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A CATEGORICAL GENEALOGY FOR THE CONGRUENCE DISTRIBUTIVE PROPERTY

DOMINIQUE BOURN

ABSTRACT. In the context of Mal'cev categories, a left exact root for the congruence distributive property is given and investigated, namely the property that there is no non trivial internal group inside the fibres of the fibration of pointed objects. Indeed, when moreover the basic category $\mathbb{C}$ is Barr exact, the two previous properties are shown to be equivalent.

Introduction

A Mal'cev category is a category $\mathbb{C}$ in which any reflexive relation is an equivalence relation [7]. When moreover it is regular, the congruence modular law holds. Accordingly, this categorical Mal'cev condition appears as a left exact root for the congruence modular property in the categorical context.

The definition of *arithmetical category* was introduced by M.C.Pedicchio in [13], as a Barr exact Mal'cev category with coequalizers such that the congruence distributive property holds. Significant examples are the categories of Boolean rings, of von Neumann regular rings, of Heyting algebras, or the dual of any (elementary) topos. Varietal examples are characterized by the existence of a ternary operation satisfying: 1) $p(x, y, y) = x$ and $p(x, x, y) = y$ (Mal'cev axiom) , 2) $p(x, y, x) = x$ (Pixley axiom), see [15].

Pedicchio showed that the previous definition is equivalent to that of a Barr exact Mal'cev category with coequalizers in which any internal groupoid is actually an equivalence relation. This characterization incites to investigate the significance and the strength of its left exact part (*): any internal groupoid is an equivalence relation, which will be the aim of this article. We shall then call *protoarithmetical* a Mal'cev category satisfying (*). Induced examples of this situation are given by the categories of any internal structures of the previous kind inside a left exact category, as the categories of topological Boolean rings, or of topological von Neumann regular rings for instance.

Once again, the fibration π of pointed objects, whose fibres are the split epimorphisms with fixed codomain, will show its powerful classification potential [3]: a category is protoarithmetical if and only if the fibres of π are antiadditive, i.e. unital and such that there is no non trivial internal group structure, this terminology being justified by the fact that an additive and antiadditive category degenerates, up to equivalence, into the

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singleton category.

Other significant characterizations will come from the notion of *connector between equivalence relations* which gives a left exact way to control centrality, see [5], and also [8] and [13]. Indeed we shall see that a category is protoarithmetical if and only if any map with abelian kernel is a monomorphism, or if and only if the following property holds: two equivalence relations R and S on a same object X are commuting if and only if $R \cap S = disX$.

When furthermore the given category is regular, we shall prove that it is protoarithmetical if and only if it satisfies a kind of congruence distributive precondition, namely: given any triple T, S, S' of equivalence relations on X, the equalities $T \cap S = disX$ and $T \cap S' = disX$ imply $T \cap (S \vee S') = disX$. When it is Barr exact, we get a characterization which improves Pedicchio's original result (we drop here the existence of coequalizers): it is protoarithmetical if and only if the congruence distributive property holds. So that this article could appear as an investigation about a categorical genealogy for the congruence distributive law.

All but the varietal examples of arithmetical categories given above are actually protomodular categories, in which there is an intrinsic notion of normal monomorphism [3]. The last section is devoted to the proof that, in an arithmetical protomodular category, a normal monomorphism has at most one retraction.

I am greatly indebted to M.Gran for having drawn Pedicchio's article to my attention, for many stimulating discussions and for his contribution to Theorem 3.13.

1. Unital and antiadditive categories

The point of this section is to set some categorical notions which the fibres of the fibration π of pointed objects will be later on assumed to possess.

1.1. Unital categories. Let us recall that a map $f : X \to Y$ is a strong epimorphism when any monomorphism whose pullback along f is an isomorphism is itself an isomorphism. In the same way, a family of coterminal maps is jointly strongly epic when any monomorphism whose pullback along all the maps of this family are isomorphisms is itself an isomorphism.

A *unital category* (see [3]) is a left exact pointed (i.e. with a zero object) category $\mathbb{C}$ such that for each pair of objects (X, Y), the canonical pair of maps is jointly strongly epic:

$$X \xrightarrow{i_X} X \times Y \xleftarrow{i_Y} Y$$

1.2. Example. The category **Mag** of unitary magmas is unital.

When it is applied to the pair (X, X), the previous axiom implies that in a unital category $\mathbb{C}$ there is, on a given object X, at most one internal binary operation with unit, which is necessarily associative and commutative. And a fortiori at most one group structure. In this very case, X is said *abelian.*

A double zero sequence in a pointed category is a diagram:

$$\begin{tikzcd} X \arrow[r, shift left, "s"] & Z \arrow[l, shift left, "f"] \arrow[r, shift left, "g"] & Y \arrow[l, shift left, "t"] \end{tikzcd}$$

where the two sequences are zero sequences (i.e. sequences factorized through the zero object 1) which, moreover, satisfy $f.s = Id_X$ and $g.t = Id_Y$.

1.3. Proposition. *Given a double zero sequence in a unital category, then the induced factorization $h = [f, g] : Z \to X \times Y$ is a strong epimorphism.*

Proof. Consider a monomorphism $j : R \to X \times Y$ whose pullback along h is an isomorphism. Then its pullbacks along $h.s = i_X$ and $h.t = i_Y$ are isomorphisms, and consequently, $\mathbb{C}$ being unital, the map j is an iso. ∎

1.4. Antiadditive categories.

1.5. Definition. *A category $\mathbb{C}$ is said antiadditive when it is unital and such that the only abelian object is the terminal object.*

1.6. Remark. The terminology is justified by the fact that any additive and antiadditive category is equivalent to the terminal category **1**.

