and $w \circ w'$, but with different meanings: $ww' = w \circ_{FG} w'$ is the concatenation in FG, while $w \circ w' = w \circ_{I\mathcal{R}} w'$ is the composition in $I\mathcal{R}$. In particular, the nerve of $I\mathcal{R}$ is defined by

$$(NI\mathcal{R})_0 = G_0\,, \quad (NI\mathcal{R})_n = \{(w_1, \ldots, w_n) : w_i \in IFG_1,\ s_0 w_i = t_0 w_{i+1}\}$$

$$d_i(w_1, \cdots, w_n) = \begin{cases} (w_2, \ldots, w_n) & i = 0 \\ (w_1, \ldots, w_i \circ w_{i+1}, \ldots, w_n) & 0 < i < n \\ (w_1, \ldots, w_{n-1}) & i = n \end{cases}$$

$$s_i(w_1, \ldots, w_n) = (w_1, \ldots, w_i, \mathrm{id}, w_{i+1}, w_n)$$

where id is the empty word $\mathrm{id} = (s_0 w_i) = (t_0 w_{i+1})$.

Here is the construction of the canonical collapsing scheme of $NI\mathcal{R}$.

- Any 0-simplex is essential, that is $E_0 = G_0$ and $R_0 = C_0 = \emptyset$.
- An n-simplex $\tau = (w_1, \ldots, w_n)$ is essential when it satisfies the following three conditions:

ES1. $w_1 \in G_1 \cap IFG_1$ (i.e. w_1 is an irreducible letter);

ES2. $w_i w_{i+1}$ $(1 \le i < n)$ is reducible;

ES3. the sub-word of $w_i w_{i+1}$ $(1 \le i < n)$ obtained by deleting the last letter is irreducible.

When a non-degenerate τ is not essential, we may have

1. $w_1 \notin G_1 \cap IFG_1$. Since the irreducible word w_1 has length $l(w_1) \ge 2$, we can write $w_1 = g w_1'$ with $g \in G_1 \cap IFG_1$. Now $\tau = (g w_1', w_2, \ldots, w_n)$ is redundant with $c_n(\tau) = (g, w_1', w_2, \ldots, w_n)$ and $\iota_n(\tau) = 1$;
2. $w_1 \in G_1 \cap IFG_1$. Let e be the largest index such that $(w_1, \ldots, w_{e-1})$ is essential (we will call this $(e-1)$-simplex the *essential front face* of τ).

2.1. If $w_{e-1}w_e$ is irreducible, then τ is collapsible.

2.2. If $w_{e-1}w_e$ is reducible, then some proper initial sub-word of $w_{e-1}w_e$ must be reducible (otherwise $(w_1, \ldots, w_{e-1}, w_e)$ would be essential). τ is redundant with $c_n(\tau) = (w_1, \ldots, w_{e-1}, w'_e, w''_e, w_{e+1}, \ldots, w_n)$ and $\iota_n(\tau) = e$ where w'_e is the smallest initial sub-word of w_e such that $w_{e-1}w'_e$ is reducible.

4.1. Proposition. *The above data define a collapsing scheme on the nerve $N(I\mathcal{R})$ of the category of normal forms of a complete rewrite system $\mathcal{R}$.*

Proof. Axiom CS1 is trivially checked.

Given a simplex $\tau = (w_1, w_2, \ldots, w_n)$, the *word of* τ is defined as $w(\tau) = w_1 w_2 \cdots w_n$. We prove that CS2 holds, showing that each descending chain $\tau_1 >_{R_n} \tau_2 >_{R_n} \cdots$ of redundant simplices gives rise to a chain of words $w(\tau_1) \succeq w(\tau_2) \succeq \cdots$; the two chains have the same length, but the second one is necessarily finite.

