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CLASSIFIERS FOR MONAD MORPHISMS AND ADJUNCTION MORPHISMS

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ABSTRACT. We provide an explicit model for the free 2-category containing n composable adjunction morphisms, comparable to the Schanuel and Street model for the free adjunction. We can extract from it an explicit model for the free 2-category containing n composable lax monad morphisms. A careful proof is given, which goes through presentations of the hom-categories of our model. We use one of these hom-categories as an indexing category to construct an extended Artin-Mazur codiagonal, whose underlying bisimplicial set has the classical Artin-Mazur codiagonal as its first column.

1. Introduction

Riehl and Verity initiated a new approach to $(\infty, 1)$-category theory, where the definitions and proofs are 2-categorical in nature. Following these authors, a good context for $(\infty, 1)$-category theory is an ∞-cosmos, which is essentially a category enriched over quasi-categories with some additional properties (see [17]).

Most of the usual models for $(\infty, 1)$-categories do form an ∞-cosmos, such as the simplicially enriched categories of quasi-categories, Segal categories, complete Segal spaces, marked simplicial sets and iterated complete Segal spaces, as Riehl and Verity prove in their article [17]. The 2-category of small categories is also an ∞-cosmos, and many models for (∞, n)-categories are as well. In a series of articles, [14, 16, 15, 17, 18], Riehl and Verity showed that much of category theory can be done in a general ∞-cosmos, and in particular in its 2-category.

One of the key ideas is to encode homotopy coherent diagrams in a simplicial category $\mathcal{K}$ as simplicial functors $\mathscr{C} \to \mathcal{K}$, where $\mathscr{C}$ is a well chosen simplicial category. This idea goes back at least to Cordier and Porter [5] and originated in earlier work of Vogt [22] on homotopy coherent diagrams. For instance, the homotopy coherent nerve is constructed in this way. In Riehl and Verity's paper [16], $\mathscr{C}$ is the universal 2-category containing the object of study, either a monad or an adjunction, and $\mathcal{K}$ is the underlying ∞-cosmos. Constructions of such universal 2-categories can generally be given by a presentation by a computad [21]. Nevertheless, some of Riehl and Verity's arguments are combinatorial in nature and thus rely heavily on the use of a very concrete description of the universal monad and the universal adjunction. The universal adjunction was described explicitly partially by Auderset in [2] and by Schanuel and Street in [19].

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Lax monad morphisms are the general notion of morphisms between monads in a 2-category. They were introduced by Street in his paper [20]. However, the readily available notion of morphisms between the homotopy coherent monads of Riehl and Verity is that of simplicial natural transformations, which correspond to strict monad morphisms in the 2-categorical case.

The main goal of this paper is to provide an explicit description of the universal 2-category containing n composable lax monad morphisms, which we call $\mathbf{Mnd}[n]$. This is a prerequisite to studying lax monad morphisms in the context of an ∞-cosmos. Based on the construction of $\mathbf{Mnd}[n]$, the author has been able to provide explicit descriptions of the universal 2-category $\mathbf{Mnd}_{\mathrm{hc}}[n]$ containing a homotopy coherent diagram of shape $[n]$ constituted of monads, lax monad morphisms and monad transformations. This is a part of the author's thesis and will be explained in a forthcoming paper.

In the first part of this paper, we review the definitions of adjunction and monad and their morphisms in a 2-category. We show that if $\mathbf{Adj}[n]$ is the free 2-category containing n composable adjunction morphisms, there is an embedding $\mathbf{Mnd}[n] \to \mathbf{Adj}[n]$.

The second part of the paper consists of the construction of an explicit model for $\mathbf{Adj}[n]$ and a detailed proof of its 2-universal property. The proof of the result is carried out in three steps. First, we give a presentation of the hom-categories of $\mathbf{Adj}[n]$ (Theorem 3.3.2). In the second step, we show that the generators and relations of the hom-categories are actually 2-categorically generated by the adjunction data and adjunction morphism data (Propositions 3.3.4, 3.3.6, 3.3.7 and 3.3.8). In the last step, we prove that these relations are verified by any sequence of adjunction morphisms.

In the last part, we provide evidence that these hom-categories can be of interest to the homotopy theorist. First, all the hom-categories of $\mathbf{Adj}[n]$ are in fact Reedy categories. As an easy application of the 2-universal property of $\mathbf{Adj}[n]$, we provide an extended bar construction in the presence of a lax morphism of monads. We also study the extended Artin-Mazur codiagonal and show that the realization of its underlying bisimplicial set is weakly equivalent to that of the Artin-Mazur codiagonal. This provides new homotopical models for the realization of a bisimplicial set, for instance by iterating the extended Artin-Mazur codiagonal.

1.1. NOTATIONS. For a category $\mathscr{C}$, $|\mathscr{C}|$ denotes its class of objects. If $A, B \in |\mathscr{C}|$, $\mathscr{C}(A, B)$ denotes the set of morphisms from A to B in $\mathscr{C}$.

For a natural number n, we will write $\mathbf{n}$ instead when we consider it as an ordinal. That is, we set $\mathbf{0} = \emptyset$ and $\mathbf{n} = \{\mathbf{0}, \dots, \mathbf{n-1}\}$.

The category Δ_+ is the category of finite ordinals and order preserving maps. Among those maps, it is well known that the cofaces $d^i : \mathbf{n-1} \to \mathbf{n}$ and codegeneracies $s^j : \mathbf{n+1} \to \mathbf{n}$ for $0 \leq i, j < n$ generate this category. The coface d^i is the only order-preserving injective map not containing i in its image, while $s^j : \mathbf{n+1} \to \mathbf{n}$ is the only order-preserving surjective map such that j is the image of two elements of $\mathbf{n+1}$. More precisely, for $k \in \mathbf{n-1}$ and $l \in \mathbf{n+1}$,

$$d^i(k) = \left\{ \begin{array}{ccc} k & : & k < i \\ k+1 & : & k \geq i \end{array} \right. , \quad s^j(l) = \left\{ \begin{array}{ccc} l & : & l \leq i \\ l-1 & : & l > i \end{array} \right. .$$

Remark that the more common notation $[n] = \{0, \dots, n\}$ for the objects of Δ is such that $\mathbf{n} = [n-1]$. This change of notation is more convenient when considering the ordinal sum, because $[n]+[m] = [n+m-1]$, whereas the ordinal sum $\mathbf{n}+\mathbf{m}$ is exactly the ordinal with $n+m$ elements and is also denoted by $\mathbf{n}+\mathbf{m}$.

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2. The relation between monads and adjunctions

In this section, we review the classical correspondence between monads and adjunctions (see Mac Lane [12]), but in the general context of monads in a 2-category that is sufficiently complete. We investigate a parallel correspondence between monad morphisms and adjunction morphisms. The theorems are phrased in term of an enriched right Kan extension (see [9]).

2.1. Monads and their Eilenberg-Moore objects of algebras. In this part, we recall briefly the definition of monad and monad morphisms in a 2-category, together with the Eilenberg-Moore object of algebras.

2.1.1. Definition. *A* monad *over an object B in a 2-category $\mathscr{C}$, is a quadruple $\mathcal{T} = (B, T, \mu, \eta)$, where*

- $B \in |\mathscr{C}|$;
- $T : B \longrightarrow B$ *is a 1-cell;*
- $\mu : T^2 \longrightarrow T$ *is a 2-cell;*
- $\eta : \mathrm{id}_B \longrightarrow T$ *is a 2-cell;*

satisfying the following properties:

- $\mu \circ T\mu = \mu \circ \mu T$,
- $\mu \circ T\eta = 1_T = \mu \circ \eta T$.

Street introduced morphisms between monads under the name *monad functors* in [20] and later used the name *monad morphisms* with Lack in [11]. Some authors call them *lax monad morphisms*. In this paper, all monad morphisms will be lax unless otherwise mentioned, thus we will drop the adjective lax, and add the adjective strict instead when necessary.

2.1.2. Definition. *Let $\mathcal{T} = (B, T, \mu, \eta)$ and $\mathcal{S} = (C, S, \rho, \iota)$ be two monads in a 2-category $\mathscr{C}$. A* monad morphism *$\mathcal{T} \longrightarrow \mathcal{S}$ is a couple (f, γ), where*

- *$f : B \longrightarrow C$ is a 1-cell of $\mathscr{C}$;*
- *$\gamma : Sf \longrightarrow fT$ is a 2-cell of $\mathscr{C}$;*

such that the following diagrams are commutative.

-
$$\begin{tikzcd} S^2 f \arrow[r, "S\gamma"] \arrow[d, "\rho f"'] & SfT \arrow[r, "\gamma T"] & fT^2 \arrow[d, "f\mu"] \\ Sf \arrow[rr, "\gamma"] & & fT \end{tikzcd}$$

-
$$\begin{tikzcd} f \arrow[r, "\iota f"] \arrow[dr, "f\eta"'] & Sf \arrow[d, "\gamma"] \\ & fT \end{tikzcd}$$

A monad morphism is said to be strict *when its 2-cell is an identity.*

Monad morphisms deserve special attention for several reasons. First, they are the only sensible morphisms to use when the base category is fixed. Second, they are the 1-cells of a 2-category $\mathbf{Mnd}(\mathscr{C})$ of monads in $\mathscr{C}$, introduced by Street in [20]. Given two monads T and S on the same object of a 2-category $\mathscr{C}$, Street also explains that a 2-cell $\lambda : TS \to ST$ is a distributive law if and only if (S, λ) is a lax monad morphism $T \to T$ and the multiplication and unit of S are respectively monad transformations $(S, \lambda)^2 \to (S, \lambda)$ and $1 \to (S, \lambda)$ in $\mathbf{Mnd}(\mathscr{C})$.

We will now define the Eilenberg-Moore object of algebras associated to a monad. Let $\mathcal{T} = (B, T, \mu, \eta)$ be a monad in a 2-category $\mathscr{C}$. For any $X \in |\mathscr{C}|$, since $\mathscr{C}(X, -)$ is a 2-functor,

$$\mathcal{T}_*(X) = (\mathscr{C}(X, B), \mathscr{C}(X, T), \mathscr{C}(X, \mu), \mathscr{C}(X, \eta))$$

is a monad in $\mathbf{Cat}$. Moreover, any 1-cell $f : X' \longrightarrow X$ induces a strict monad morphism $\mathscr{C}(f, B) : \mathcal{T}_*(X) \longrightarrow \mathcal{T}_*(X')$ and thus a functor $\mathscr{C}(X, B)^{\mathcal{T}_*(X)} \longrightarrow \mathscr{C}(X', B)^{\mathcal{T}_*(X')}$ between the categories of algebras. Also, one can check that any 2-cell induces a natural transformation between these functors. This defines a 2-functor

$$\mathscr{C}(-, B)^{\mathcal{T}_*(-)} : \mathscr{C}^{\mathrm{op}} \longrightarrow \mathbf{Cat}. \tag{1}$$

2.1.3. DEFINITION. *Let $\mathcal{T} = (B, T, \mu, \eta)$ be a monad in a 2-category $\mathscr{C}$. The* Eilenberg-Moore object of algebras *of $\mathcal{T}$, if it exists, is a representing object $B^{\mathcal{T}}$ of the 2-functor (1).*

The universal 2-category containing a monad, also called *the free monad* by Schanuel and Street in [19], is a 2-category **Mnd** such that there is a natural bijection between the monads in a 2-category $\mathscr{C}$ and the 2-functors $\mathbf{Mnd} \to \mathscr{C}$. Remark that by the Yoneda lemma, this property determines **Mnd** up to isomorphism. Its existence can be derived by constructing it by generators and relations, using presentations by computads as in [21]. A short introduction to computads is provided in Appendix A and coequalizer (3) below will provide an example of such a construction.

As in category theory, an adjunction in a 2-category induces a monad. Thus, there is a 2-functor $i : \mathbf{Mnd} \to \mathbf{Adj}$, where **Adj** is the universal 2-category containing an adjunction, also called *the free adjunction* by Schanuel and Street in [19]. Let us denote by B the domain of the left adjoint in **Adj** and A its codomain. These two objects are distinct and the only objects of **Adj**. Auderset in [2, Proposition 2.2] showed that i is actually fully faithful. The 2-category **Mnd** can thus be described as the full sub 2-category of **Adj** generated by B. Moreover, in [2, Theorem 4.4], Auderset shows the following.

2.1.4. PROPOSITION. *Let $\mathcal{T}$ be a monad over B in a 2-category $\mathscr{C}$ and $\mathbb{T} : \mathbf{Mnd} \longrightarrow \mathscr{C}$ the corresponding 2-functor. When $\mathscr{C}$ is complete, the enriched right Kan extension of the 2-functor $\mathbb{T} : \mathbf{Mnd} \longrightarrow \mathscr{C}$ along $i : \mathbf{Mnd} \longrightarrow \mathbf{Adj}$ exists and is the free forgetful adjunction in $\mathscr{C}$:*

$$B \underset{u}{\overset{f}{\rightleftarrows}} B^{\mathcal{T}}. \tag{2}$$

It follows that $B^{\mathcal{T}}$ can be expressed as the weighted limit $\{\mathbf{Adj}(A, i(-)), \mathbb{T}\}$.

Remark that the assumption of $\mathscr{C}$ being complete is made only to ensure the existence of any enriched right Kan extension with target $\mathscr{C}$, and is stronger than needed. It can be weakened to the existence of only those limits needed to construct that particular right Kan extension. We will not explore this aspect, but *finitely complete* is a sufficient hypothesis (see [10, page 92] for instance).

2.1.5. EXAMPLE. Let $\mathscr{C} = \mathbf{Cat}$ and let $\mathscr{B} \underset{R}{\overset{L}{\rightleftarrows}} \mathscr{A}$ be an adjunction with $L \dashv R$, unit η and counit ϵ. This determines a 2-functor $\mathbb{A} : \mathbf{Adj} \to \mathbf{Cat}$, and thus also a monad $\mathbb{A} \circ i : \mathbf{Mnd} \to \mathbf{Cat}$, which is $\mathcal{T} = (\mathscr{B}, RL, R\epsilon L, \eta)$.

The object of algebras $\mathscr{B}^{\mathcal{T}}$ is the usual category of algebras over $\mathcal{T}$, and the adjunction (2) is the usual free-forgetful adjunction. Let us describe the 2-natural transformation
$\begin{tikzcd} \mathbf{Adj} \arrow[r, bend left, "\mathbb{A}"] \arrow[r, bend right, "\mathrm{Ran}_i(\mathbb{A}\circ i)"'] \arrow[r, phantom, "\alpha\Downarrow"] & \mathbf{Cat} \end{tikzcd}$ given by the universal property of the enriched right Kan extension. It is given by $\alpha_B = \mathrm{id}_{\mathscr{B}}$ and $\alpha_A = \mathrm{Can}_{\mathbb{A}} : \mathscr{A} \to \mathscr{B}^{\mathcal{T}}$ on the other object, where $\mathrm{Can}_{\mathbb{A}}$ is defined by

- $\mathrm{Can}_{\mathbb{A}}(A) = (RA, R\epsilon)$ for all $A \in |\mathscr{A}|$;
- $\mathrm{Can}_{\mathbb{A}}(f) = Rf$ for all $f \in \mathrm{Mor}\, \mathscr{A}$.