1.7. Example. The category **Imag** of unitary idempotent magmas is antiadditive.

1.8. Naturally antidditive category. Suppose now $\mathbb{C}$ only pointed with products. Suppose moreover that for each object X the following square is a pushout:

$$\begin{tikzcd} X \times X \arrow[r, "\tau_{X\times X}"] & 1 \\ X \arrow[u, "s_0"] \arrow[r, "\tau_X"'] & 1 \arrow[u] \end{tikzcd}$$

1.9. Proposition. *In such a category $\mathbb{C}$, the only object X endowed with a group structure is the terminal object* 1.

Proof. Let X be an object endowed with a group structure, and $d : X \times X \to X$ denote the division map $d(x, y) = x^{-1}.y$. Now consider the following commutative square:

$$\begin{tikzcd} X \times X \arrow[r, "d"] & X \\ X \arrow[u, "s_0"] \arrow[r, "\tau_X"'] & 1 \arrow[u] \end{tikzcd}$$

According to the given property of $\mathbb{C}$, the initial map $\alpha_X : 1 \to X$ is such that $\alpha_X.\tau_{X\times X} = d$. But d is a split epi, and we can check that: $\alpha_X.\tau_X.d = \alpha_X.\tau_{X\times X} = d$. Consequently $\alpha_X.\tau_X = Id_X$ and X is isomorphic to 1. ∎

1.10. Definition. *A category $\mathbb{C}$ is naturally antiadditive when it is unital and such that, for each object X, the following square is a pushout:*

$$\begin{tikzcd} X \times X \arrow[r, "\tau_{X\times X}"] & 1 \\ X \arrow[u, "s_0"] \arrow[r, "\tau_X"'] & 1 \arrow[u] \end{tikzcd}$$

1.11. Example. The category **Imon** of unitary idempotent monoids is naturally antiadditive.

Of course, a naturally antiadditive category is antiadditive.

2. Mal'cev and protoarithmetical categories

2.1. Mal'cev categories and fibration of pointed objects. Let us recall the following:

2.2. Definition. *(see [7]) A category $\mathbb{C}$ is said Mal'cev when it is left exact and such that any reflexive relation is an equivalence relation.*

Now let us consider any left exact category $\mathbb{C}$. We denote by $Pt\mathbb{C}$ the category whose objects are the split epimorphisms in $\mathbb{C}$ with a given splitting and morphisms the commutative squares between these data. We denote by π: $Pt\mathbb{C} \to \mathbb{C}$ the functor associating its codomain with any split epimorphism. Since the category $\mathbb{C}$ has pullbacks, the functor π is a fibration which is called the *fibration of pointed objects*.

This fibration π has important classification properties; we shall be, here, particularly interested by the following one [3]:

2.3. Theorem. *A category $\mathbb{C}$ is Mal'cev if and only if the fibration π is unital, i.e. if and only if each fibre of π is unital.*

In the Mal'cev context, let us recall that, given two reflexive relations R and S on the same object X, then if $R \times_X S$ is defined by the following pullback :

$$\begin{tikzcd} R \times_X S \arrow[r, "p_1"] \arrow[d, "p_0"'] & S \arrow[d, "d_0"] \\ R \arrow[r, "d_1"'] & X \end{tikzcd}$$

there is *at most one map* p: $R \times_X S \to X$,which satisfies $p(x, x, y) = y$ and $p(x, y, y) = x$ whenever these terms are defined.

When such a map p exists, it is called the connector between R and S or, equivalently, R and S are said commuting [5], see also [8] and [13].

2.4. Remark. 1) An object X is endowed with an internal Mal'cev operation if and only if the coarse relation grX (i.e. the kernel equivalence of the terminal map $\tau_X : X \to 1$) is commuting with itself. This is, in particular, the case for any object endowed with an internal group structure.

2) The category $\mathbb{C}$ being Mal'cev, when R and S are commuting, we have necessarily: $xSp(x,y,z)$ and $zRp(x,y,z)$. If we define $\pi_0 : R \times_X S \to S$ and $\pi_1 : R \times_X S \to R$ by $\pi_0(x,y,z) = (x, p(x,y,z))$ and $\pi_1(x,y,z) = (p(x,y,z), z)$ we obtain a double relation in $\mathbb{C}$, see [5]:

$$\begin{tikzcd}
R \times_X S \arrow[r, shift left, "p_1"] \arrow[r, shift right, "\pi_0"'] \arrow[d, shift right, "p_0"'] \arrow[d, shift left, "\pi_1"] & S \arrow[d, shift right, "d_0"'] \arrow[d, shift left, "d_1"] \\
R \arrow[r, shift left, "d_1"] \arrow[r, shift right, "d_0"'] & X
\end{tikzcd}$$

Because of the unicity of the connector in the Mal'cev context, we are allowed to give the following:

2.5. Definition. *A map $f : X \to Y$ in a Mal'cev category $\mathbb{C}$ is said to have an abelian kernel when its kernel equivalence $R[f]$ is commuting with itself:*

$$\begin{tikzcd}
R[f] \arrow[r, shift left, "p_0"] \arrow[r, shift right, "p_1"'] & X \arrow[r, "f"] & Y
\end{tikzcd}$$

We shall derive now a significant property:

2.6. Theorem. *In a Mal'cev category $\mathbb{C}$, if we have $R \cap S = disX$, then the relations R and S are commuting.*

Proof. Let us define $R \square S$ by the following pullback:

$$\begin{tikzcd}
R \square S \arrow[r] \arrow[d] & S \times S \arrow[d, "{[d_0 \times d_0, d_1 \times d_1]}"] \\
R \times R \arrow[r, "{[d_0,d_1] \times [d_0,d_1]}"'] & X \times X \times X \times X
\end{tikzcd}$$

it corresponds to the subobject of X^4 consisting of the quadruples (x, x', y, y') such that xRx', yRy', xSy and $x'Sy'$ and determines a double relation on X:

$$\begin{tikzcd}
R \square S \arrow[r, shift left, "p_1"] \arrow[r, shift right, "p_0"'] \arrow[d, shift right, "p_0"'] \arrow[d, shift left, "p_1"] & S \arrow[d, shift right, "d_0"'] \arrow[d, shift left, "d_1"] \\
R \arrow[r, shift left, "d_1"] \arrow[r, shift right, "d_0"'] & X
\end{tikzcd}$$