We introduce the relation $>$ on the set FG_1 of words saying that $w > w'$ if and only if either $w \supset w'$ (i.e. w' is a proper sub-word of w) or there exists a reduction $q \in wR$ such that $q : w \Rightarrow w'$. Observe that we have

$$w_1 \supset w_2 \xRightarrow{q} w_3 \qquad \text{if and only if} \qquad w_1 = uw_2v \xRightarrow{uqv} uw_3v \supset w_3 \ ;$$

in any chain $w > w_1 > w_2 > \cdots$, we can thus move the symbols $\Rightarrow$ on the left and obtain a chain of the same length

$$w_1 \Rightarrow w'_2 \Rightarrow \cdots \Rightarrow w'_k \supset w'_{k+1} \supset \cdots \supset w'_{k+h} \qquad (k, h \leq +\infty)\,.$$

Since the rewrite system $\mathcal{R}$ is terminating, there is no infinite chain of reductions and k must be finite. Any word $w \in FG_1$ has finite length, so h is finite too. The relation $>$ and its transitive closure $\succ$ are terminating.

Let τ and τ' be two redundant n-simplices with $\tau >_{R_n} \tau'$; if $\sigma = c_n(\tau) = (w_1, \ldots, w_{n+1})$, then, for an index $j \neq \iota_n(\tau)$,

$$\tau = d_{\iota_n(\tau)}(\sigma) = (w_1, \ldots, w_{i-1}, w_i w_{i+1}, w_{i+2}, \ldots, w_{n+1})\ ,$$
$$\tau' = d_j(\sigma) = \begin{cases} (w_2, \ldots, w_{n+1}) & j = 0, \\ (w_1, \ldots, w_j \circ w_{j+1}, \ldots, w_{n+1}) & 0 < j < n+1, \\ (w_1, \ldots, w_n) & j = n+1. \end{cases}$$

If $j = 0$ or $j = n+1$, then $w(\tau) \supset w(\tau')$ and $w(\tau) \succ w(\tau')$. If $0 < j < n+1$ and $w_j w_{j+1}$ is reducible, there exists a chain of reductions $q : w_j w_{j+1} \Rightarrow\Rightarrow \cdots \Rightarrow w_j \circ w_{j+1}$. So $(w_1 \cdots w_{j-1}) q (w_{j+2} \cdots w_{n+1})$ is a derivation $w_1 \cdots w_{j-1}(w_j w_{j+1}) w_{j+2} \cdots w_{n+1} \Rightarrow w_1 \cdots w_{j-1}(w_j \circ w_{j+1}) w_{j+2} \cdots w_{n+1}$ and $w(\tau) \succ w(\tau')$.

If $0 < j < n+1$ and $w_j w_{j+1}$ is irreducible, we have $w(\tau) = w(\tau')$. Observe that, since $(w_1, \ldots, w_{i-1}, w_i)$ is the essential front face of σ, we may have $w_j w_{j+1}$ irreducible only if $i < j < n+1$ and thus the essential front face of τ' has at least dimension i, while the one of τ is $(w_1, \ldots, w_{i-1})$.

From a descending chain $\tau_1 >_{R_n} \tau_2 >_{R_n} \cdots$ of redundant simplices, we get a chain of words $w(\tau_1) \succeq w(\tau_2) \succeq \cdots$, where $\succeq$ is the reflexive closure of $\succ$. Since $\succ$ is terminating, the chain of words must be of the form $w(\tau_1) \succeq \cdots \succeq w(\tau_k) = w(\tau_{k+1}) = \cdots$ with k finite. On the other hand we have observed that if $\tau >_{R_n} \tau'$ implies $w(\tau) = w(\tau')$, then the dimension of the essential front face of τ' is higher than the dimension of the essential front face of τ. Since such dimension is bounded by n, the chain $w(\tau_1) \succeq \cdots \succeq w(\tau_k) = w(\tau_{k+1}) = \cdots = w(\tau_{k+h})$ is finite and CS2 holds. ∎

4.2. Example. The category $\mathbf{\Delta}_e$ of ordinal numbers and order preserving surjective functions has the following presentation. The graph G has G_0 as the set of ordinal numbers, while the set G_1 consist of the order preserving elementary surjective functions

$$\sigma_i(x) = \begin{cases} x & (x \leq i)\ ; \\ x-1 & (x > i)\ . \end{cases}$$