2.2. Relations between monad morphisms and adjunction morphisms. Our goal is to generalize the results of the previous section so as to include monad morphisms in the picture. We introduce the following definitions.

2.2.1. Definition. *The* 2*-category* $\mathbf{Mnd}[n]$ *is the universal* 2*-category containing* n *composable monad morphisms (with* $\mathbf{Mnd}[0] = \mathbf{Mnd}$*).*

2.2.2. Definition. *An* adjunction morphism *from an adjunction* $B \underset{r}{\overset{l}{\underset{\perp}{\rightleftarrows}}} A$ *to an adjunction* $B' \underset{r'}{\overset{l'}{\underset{\perp}{\rightleftarrows}}} A'$ *in a* 2*-category* $\mathscr{C}$ *is a pair of* 1*-cells* $b : B \longrightarrow B'$ *and* $a : A \longrightarrow A'$ *such that* $br = r'a$.

2.2.3. Definition. *The* 2*-category* $\mathbf{Adj}[n]$ *is the universal* 2*-category containing* n *composable adjunction morphisms (with* $\mathbf{Adj}[0] = \mathbf{Adj}$*).*

As before, by Yoneda's lemma, $\mathbf{Mnd}[n]$ and $\mathbf{Adj}[n]$ are determined up to isomorphism by their 2-universal property. Their existence can be proven using presentation by computads. For instance, the coequalizer $\mathcal{ADJ}[n]$ displayed in (3) below satisfies the universal property of $\mathbf{Adj}[n]$. We notationally distinguish this coequalizer from the explicit model that we build in Section 3.2, which we denote by $\mathbf{Adj}[n]$. We prove in Theorem 3.3.9 that $\mathcal{ADJ}[n] \cong \mathbf{Adj}[n]$. Hence, this is just a notational convenience made for the proof of Theorem 3.3.9. Observe that sequences of $n+1$ monads in $\mathscr{C}$ together with n composable monad morphisms between them are in bijective correspondence with functors $[n] \to \mathbf{Lax}(\mathbf{Mnd}, \mathscr{C})$, where $\mathbf{Lax}(\mathbf{Mnd}, \mathscr{C})$ denotes the category whose objects are 2-functors $\mathbf{Mnd} \to \mathscr{C}$ and whose morphisms are lax natural transformations. A short introduction to lax natural transformations is provided in Appendix B.

This implies that

$$\mathbf{Mnd}[n] \cong \mathbf{Mnd} \otimes_{\mathbf{Lax}} [n],$$

where $\otimes_{\mathbf{Lax}}$ denotes the lax Gray tensor product on 2-$\mathbf{Cat}$, as defined in [7]. This follows from the Yoneda lemma and the bijection

$$2\text{-}\mathbf{Cat}(\mathbf{Mnd} \otimes_{\mathbf{Lax}} [n], \mathscr{C}) \cong 2\text{-}\mathbf{Cat}([n], \mathbf{Lax}(\mathbf{Mnd}, \mathscr{C})).$$

On the other hand, $\mathbf{Adj}[n]$ cannot be expressed as $\mathbf{Adj} \otimes_{\mathbf{Lax}} [n]$, because in $\mathbf{Adj}[n]$ the squares involving the right adjoints commute *on the nose* rather than up to a 2-cell isomorphism.

We construct now the 2-category $\mathcal{ADJ}[n]$ using a presentation by a computad. A similar construction can be performed to obtain $\mathbf{Mnd}[n]$. A reader not familiar with presentations by computads can consult Appendix A.

We define a computad $\mathrm{Adj}[n]$ as follows. Its graph $\mathrm{GrAdj}[n]$ is given by

$$\begin{tikzcd}
(B,n) \arrow[r, shift left, "L_n"] \arrow[d, "B_n"] & (A,n) \arrow[l, shift left, "R_n"] \arrow[d, "A_n"] \\
(B,n-1) \arrow[r, shift left, "L_{n-1}"] \arrow[d, "B_{n-1}"] & (A,n-1) \arrow[l, shift left, "R_{n-1}"] \arrow[d, "A_{n-1}"] \\
\vdots \arrow[d, "B_1"] & \vdots \arrow[d, "A_1"] \\
(B,0) \arrow[r, shift left, "L_0"] & (A,0) \arrow[l, shift left, "R_0"]
\end{tikzcd}$$

and non-trivial graphs $\mathrm{Adj}[n](X,Y)$ are given by $\mathrm{Adj}[n]((B,i),(B,i)) =$ $\begin{tikzcd}(B,i) \arrow[r, bend left, "\emptyset"] \arrow[r, bend right, "R_iL_i"'] \arrow[r, phantom, "\eta_i\Downarrow"] & (B,i)\end{tikzcd}$ and $\mathrm{Adj}[n]((A,i),(A,i)) =$ $\begin{tikzcd}(A,i) \arrow[r, bend left, "L_iR_i"] \arrow[r, bend right, "\emptyset"'] \arrow[r, phantom, "\epsilon_i\Downarrow"] & (A,i)\end{tikzcd}$. The computad $\mathrm{Rel}[n]$ has underlying graph

$$\begin{tikzcd}
(B,n) \arrow[r, shift left, "L_n"] \arrow[d, "B_n"] & (A,n) \arrow[l, shift left, "R_n"] \arrow[d, "A_n"] \arrow[dl, "C_n"] \\
(B,n-1) \arrow[r, shift left, "L_{n-1}"] \arrow[d, "B_{n-1}"] & (A,n-1) \arrow[l, shift left, "R_{n-1}"] \arrow[d, "A_{n-1}"] \arrow[dl, "C_{n-1}"] \\
\vdots \arrow[d, "B_1"] & \vdots \arrow[d, "A_1"] \arrow[dl, "C_1"] \\
(B,0) \arrow[r, shift left, "L_0"] & (A,0) \arrow[l, shift left, "R_0"]
\end{tikzcd}$$

and non-trivial graphs $\mathrm{Rel}[n](X,Y)$ are given by $\mathrm{Rel}[n]((B,i),(A,i)) =$ $\begin{tikzcd}(B,i) \arrow[r, bend left, "L_i"] \arrow[r, bend right, "L_i"'] \arrow[r, phantom, "\lambda_i\Downarrow"] & (A,i)\end{tikzcd}$ and $\mathrm{Rel}[n]((A,i),(B,i)) =$ $\begin{tikzcd}(A,i) \arrow[r, bend left, "R_i"] \arrow[r, bend right, "R_i"'] \arrow[r, phantom, "\rho_i\Downarrow"] & (B,i)\end{tikzcd}$.

We define two 2-functors $M, N : \mathcal{F}(\mathrm{Rel}[n]) \to \mathcal{F}(\mathrm{Adj}[n])$ that are identities on objects, by $M(S) = N(S) = S$ for $S \in \{L_i, R_i, A_i, B_i\}$, $M(C_i) = B_i \cdot R_i$, $N(C_i) = R_{i-1} \cdot A_i$, $M(\rho_i) = (\epsilon_i R_i) \circ (R_i \eta_i)$, $N(\rho_i) = 1_{R_i}$, $M(\lambda_i) = (L_i \epsilon_i) \circ (\eta_i L_i)$, $N(\lambda_i) = 1_{L_i}$.

The 2-category $\mathcal{ADJ}[n]$ is the coequalizer (in 2-**Cat**) in the diagram

$$\mathcal{F}(\mathrm{Rel}[n]) \underset{N}{\overset{M}{\rightrightarrows}} \mathcal{F}(\mathrm{Adj}[n]) \longrightarrow \mathcal{ADJ}[n]. \tag{3}$$

The universal properties of the free 2-category on a computad and of the coequalizer imply that $\mathcal{ADJ}[n]$ satisfies the 2-universal property we were looking for.

2.2.4. Proposition. *An adjunction morphism between two adjunctions induces a monad morphism between the induced monads.*

Proof. Let $b : B \longrightarrow B'$ and $a : A \longrightarrow A'$ form an adjunction morphism from an adjunction $\begin{tikzcd} B \arrow[r, shift left, "l"] \arrow[r, phantom, "\perp"] & A \arrow[l, shift left, "r"] \end{tikzcd}$ to an adjunction $\begin{tikzcd} B' \arrow[r, shift left, "l'"] \arrow[r, phantom, "\perp"] & A' \arrow[l, shift left, "r'"] \end{tikzcd}$.

Let β be the mate of the identity $br \to r'a$, that is, the composite

$$\begin{tikzcd}
B \arrow[r, "l"] \arrow[dr, "1"'] & A \arrow[r, "a"] \arrow[d, "r"] \arrow[dl, phantom, "\Uparrow\eta"] & A' \arrow[d, "r'"] \arrow[dl, equal] \arrow[dr, "1"] & \\
& B \arrow[r, "b"'] & B' \arrow[r, "l'"'] \arrow[ur, phantom, "\Uparrow\epsilon'"] & A'.
\end{tikzcd}$$

We claim that $(b, r'\beta)$ is a monad morphism $(rl, r\epsilon l, \eta) \to (r'l', r'\epsilon' l', \eta')$.

Remark first that $r'\beta : r'l'b \longrightarrow r'al = brl$, as desired. The compatibility with the unit is given by the triangle identity of the second adjunction and the interchange law

$$\begin{tikzcd}
B \arrow[r, "l"] \arrow[dr, "1"'] & A \arrow[r, "a"] \arrow[d, "r"] \arrow[dl, phantom, "\Uparrow\eta"] & A' \arrow[d, "r'"] \arrow[dl, equal] \arrow[dr, "1"] & \\
& B \arrow[r, "b"'] & B' \arrow[r, "l'"] \arrow[ur, phantom, "\Uparrow\epsilon'"] \arrow[dr, "1"'] & A' \arrow[d, "r'"] \arrow[dl, phantom, "\Uparrow\eta'"] \\
& & & B'
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
B \arrow[r, "l"] \arrow[dr, "1"'] & A \arrow[r, "a"] \arrow[d, "r"] \arrow[dl, phantom, "\Uparrow\eta"] & A' \arrow[d, "r'"] \arrow[dl, equal] \\
& B \arrow[r, "b"'] & B'.
\end{tikzcd}$$

The compatibility condition with the multiplication is given by the triangle identity of the first adjunction and the interchange law

$$\begin{tikzcd}
B \arrow[r, "1"] \arrow[d, "l"'] \arrow[d, phantom, shift left=3, "\eta"] & B \arrow[r, "b"] \arrow[d, equal] & B' \arrow[d, "l'"] \arrow[d, phantom, shift right=3, "\epsilon'"] \\
A \arrow[r, "a"] \arrow[ur, "r"'] \arrow[d, "r"] \arrow[dd, bend right=60, "1"'] & A' \arrow[r, "1"] \arrow[ur, "r'"] & A' \arrow[d, "r'"] \arrow[dll, equal] \\
B \arrow[r, "1"] \arrow[d, "l"'] \arrow[d, phantom, shift left=3, "\eta"] \arrow[l, phantom, "\Leftarrow\epsilon"] & B \arrow[r, "b"] \arrow[d, equal] & B' \arrow[d, "l'"] \arrow[d, phantom, shift right=3, "\epsilon'"] \\
A \arrow[r, "a"] \arrow[ur, "r"'] \arrow[d, "r"'] & A' \arrow[r, "1"] \arrow[ur, "r'"] & A' \arrow[d, "r'"] \arrow[dll, equal] \\
B \arrow[rr, "b"'] & & B'
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
B \arrow[r, "1"] \arrow[d, "l"'] \arrow[d, phantom, shift left=3, "\eta"] & B \arrow[r, "b"] \arrow[d, equal] & B' \arrow[d, "l'"] \arrow[d, phantom, shift right=3, "\epsilon'"] \\
A \arrow[r, "a"] \arrow[ur, "r"'] & A' \arrow[r, "1"] \arrow[ur, "r'"] & A' \arrow[d, "r'"] \arrow[dd, bend right=60, "1"'] \\
& & B' \arrow[d, "l'"] \arrow[l, phantom, "\Leftarrow\epsilon'"] \\
& & A' \arrow[d, "r'"] \\
& & B'.
\end{tikzcd}$$

■

Proposition 2.2.4 implies that there is an induced 2-functor $j : \mathbf{Mnd}[n] \to \mathbf{Adj}[n]$, which satisfies the following important property.

2.2.5. Proposition. *Let $\mathbb{T} : \mathbf{Mnd}[n] \longrightarrow \mathbf{Cat}$ be the 2-functor induced by the n composable morphisms of monads $\mathcal{T}_n \xrightarrow{(B_n,\gamma_n)} \dots \xrightarrow{(B_1,\gamma_1)} \mathcal{T}_0$ where $\mathcal{T}_i$ is a monad (T_i, μ_i, η_i) on a category $\mathscr{B}_i$. There is a diagram*

$$\begin{tikzcd}
\mathscr{B}_n \arrow[r, shift left, "\mathcal{F}_n"] \arrow[d, "B_n"] & \mathscr{B}_n^{\mathcal{T}_n} \arrow[l, shift left, "U_n"] \arrow[d, "\overline{(B_n,\gamma_n)}"] \\
\mathscr{B}_{n-1} \arrow[r, shift left, "\mathcal{F}_{n-1}"] \arrow[d] & \mathscr{B}_{n-1}^{\mathcal{T}_{n-1}} \arrow[l, shift left, "U_{n-1}"] \arrow[d] \\
\vdots \arrow[d, "B_1"] & \vdots \arrow[d, "\overline{(B_1,\gamma_1)}"] \\
\mathscr{B}_0 \arrow[r, shift left, "\mathcal{F}_0"] & \mathscr{B}_0^{\mathcal{T}_0} \arrow[l, shift left, "U_0"]
\end{tikzcd}$$

which determines a 2-functor $\mathbb{A} : \mathbf{Adj}[n] \longrightarrow \mathbf{Cat}$. Moreover, this 2-functor is the enriched right Kan extension of $\mathbb{T}$ along $j : \mathbf{Mnd}[n] \to \mathbf{Adj}[n]$ and $\mathbb{A}j = \mathbb{T}$ (the natural transformation is the identity).

Proof. Let $0 < i \leq n$ be an integer. Recall that if (X, m) is a $\mathcal{T}_i$-algebra, then

$$\overline{(B_i, \gamma_i)}(X, m) = (B_i X, B_i(m) \circ (\gamma_i)_X),$$

and for f a morphism of algebras, $\overline{(B_i, \gamma_i)}(f) = B_i(f)$, thus the squares involving the right adjoints commute on the nose. This determines n composable adjunction morphisms and thus a 2-functor $\mathbb{A} : \mathbf{Adj}[n] \longrightarrow \mathbf{Cat}$.

We compute now the monad morphism $(B_i, \tilde{\gamma}_i)$ induced by the morphism of adjunction

$$\begin{tikzcd}
\mathscr{B}_i \arrow[r, shift left, "\mathcal{F}_i"] \arrow[d, "B_i"] & \mathscr{B}_i^{\mathcal{T}_i} \arrow[l, shift left, "U_i"] \arrow[d, "\overline{(B_i,\gamma_i)}"] \\
\mathscr{B}_{i-1} \arrow[r, shift left, "\mathcal{F}_{i-1}"] & \mathscr{B}_{i-1}^{\mathcal{T}_{i-1}} \arrow[l, shift left, "U_{i-1}"]
\end{tikzcd}.$$

Let us recall the unit and counit of the free forgetful adjunction on level i.