Let us denote by $\alpha : R\Box S \to R\times_X S$ the induced factorization, and by (π, σ) the following split epi: $\pi\colon R\Box S \xrightarrow{p_1} S \xrightarrow{d_0} X$.Then the following commutative square:

$$\begin{tikzcd} R\Box S \arrow[d, "p_0"'] & S \arrow[l, "s_0"'] \arrow[d, "d_0"] \\ R & X \arrow[l, "s_0"] \end{tikzcd}$$

determines a zero sequence in the fibre $Pt_X\mathbb{C}$:

$$(S, (d_0, d_1)) \xrightarrow{s_0} (R\Box S, (\pi, \sigma)) \xrightarrow{p_0} (R, (d_1, s_0))$$

and the following one:

$$\begin{tikzcd} R\Box S \arrow[r, "p_1"] & S \\ R \arrow[u, "s_0"] \arrow[r, "d_1"'] & X \arrow[u, "s_0"'] \end{tikzcd}$$

completes a double zero sequence in $Pt_X\mathbb{C}$:

$$\begin{tikzcd} (S, (d_0, d_1)) \arrow[r, shift left, "s_0"] & (R\Box S, (\pi, \sigma)) \arrow[l, shift left, "p_1"] \arrow[r, shift left, "p_0"] & (R, (d_1, s_0)) \arrow[l, shift left, "s_0"] \end{tikzcd}$$

Consequently, the category $\mathbb{C}$ being Mal'cev and the fibre $Pt_X\mathbb{C}$ unital, the factorization α is a strong epi. It is a mono since $R \cap S = disX$. Accordingly, the following square is a pullback:

$$\begin{tikzcd} R\Box S \arrow[r, "p_1"] \arrow[d, "p_0"'] & S \arrow[d, "d_0"] \\ R \arrow[r, "d_1"'] & X \end{tikzcd}$$

and the map $d_1.p_0 : R\Box S \to S \to X$ produces the expected connector. ■

2.7. Protoarithmetical categories. Let us get now to the heart of the matter:

2.8. Definition. *A category $\mathbb{C}$ is protoarithmetical when it is left exact and such that the fibration π of pointed objects is antiadditive, i.e. when any of its fibres is antiadditive .*

2.9. Remark. A protoarithmetical category is consequently Mal'cev.

2.10. Example. The categories **Bool** of Boolean algebras (or Boolean rings), **VoN** of von Neumann regular rings and **Hey** of Heyting algebras are protoarithmetical. The dual of any elementary topos $\mathbb{E}$ is protoarithmetical. A variety $\mathbb{V}$ is protoarithmetical if and only if there is a ternary operation p such that: 1) $p(x,y,y) = x$ and $p(x,x,y) = y$ ($\mathbb{V}$ Mal'cev) , 2) $p(x,y,x) = x$ (Pixley axiom), see [15].

Let $F : \mathbb{C} \to \mathbb{D}$ be a pullback preserving and conservative functor, then if $\mathbb{D}$ is protoarithmetical, such is $\mathbb{C}$. Consequently any slice or coslice of a protoarithmetical category is protoarithmetical. Any category of algebras $A\lg T$ associated with a monad (T, λ, μ) on a protoarithmetical category is protoarithmetical.

2.11. Theorem. *Given a Mal'cev category $\mathbb{C}$, the following conditions are equivalent :*

1) it is protoarithmetical

2) any map with abelian kernel is a monomorphism.

3) any internal groupoid is an equivalence relation

4) two equivalence relations R and S on a same object X are commuting if and only if $R \cap S = disX$.

Proof. 1)⇒2) : Suppose $\mathbb{C}$ protoarithmetical. Let $f : X \to Y$ be a map with abelian kernel. The connector $p : X \times_Y X \times_Y X \to X$ between $R[f]$ and $R[f]$ determines an internal group structure on the object $(p_0, s_0), p_0$: $X \times_Y X \to X$ in the fibre $Pt_X\mathbb{C}$ above X, by setting : $(x,y) * (x,z) = (x, p(y,x,z))$, see [2]. The fibre $Pt_X\mathbb{C}$ being antiadditive, p_0 is an isomorphism and f is a mono.

2)⇒3) : Consider a groupoid in $\mathbb{C}$ whose object of objects is X and object of morphisms W. The kernel equivalence $R[[d_0, d_1]]$ of the map $[d_0, d_1] : W \to X \times X$ is given by the object of "pairs of parallel arrows". The object $R[[d_0, d_1]] \times_W R[[d_0, d_1]]$ is just the object of "triple of parallel arrows". Now the groupoid structure determines a canonical connector between $R[[d_0, d_1]]$ and itself, which is given by the map p representing the operation associating $\alpha.\beta^{-1}.\gamma$ with any triple (α, β, γ) of parallel arrows. Consequently, in any left exact category $\mathbb{C}$, the map $[d_0, d_1]$ has always an abelian kernel. When $\mathbb{C}$ satisfies condition 2, this map is then a monomorphism, and the given groupoid is an equivalence relation.

3)⇒4) : Consider R and S two commuting equivalence relations on X, and their connector $p : R \times_X S \to X$. It determines a groupoid structure whose object of objects is X and object of morphisms is $R \times_X S$, with $d_0(x,y,z) = z, d_1(x,y,z) = x$ and $(u,v,x).(x,y,z) = (u, p(y,x,v), z)$. This groupoid is thus an equivalence relation, which means that when we have $xRySz$ and $xRy'Sz$, this implies $y = y'$. In particular xRy and xSy imply $xRySy$ and $xRxSy$, and consequently $x = y$. Whence $R \cap S = disX$.