We obtain the free category on G by considering the concatenations of these functions. Observe that an order preserving surjective function may be written in different ways as the composition of these elementary functions; for any of these possibilities we want to choose a "normal form". On the free category FG, consider the set of rewrite rules given by

$$\sigma_i \sigma_j \Rightarrow \sigma_{j-1}\sigma_i \qquad (i < j)\ .$$

We obtain in this way a complete rewrite system whose normal forms are words of the type

$$\sigma_{i_1}\sigma_{i_2}\cdots\sigma_{i_s} \qquad \text{with} \qquad i_1 \geq i_2 \geq \cdots \geq i_s\ .$$

The quotient category is the category $\mathbf{\Delta}_e$. For the canonical collapsing scheme of this rewrite system, we have that all the ordinal numbers are the essential 0-simplices. The essential 1-simplices are the elementary functions σ_i, while all the normal forms of length greater than one are redundant 1-simplices. For example $\sigma_3\sigma_2\sigma_1$ is redundant and its collapsible is the 2-simplex $(\sigma_3, \sigma_2\sigma_1)$. The essential n-simplex has the form

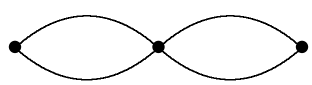
$$(\sigma_{i_1}, \sigma_{i_2}, \cdots, \sigma_{i_n}) \qquad \text{with} \qquad i_1 < i_2 < \cdots < i_n\ .$$

4.3. Example. A free category $\mathbf{C}$ is presented by a rewrite system (G, R) with R empty. $(G, \emptyset)$ is surely complete and the collapsing scheme is very simple. Since no word of FG_1 is reducible, after condition ES2, there are no essential n-simplices with $n > 1$ and we have

$$E_0 = G_0\ , \quad E_1 = G_1\ , \quad E_n = \emptyset\ (n \geq 2).$$

Using other words, *the classifying space of a free category generated by the graph G has the homotopy type of the topological graph $|G|$ (the geometric realization of G).*

For example, the category $\mathbf{C}$ in Example 2.3 is the free category on the graph pictured there and thus $B\mathbf{C}$ has the same homotopy type as



4.4. Proposition. *There exists a functor K such that the following diagram commutes*

$$\begin{tikzcd} \mathbf{CRS} \arrow[r, "K"] \arrow[d, "I"'] & \mathcal{C}\mathbf{Set}^{\mathbf{\Delta}^{op}} \arrow[d, "U"] \\ \mathbf{Cat} \arrow[r, "N"'] & \mathbf{Set}^{\mathbf{\Delta}^{op}} \end{tikzcd}$$

Proof. The functor K assigns to a complete rewrite system the collapsing scheme defined above. Now remember that a morphism of rewrite systems preserves the length of words and sends reducible words to reducible words. A morphism of complete rewrite systems sends also irreducible to irreducible and the rest of the proof is immediate. ∎

References

[BG84] K. S. Brown and R. Geoghegan, *An infinite-dimensional torsion-free FP_∞ group*, Invent. Math. **77** (1984), 367–381.

[Bro92] K. S. Brown, *The geometry of rewriting systems: a proof of the Anick-Groves-Squier theorem*, Algorithms and classification in combinatorial group theory (Berkeley, CA, 1989), Springer, MSRI 23, New York, 1992, pp. 137–163.

[Cit00] M. G. Citterio, *Classifying spaces of categories and term rewriting*, Ph.D. thesis, II University of Rome "Tor Vergata", March 2000.

[ES87] S. Eilenberg and R. Street, *Rewrite systems, algebraic structures and higher-order categories*, Handwritten manuscript, 1987.

[Mil57] J. Milnor, *The geometric realization of a semi-simplicial complex*, Ann. Math. **65** (1957), no. 2, 357–362.

[Str96] R. Street, *Categorical structures*, Handbook of algebra, Vol. 1, North-Holland, Amsterdam, 1996, pp. 529–577.

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