- The unit is $\eta_i : 1_{\mathscr{B}_i} \to U_i \mathcal{F}_i = T_i$.
- The counit ϵ_i is given on the $\mathcal{T}_i$-algebra (X, m) by $\epsilon_{(X,m)} = m : T_i X \to X$.

Thus, the 2-cell $\tilde{\gamma}_i$ that makes $(B_i, \tilde{\gamma}_i)$ the associated monad morphism $\mathcal{T}_i \to \mathcal{T}_{i-1}$ at the object $X \in \mathscr{B}_i$ is the composite

$$T_{i-1} B_i(X) \xrightarrow{(T_{i-1} B_i \eta_i)_X} T_{i-1} B_i(T_i X) \xrightarrow{(\gamma_i T_i)_X} B_i T_i^2 X \xrightarrow{(B_i \mu_i)_X} B_i T_i X.$$

Since (B_i, γ_i) is a monad morphism, the unit condition implies that this is equal to the composite

$$T_{i-1}B_i(X) \xrightarrow{(T_{i-1}\eta_{i-1}B_i)_X} T_{i-1}^2 B_i(X) \xrightarrow{(T_{i-1}\gamma_i)_X} T_{i-1}B_i(T_iX) \xrightarrow{(\gamma_i T_i)_X} B_iT_i^2X \xrightarrow{(B_i\mu_i)_X} B_i\mathcal{T}_iX.$$

Now the compatibility with the multiplications implies that this is equal to

$$T_{i-1}B_i(X) \xrightarrow{(T_{i-1}\eta_{i-1}B_i)_X} T_{i-1}^2 B_i(X) \xrightarrow{(\mu_{i-1}B_i)_X} T_{i-1}B_i(X) \xrightarrow{(\gamma_i)_X} B_iT_iX$$

and thus $\tilde{\gamma}_i = \gamma_i$. This shows that $\mathbb{A}j = \mathbb{T}$.

We are going to use the fact that the enriched right Kan extension can be recognized by its universal property. Its existence is a consequence of **Cat** being complete as a 2-category. Let

$$\begin{tikzcd} \mathscr{C}_i \arrow[r, shift left, "L_i"] \arrow[d, "C_i"] & \mathscr{D}_i \arrow[l, shift left, "R_i"] \arrow[d, "D_i"] \\ \mathscr{C}_{i-1} \arrow[r, shift left, "L_{i-1}"] & \mathscr{D}_{i-1} \arrow[l, shift left, "R_{i-1}"] \end{tikzcd}$$

be a morphism of adjunctions for all $i = 1, \ldots, n$ where the adjunction $L_i \dashv R_i$ has unit $\tilde{\eta}_i$ and counit $\tilde{\epsilon}_i$. The adjunction $L_i \dashv R_i$ determines a 2-functor $\mathbb{B}_i : \mathbf{Adj} \to \mathbf{Cat}$. The whole structure determines a 2-functor $\mathbb{B} : \mathbf{Adj}[n] \longrightarrow \mathbf{Cat}$. Let $\mathcal{P}_i$ be the monad on $\mathscr{C}_i$ induced by the adjunction $L_i \dashv R_i$. By 2.2.4, the monad morphism $\mathcal{P}_i \to \mathcal{P}_{i-1}$ induced by the adjunction morphism (C_i, D_i) is (C_i, δ_i), where

$$\delta_i = R_{i-1}((\tilde{\epsilon}_{i-1}D_iL_i) \circ (L_{i-1}C_i\tilde{\eta}_i)).$$

Let $\psi : \mathbb{B}j \longrightarrow \mathbb{T}$ be a 2-natural transformation. We thus have functors $\psi_i : \mathscr{C}_i \longrightarrow \mathscr{B}_i$ such that

(i) $\psi_iR_iL_i = T_i\psi_i$ and $\psi_i\tilde{\eta}_i = \eta_i\psi_i$, $\psi_iR_i\tilde{\epsilon}_iL_i = \mu_i\psi_i$;

(ii) $\psi_{i-1}C_i = B_i\psi_i$ and $\gamma_i\psi_i = \psi_{i-1}R_{i-1}((\tilde{\epsilon}_{i-1}D_iL_i) \circ (L_{i-1}C_i\tilde{\eta}_i))$.

We need to show that there is a unique 2-natural transformation $\omega : \mathbb{B} \longrightarrow \mathbb{A}$ such that $\omega j = \psi$. Said differently, we have to show that ψ extends uniquely to a 2-natural transformation $\mathbb{B} \longrightarrow \mathbb{A}$.

The relations (i) above show that ψ_i is a strict monad morphism from the monad to the monad $\mathcal{T}_i$ and thus induces a functor $(\mathscr{C}_i)^{\mathcal{P}_i} \longrightarrow (\mathscr{B}_i)^{\mathcal{T}_i}$. Precomposing with $\mathrm{Can}_{\mathbb{B}_i} : \mathscr{D}_i \longrightarrow (\mathscr{C}_i)^{\mathcal{P}_i}$ (see Example 2.1.5) yields $\rho_i : \mathscr{D}_i \longrightarrow (\mathscr{B}_i)^{\mathcal{T}_i}$.

We are now going to show that the 2-transformation $\omega : \mathbb{B} \to \mathbb{A}$ given on level i by ψ_i, ρ_i is 2-natural. It is enough to check naturality with respect to the generating 1-cells and 2-cells. To deal with naturality with respect to 1-cells, it is enough to show that in the diagram

$$\begin{tikzcd}
 & \mathscr{B}_i \arrow[rr, shift left, "\mathcal{F}_i"] \arrow[dd, "B_i"] & & \mathscr{B}_i^{\mathcal{T}_i} \arrow[ll, shift left, "U_i"] \arrow[dd, "\overline{(B_i,\gamma_i)}"] \\
\mathscr{C}_i \arrow[ur, "\psi_i"] \arrow[rr, shift left, "L_i"] \arrow[dd, "C_i"'] & & \mathscr{D}_i \arrow[ll, shift left, "R_i"] \arrow[ur, "\rho_i"] \arrow[dd, "D_i"] & \\
 & \mathscr{B}_{i-1} \arrow[rr, shift left, "\mathcal{F}_{i-1}"] & & \mathscr{B}_{i-1}^{\mathcal{T}_{i-1}}, \arrow[ll, shift left, "U_{i-1}"] \\
\mathscr{C}_{i-1} \arrow[ur, "\psi_{i-1}"] \arrow[rr, shift left, "L_{i-1}"] & & \mathscr{D}_{i-1} \arrow[ll, shift left, "R_{i-1}"] \arrow[ur, "\rho_{i-1}"] &
\end{tikzcd}$$

all squares other than the front and back squares are commutative. Example 2.1.5 implies that the top and bottom squares are commutative, while the left square is commutative by definition. For the right square, relation (ii) implies that $(\psi_{i-1}, 1) \circ (C_i, \delta_i) = (B_i, \gamma_i) \circ (\psi_i, 1)$, and thus the right square of the diagram

$$\begin{tikzcd}
\mathscr{D}_i \arrow[r, "\mathrm{Can}_{\mathbb{B}_i}"] \arrow[d, "D_i"'] \arrow[rr, bend left, "\rho_i"] & \mathscr{C}_i^{\mathcal{P}_i} \arrow[r, "\overline{(\psi_i,1)}"] \arrow[d, "\overline{(C_i,\delta_i)}"'] & \mathscr{B}_i^{\mathcal{T}_i} \arrow[d, "\overline{(B_i,\gamma_i)}"] \\
\mathscr{D}_{i-1} \arrow[r, "\mathrm{Can}_{\mathbb{B}_{i-1}}"] \arrow[rr, bend right, "\rho_{i-1}"'] & \mathscr{C}_{i-1}^{\mathcal{P}_{i-1}} \arrow[r, "\overline{(\psi_{i-1},1)}"] & \mathscr{B}_{i-1}^{\mathcal{T}_{i-1}}
\end{tikzcd}$$

is commutative. The triangle identity of the adjunction $L_i \dashv R_i$ implies that the left square commutes also. Finally, we need to check naturality with respect to the counits. Recall that the counit ϵ_i of the free-forgetful adjunction is given by $(\epsilon_i)_{(B_i,m_i)} = m_i$. Thus,

$$\epsilon_i \rho_i = \psi_i R_i \tilde{\epsilon}_i = U_i \rho_i \tilde{\epsilon}_i = \rho_i \tilde{\epsilon}_i.$$

Remark that the equation $U_i \rho_i = R_i \psi_i$ determines the underlying object of the $\mathcal{T}_{i-1}$-algebra $\rho_i(D)$ for all $D \in |\mathscr{D}_i|$, and also how ρ_i acts on morphisms. The equation $\epsilon_i \rho_i = \rho_i \tilde{\epsilon}_i$ determines the multiplication of $\rho_i(D)$, establishing uniqueness. ∎

The proposition below is a general result of enriched category theory. One can find it in a slightly different form together with its proof in [9, Paragraph 1.10].

2.2.6. Proposition. *Let $\mathscr{V}$ be a symmetric monoidal category and let $\mathscr{X}, \mathscr{C}$ be $\mathscr{V}$-categories. Let $F : \mathscr{X} \to [\mathscr{C}^{\mathrm{op}}, \mathscr{V}]$ be a $\mathscr{V}$-functor such that its image is contained in the essential image of the Yoneda embedding $Y : \mathscr{C} \to [\mathscr{C}^{\mathrm{op}}, \mathscr{V}]$. For each $X \in |\mathscr{X}|$, choose a representation of $F(X)$, $\alpha_X : \mathscr{C}(-, KX) \to F(X)$ (α_X is a $\mathscr{V}$-natural isomorphism).*

Then, there exists a unique $\mathscr{V}$-functor $K : \mathscr{X} \to \mathscr{C}$ such that

$$\alpha : YK \to F$$

is $\mathscr{V}$-natural.

2.2.7. Definition. *Let $F : \mathscr{X} \to [\mathscr{C}^{\mathrm{op}}, \mathscr{V}]$ be a $\mathscr{V}$-functor such that its image is contained in the essential image of the Yoneda embedding $Y : \mathscr{C} \to [\mathscr{C}^{\mathrm{op}}, \mathscr{V}]$.* A representation *of F is defined to be the choice of a $\mathscr{V}$-functor $K : \mathscr{X} \to \mathscr{C}$ together with a $\mathscr{V}$-isomorphism $YK \to F$.*

2.2.8. Theorem. *Let $\mathscr{C}$ be a complete 2-category and $\mathbb{T} : \mathbf{Mnd}[n] \longrightarrow \mathscr{C}$ a 2-functor corresponding to n composable morphisms of monads $\mathcal{T}_n \xrightarrow{(b_n,\gamma_n)} \dots \xrightarrow{(b_1,\gamma_1)} \mathcal{T}_0$ where $\mathcal{T}_i$ is a monad (T_i, μ_i, η_i) on an object B_i.*

Then, the enriched right Kan extension of $\mathbb{T}$ along j is given by a representation of the diagram

$$\begin{tikzcd}
\mathscr{C}(-,B_n) \arrow[r, shift left, "\mathscr{C}(-,f_n)"] \arrow[d, "\mathscr{C}(-,b_n)"] & \mathscr{C}(-,B_n)^{\mathcal{T}_{n*}(-)} \arrow[l, shift left, "\mathscr{C}(-,u_n)"] \arrow[d, "\overline{\mathscr{C}(-,b_n),\mathscr{C}(-,\gamma_n)}"] \\
\mathscr{C}(-,B_{n-1}) \arrow[r, shift left, "\mathscr{C}(-,f_{n-1})"] \arrow[d] & \mathscr{C}(-,B_{n-1})^{\mathcal{T}_{n-1*}(-)} \arrow[l, shift left, "\mathscr{C}(-,u_{n-1})"] \arrow[d] \\
\vdots \arrow[d] & \vdots \arrow[d] \\
\mathscr{C}(-,B_0) \arrow[r, shift left, "\mathscr{C}(-,f_0)"] & \mathscr{C}(-,B_0)^{\mathcal{T}_{0*}(-)} \arrow[l, shift left, "\mathscr{C}(-,u_0)"]
\end{tikzcd} \tag{4}$$

which can be chosen to be

$$\begin{tikzcd}
B_n \arrow[r, shift left, "f_n"] \arrow[d, "b_n"] & B_n^{\mathcal{T}_n} \arrow[l, shift left, "u_n"] \arrow[d] \\
B_{n-1} \arrow[r, shift left, "f_{n-1}"] \arrow[d] & B_{n-1}^{\mathcal{T}_{n-1}} \arrow[l, shift left, "u_{n-1}"] \arrow[d] \\
\vdots \arrow[d] & \vdots \arrow[d] \\
B_0 \arrow[r, shift left, "f_0"] & B_0^{\mathcal{T}_0}. \arrow[l, shift left, "u_0"]
\end{tikzcd}$$

The lemma below is the key of the proof of Theorem 2.2.8.

2.2.9. Lemma. *Let* $\begin{tikzcd} \mathscr{A} \arrow[r, "F"] \arrow[d, "J"] & {[\mathscr{C}^{\mathrm{op}}, \mathscr{V}]} \\ \mathscr{B} \end{tikzcd}$ *be a diagram of $\mathscr{V}$-functors. The enriched right Kan extension of F along J can be obtained as the adjunct of the composite*

$$\mathscr{C}^{\mathrm{op}} \longrightarrow [\mathscr{A}, \mathscr{V}] \xrightarrow{\mathrm{Ran}_J} [\mathscr{B}, \mathscr{V}].$$

Proof of lemma 2.2.9. Consider the diagram

$$\begin{tikzcd}
{[\mathscr{A},[\mathscr{C}^{\mathrm{op}},\mathscr{V}]]} \arrow[r, shift left, "\cong"] \arrow[d, shift left, "\mathrm{Ran}_J"] & {[\mathscr{C}^{\mathrm{op}},[\mathscr{A},\mathscr{V}]]} \arrow[l, shift left] \arrow[d, shift left, "{[1,\mathrm{Ran}_J]}"] \\
{[\mathscr{B},[\mathscr{C}^{\mathrm{op}},\mathscr{V}]]} \arrow[r, shift left, "\cong"] \arrow[u, shift left, "J^*"] & {[\mathscr{C}^{\mathrm{op}},[\mathscr{B},\mathscr{V}]]} \arrow[l, shift left] \arrow[u, shift left, "{[1,J^*]}"]
\end{tikzcd}$$

Since the diagram of left adjoints commutes, the diagram of right adjoints commutes up to isomorphism. ∎

Proof of theorem 2.2.8. Let $\mathrm{Ran}_j\mathbb{T} : \mathbf{Adj}[n] \to \mathscr{C}$ be the enriched right Kan extension. Since the Yoneda embedding $Y : \mathscr{C} \to [\mathscr{C}^{\mathrm{op}}, \mathbf{Cat}]$ preserves weighted limits, $Y\mathrm{Ran}_j\mathbb{T} \cong \mathrm{Ran}_j(Y\mathbb{T})$. By the uniqueness statement of Proposition 2.2.6, we only have to show that for $X \in |\mathbf{Adj}[n]|$, $\mathrm{Ran}_j(Y\mathbb{T})(X) : \mathscr{C}^{\mathrm{op}} \to \mathbf{Cat}$ is the corresponding representable.