4)⇒1) : Consider an internal group structure on an object $(f,s), f : X \to Z$ in the fibre $Pt_Z\mathbb{C}$ above Z. According to Remark 2.4, this means that $R[f]$ is commuting with itself in this fibre. Consequently $R[f] \cap R[f] = disX$. Then $R[f] = disX$ and f is a mono which, being split, is an iso. The fibre $Pt_Z\mathbb{C}$ above Z is then antiadditive. ■

2.12. Remark. The proof of 1)⇒2) emphasizes the fact that the protoarithmetical axiom appears to be strongly related to idempotency. Indeed, consider any ternary operation

p on a set X, and define on $X \times X$ a binary operation $*$ (see [2]) on the model of the one given in the proof of 1)$\Rightarrow$2). Then the ternary operation p satisfies the Pixley axiom $p(x, y, x) = x$ if and only if $(x, y) * (x, y) = (x, p(y, x, y)) = (x, y)$, i.e. if and only if this binary operation $*$ is idempotent.

In any category, if we are given a product $X \times Y$, then the kernel equivalences $R[p_X]$ and $R[p_Y]$ of the projections are canonically linked by a connector given by $p(x, x', y, y') = (x', y)$, see [5] . In any pointed protoarithmetical category $\mathbb{C}$, we have the remarkable result following which this is a characteristic property of the product:

2.13. Theorem. *If $\mathbb{C}$ is a pointed protoarithmetical category, then a double zero sequence:*

$$\begin{tikzcd} X \arrow[r, shift left, "s"] & Z \arrow[l, shift left, "f"] \arrow[r, shift left, "g"] & Y \arrow[l, shift left, "t"] \end{tikzcd}$$

is isomorphic to the canonical double zero sequence associated with the product $X \times Y$ if and only if the equivalence relations $R[f]$ and $R[g]$ are commuting.

Proof. According to Proposition 1.3, the map $[f, g] : Z \to X \times Y$ is a strong epimorphism. It is also a mono since $R[[f, g]] = R[f] \cap R[g] = disZ$, according to the previous theorem. Consequently it is an isomorphism. ∎

2.14. Naturally Mal'cev and protoarithmetical categories. A *Naturally Mal'cev* category is a left exact category $\mathbb{C}$ such that each object X is endowed with a natural Mal'cev operation $p_X : X \times X \times X \to X$, see [11]. Once again, the fibration π is classifying: a left exact category $\mathbb{C}$ is Naturally Mal'cev if and only if the fibration π is additive, i.e. if it has its fibres additive [3]. This implies, in particular, $\mathbb{C}$ is Mal'cev.

2.15. Proposition. *(See also [13]) A Mal'cev $\mathbb{C}$ category is Naturally Mal'cev if and only if any pair R and S of equivalence relations on the same object X is commuting.*

Proof. Let R and S be two equivalence relations on the same object X. Then $R \times_X S$ is a subobject of $X \times X \times X$ and the following map produces the connector:

$$R \times_X S \hookrightarrow X \times X \times X \xrightarrow{p_X} X$$

Conversely, the coarse relation grX on any object X being commuting with itself, X is naturally endowed with a Mal'cev operation, see Remark 2.4. ∎

2.16. Theorem. *A category $\mathbb{C}$ which is Naturally Mal'cev and protoarithmetical is a poset.*

Proof. The fibres $Pt_X\mathbb{C}$, being additive and antiadditive, are equivalent to the terminal category, which means that the only split epimorphisms in $\mathbb{C}$ are the isomorphisms. Now the kernel relation of any map f is trivial, and thus f is a monomorphism. This is the case in particular for the terminal map. Consequently $\mathbb{C}$ is a poset ∎

3. Regular and exact protoarithmetical categories

Let us recall that the category $\mathbb{C}$ is regular [1] when the regular epimorphisms are stable by pullback and any effective equivalence relation admits a quotient. It is Barr exact when moreover any equivalence relation is effective.

3.1. Regular Mal'cev categories. In the Mal'cev context, regular epimorphisms have a strong stability property:

3.2. Theorem. *Let $\mathbb{C}$ be a regular Mal'cev category. Given two commutative squares of (vertical) split epimorphisms:*

$$\begin{tikzcd} A \arrow[r, "g"] \arrow[d, "\alpha"'] & B \arrow[d, "\beta"] & & C \arrow[r, "h"] \arrow[d, "\gamma"'] & D \arrow[d, "\delta"] \\ X \arrow[r, "f"'] & Y & & X \arrow[r, "f"'] & Y \end{tikzcd}$$

when the maps f, g and h are regular epimorphisms, then the induced factorization $k : A \times_X C \to B \times_Y D$ is a regular epimorphism.

Proof. Consider $R[f]$, $R[g]$ and $R[h]$ the kernel relations of the maps f, g and h. Then the kernel relation $R[k]$ is nothing but $R[g] \times_{R[f]} R[h]$. Now let us denote by $\rho : A \times_X C \to Q$ the quotient of $R[k]$ and $i : Q \to B \times_Y D$ the monomorphic factorization of k, satisfying $k = i.\rho$.

On the other hand, when the maps f, g and h are regular epimorphisms, the following double zero sequence in the fibre $Pt_X\mathbb{C}$:

$$A \rightleftarrows A \times_X C \rightleftarrows C$$

is naturally extended into a double zero sequence in the fibre $Pt_Y\mathbb{C}$:

$$B \rightleftarrows Q \rightleftarrows D$$

But the category $\mathbb{C}$ is Mal'cev, the fibre $Pt_Y\mathbb{C}$ is unital, and consequently the factorization $i : Q \to B \times_Y D$ of this double zero sequence is a regular epi. Thus i is an isomorphism and k a regular epi. ∎

From that, we can deduce:

3.3. Theorem. *Given a regular epimorphism $f : X \to Y$ in a regular Mal'cev category $\mathbb{C}$, if two equivalence relations T and S on X are commuting, then their images $f(T)$ and $f(S)$ are commuting.*

Proof. Let us denote by $f_T : T \to f(T)$ and $f_S : S \to f(S)$ the induced maps. According to the previous result and the notations of Remark 2.4, the factorization $\varphi : T \times_X S \to f(T) \times_Y f(S)$ is a regular epimorphism. Let p denote the connector between T and S. We are now going to prove that the map $f.p : T \times_X S \to Y$ factorizes through φ, and thus produces a connector q between $f(T)$ and $f(S)$. We must show that the map $f.p$ coequalizes the kernel relation $R[\varphi]$. But the following square is a pullback of split epimorphisms, and, the category $\mathbb{C}$ being Mal'cev, the induced sections σ and σ' of $R(p_0)$ and $R(p_1)$ are jointly strongly epic :

$$\begin{tikzcd}
R[f_T] \times_{R[f]} R[f_S] = R[\varphi] \arrow[r, "R(p_1)"] \arrow[d, "R(p_0)"'] & R[f_S] \arrow[d, "R(d_0)"] \\
R[f_T] \arrow[r, "R(d_1)"'] & R[f]
\end{tikzcd}$$

It is then sufficient to check that $(f.p.p_0).\sigma = (f.p.p_1).\sigma$ and $(f.p.p_0).\sigma' = (f.p.p_1).\sigma'$. But $f.p.p_0.\sigma = f.p.s_{R,0}.p_0 = f.d_0.p_0$ since $p(x,y,y) = x$ means $p.s_{R,0} = d_0$. And $f.d_0.p_0 = d_0.f_R.p_0 = d_0.f_R.p_1 = f.d_0.p_1 = f.p.s_{R,0}.p_1 = f.p.p_1.\sigma$. The same equalities hold for σ'. ∎

In addition, when the equivalence relation T is such that $R[f] \subset T$, then $f^{-1}.f(T) = T$, see [6] for instance. As a consequence, we have:

3.4. Proposition. *In a regular Mal'cev category, when $R[f] \subset T$ and $R[f] \subset S$, we have $f(T \cap S) = f(T) \cap f(S)$.*

Proof. The relation $T \cap S$ is nothing but the inverse image $s_0^{-1}(T \times S)$ where $s_0 : X \to X \times X$ is the diagonal. Now, since we have $R[f] \subset T$ and $R[f] \subset S$, then $f^{-1}.f(T) = T$, $f^{-1}.f(S) = S$ and consequently $(f \times f)^{-1}(f(T) \times f(S)) = T \times S$. Therefore we obtain: $T \cap S = s_0^{-1}(T \times S) = s_0^{-1}(f \times f)^{-1}(f(T) \times f(S)) = f^{-1}s_0^{-1}(f(T) \times f(S)) = f^{-1}(f(T) \cap f(S))$ and, accordingly, $f(T \cap S) = f(T) \cap f(S)$. ∎

On the other hand, as soon as $\mathbb{C}$ is regular Mal'cev, the composite $T.S$ of two equivalence relations T and S is an equivalence relation, and moreover we have $T \vee S = T.S = S.T$. More precisely the object $T \vee S$ is given by the canonical epi-mono factorization of the map $[d_0.p_0, d_1.p_1] : T \times_X S \to X \times X$, where $T \times_X S$ is defined as in Remark 2.4. We have moreover the following result, see [5]:

3.5. Proposition. *In a regular Mal'cev category, when the pairs (T, S) and (T, S') of equivalence relations are commuting, then the pair $(T, S \vee S')$ is commuting.*

Now Theorem 3.2 gives us the following:

3.6. Proposition. *Given a regular epimorphism $f : X \to Y$ in a regular Mal'cev category $\mathbb{C}$, T and S two equivalence relations on X, then $f(T \vee S) = f(T) \vee f(S)$.*

Proof. Consider the following commutative square, where the vertical maps are the regular epimorphisms defining the joins:

$$\begin{tikzcd}
T \times_X S \arrow[r, "\varphi"] \arrow[d] & f(T) \times_Y f(S) \arrow[d] \\
T \vee S \arrow[r] & f(T) \vee f(S)
\end{tikzcd}$$

then the fact that the lower map is a regular epi is a consequence of the fact that the upper factorization $\varphi : T \times_X S \to f(T) \times_Y f(S)$ is a regular epi. ∎

3.7. Regular protoarithmetical categories. We have then the following characterization:

3.8. Theorem. *Given a regular category $\mathbb{C}$, it is protoarithmetical if and only if the fibration π is naturally antiadditive.*

Proof. This comes from the fact that in any pointed regular Mal'cev category, the object A is the abelian object associated with X if and only if the following square is a pushout, see [4]:

$$\begin{tikzcd} X \times X \arrow[r, "h"] & A \\ X \arrow[u, "s_0"] \arrow[r, "\tau_X"'] & 1 \arrow[u] \end{tikzcd}$$

■

And also the following link with a kind of congruence distributive precondition:

3.9. Theorem. *Given a regular Mal'cev category $\mathbb{C}$, it is protoarithmetical if and only if, given any triple T, S, S' of equivalence relations on X, the equalities $T \cap S = disX$ and $T \cap S' = disX$ imply $T \cap (S \vee S') = disX$.*

Proof. Suppose that $T \cap S = disX$ and $T \cap S' = disX$. Then, the category $\mathbb{C}$ being Mal'cev, the pairs (T, S) and(T, S') are commuting, therefore the pair $(T, S \vee S')$ is commuting. Now $\mathbb{C}$ is protoarithmetical, thus $T \cap (S \vee S') = disX$.