By the previous lemma, $\mathrm{Ran}_j(Y\mathbb{T})$ is isomorphic to the adjunct of the composite

$$\mathscr{C}^{\mathrm{op}} \xrightarrow{\mathscr{C}(-,\mathbb{T}(-))} [\mathbf{Mnd}[n], \mathbf{Cat}] \xrightarrow{\mathrm{Ran}_j} [\mathbf{Adj}[n], \mathbf{Cat}],$$

which we denote by $\mathrm{ad}_1(\mathrm{Ran}_j \circ \mathscr{C}(-,\mathbb{T}(-)))$. We also denote by $\mathrm{ad}_2(\mathrm{Ran}_j \circ \mathscr{C}(-,\mathbb{T}(-)))$ its adjunct $\mathscr{C}^{\mathrm{op}} \times \mathbf{Adj}[n] \to \mathbf{Cat}$.

By Proposition 2.2.5 and its proof, we can describe

$$\mathrm{Ran}_j : [\mathbf{Mnd}[n], \mathbf{Cat}] \to [\mathbf{Adj}[n], \mathbf{Cat}].$$

To a 2-functor $\mathbb{X} : \mathbf{Mnd}[n] \to \mathbf{Cat}$, it associates its enriched right Kan extension given by morphisms of adjunctions induced by the monad morphisms, between the corresponding free-forgetful adjunctions.

To a 2-natural transformation $\alpha : \mathbb{X} \Rightarrow \mathbb{X}'$, it associates the 2-natural transformation $\mathrm{Ran}_j\alpha : \mathrm{Ran}_j\mathbb{X} \Rightarrow \mathrm{Ran}_j\mathbb{X}'$ determined by the universal property of $\mathrm{Ran}_j\mathbb{X}'$ applied to the couple $(\mathrm{Ran}_j\mathbb{X}, \alpha \cdot j)$. From the proof, this extension of α is given by the induced (strict) morphisms of monads.

For a modification $\chi : \alpha \to \alpha'$, its extension $\mathrm{Ran}_j\chi : \mathrm{Ran}_j\alpha \to \mathrm{Ran}_j\alpha'$ is given on the algebras object by the same natural transformation as on the base object. This is the only possible choice because of the relation it must satisfy with respect to the forgetful functor, and thus a valid choice.

As a consequence,

$$\mathrm{ad}_2(\mathrm{Ran}_j \circ \mathscr{C}(-,\mathbb{T}(-))) : \mathscr{C}^{\mathrm{op}} \times \mathbf{Adj}[n] \to \mathbf{Cat}$$

is the diagram (4). ∎

2.2.10. Remark. The theorem above can be interpreted with target category $\mathscr{C}^{\mathrm{op}}, \mathscr{C}^{\mathrm{co}}$ and $\mathscr{C}^{\mathrm{coop}}$ instead of $\mathscr{C}$. In the first case, it gives the morphism between the Kleisli objects induced by an oplax monad morphism. In the second case, it gives the morphism between the Eilenberg-Moore coalgebra objects induced by an oplax comonad morphism. The last case gives the morphism between the co-Kleisli objects induced by a lax comonad morphism.

2.2.11. Corollary. *The* 2*-functor* $j : \mathbf{Mnd}[n] \longrightarrow \mathbf{Adj}[n]$ *is fully faithful.*

Proof. This is a consequence of the previous proposition and the dual of [9, Proposition 4.23]. ■

3. An explicit description of the 2-category $\mathbf{Adj}[n]$

The goal of this section is to construct an explicit model for the 2-category $\mathbf{Adj}[n]$, and thus derive one for $\mathbf{Mnd}[n]$, thanks to Corollary 2.2.11.

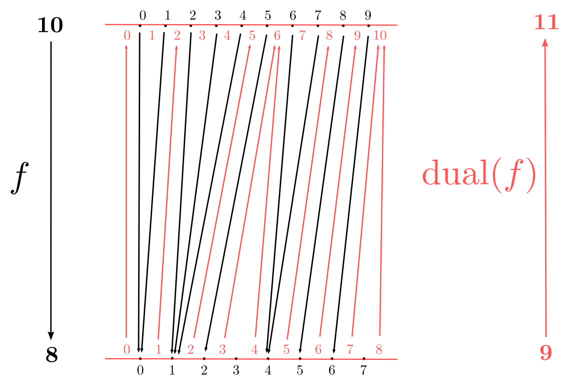
Let us first review the combinatorial model of $\mathbf{Adj}$ provided by Auderset, Schanuel and Street, because we are going to build on top of this construction.

3.1. Review of the explicit model of $\mathbf{Adj}$. In [2], Auderset defines $\mathbf{Adj}$ up to isomorphism by its universal property, computes half of the hom-categories, and shows that the other half are duals of the computed ones. The 2-category $\mathbf{Adj}$ has exactly two objects, A and B, and Auderset provided a quotient-free description of the hom-categories, which are

- $\mathbf{Adj}(B, B) = \Delta_+$, the category of possibly empty finite ordinals with non-decreasing maps;
- $\mathbf{Adj}(B, A) = \Delta_{-\infty}$, the category of non-empty finite ordinals with non-decreasing maps that preserve the minimal element;
- $\mathbf{Adj}(A, B) = \Delta_{-\infty}^{\mathrm{op}}$;
- $\mathbf{Adj}(A, A) = \Delta_+^{\mathrm{op}}$.

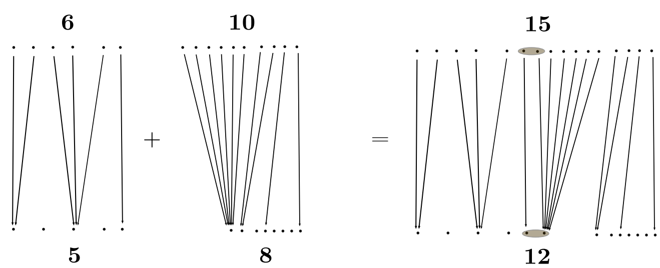
Unfortunately, the composition maps are not easy to express from this point of view, for instance $\Delta_{-\infty}^{\mathrm{op}} \times \Delta_+ \longrightarrow \Delta_{-\infty}^{\mathrm{op}}$ is not clear in Auderset's article.

There is a well-known isomorphism $\mathrm{dual} : \Delta_+^{\mathrm{op}} \longrightarrow \Delta_{-\infty,+\infty}$, (that one can call *Stone duality for intervals*, following for instance Andrade's thesis [1, page 206]) where the codomain is the category of non-empty finite ordinals with non-decreasing maps that preserve the minimal and maximal elements. Another name for this correspondence is *interval representation*, because one can have the following picture in mind.



In [19], Schanuel and Street use Stone duality for intervals and express **Adj** in the following way.

- $\mathbf{Adj}(B, B)$ and $\mathbf{Adj}(B, A)$ are as before;
- $\mathbf{Adj}(A, B) = \Delta_{+\infty}$, the category of non-empty finite ordinals with non-decreasing maps that preserves the maximal element;
- $\mathbf{Adj}(A, A) = \Delta_{-\infty,+\infty}$;
- the composition $\mathbf{Adj}(Y, Z) \times \mathbf{Adj}(X, Y) \longrightarrow \mathbf{Adj}(X, Z)$ is given by ordinal sum when $Y = B$. When $Y = A$, it is given by a quotient of the ordinal sum, where the minimal element of an object of $\mathbf{Adj}(X, Y)$ is identified with the maximal element of the object of $\mathbf{Adj}(Y, Z)$ in the ordinal sum. One can picture an example of such a composition in the following way,

  where on the right the dark grey region is considered as a point.

The adjunction data is given as follows:

- the left adjoint is $\mathbf{1} \in \mathbf{Adj}(B, A)$ while the right adjoint is $\mathbf{1} \in \mathbf{Adj}(A, B)$;
- the unit of the adjunction is the unique map $\mathrm{id}_B = \mathbf{0} \to \mathbf{1}$;
- the counit of the adjunction is the unique map $\mathbf{2} \to \mathbf{1} = \mathrm{id}_A$.

3.2. **Adj**$[n]$, THE EXPLICIT CONSTRUCTION. In order to build an explicit model, we will use lax comma objects. A brief introduction to them can be found in Appendix B.

Let **TOFS** be the full sub 2-category of **Cat** generated by the Totally Ordered Finite Sets. The 1-cells are non-decreasing maps whereas the 2-cells are given by the order on the 1-cells, as follows. For $f, g : S \to T$, $f \leq g$ if and only if for all $s \in S$, $f(s) \leq g(s)$.

Let $\mathbf{TOFS}_{+\infty}$ be the locally full sub 2-category containing all objects but the empty set and only the 1-cells that preserve the maximal element.

For $X, Y \in |\mathbf{Adj}|$, we define categories $\mathbf{TOFS}_{X,Y}$ by setting

$$\mathbf{TOFS}_{X,Y} = \begin{cases} \mathbf{TOFS} & X \neq A; \\ \mathbf{TOFS}_{+\infty} & X = A. \end{cases}$$

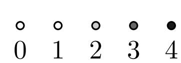
Remark that we have an inclusion $i_{X,Y} : \mathbf{Adj}(X,Y) \to \mathbf{TOFS}_{X,Y}$. We write $\cdot_{\mathbf{Adj}}$ for the horizontal composition in **Adj**. For two integers $y \leq x$, we write $[y, x]$ for the set $[y, x] = \{y, y + 1, \ldots x\}$, which is naturally ordered.

Let us now introduce our explicit model for $\mathbf{Adj}[n]$.

- The set of objects is $\{(X, k) : X \in |\mathbf{Adj}|, k \in \{0, \ldots, n\}\}$.

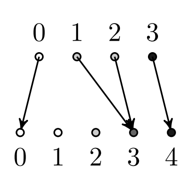
- For the hom-categories, we set $\mathbf{Adj}[n]((X, x), (Y, y)) = \emptyset$ if $x < y$. For $x \geq y$, we define
$$\mathbf{Adj}[n]((X, x), (Y, y)) = i_{X,Y} \downarrow^l [y, x],$$
where $[y, x]$ denotes the inclusion $* \to \mathbf{TOFS}_{X,Y}$ hitting that object. (Note that we keep only the underlying category of this 2-category: objects and 1-cells, which become respectively 1-cells and 2-cells of $\mathbf{Adj}[n]$).

  More explicitly, a 1-cell from $(X, x) \to (Y, y)$ is a non-decreasing map $\mu : \mathbf{m} \to [y, x]$, preserving the maximal element if $X = A$. One can think of it as a colored ordinal, with colors belonging to the set $[y, x]$. For instance, the picture below shows a coloring of **5**.



  A 2-cell from $\mu : \mathbf{m} \to [y, x]$ to $\mu' : \mathbf{m}' \to [y, x]$ is a non-decreasing map $f : \mathbf{m} \to \mathbf{m}'$ verifying some minimal/maximal element preservation condition depending on X and Y, and such that $\mu \leq \mu' f$.

$$\begin{tikzcd} \mathbf{m} \arrow[r, "f"] \arrow[d, "\mu"'] \arrow[dr, phantom, "\Rightarrow"] & \mathbf{m}' \arrow[dl, "\mu'"] \\ {[y,x]} & \end{tikzcd}$$



- The composition is defined as follows. If $X, Y, Z \in |\mathbf{Adj}|$ and $x \geq y \geq z$, we define the composition

  $$C_{X,x,Y,y,Z,z} : \mathbf{Adj}[n]((Y,y),(Z,z)) \times \mathbf{Adj}[n]((X,x),(Y,y)) \longrightarrow \mathbf{Adj}[n]((X,x),(Z,z))$$

  using Proposition B.5 and the diagram

  $$\begin{tikzcd}
  & \mathbf{Adj}[n]((Y,y),(Z,z)) \times \mathbf{Adj}[n]((X,x),(Y,y)) \arrow[dl] \arrow[dr] & \\
  \mathbf{Adj}(X,Z) \arrow[dr, "i_{X,Z}"'] \arrow[rr, phantom, "\overset{\Longrightarrow}{\theta}"] & & * \arrow[dl, "{[z,x]}"] \\
  & \mathbf{TOFS}_{X,Z} &
  \end{tikzcd}$$

  where the top left morphism is given by projection onto the underlying $\mathbf{Adj}$ categories and composition in $\mathbf{Adj}$. A lax natural transformation θ fitting into the diagram is just a collection of 1-cells $\theta_{(\mathbf{m},\mu),(\mathbf{m}',\mu)} : \mathbf{m} \cdot_{\mathbf{Adj}} \mathbf{m}' \longrightarrow [z,x]$ indexed by objects of the top category of the diagram, that is, pairs of objects $\mu : \mathbf{m} \to [z,y]$, $\mu' : \mathbf{m}' \to [y,x]$ satisfying a maximal-element-preserving condition depending respectively on Y and X. The 2-cells don't have to be specified, because there is either one or no 2-cells between two 1-cell in $\mathbf{TOFS}$. The 2-cells, if they exist, can be defined in only one way, and all diagram will be automatically commutative. Thus, in this particular case, 2-cells are equivalent to a condition on the 1-cells, which is that for all 1-cells $f : (\mathbf{l},\lambda) \to (\mathbf{l}',\lambda') \in \mathbf{Adj}[n]((Y,y),(Z,z))$ and $g : (\mathbf{m},\mu) \to (\mathbf{m}',\mu') \in \mathbf{Adj}[n]((X,x),(Y,y))$,

  $$\theta_{(\mathbf{l}',\lambda'),(\mathbf{m}',\mu')}(f \cdot_{\mathbf{Adj}} g) \geq \theta_{(\mathbf{l},\lambda),(\mathbf{m},\mu)}.$$

  We define $\theta_{(\mathbf{l},\lambda),(\mathbf{m},\mu)}$ to be the composite

  $$\mathbf{l} \cdot_{\mathbf{Adj}} \mathbf{m} \xrightarrow{(d^{l-1})^{\delta_{YA}}} \mathbf{l} + \mathbf{m} \xrightarrow{\lambda+\mu} [z,x] = [z,y] \cup [y,x].$$

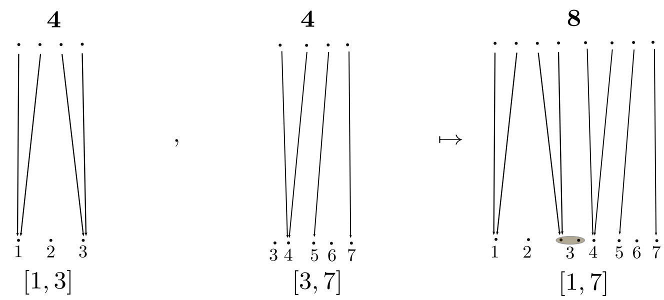
  By δ_{YA}, we denote the Kronecker delta. Its value is 1 if $Y = A$ and 0 otherwise. As the notation suggests, we let $(d^{l-1})^1 = d^{l-1}$ and $(d^{l-1})^0 = \mathrm{id}$. To check the condition when $Y = A$, one can draw the following diagram

  $$\begin{tikzcd}
  \mathbf{l} + \mathbf{m} - \mathbf{1} \arrow[r, "d^{l-1}"] \arrow[ddr, bend right, "f \cdot_{\mathbf{Adj}} g"'] \arrow[dr, phantom, "="] & \mathbf{l} + \mathbf{m} \arrow[r, "\lambda+\mu"] \arrow[d, "f+g"] \arrow[dr, phantom, "\Leftarrow"] & {[z,x]} \\
  & \mathbf{l}' + \mathbf{m}' \arrow[ur, "\lambda'+\mu'"'] \arrow[d, shift right, "s^{l'-1}"'] & \\
  & \mathbf{l}' + \mathbf{m}' - \mathbf{1} \arrow[u, shift right, "d^{l'-1}"'] &
  \end{tikzcd}$$

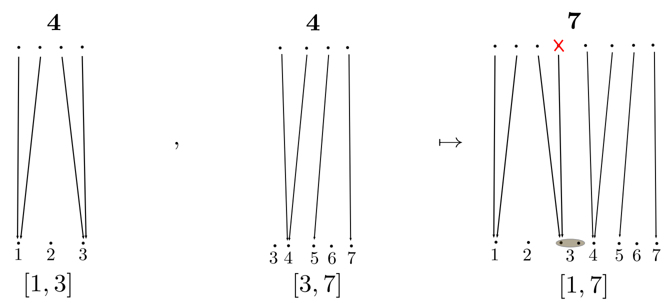
and remark that id $\leq d^{l'-1}s^{l'-1}$. One can consider the right triangle of the same diagram when $Y = B$.