Conversely, consider any internal group structure on an object (f, s), $f : X \to Z$ in the fibre $Pt_Z\mathbb{C}$ above Z, and denote $m : X \times_Z X \to X$ the multiplication, p_0 and p_1 the projections. Then we have $R[m] \cap R[p_0] = dis(X \times_Z X)$ and $R[m] \cap R[p_1] = dis(X \times_Z X)$, as for any internal group structure. Consequently we have: $R[m] = R[m] \cap gr(X \times_Z X) = R[m] \cap (R[p_0] \vee R[p_1]) = dis(X \times_Z X)$. Thus the multiplication m is a mono, and therefore an iso, which implies the same result for p_0 and p_1. Accordingly, the map f is a mono, and, as a split epi, an iso. The only group structure in $Pt_Z\mathbb{C}$ is therefore the terminal object, and $Pt_Z\mathbb{C}$ is antiadditive. ■

3.10. Arithmetical categories. In the Barr exact context, improving the original result of [14] (we drop here the existence of coequalizers), we shall prove a particularly significant characterization, namely the congruence distributive property.

3.11. Theorem. *A Barr exact Mal'cev category is protoarithmetical if and only if the lattice of equivalence relations on any object X is distributive.*

Proof. Suppose the Barr exact Mal'cev category $\mathbb{C}$ satisfies the congruence distributive property. Then it satisfies the conditions of the previous theorem, and thus it is protoarithmetical.

Conversely, let T, S, S' be three equivalence relations on the same object X in any regular Mal'cev category. We have always $(T \cap S) \vee (T \cap S') \subset T \cap (S \vee S')$.

Suppose moreover $\mathbb{C}$ Barr exact. Then any equivalence relation has an effective quotient. Suppose now it is protoarithmetical. Consider $\varphi : X \to Q$, $\rho : X \to D$ and $\rho' : X \to D'$ the effective quotients of $(T \cap S) \vee (T \cap S')$, $T \cap S$ and $T \cap S'$. Of course, there are epimorphic factorizations τ and τ' , such that $\tau.\rho = \varphi = \tau'.\rho'$.

Now, since $T \cap S \subset T$ and $T \cap S \subset S$, we have $\rho(T) \cap \rho(S) = \rho(T \cap S) = disD$. The same equality holds for the pair (T, S'). Therefore the pairs $(\rho(T), \rho(S))$ and $(\rho'(T), \rho'(S'))$ are commuting. Consequently the pairs $(\tau.\rho(T), \tau.\rho(S))$ and $(\tau'.\rho'(T), \tau'.\rho'(S'))$ are commuting. The category $\mathbb{C}$ being protoarithmetical, we have $\tau.\rho(T) \cap \tau.\rho(S) = disQ$ and $\tau'.\rho'(T) \cap \tau'.\rho'(S') = disQ$ and thus $\varphi(T) \cap \varphi(S) = disQ$ and $\varphi(T) \cap \varphi(S') = disQ$. The category $\mathbb{C}$ being regular protoarithmetical, we have then $disQ = \varphi(T) \cap (\varphi(S) \vee \varphi(S')) = \varphi(T) \cap \varphi(S \vee S')$. Thus certainly $\varphi(T \cap (S \vee S')) = disQ$, since $\varphi(T \cap (S \vee S')) \subset \varphi(T) \cap \varphi(S \vee S')$. Consequently $T \cap (S \vee S') \subset R[\varphi] = (T \cap S) \vee (T \cap S')$. ■

Following M.C.Pedicchio, see [14], and according to the previous improvement, we shall propose the following straightened up definition:

3.12. Definition. *A category $\mathbb{C}$ is said arithmetical when it is Barr exact Mal'cev and satisfies the congruence distributive property. In other words, an arithmetical category is a Barr exact protoarithmetical category*

On the other hand, when $\mathbb{C}$ is Barr exact and Mal'cev, the category $Rel\mathbb{C}$ of internal equivalence relations in $\mathbb{C}$ is not generally Barr exact. This is precisely the case when $\mathbb{C}$ is arithmetical.

3.13. Theorem. *Let $\mathbb{C}$ be a Barr exact Mal'cev category. It is arithmetical if and only if $Rel\mathbb{C}$ is Barr exact.*

Proof. Consider an internal groupoid $\underline{X_1}$ in any category $\mathbb{C}$:

$$\begin{tikzcd} X_1 \arrow[r, shift left=2, "d_1"] \arrow[r, shift right=2, "d_0"'] & X_0 \arrow[l] \end{tikzcd}$$

It determines canonically an exact sequence in the category Grd$\mathbb{C}$ of internal groupoids in $\mathbb{C}$ (in the following diagram, the groupoids are vertical):

$$\begin{tikzcd} R[[d_0]] \arrow[r, shift left, "d_3"] \arrow[r, shift right, "p_2"'] \arrow[d, shift right, "p_0"'] \arrow[d, shift left, "p_1"] & R[d_0] \arrow[r, "d_2"] \arrow[d, shift right, "p_0"'] \arrow[d, shift left, "p_1"] & X_1 \arrow[d, shift right, "d_0"'] \arrow[d, shift left, "d_1"] \\ R[d_0] \arrow[r, shift left, "d_2"] \arrow[r, shift right, "p_1"'] & X_1 \arrow[r, "d_1"'] & X_0 \end{tikzcd}$$

where the map $d_2 : R[d_0] \to X_1$ represents the division map $d_2(\varphi, \psi) = \psi.\varphi^{-1}$. Consequently the induced internal functor $\delta_1 : \underline{R}[d_0] \to \underline{X_1}$ is a regular epi in Grd$\mathbb{C}$. But observe that $\underline{R}[d_0]$ and $\underline{R}[[d_0]]$ lie in $Rel\mathbb{C}$ and determine an equivalence relation $\underline{R}[[d_0]] \rightrightarrows \underline{R}[d_0]$ in this category. In other words, any object of $Grd\mathbb{C}$ can be presented as the coequalizer of an equivalence relation in $Rel\mathbb{C}$.

On the other hand, the category $\mathbb{C}$ being Barr exact and Mal'cev, the category $Grd\mathbb{C}$ is always Barr exact, see [9].

Suppose now $Rel\mathbb{C}$ is Barr exact. Let us denote by $q_1 : \underline{R}[d_0] \to \underline{\Sigma}$ its effective quotient in $Rel\mathbb{C}$. But $\underline{\Sigma}$, being an equivalence relation in $\mathbb{C}$, is a particular kind of groupoid and produces a factorization $f_1 : \underline{X_1} \to \underline{\Sigma}$ in $Grd\mathbb{C}$ such that $f_1.\delta_1 = q_1$. But the equivalence relation $\underline{R}[[d_0]] \rightrightarrows \underline{R}[d_0]$ in $Grd\mathbb{C}$ is the kernel relation of q_1 and of δ_1 . Consequently $f_1 : \underline{X_1} \to \underline{\Sigma}$ is a mono, and $\underline{X_1}$ is an equivalence relation since $\underline{\Sigma}$ is an equivalence relation.

Conversely suppose $\mathbb{C}$ arithmetical. Then the category $Rel\mathbb{C}$ is equivalent to the category $Grd\mathbb{C}$ of internal groupoids in $\mathbb{C}$. We recall that the category $\mathbb{C}$ being Barr exact and Mal'cev, the category $Grd\mathbb{C}$ is always Barr exact. Consequently $Rel\mathbb{C} \backsimeq Grd\mathbb{C}$ is Barr exact. ∎

3.14. Remark. Accordingly, when $\mathbb{C}$ is Barr exact and arithmetical, so is $Rel\mathbb{C}$.

3.15. Arithmetical protomodular categories. A left exact category $\mathbb{C}$ is protomodular when the fibration π has its change of base functors conservative, i.e. reflecting the isomorphisms, see [3]. A protomodular category is necessarily Mal'cev. The main fact is that, in a protomodular category, there is an intrinsic notion of normal subobject.

Let us recall, indeed, that in any left exact category $\mathbb{C}$ a map $j : I \to X$ is *normal to an equivalence relation* R when $j^{-1}(R)$ is the coarse relation grI on I and when moreover the induced map $grI \to R$ in $Rel\mathbb{C}$ is fibrant, i.e. when any commutative square in the following diagram is a pullback:

$$\begin{tikzcd} I \times I \arrow[r, "\bar{j}"] \arrow[d, shift right, "p_0"'] \arrow[d, shift left, "p_1"] & R \arrow[d, shift right, "d_0"'] \arrow[d, shift left, "d_1"] \\ I \arrow[r, "j"'] & X \end{tikzcd}$$

This implies that j is necessarily a monomorphism. This definition gives an intrinsic way to express that I is an equivalence class of R. But when $\mathbb{C}$ is moreover protomodular, the map j is normal to at most one equivalence relation and consequently the fact to be normal, in this kind of category, becomes a property [3].

All but the varietal examples above are not only protoarithmetical, but also protomodular. This is the case in particular for the dual $\mathbb{E}^{op}$ of any elementary topos $\mathbb{E}$, in the same way as for the dual $(\mathrm{Pt}_1\mathbb{E})^{op}$ of the category of pointed objects in the given topos $\mathbb{E}$. We are now going to prove in full generality a very strong property which holds in the dual of the category $Sets^*$ of pointed sets, for instance.

3.16. Theorem. *In any regular protoarithmetical protomodular category $\mathbb{C}$, a normal monomorphism has at most one retraction.*

Proof. Let f_0 and f_1 be two retractions of the normal monomorphism j, and R the equivalence relation to which j is normal. As a fibrant morphism, the induced map $grI \to R$ in the category $Rel\mathbb{C}$ of equivalence relations in the protomodular category $\mathbb{C}$ is cocartesian with respect to the forgetful functor $Rel\mathbb{C} \to \mathbb{C}$, see [5]. Thus f_0 and f_1

extend to morphisms in $Rel\mathbb{C}$ which are themselves fibrant, and produce the following diagram which determines an internal reflexive graph in $Rel\mathbb{C}$:

$$\begin{tikzcd} R \arrow[r, shift left=2, "\tilde{f}_1"] \arrow[r, shift right=2, "\tilde{f}_0"'] \arrow[d, shift right, "d_0"'] \arrow[d, shift left, "d_1"] & I \times I \arrow[l, "\tilde{j}"'] \arrow[d, shift right, "p_0"'] \arrow[d, shift left, "p_1"] \\ X \arrow[r, shift left=2, "f_1"] \arrow[r, shift right=2, "f_0"'] & I \arrow[l, "j"'] \end{tikzcd}$$

Now consider the following epi-mono factorizations of the maps $[f_0, f_1]$ and $[\widetilde{f_0}, \widetilde{f_1}]$:

$$X \xrightarrow{\phi} S \xrightarrow{[d_0,d_1]} I \times I$$

$$R \xrightarrow{\tilde{\phi}} \tilde{S} \xrightarrow{[\tilde{d}_0,\tilde{d}_1]} I \times I \times I \times I$$

These factorizations produce the following double relation, where S, as a reflexive relation, is an equivalence relation:

$$\begin{tikzcd} \tilde{S} \arrow[r, shift left=2, "\tilde{d}_1"] \arrow[r, shift right=2, "\tilde{d}_0"'] \arrow[d, shift right, "d_0"'] \arrow[d, shift left, "d_1"] & I \times I \arrow[l] \arrow[d, shift right, "p_0"'] \arrow[d, shift left, "p_1"] \\ S \arrow[r, shift left=2, "d_1"] \arrow[r, shift right=2, "d_0"'] & I \arrow[l] \end{tikzcd}$$