We provide below some examples of how 1-cells compose, first when $Y = B$ and then when $Y = A$.

$$\mathbf{Adj}[n]((B,3),(A,1)) \times \mathbf{Adj}[n]((A,7),(B,3)) \to \mathbf{Adj}[n]((A,7),(A,1))$$



$$\mathbf{Adj}[n]((A,3),(B,1)) \times \mathbf{Adj}[n]((A,7),(A,3)) \to \mathbf{Adj}[n]((A,7),(B,1))$$



The composition is basically given by ordinal sum, with the exception that the crossed point is discarded and the two points in the dark grey region are identified. The 2-cells compose exactly as in **Adj**.

- The identities are $1_{(B,k)} = \mathbf{0} \to \{k\}$ and $1_{(A,k)} = \mathbf{1} \to \{k\}$, for all $k \in \{0, \ldots, n\}$.

3.2.1. Proposition. *The 2-category* $\mathbf{Adj}[n]$ *is well defined: its composition is associative.*

Proof. Let $(X, x), (Y, y), (Z, z)$ and (W, w) be objects of $\mathbf{Adj}[n]$ with $x \geq y \geq z \geq w$. Remark that the lax natural transformation associated to the 2-functor

$$\begin{tikzcd}
\mathbf{Adj}[n]((Z,z),(W,w)) \times \mathbf{Adj}[n]((Y,y),(Z,z)) \times \mathbf{Adj}[n]((X,x),(Y,y)) \arrow[d] \\
\mathbf{Adj}[n]((Z,z),(W,w)) \times \mathbf{Adj}[n]((X,x),(Z,z)) \arrow[d] \\
\mathbf{Adj}[n]((X,x),(W,w))
\end{tikzcd}$$

has value on a triple of objects $\rho : \mathbf{r} \to [w, z]$ $\mu : \mathbf{m} \to [z, y]$ $\pi : \mathbf{p} \to [y, x]$ given by the composite

$$\mathbf{r} \cdot_{\mathbf{Adj}} (\mathbf{m} \cdot_{\mathbf{Adj}} \mathbf{p}) \xrightarrow{(d^{r-1})^{\delta_{ZA}}} \mathbf{r} + \mathbf{m} \cdot_{\mathbf{Adj}} \mathbf{p} \xrightarrow{\rho + (\mu \cdot \pi)} [w, z] \cup [z, x] = [w, x].$$

which can be further decomposed into

$$\mathbf{r} \cdot_{\mathbf{Adj}} (\mathbf{m} \cdot_{\mathbf{Adj}} \mathbf{p}) \xrightarrow{(d^{r-1})^{\delta_{ZA}}} \mathbf{r} + \mathbf{m} \cdot_{\mathbf{Adj}} \mathbf{p} \xrightarrow{(d^{r+m-1})^{\delta_{YA}}} \mathbf{r} + \mathbf{m} + \mathbf{p} \xrightarrow{\nu + \mu + \pi} [w, x].$$

On the other hand, the lax natural transformation associated to the 2-functor

$$\begin{array}{c} \mathbf{Adj}[n]((Z, z), (W, w)) \times \mathbf{Adj}[n]((Y, y), (Z, z)) \times \mathbf{Adj}[n]((X, x), (Y, y)) \\ \downarrow \\ \mathbf{Adj}[n]((Y, y), (W, w)) \times \mathbf{Adj}[n]((X, x), (Y, y)) \\ \downarrow \\ \mathbf{Adj}[n]((X, x), (W, w)) \end{array}$$

has value on the same triple of objects given by the composite

$$(\mathbf{r} \cdot_{\mathbf{Adj}} \mathbf{m}) \cdot_{\mathbf{Adj}} \mathbf{p} \xrightarrow{(d^{r \cdot_{\mathbf{Adj}} m-1})^{\delta_{YA}}} (\mathbf{r} \cdot_{\mathbf{Adj}} \mathbf{m}) + \mathbf{p} \xrightarrow{(\rho \cdot \mu) + \pi} [w, y] \cup [y, x] = [w, x].$$

which can be further decomposed in

$$(\mathbf{r} \cdot_{\mathbf{Adj}} \mathbf{m}) \cdot_{\mathbf{Adj}} \mathbf{p} \xrightarrow{(d^{r \cdot_{\mathbf{Adj}} m-1})^{\delta_{YA}}} (\mathbf{r} \cdot_{\mathbf{Adj}} \mathbf{m}) + \mathbf{p} \xrightarrow{(d^{r-1})^{\delta_{ZA}}} \mathbf{r} + \mathbf{m} + \mathbf{p} \xrightarrow{\rho + \mu + \pi} [w, x].$$

It is now an easy consequence of the cosimplicial identities that the lax natural transformations are equal, and thus from the universal property we deduce that the two 2-functors are equal. ∎

3.3. Proof of the 2-universal property of $\mathbf{Adj}[n]$. Remark that the full sub 2-category of $\mathbf{Adj}[n]$ generated by the objects (A, k) and (B, k) is isomorphic to $\mathbf{Adj}$ for all $k \in \{0, \ldots n\}$. Indeed, 1-cells $(X, k) \to (Y, k)$ are maps $\mathbf{n} \to [k, k] = \{k\}$, with $\mathbf{n} \in \mathbf{Adj}(X, Y)$. Since $\{k\}$ is a terminal object in $\mathbf{TOFS}$, $\mathbf{Adj}[n]((X, k), (Y, k)) \cong \mathbf{Adj}(X, Y)$. We denote by $l_k : (B, k) \to (A, k)$ and $r_k : (A, k) \to (B, k)$ the 1-cells given by the unique map $\mathbf{1} \to \{k\}$. They are respectively the left and right adjoint of the adjunction on level k. Moreover, we have $b_k : (B, k) \to (B, k-1)$ and $a_k : (A, k) \to (A, k-1)$, respectively given by the unique 1-cells $\mathbf{0} \to [k-1, k]$ and $\mathbf{1} \to [k-1, k]$ (a_k must preserve the maximal element). Remark that precomposing b_k with the right adjoint r_k yield $\mathbf{1} \to [k-1, k] : (A, k) \to (B, k-1)$ which is also the result of post-composing a_k with the right adjoint r_{k-1}. We thus have a chain of n adjunctions morphisms in our 2-category, as expected.

The next step is to show that they generate all data in $\mathbf{Adj}[n]$ (together with the adjunctions) and that the relations that they satisfy are also satisfied by all such data in any 2-category. For this, let us take a closer look at the 1-category

$$\mathbf{Adj}[n]((X,x),(Y,y)) = i_{X,Y} \downarrow^l [y,x].$$

The first observation is that it admits two interesting sub 2-categories. The first one, which we denote by $i_{X,Y} \downarrow^l_{ic} [y,x]$, is generated by the 1-cells of the form

$$\begin{tikzcd} \mathbf{m} \arrow[r, "1"] \arrow[d, "\mu"'] \arrow[dr, phantom, "\overset{\theta}{\nearrow}"] & \mathbf{m} \arrow[d, "\mu'"] \\ {[y,x]} \arrow[r, "1"] & {[y,x]} \end{tikzcd}$$

that is with the 1-cell component being an identity. The subscript ic stands for "identity component". The second one, which we denote by $i_{X,Y} \downarrow^l_s [y,x]$ is generated by the 1-cells of the form

$$\begin{tikzcd} \mathbf{m} \arrow[r, "f"] \arrow[d, "\mu"'] \arrow[dr, phantom, "="] & \mathbf{m}' \arrow[d, "\mu'"] \\ {[y,x]} \arrow[r, "1"] & {[y,x]}. \end{tikzcd}$$

The subscript s stands for "strict". Any map

$$\begin{tikzcd} \mathbf{m} \arrow[r, "f"] \arrow[d, "\mu"'] \arrow[dr, phantom, "\overset{\theta}{\nearrow}"] & \mathbf{m}' \arrow[d, "\mu'"] \\ {[y,x]} \arrow[r, "1"] & {[y,x]} \end{tikzcd}$$

can be uniquely factored as the composite of a morphism in $i_{X,Y} \downarrow^l_{ic} [y,x]$ followed by a morphism in $i_{X,Y} \downarrow^l_s [y,x]$.

$$\begin{tikzcd} \mathbf{m} \arrow[r, "1"] \arrow[d, "\mu"'] \arrow[dr, phantom, "\overset{\theta}{\nearrow}"] & \mathbf{m} \arrow[r, "f"] \arrow[d, "\mu' f"] \arrow[dr, phantom, "="] & \mathbf{m}' \arrow[d, "\mu'"] \\ {[y,x]} \arrow[r, "1"] & {[y,x]} \arrow[r, "1"] & {[y,x]} \end{tikzcd} \tag{5}$$

Actually, $(\mathrm{Mor}\ i_{X,Y} \downarrow^l_{ic} [y,x], \mathrm{Mor}\ i_{X,Y} \downarrow^l_s [y,x])$ is an orthogonal factorization system. Indeed, we showed that any morphism can be factored uniquely.

Since the 2-category $\mathbf{TOFS}_{X,Y}$ is locally a finite poset, the category $i_{X,Y} \downarrow^l_{ic} [y,x]$ is a poset, thus all diagrams commute, and it is generated by pairs of consecutive elements. Recall that two elements $p_1 < p_2$ are said to be *consecutive* in a poset P if the subset $\{p \in P : p_1 < p < p_2\}$ is empty.

Let us have a closer look at this poset. Its elements are non-decreasing functions $\mu : \mathbf{m} \to [y, x]$, preserving the maximal element if $X = A$. Functions are comparable only if they have the same domain, and $\mu \leq \mu'$ if and only if $\mu(k) \leq \mu'(k)$ for all k in the domain. Thus, the poset is a disjoint union

$$i_{X,Y} \downarrow^l_{ic} [y, x] = \coprod_{\mathbf{m} \in \mathbf{Adj}(X,Y)} \mathcal{P}_{\mathbf{m}},$$

where $\mathcal{P}_{\mathbf{m}} = \{\mu : \mathbf{m} \to [y, x]\}$. Each set $\mathcal{P}_{\mathbf{m}}$ also admits a metric, given by

$$d(\mu, \mu') = \sum_{i \in \mathbf{m}} |\mu(i) - \mu'(i)|.$$

If $\mu \leq \mu' \leq \mu''$ then $d(\mu, \mu') + d(\mu', \mu'') = d(\mu, \mu'')$. Remark that this implies that if $d(\mu, \mu') = 1$, then μ and μ' are consecutive. The converse statement is also true. This means that $i_{X,Y} \downarrow^l_{ic} [y, x]$ is generated by the morphisms

$$\begin{tikzcd} \mathbf{m} \arrow[r, "1"] \arrow[d, "p^i(\mu)"'] \arrow[dr, phantom, "\nearrow\!\!\!\nearrow"] & \mathbf{m} \arrow[d, "\mu"] \\ {[y, x]} \arrow[r, "1"] & {[y, x]} \end{tikzcd}$$

for $y < i \leq x$ and $\mu^{-1}(i) \neq \emptyset$, and where $p^i(\mu)(k) = \mu(k) - \delta_{\min \mu^{-1}(i), k}$ for all $k \in \mathbf{m}$. The object $p^i(\mu)$ can be thought of as the predecessor of μ obtained by moving the minimal member of the i-th fiber to the $i - 1$-th fiber. Let us name this morphism $\tau^i_\mu : p^i(\mu) \to \mu$. Since $p^i(\mu)$ and μ are at distance one, they are consecutive. All pairs of consecutive elements are of this form, and thus $i_{X,Y} \downarrow^l_{ic} [y, x]$ is generated by these elements.

Let us focus now on $i_{X,Y} \downarrow^l_s [y, x]$. Since the 2-cells are identities there, the square

$$\begin{tikzcd} \mathbf{m} \arrow[r, "f"] \arrow[d, "\mu"'] \arrow[dr, phantom, "="] & \mathbf{m}' \arrow[d, "\mu'"] \\ {[y, x]} \arrow[r, "1"] & {[y, x]} \end{tikzcd}$$

commutes on the nose. Thus, the domain object is determined by f and the codomain. We denote this morphism by $f_{\mu'} : (\mathbf{m}, \mu' f) \to (\mathbf{m}', \mu')$. Generators of $i_{X,Y} \downarrow^l_s [y, x]$ are given by maps $d^i_\mu : (\mathbf{m} - \mathbf{1}, \mu d^i) \to (\mathbf{m}, \mu)$ and $s^j_\mu : (\mathbf{m} + \mathbf{1}, \mu s^j) \to (\mathbf{m}, \mu)$ for i, j such that $\delta_{YA} \leq i < m - \delta_{XA}$ and $0 \leq j \leq m - 1$. The conditions express that the morphisms may have to preserve the minimal or maximal element or both, depending on X and Y.

The fact that the maps listed above generate $i_{X,Y} \downarrow^l_s [y, x]$ is a consequence of the decomposition lemma of Gabriel and Zisman [6, Lemma 2.2 page 24]. This lemma explains that a non-decreasing map between ordinals can be uniquely factored as a sequence of codegeneracies followed by cofaces, when one sets appropriate constraints on their indices. It can be generalized as follows.