Assume now that the following square is a pullback:

$$\begin{tikzcd} \tilde{S} \arrow[r, "\tilde{d}_1"] \arrow[d, "d_0"'] & I \times I \arrow[d, "p_0"] \\ S \arrow[r, "d_1"'] & I \end{tikzcd}$$

then the relations S and grI are commuting by means of the following connector: $p_1.\widetilde{d_0} : \widetilde{S} \to I \times I \to I$, whence $S = S \cap grI = disI$ according to Theorem 2.11. Consequently S is the discrete equivalence on I, and :

$$f_0 = d_0.\phi = d_1.\phi = f_1$$

■

The fact that the square in question is a pullback is a consequence of the following lemma:

3.17. Lemma. *Let $\mathbb{C}$ be a regular Mal'cev category. Given any reflexive graph in $Rel\mathbb{C}$ (in the following diagram the relations are vertical and the graph horizontal):*

$$\begin{tikzcd} R \arrow[r, shift left=2, "\tilde{f}_1"] \arrow[r, shift right=2, "\tilde{f}_0"'] \arrow[d, shift right, "d_0"'] \arrow[d, shift left, "d_1"] & T \arrow[l] \arrow[d, shift right, "d_0"'] \arrow[d, shift left, "d_1"] \\ X \arrow[r, shift left=2, "f_1"] \arrow[r, shift right=2, "f_0"'] & I \arrow[l] \end{tikzcd}$$

consider the canonical epi-mono factorizations of the maps $[f_0, f_1]$ and $[\widetilde{f_0}, \widetilde{f_1}]$;

$$X \xrightarrow{\phi} S \xrightarrow{[d_0,d_1]} I \times I$$

$$R \xrightarrow{\tilde{\phi}} \tilde{S} \xrightarrow{[\tilde{d}_0,\tilde{d}_1]} T \times T$$

and the associated double relation:

$$\begin{tikzcd} \tilde{S} \arrow[r, shift left=2, "\tilde{d}_1"] \arrow[r, shift right=2, "\tilde{d}_0"'] \arrow[d, shift right, "d_0"'] \arrow[d, shift left, "d_1"] & T \arrow[l] \arrow[d, shift right, "d_0"'] \arrow[d, shift left, "d_1"] \\ S \arrow[r, shift left=2, "d_1"] \arrow[r, shift right=2, "d_0"'] & I \arrow[l] \end{tikzcd}$$

When the map $f_0 : R \to T$ in $Rel\mathbb{C}$ is fibrant, then the map $d_0 : \widetilde{S} \to T$ in $Rel\mathbb{C}$ is also fibrant.

Proof. It will be sufficient to prove that the following square is a pullback:

$$\begin{tikzcd} R \arrow[r, "\tilde{\phi}"] \arrow[d, "d_0"'] & \tilde{S} \arrow[d, "d_0"] \\ X \arrow[r, "\phi"'] & S \end{tikzcd}$$

Indeed, since the category $\mathbb{C}$ is regular and the following rectangle is a pullback (the map $f_0 : R \to T$ being fibrant in $Rel\mathbb{C}$), the right hand side square will be a pullback:

$$\begin{tikzcd} R \arrow[r, "\tilde{\phi}"] \arrow[d, "d_0"'] & \tilde{S} \arrow[r, "\tilde{d}_0"] \arrow[d, "d_0"] & T \arrow[d, "d_0"] \\ X \arrow[r, "\phi"'] & S \arrow[r, "d_0"'] & I \end{tikzcd}$$

Now, on the one hand, the pair $(d_0, \tilde{\phi})$ is jointly monic since the previous rectangle is a pullback. On the other hand the factorization ψ towards the pullback of d_0 along ϕ induced by this same pair is a regular epimorphism. Indeed $\mathbb{C}$ is Mal'cev regular and, in the square in question, the vertical maps are split epis and the horizontal ones are regular epis, see [10]. Consequently ψ is an isomorphism, and the square in question a pullback. ∎

The meaning of the previous theorem culminates when the basic category is pointed:

3.18. Proposition. *When a pointed regular protomodular category $\mathbb{C}$ is protoarithmetical, then the canonical injections $i_A : A \to A \times B$ have the projections as only retractions.*

Proof. Clearly, in any pointed category $\mathbb{C}$, the canonical injections $i_A : A \to A \times B$ are normal to the kernel relations associated to the projections $p_B : A \times B \to B$. According to the previous theorem, when moreover $\mathbb{C}$ is protomodular, they have the projections as only retractions. ■

On the other hand, the pair (i_A, i_B) being jointly strongly epic, a map $\varphi : A \times B \to Z$ is uniquely determined by the pair of maps (g, h), $g : A \to Z$ and $h : B \to Z$, with $g = \varphi.i_A$ and $h = \varphi.i_B$. When such a map φ exists, let us say that the maps g and h commute, see [12] and [5]. Accordingly, the previous theorem means that, in any pointed regular protoarithmetical and protomodular category, the only maps which commute with Id_A are the null maps (i.e. the maps that factorize through the zero object).

3.19. Remark. The property involved in the last proposition is very strong and timely dealing with the beginning of this article: in any pointed category $\mathbb{C}$ such that the canonical injections $i_A : A \to A \times B$ have the projections as only retractions, the only monoid object is the terminal object. Indeed the multiplication $\mu : X \times X \to X$ of this monoid is a retraction of the two canonical injections and consequently $p_0 = \mu = p_1$. The terminal map $\tau_X : X \to 1$ is then a mono and thus an iso.

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Université du Littoral
BP699
62228 Calais Cedex France
Email: `bourn@lmpa.univ-littoral.fr`

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