3.3.1. THEOREM. *A morphism $f : (\mathbf{l}, \lambda) \to (\mathbf{m}, \mu)$ in the category $\mathbf{Adj}[n](X, x)(Y, y)$ can be written in a unique way as a composition*

$$\begin{pmatrix}\mathbf{l}\\ \lambda\end{pmatrix} \xrightarrow{\tau_\bullet^{\lambda(l-1)+1}} \cdots \longrightarrow \cdots \xrightarrow{\tau_\bullet^{\mu f(l-1)}} \begin{pmatrix}\mathbf{l}\\ \mu f - \sum_{i=0}^{l-2} k_i\delta_{ix}\end{pmatrix} \xrightarrow{1}$$

$$\longrightarrow \cdots \longrightarrow \cdots \longrightarrow \cdots \longrightarrow \cdots \xrightarrow{1}$$

$$\longrightarrow \begin{pmatrix}\mathbf{l}\\ \mu f - k_0\delta_{0x}\end{pmatrix} \xrightarrow{\tau_\bullet^{\lambda(0)+1}} \cdots \longrightarrow \begin{pmatrix}\mathbf{l}\\ \mu f - \delta_{0x}\end{pmatrix} \xrightarrow{\tau_{\mu f}^{\mu f(0)}} \begin{pmatrix}\mathbf{l}\\ \mu f\end{pmatrix} \xrightarrow{1}$$

$$\longrightarrow \begin{pmatrix}\mathbf{l}\\ \mu f\end{pmatrix} \xrightarrow{s_\bullet^{j_1}} \cdots \longrightarrow \begin{pmatrix}\mathbf{m-s+1}\\ \mu d^{i_s}\cdots d^{i_1}s^{j_t}\end{pmatrix} \xrightarrow{s_\bullet^{j_t}} \begin{pmatrix}\mathbf{m-s}\\ \mu d^{i_s}\cdots d^{i_1}\end{pmatrix} \xrightarrow{1}$$

$$\longrightarrow \begin{pmatrix}\mathbf{m-s}\\ \mu d^{i_s}\cdots d^{i_1}\end{pmatrix} \xrightarrow{d^{i_1}_{\mu d^{i_s}\cdots d^{i_2}}} \cdots \xrightarrow{d^{i_{s-1}}_{\mu d^{i_s}}} \begin{pmatrix}\mathbf{m-1}\\ \mu d^{i_s}\end{pmatrix} \xrightarrow{d^{i_s}_{\mu}} \begin{pmatrix}\mathbf{m}\\ \mu\end{pmatrix}$$

with

(*i*) $m - \delta_{XA} > i_s > \ldots > i_1 \geq \delta_{YA}$;

(*ii*) $0 \leq j_t < \ldots < j_1 < n$;

(*iii*) $n - t + s = m$;

(*iv*) $k_i = \mu f(i) - \lambda(i)$.

and where the $\bullet$ are place holders for indices easily determined but unpleasant to write down.

PROOF. We already stated that f can be decomposed in a unique way as the composition of a morphism in $i_{X,Y} \downarrow^l_{ic} [y, x]$ followed by a morphisms of $i_{X,Y} \downarrow^l_s [y, x]$. Remark that the decomposition of a morphism as a composite of generators in $i_{X,Y} \downarrow^l_{ic} [y, x]$ is determined by the objects of the sequence, since it is a poset. The uniqueness of the decomposition in $i_{X,Y} \downarrow^l_s [y, x]$ follows from the Gabriel-Zisman lemma [6, Lemma 2.2 page 24]. ∎

Let us now describe relations between the τ^i_μ and the d^i_μ, s^j_μ. Let $\mu : \mathbf{m} \to [y, x]$ and $y < j \leq x$ be such that $\mu^{-1}j \neq \emptyset$. Said differently, μ is such that τ^j_μ is a well defined morphism of $\mathbf{Adj}[n]((X, x), (Y, y))$. Let $w = \min \mu^{-1}(j)$.

Let us consider first $p^j(\mu) \circ f$, when $f : \mathbf{l} \to \mathbf{m}$. Remark that for all $k \in \mathbf{l}$,

$$[p^j(\mu) \circ f](k) = \mu \circ f(k) - \chi_{f^{-1}(w)}(k),$$

where χ_A denotes the characteristic function of the subset A. If $f^{-1}(w) = \emptyset$, remark that $p^j(\mu)f = \mu f$. In particular, if $i = w$, one obtains

$$d^i_\mu = \tau^j_\mu \circ d^i_{p^j(\mu)}.$$

If $f^{-1}(w) \neq \emptyset$, then $\min(\mu \circ f)^{-1}(j) = \min f^{-1}(w)$. Thus, $p^j(\mu \circ f)$ exists. Moreover, for all $k \in \mathbf{l}$, $p^j(\mu \circ f)(k) = \mu \circ f(k) - \delta_{\min f^{-1}(w),k}$. As a consequence, if $f^{-1}(w)$ contains a unique element, then $p^j(\mu) \circ f = p^j(\mu \circ f)$, which implies that the two following composites are equal.

$$(\mathbf{l}, p^j(\mu f)) \xrightarrow{\tau^j_{\mu f}} (\mathbf{l}, \mu f) \xrightarrow{f_\mu} (\mathbf{m}, \mu) \qquad (\mathbf{l}, p^j(\mu) \circ f) \xrightarrow{f_{p^j(\mu)}} (\mathbf{m}, p^j(\mu)) \xrightarrow{\tau^j_\mu} (\mathbf{m}, \mu)$$

$$\begin{tikzcd}
\mathbf{l} \arrow[r, "1"] \arrow[d, "p^j(\mu f)"'] & \mathbf{l} \arrow[r, "f"] \arrow[d, "\mu f"] & \mathbf{m} \arrow[d, "\mu"] \\
{[y,x]} \arrow[r, "1"] \arrow[ur, phantom, "\Nearrow"] & {[y,x]} \arrow[r, "1"] & {[y,x]}
\end{tikzcd}
=
\begin{tikzcd}
\mathbf{l} \arrow[r, "f"] \arrow[d, "p^j(\mu)\circ f"] & \mathbf{m} \arrow[r, "1"] \arrow[d, "p^j(\mu)"'] & \mathbf{m} \arrow[d, "\mu"] \\
{[y,x]} \arrow[r, "1"] & {[y,x]} \arrow[r, "1"] \arrow[ur, phantom, "\Nearrow"] & {[y,x]}
\end{tikzcd}$$

This is in particular true when $f = d^i, s^i$ with $\min \mu^{-1}(j) \neq i$, and thus we have

- $s^i_\mu \circ \tau^j_{\mu s^i} = \tau^j_\mu \circ s^i_{p^j(\mu)}$;
- $d^i_\mu \circ \tau^j_{\mu d^i} = \tau^j_\mu \circ d^i_{p^j(\mu)}$.

If $\min f^{-1}(w) = k_0$ and $k_0 + 1 \in f^{-1}(w)$, then $p^j(\mu f)(k_0) = j - 1$ and $p^j(\mu f)(k_0 + 1) = j$. Thus, $p^j(p^j(\mu f))$ is defined, and, for all $k \in \mathbf{l}$,

$$p^j(p^j(\mu f))(k) = \mu f(k) - \delta_{k_0,k} - \delta_{k_0+1,k}.$$

Thus, if $f^{-1}(w)$ contains exactly two elements, $p^j(p^j(\mu f)) = p^j(\mu) \circ f$. As a consequence, when $f = s^i$ and $i = w$, the diagram

$$\begin{tikzcd}
p^j(p^j(\mu s^i)) \arrow[r, "\tau^j_{p^j(\mu s^i)}"] \arrow[d, "s^i_{p^j(\mu)}"] & p^j(\mu s^i) \arrow[r, "\tau^j_{\mu s^i}"] & \mu s^i \arrow[d, "s^i_\mu"] \\
p^j(\mu) \arrow[rr, "\tau^j_\mu"] & & \mu
\end{tikzcd} \tag{6}$$

is commutative. Remark also that if $y < i \leq x$, then

$$p^j(\mu)^{-1}(i) = \big(\mu^{-1}(i) \setminus \{\min \mu^{-1}(j)\}\big) \cup \big(\{\min \mu^{-1}(j)\} \cap \mu^{-1}(i+1)\big)$$

Thus, if $j \neq i, i+1$, $p^i p^j(\mu)$ exists if and only if $\mu^{-1}(i) \neq \emptyset \neq \mu^{-1}(j)$ and for $k \in \mathbf{m}$,

$$p^i(p^j(\mu))(k) = \mu(k) - \delta_{\min \mu^{-1}(i),k} - \delta_{\min \mu^{-1}(j),k}.$$

Remark that $p^i p^{i+1}(\mu)$ exists if and only if $\mu^{-1}(i+1) \neq \emptyset$. If also $\mu^{-1}(i) \neq \emptyset$, then

$$p^i(p^{i+1}(\mu))(k) = \mu(k) - \delta_{\min \mu^{-1}(i),k} - \delta_{\min \mu^{-1}(i+1),k}.$$

Thus, when τ^i_μ and τ^j_μ both exist, we have the following equality:

$$\tau^i_\mu \circ \tau^j_{p^i(\mu)} = \tau^j_\mu \circ \tau^i_{p^j(\mu)}.$$

We have now almost proved the following theorem.

3.3.2. THEOREM. *The category* $\mathbf{Adj}[n]((X,x)(Y,y))$ *admits the following presentation.*

(*i*) *Objects are non-decreasing maps* $\mathbf{m} \to [y,x]$ *that preserve the maximal element if* $X = A$.

(*ii*) *Generating maps are given as follows.*

cofaces: $d^i_\mu : (\mathbf{m-1}, \mu d^i) \to (\mathbf{m}, \mu)$ *for* $\delta_{YA} \leq i < m - \delta_{XA}$ *and all objects* $(\mathbf{m}, \mu)$ *with* $\mathbf{m-1} \in |\mathbf{Adj}(X,Y)|$;

codegeneracies: $s^i_\mu : (\mathbf{m+1}, \mu s^i) \to (\mathbf{m}, \mu)$ *for* $0 \leq i < m$ *and all objects* $(\mathbf{m}, \mu)$;

transfers: $\tau^i_\mu : (\mathbf{m}, p^i(\mu)) \to (\mathbf{m}, \mu)$, *for all objects* $(\mathbf{m}, \mu)$ *and* $y < i \leq x$ *with* $\mu^{-1}(i) \neq \emptyset$ *and* $\min \mu^{-1}(i) \neq m-1$ *if* $X = A$.

(*iii*) *The relations are given by requiring all the diagrams below to commute, under the condition that all generators exist in the category.*

- *For* $i < j$,

$$\begin{tikzcd} (\mathbf{m-2}, \mu d^j d^i) \arrow[r, "d^i_{\mu d^j}"] \arrow[d, "d^{j-1}_{\mu d^i}"'] & (\mathbf{m-1}, \mu d^j) \arrow[d, "d^j_\mu"] \\ (\mathbf{m-1}, \mu d^i) \arrow[r, "d^i_\mu"'] & (\mathbf{m}, \mu) \end{tikzcd} \qquad \begin{tikzcd} (\mathbf{m+2}, \mu s^i s^j) \arrow[r, "s^j_{\mu s^i}"] \arrow[d, "s^i_{\mu s^{j-1}}"'] & (\mathbf{m+1}, \mu s^i) \arrow[d, "s^i_\mu"] \\ (\mathbf{m+1}, \mu s^{j-1}) \arrow[r, "s^{j-1}_\mu"'] & (\mathbf{m}, \mu) \end{tikzcd} . \tag{7}$$

$$\begin{tikzcd} (\mathbf{m}, \mu s^j d^i) \arrow[r, "d^i_{\mu s^j}"] \arrow[d, "s^{j-1}_{\mu d^i}"'] & (\mathbf{m+1}, \mu s^j) \arrow[d, "s^j_\mu"] \\ (\mathbf{m+1}, \mu d^i) \arrow[r, "d^i_\mu"'] & (\mathbf{m}, \mu) \end{tikzcd} . \tag{8}$$

- *For* $i > j+1$

$$\begin{tikzcd} (\mathbf{m}, \mu s^j d^i) \arrow[r, "d^i_{\mu s^j}"] \arrow[d, "s^j_{\mu d^{i-1}}"'] & (\mathbf{m+1}, \mu s^j) \arrow[d, "s^j_\mu"] \\ (\mathbf{m+1}, \mu d^{i-1}) \arrow[r, "d^{i-1}_\mu"'] & (\mathbf{m}, \mu) \end{tikzcd} . \tag{9}$$

- $$\begin{tikzcd} (\mathbf{m+1}, \mu s^i) \arrow[dr, "s^i_\mu"'] & (\mathbf{m}, \mu) \arrow[l, "d^{i+1}_{\mu s^i}"'] \arrow[r, "d^i_{\mu s^i}"] \arrow[d, "1"] & (\mathbf{m+1}, \mu s^i) \arrow[dl, "s^i_\mu"] \\ & (\mathbf{m}, \mu) & \end{tikzcd} \tag{10}$$

- $$\begin{tikzcd} (\mathbf{m}, p^i p^j(\mu)) \arrow[r, "\tau^j_{p^i(\mu)}"] \arrow[d, "\tau^i_{p^j(\mu)}"] & (\mathbf{m}, p^i(\mu)) \arrow[d, "\tau^i_\mu"] \\ (\mathbf{m}, p^j(\mu)) \arrow[r, "\tau^j_\mu"] & (\mathbf{m}, \mu) \end{tikzcd} \tag{11}$$

- *If* $i \neq \min \mu^{-1}(j)$,

$$\begin{tikzcd} (\mathbf{m+1}, p^j(\mu s^i)) \arrow[r, "\tau^j_{\mu s^i}"] \arrow[d, "s^i_{p^j(\mu)}"'] & (\mathbf{m+1}, \mu s^i) \arrow[d, "s^i_\mu"] & (\mathbf{m-1}, p^j(\mu d^i)) \arrow[r, "\tau^j_{\mu d^i}"] \arrow[d, "d^i_{p^j(\mu)}"'] & (\mathbf{m-1}, \mu d^i) \arrow[d, "d^i_\mu"] \\ (\mathbf{m}, \mu p^j(\mu)) \arrow[r, "\tau^j_\mu"'] & (\mathbf{m}, \mu) & (\mathbf{m}, \mu p^j(\mu)) \arrow[r, "\tau^j_\mu"'] & (\mathbf{m}, \mu) \end{tikzcd} . \tag{12}$$

- *If* $i = \min \mu^{-1}(j)$,

$$\begin{tikzcd} (\mathbf{m}-1, \mu d^i) \arrow[r, "d^i_{p^j(\mu)}"] \arrow[d, "d^i_\mu"'] & (\mathbf{m}, p^j(\mu)) \arrow[dl, "\tau^j_\mu"] \\ (\mathbf{m}, \mu) & \end{tikzcd} \tag{13}$$

$$\begin{tikzcd} (\mathbf{m+1}, p^j(p^j(\mu s^i))) \arrow[r, "\tau^j_{p^j(\mu s^i)}"] \arrow[d, "s^i_{p^j(\mu)}"] & (\mathbf{m+1}, p^j(\mu s^i)) \arrow[r, "\tau^j_{\mu s^i}"] & (\mathbf{m+1}, \mu s^i) \arrow[d, "s^i_\mu"] \\ (\mathbf{m}, p^j(\mu)) \arrow[rr, "\tau^j_\mu"] & & (\mathbf{m}, \mu) \end{tikzcd} \tag{14}$$

3.3.3. Remark.

- The first five diagrams of the statement of the Theorem 3.3.2 are the obvious generalizations of the cosimplicial identities. The last two identities should remind the reader of the relations satisfied by monad morphisms.
- One can think of $\mathbf{Adj}[n]((X,x)(Y,y))$ as consisting essentially of a product of $|[y,x]| = y - x + 1$ copies of Δ_+, which are related by the transfers.
- The presentation of $\mathbf{Adj}[1]((B,1)(B,0))$ given in Theorem 3.3.2 is leveraged in Proposition 4.2.2 to describe functors $(\Delta \downarrow^l \mathbf{2})^{\mathrm{op}} \to \mathbf{Set}$ as biaugmented bisimplicial sets with more structure.

Proof of theorem 3.3.2. Let us write $\mathcal{F}$ for the free category generated by the objects and generators above, and $\sim$ for the congruence generated by the relations. Since the relations are verified in $\mathbf{Adj}[n]((X,x)(Y,y))$ there is a functor $T : \mathcal{F}/_\sim \to \mathbf{Adj}[n]((X,x)(Y,y))$. The functor T is full since the second category is generated by those morphisms as well.

Let us show that the relations (11) imply that the subcategory generated by the transfers in $\mathcal{F}/_\sim$ is a poset. Recall that there is a natural order on the objects, given by $\mu \leq \mu'$ if and only if μ and μ' have the same domain $\mathbf{m}$ and $\mu(i) \leq \mu'(i)$ for all $i \in \mathbf{m}$. We will show that this subcategory is exactly the category associated to this poset. We thus consider the subcategory $\mathscr{C}_m$ of $\mathcal{F}/_\sim$ generated by the objects $\mu : \mathbf{m} \to [y,x]$ for a fixed $\mathbf{m}$ and the transfers between them. Recall that there is also a metric given by

$$d(\mu,\mu') = \sum_{i \in \mathbf{m}} |\mu(i) - \mu'(i)|.$$

Recall also that if $\mu \leq \mu' \leq \mu''$ then $d(\mu,\mu') + d(\mu',\mu'') = d(\mu,\mu'')$. Remark that by construction, there is a transfer $\mu \to \mu'$ in $\mathscr{C}_m$ if and only if $\mu \leq \mu'$ and $d(\mu,\mu') = 1$ and furthermore this transfer is unique. By 3.3.1, there is a map $\mu \to \mu'$ in $\mathscr{C}_m$ if and only if $\mu \leq \mu'$.

Let $f, g : \mu \to \mu'$ be two morphisms in $\mathscr{C}_m$. We prove by induction on $d(\mu,\mu')$ that $f = g$. The case $d(\mu,\mu') = 2$ is a direct consequence of the commutativity of (11). We suppose that the result is proved for $2 \leq d(\mu,\mu') < n$, and we prove it for $d(\mu,\mu') = n$.

Let us consider a decomposition of f, g into generators, and name them $f = f_n \cdots f_1$, $g = g_n \cdots g_1$. Remark that the number of generators is the same as the distance from μ to μ'. Now, consider $f_1 : \mu \to \hat{\mu}$, and $g_1 : \mu \to \check{\mu}$. If $f_1 = g_1$, then the induction hypothesis implies that $f_n \cdot f_2 = g_n \cdots g_2$ and thus $f = g$. If not, then $\hat{\mu}$ and $\check{\mu}$ are not comparable. Consider a diagram $\hat{\mu} \xrightarrow{i} \sup(\hat{\mu},\check{\mu}) \xleftarrow{j} \check{\mu}$. Consider also a map $k : \sup(\hat{\mu},\check{\mu}) \to \mu'$. By the induction hypothesis, $f_n \cdots f_2 = k \cdot i$ and $g_n \cdots g_2 = k \cdot j$. By the case $n = 2$, and since $d(\mu, \sup(\hat{\mu},\check{\mu})) = 2$, $if_1 = jg_1$.

We are now ready to show that T is also faithful. Indeed, suppose that $w, w' \in \mathcal{F}$ are morphisms such that $T(w) = T(w')$. The relations given above are enough to find other morphisms $v, v' \in \mathcal{F}$, with $v \sim w$ and $v' \sim w'$ such that v, v' are words in the same form as in Theorem 3.3.1. By uniqueness, $T(v) = T(v')$ if and only if $v = v'$. Thus, $T(w) = T(w') \Rightarrow w \sim w'$. ∎

We are now going to show that the objects, generators and relations of Theorem 3.3.2, which are respectively 1-cells, 2-cells and their relations in $\mathbf{Adj}[n]$, are 2-categorically generated by the adjunctions and the commutative squares involving the right adjoints in $\mathbf{Adj}[n]$. Let us write $l_k \dashv r_k; \epsilon_k, \eta_k$ for the data corresponding to the k-th adjunction, and $a_k : (A,k) \to (A,k-1)$, $b_k : (B,k) \to (B,k-1)$ for the adjunction morphism from the k-th adjunction to the $(k-1)$-st. The next proposition describes how the 1-cells r_k, l_k, a_i, b_i for $0 \leq k \leq n$ and $1 \leq i \leq n$ generate all 1-cells of $\mathbf{Adj}[n]$.

3.3.4. Proposition. *Let $\mu : \mathbf{m} \to [y,x]$ be a 1-cell $(X,x) \to (Y,y)$, and let $k_l = |\mu^{-1}(l)|$ for all $y \leq l \leq x$.*

(*i*) *If* $X = Y = B$,

$$\mu = \begin{tikzcd} (B,x) \arrow[loop above, "(r_x l_x)^{k_x}"] \arrow[r, "b_x"] & (B,x-1) \arrow[loop above, "(r_{x-1} l_{x-1})^{k_{x-1}}"] \arrow[r] & \dots \arrow[r, "b_{y+1}"] & (B,y). \arrow[loop above, "(r_y l_y)^{k_y}"] \end{tikzcd} \tag{15}$$

(*ii*) *If* $X = B, Y = A$, *there exists* $t \in [y, x]$ *such that* $k_t > 0$, *since* $\mathbf{m} > 0$ *in this case. Then,*

$$\mu = \begin{tikzcd} (B,x) \arrow[loop above, "(r_x l_x)^{k_x}"] \arrow[r, "b_x"] & (B,x-1) \arrow[loop above, "(r_{x-1} l_{x-1})^{k_{x-1}}"] \arrow[r] & \dots \arrow[r, "b_{t+1}"] & (B,t) \arrow[loop above, "(r_t l_t)^{k_t - 1}"] \arrow[d, "l_t"] & & & \\ & & & (A,t) \arrow[r, "a_t"] & (A,t-1) \arrow[loop below, "(l_{t-1} r_{t-1})^{k_{t-1}}"] \arrow[r] & \dots \arrow[r, "a_{y+1}"] & (A,y). \arrow[loop below, "(l_y r_y)^{k_y}"] \end{tikzcd} \tag{16}$$

(*iii*) *If* $X = A$, *we have*

$$\mu = \begin{tikzcd} & & & (B,y). \\ (A,x) \arrow[loop below, "(l_x r_x)^{k_x - 1}"] \arrow[r, "a_x"] & (A,x-1) \arrow[loop below, "(l_{x-1} r_{x-1})^{k_{x-1}}"] \arrow[r] & \dots \arrow[r, "a_{y+1}"] & (A,y) \arrow[loop below, "(l_y r_y)^{k_y}"] \arrow[u, "r_y^{\delta_{YB}}"'] \end{tikzcd} \tag{17}$$

PROOF. We leave the proofs of equations (15) and (17) to the reader, as a good exercise to become familiar with the composition. We prove now equation (16).

Define v to be the minimum of $\mu^{-1}[t, x]$ and consider the map $\lambda : \mathbf{m} - \mathbf{v} - \mathbf{1} \to [t, x]$ induced by the restriction of μ to the set $\{v + 1, \dots, m - 1\}$. We consider λ as a 1-cell $(B, x) \to (B, t)$. Since $|\lambda^{-1}(l)| = k_l$ for $l > t$ and $|\lambda^{-1}(t)| = k_t - 1$, equation (15) implies that

$$\lambda = (r_t l_t)^{k_t - 1} b_{t+1} \cdots b_x \cdot (r_x l_x)^{k_x}.$$

Also, $a_t l_t : \mathbf{1} \to [t - 1, t]$ is the map which contains t in its image (direct computation). As a consequence,

$$a_t l_t \lambda = a_t l_t (r_t l_t)^{k_t - 1} b_{t+1} \cdots b_x \cdot (r_x l_x)^{k_x}$$

is the map $\mathbf{m} - \mathbf{v} \to [t - 1, x]$ induced by the restriction of μ to the set $\{v, \dots, m - 1\}$. It is a 1-cell $(B, x) \to (A, t - 1)$. Let $\rho : \mathbf{v} + \mathbf{1} \to [y, t - 1]$ be defined by $\rho(w) = \mu(w)$ for

$w < v$ and $\rho(v) = t-1$. We consider it as a 1-cell $(A, t-1) \to (A, y)$. Since $|\rho^{-1}(l)| = k_l$ for $l < t-1$ and $|\rho^{-1}(t-1)| = k_{t-1}+1$, equation (17) implies that

$$\rho = (l_y r_y)^{k_y} \cdot a_{y+1} \cdots a_{t-1} \cdot (l_{t-1} r_{t-1})^{k_{t-1}}.$$

Now, $\mu = \rho \cdot \lambda$, since we are composing at $(A, t-1)$. ■

3.3.5. Corollary. *The underlying category of* $\mathbf{Adj}[n]$ *admits the following presentation.*

- *Objects are pairs* (X, k), $k \in \{0, \dots, n\}$, $X \in \{A, B\}$.
- *Generators are* $l_k : (B, k) \to (A, k)$, $r_k : (A, k) \to (B, k)$, *for all* $k \in \{0, \dots, n\}$ *and* $a_k : (A, k) \to (A, k-1)$, $b_k : (B, k) \to (B, k-1)$, *for all* $k \in \{1, \dots, n\}$.
- *Relations are given by* $b_{k-1} r_k = r_{k-1} a_k$ *for all* $k \in \{1, \dots, n\}$.

Proof. Let $\mathcal{F}$ be the category generated by the objects and generators above, and $\sim$ the congruence relation generated by the relations above. Since the corresponding 1-cells in the underlying category of $\mathbf{Adj}[n]$ verify the relations, there is a unique functor $T : \mathcal{F}/_{\sim} \to \mathbf{Adj}[n]$ which is bijective on objects. By Proposition 3.3.4, T is full: all 1-cells are the image of a composite of the generators.

Let w be a morphism in $\mathcal{F}((X, x), (Y, y))$. Remark that the relations enables us to find $C(w) \in \mathcal{F}((X, x), (Y, y))$ with $C(w) \sim w$ where $C(w)$ is of the form of the right-hand side of one of the equations (15), (16), (17).

If w, w' are two morphisms in $\mathcal{F}((X, j), (Y, i))$ such that $T(w) = T(w')$, then $TC(w) = TC(w')$. Equations (15), (16), (17) imply that $C(w) = C(w')$, and thus $w \sim w'$. As a consequence, T is faithful and an isomorphism of categories. ■

We provide in the following two propositions explicit formulas for the codegeneracies and cofaces in the different hom-categories of $\mathbf{Adj}[n]$.

3.3.6. Proposition. *Let* $\mu : \mathbf{m} \to [y, x]$ *be a* 1*-cell* $(X, x) \to (Y, y)$ *and* $0 \le j < m$. *Let* $t = \mu(j)$, $k_l = |\mu^{-1}(l)|$ *for* $l \in [y, x]$, $k_t^0 = |\{u \in \mathbf{m} : \mu(u) = t, u < j\}|$ *and* $k_t^1 = |\{u \in \mathbf{m} : \mu(u) = t, u > j\}|$.

(*i*) *If* $X = Y = B$,

$$s^j_\mu = \begin{tikzcd} (B,x) \arrow[loop above, "(r_x l_x)^{k_x}"] \arrow[r, "b_x"] & \dots \arrow[r, "b_{t+1}"] & (B,t) \arrow[loop above, "(r_t l_t)^{k_t^1}"] \arrow[r, bend left, "(r_t l_t)^2"] \arrow[r, bend right, "r_t l_t"'] \arrow[r, phantom, "\Downarrow r_t\epsilon_t l_t"] & (B,t) \arrow[loop above, "(r_t l_t)^{k_t^0}"] \arrow[r, "b_t"] & \dots \arrow[r, "b_{y+1}"] & (B,y). \arrow[loop above, "(r_y l_y)^{k_y}"] \end{tikzcd} \tag{18}$$

(ii) *If* $X = B, Y = A$,

$$s^j_\mu = \begin{tikzcd} (B,x) \arrow[loop above, "(r_x l_x)^{k_x}"] \arrow[r, "b_x"] & \dots \arrow[r, "b_{t+1}"] & (B,t) \arrow[d, "l_t"] & & & & \\ & & (A,t) \arrow[loop below, "(l_t r_t)^{k^1_t}"] \arrow[r, bend left, "l_t r_t"] \arrow[r, bend right, "1"'] \arrow[r, phantom, "\Downarrow\epsilon_t"] & (A,t) \arrow[loop below, "(l_t r_t)^{k^0_t}"] \arrow[r, "a_t"] & \dots \arrow[r, "a_{y+1}"] & (A,y). \arrow[loop below, "(l_y r_y)^{k_y}"] \end{tikzcd} \tag{19}$$

(iii) *If* $X = A$,

$$s^j_\mu = \begin{tikzcd} & & & & & (B,y). \\ (A,x) \arrow[loop below, "(l_x r_x)^{k_x - 1}"] \arrow[r, "a_x"] & \dots \arrow[r, "a_{t+1}"] & (A,t) \arrow[loop below, "(l_t r_t)^{1+k^1_t}"] \arrow[r, bend left, "l_t r_t"] \arrow[r, bend right, "1"'] \arrow[r, phantom, "\Downarrow\epsilon_t"] & (A,t) \arrow[loop below, "(l_t r_t)^{k^0_t}"] \arrow[r, "a_t"] & \dots \arrow[r, "a_{y+1}"] & (A,y) \arrow[loop below, "(l_y r_y)^{k_y}"] \arrow[u, "r_y^{\delta_{YB}}"'] \end{tikzcd} \tag{20}$$

Proof. We prove (18) and leave the other verifications to the reader. By (15), the 1-cell $(r_y l_y)^{k_y} \cdot b_{y+1} \cdots b_t \cdot (r_t l_t)^{k^0_t} : (B,t) \to (B,y)$ is the unique non-decreasing map

$$\nu : \mathbf{k_y} + \ldots + \mathbf{k_{t-1}} + \mathbf{k^0_t} \to [y,t]$$

such that $|\nu^{-1}(l)| = k_l$, for $y \leq l < t$ and $|\nu^{-1}(t)| = k^0_t$. Similarly, by (15) the 1-cell $(r_t l_t)^{k^1_t} \cdot b_{t+1} \cdots b_x \cdot (r_x l_x)^{k_x} : (B,x) \to (B,t)$ is the unique non-decreasing map

$$\nu' : \mathbf{k^1_t} + \mathbf{k_{t+1}} + \ldots + \mathbf{k_x} \to [t,x]$$

such that $|\nu'^{-1}(l)| = k_l$, for $t < l \leq x$ and $|\nu'^{-1}(t)| = k^1_t$. Remark also by direct computation that $(r_t \epsilon_t l_t)$ is the unique map

$$(s^0, \mathrm{id}) : (\mathbf{2} \to \{t\}) \to (\mathbf{1} \to \{t\}).$$

As a consequence, the 2-cell of the proposition is

$$\nu \cdot (s^0, \mathrm{id}) \cdot \nu' = s^{\sum_l^{t-1} k_l + k^0_t} : \left(\sum_{l=y}^{x} \mathbf{k_l} + \mathbf{1} \to [y,x] \right) \to \left(\sum_{l=y}^{x} \mathbf{k_l} \to [y,x] \right)$$

where the second 1-cell is μ. The last step is to check that $j = \sum_l^{t-1} k_l + k^0_t$. ∎

3.3.7. Proposition. *Let $\mu : \mathbf{m} \to [y,x]$ be a 1-cell $(X,x) \to (Y,y)$ and $0 \le j < m$. Let $t = \mu(j)$, $k_l = |\mu^{-1}(l)|$ for $l \in [y,x]$, and define $k_t^0 = |\{u \in \mathbf{m} : \mu(u) = t, u < j\}|$, $k_t^1 = |\{u \in \mathbf{m} : \mu(u) = t, u > j\}|$.*

(i) If $X = Y = B$, we have

$$d_\mu^j = \begin{tikzcd} (B,x) \arrow[r, "b_x"] \arrow[loop above, "(r_xl_x)^{k_x}"] & \dots \arrow[r, "b_{t+1}"] & (B,t) \arrow[r, bend left, "1"] \arrow[r, bend right, "r_tl_t"'] \arrow[r, phantom, "\Downarrow\eta_t"] \arrow[loop above, "(r_tl_t)^{k_t^1}"] & (B,t) \arrow[r, "b_t"] \arrow[loop above, "(r_tl_t)^{k_t^0}"] & \dots \arrow[r, "b_{y+1}"] & (B,y). \arrow[loop above, "(r_yl_y)^{k_y}"] \end{tikzcd}$$

(ii) If $X = B, Y = A$, let $s = \mu(j-1)$. We have

$$d_\mu^j = \begin{tikzcd} (B,x) \arrow[r, "b_x"] \arrow[loop above, "(r_xl_x)^{k_x}"] & \dots \arrow[r, "b_{t+1}"] & (B,t) \arrow[r, bend left, "1"] \arrow[r, bend right, "r_tl_t"'] \arrow[r, phantom, "\Downarrow\eta_t"] \arrow[loop above, "(r_tl_t)^{k_t^1}"] & (B,t) \arrow[r, "b_t"] \arrow[loop above, "(r_tl_t)^{k_t^0}"] & \dots \arrow[r] & (B,s) \arrow[d, "l_s"] \arrow[loop above, "(r_sl_s)^{k_s-1}"] & & \\ & & & & & (A,t) \arrow[r, "a_t"] & \dots \arrow[r, "a_{y+1}"] & (A,y). \arrow[loop below, "(l_yr_y)^{k_y}"] \end{tikzcd}$$

(iii) If $X = A$, let $s = \begin{cases} \mu(j-1) & Y = A \\ y & Y = B \end{cases}$. We have

$$d_\mu^j = \begin{tikzcd} & & (B,t) \arrow[r, bend left, "1"] \arrow[r, bend right, "r_tl_t"'] \arrow[r, phantom, "\Downarrow\eta_t"] & (B,t) \arrow[r, "b_t"] \arrow[loop above, "(r_tl_t)^{k_t^0}"] & \dots \arrow[r, "b_{s+1}"] & (B,s) \arrow[d, "l_s"] \arrow[loop above, "(r_sl_s)^{k_s-1}"] & & (B,y). \\ (A,x) \arrow[r, "a_x"] \arrow[loop below, "(l_xr_x)^{k_x-1}"] & \dots \arrow[r, "a_{t+1}"] & (A,t) \arrow[u, "r_t"] \arrow[loop below, "(r_tl_t)^{k_t^1}"] & & & (A,s) \arrow[r] & \dots \arrow[r, "a_{y+1}"] & (A,y) \arrow[u, "r_y^{\delta_{YB}}"'] \arrow[loop below, "(l_yr_y)^{k_y}"] \end{tikzcd}$$

Proof. Left to the reader. ∎

We describe now how the transfers are generated. For this, compute the mate of the identity map $b_ir_i \Rightarrow r_{i-1}a_i$, that is $(\epsilon_{i-1}a_il_i) \circ (l_{i-1}b_i\eta_i) : l_{i-1}b_i \Rightarrow a_il_i$. The result is the map : $(d^i : \mathbf{1} \to [i-1,i]) \Rightarrow (d^{i-1} : \mathbf{1} \to [i-1,i])$ where the first map hits $i-1$ and the second i. This is the transfer $\tau^i_{d^{i-1}} \in \mathbf{Adj}[n]((B,k)(A,k-1))$. Let us call $m_i : l_{i-1}b_i \Rightarrow a_il_i$ the mate of the identity map $b_ir_i \Rightarrow r_{i-1}a_i$.

All the other transfers can be obtained by pre or post composing this 2-cell with appropriate 1-cells, as the following proposition shows.

3.3.8. Proposition. *Let $\mu : \mathbf{m} \to [y,x]$ be a 1-cell $(X,x) \to (Y,y)$ and $y < t \leq x$. Let $k_l = |\mu^{-1}(l)|$ for $l \in [y,x]$.*

(i) *If $X = Y = B$, $\tau^t_\mu =$*

$$
\begin{tikzcd}
(B,x) \arrow[loop above, "(r_xl_x)^{k_x}"] \arrow[r, "b_x"] & \dots \arrow[r, "b_{t+1}"] & (B,t) \arrow[loop above, "(r_tl_t)^{k_t-1}"] \arrow[r, "1"] \arrow[d, "l_t"'] \arrow[d, phantom, shift left=3, "\Downarrow \eta_t"] & (B,t) \arrow[r, "b_t"] \arrow[d, phantom, "||"] & (B,t-1) \arrow[d, "l_{t-1}"] \arrow[d, phantom, shift right=3, "\Downarrow \epsilon_t"'] & (B,t-1) \arrow[loop above, "(r_{t-1}l_{t-1})^{k_{t-1}}"] \arrow[r, "b_{t-1}"] & \dots \arrow[r, "b_{y+1}"] & (B,y). \arrow[loop above, "(r_yl_y)^{k_y}"] \\
 & & (A,t) \arrow[r, "a_t"] \arrow[ur, "r_t"'] & (A,t-1) \arrow[r, "1"'] \arrow[ur, "r_{t-1}"] & (A,t-1) \arrow[ur, "r_{t-1}"] & & &
\end{tikzcd}
$$

(ii) *If $X = B, Y = A$, We have*

$$
\tau^t_\mu = \begin{tikzcd}
(B,x) \arrow[loop above, "(r_xl_x)^{k_x}"] \arrow[r, "b_x"] & \dots \arrow[r, "b_{t+1}"] & (B,t) \arrow[loop above, "(r_tl_t)^{k_t-1}"] \arrow[r, "1"] \arrow[d, "l_t"'] \arrow[d, phantom, shift left=3, "\Downarrow \eta_t"] & (B,t) \arrow[r, "b_t"] \arrow[d, phantom, "||"] & (B,t-1) \arrow[d, "l_{t-1}"] \arrow[d, phantom, shift right=3, "\Downarrow \epsilon_t"'] & & \\
 & & (A,t) \arrow[r, "a_t"] \arrow[ur, "r_t"'] & (A,t-1) \arrow[r, "1"'] \arrow[ur, "r_{t-1}"] & (A,t-1) \arrow[loop below, "(l_{t-1}r_{t-1})^{k_{t-1}}"] \arrow[r, "a_{t-1}"] & \dots \arrow[r, "a_{y+1}"] & (A,y). \arrow[loop below, "(l_yr_y)^{k_y}"]
\end{tikzcd} \tag{21}
$$

(iii) *If $X = A$,*

$$
\begin{tikzcd}
 & & & (B,t) \arrow[r, "1"] \arrow[d, "l_t"'] \arrow[d, phantom, shift left=3, "\Downarrow \eta_t"] & (B,t) \arrow[r, "b_t"] \arrow[d, phantom, "||"] & (B,t-1) \arrow[d, "l_{t-1}"] \arrow[d, phantom, shift right=3, "\Downarrow \epsilon_t"'] & & (B,y). \\
(A,x) \arrow[loop below, "(l_xr_x)^{k_x-1}"] \arrow[r, "a_x"] & \dots \arrow[r, "a_{t+1}"] & (A,t) \arrow[loop below, "(l_tr_t)^{k_t-1}"] \arrow[ur, "r_t"] & (A,t) \arrow[r, "a_t"] \arrow[ur, "r_t"'] & (A,t-1) \arrow[r, "1"'] \arrow[ur, "r_{t-1}"] & (A,t-1) \arrow[loop below, "(l_{t-1}r_{t-1})^{k_{t-1}}"] \arrow[r, "a_{t-1}"] & \dots \arrow[r, "a_{y+1}"] & (A,y) \arrow[loop below, "(l_yr_y)^{k_y}"] \arrow[u, "(r_y)^{\delta_{YB}}"']
\end{tikzcd} \tag{22}
$$

Proof. It is enough to check that the domain and codomain are as expected, since the subcategory $i_{X,Y} \downarrow^l_{ic} [y,x]$ is a poset. One can compute the domain and codomain from Proposition 3.3.4. ∎

3.3.9. Theorem. *The 2-functors $\mathbf{Adj}[n] \to \mathscr{C}$ are in bijective correspondence with sequences of n composable adjunctions morphisms in $\mathscr{C}$.*

Proof. Let $\mathcal{ADJ}[n]$ be the 2-category presented by computads as in coequalizer (3). Let us denote $\tilde{l}_i = [L_i], \tilde{r}_i = [R_i], \tilde{a}_i = [A_i]$ and $\tilde{b}_i = [B_i]$ the generating 1-cells of $\mathcal{ADJ}[n]$. Let also $\tilde{\epsilon}_i$ and $\tilde{\eta}_i$ be the counit and unit of $\tilde{l}_i \dashv \tilde{r}_i$ respectively. We denote by $\tilde{m}_i : \tilde{l}_{i-1}\tilde{b}_i \Rightarrow \tilde{a}_i\tilde{l}_i$ the mate of the identity $\tilde{b}_i\tilde{r}_i \Rightarrow \tilde{r}_{i-1}\tilde{a}_i$.

Since $\mathbf{Adj}[n]$ has n composable adjunction morphisms, this determines a unique 2-functor sending the adjunction $\tilde{l}_i \dashv \tilde{r}_i, (\tilde{\epsilon}_i, \tilde{\eta}_i)$ to the adjunction $l_i \dashv r_i(\epsilon_i, \eta_i)$, $\tilde{b}_i$ to b_i and $\tilde{a}_i$ to a_i. We denote this 2-functor by

$$T : \mathcal{ADJ}[n] \longrightarrow \mathbf{Adj}[n].$$

Remark that by the Corollary 3.3.5, and the fact that the underlying category functor 2-$\mathbf{Cat} \to \mathbf{Cat}$ preserves colimits (it is a left adjoint), T is bijective on objects and 1-cells.

To prove that it is locally full and faithful, we are going to show that T has an inverse locally. To do so, we use Theorem 3.3.2 and Propositions 3.3.6, 3.3.7 and 3.3.8. More precisely for $(X,x),(Y,y)$ objects of $\mathbf{Adj}[n]$, we use the formulas of the propositions to determine $S_{(X,x)(Y,y)} : \mathbf{Adj}[n]((X,x)(Y,y)) \to \mathcal{ADJ}[n]((X,x)(Y,y))$ on the objects and generators, such that $T_{(X,x)(Y,y)}S_{(X,y)(Y,y)} = 1_{\mathbf{Adj}[n]((X,x),(Y,y))}$.

To prove that S extends to a functor, we need to show that the images of the generators of $\mathbf{Adj}[n]((X,x)(Y,y))$ in $\mathcal{ADJ}[n]$ satisfy the relations of 3.3.2.

- The cosimplicial identities (7), (8), (9) are satisfied by the images of the corresponding generators in $\mathcal{ADJ}[n]$ because the interchange law holds in $\mathcal{ADJ}[n]$.

Identity (7) Let $i < j$. The case $\mu(i) = \mu(j)$ is well known. Let $\mu(i) = s < t = \mu(j)$. The interchange law of the pasting diagram

$$\begin{tikzcd}
(B,t) \arrow[loop above, "(\tilde{r}_t\tilde{l}_t)^{k_t^1}"] \arrow[r, bend left, "1"] \arrow[r, bend right, "\tilde{r}_t\tilde{l}_t"'] \arrow[r, phantom, "\Downarrow\tilde{\eta}_t"] & (B,t) \arrow[loop above, "(\tilde{r}_t\tilde{l}_t)^{k_t^0}"] \arrow[r, "b_t"] & \cdots \arrow[r, "b_{s+1}"] & (B,s) \arrow[loop above, "(\tilde{r}_s\tilde{l}_s)^{k_s^1}"] \arrow[r, bend left, "1"] \arrow[r, bend right, "\tilde{r}_s\tilde{l}_s"'] \arrow[r, phantom, "\Downarrow\tilde{\eta}_s"] & (B,s) \arrow[loop above, "(\tilde{r}_s\tilde{l}_s)^{k_s^0}"]
\end{tikzcd}$$

implies the commutativity of the first diagram in all cases. The interchange law of the pasting diagram

$$\begin{tikzcd}
(A,t) \arrow[loop above, "(\tilde{l}_t\tilde{r}_t)^{k_t^1}"] \arrow[r, bend left, "\tilde{l}_t\tilde{r}_t"] \arrow[r, bend right, "1"'] \arrow[r, phantom, "\Downarrow\tilde{\epsilon}_t"] & (A,t) \arrow[loop above, "(\tilde{l}_t\tilde{r}_t)^{k_t^0}"] \arrow[r, "a_t"] & \cdots \arrow[r, "a_{s+1}"] & (A,s) \arrow[loop above, "(\tilde{l}_s\tilde{r}_s)^{k_s^1}"] \arrow[r, bend left, "\tilde{l}_s\tilde{r}_s"] \arrow[r, bend right, "1"'] \arrow[r, phantom, "\Downarrow\tilde{\epsilon}_s"] & (A,s) \arrow[loop above, "(\tilde{l}_s\tilde{r}_s)^{k_s^0}"]
\end{tikzcd}$$

implies the commutativity of the second diagram in all cases.

Identity (8) Let $i < j$. The case $\mu(i) = \mu(j)$ is well known. Let $\mu(j) = s > t = \mu(i)$. The interchange law of the pasting diagram

$$\begin{tikzcd}
 & & & (B,t) \arrow[r, bend left, "1"] \arrow[r, bend right, "\tilde{r}_t\tilde{l}_t"'] \arrow[r, phantom, "\Downarrow\tilde{\eta}_t"] & (B,t) \arrow[loop above, "(\tilde{r}_t\tilde{l}_t)^{k_t^0}"] \\
(A,s) \arrow[loop below, "(\tilde{l}_s\tilde{r}_s)^{k_s^1}"] \arrow[r, bend left, "\tilde{l}_s\tilde{r}_s"] \arrow[r, bend right, "1"'] \arrow[r, phantom, "\Downarrow\tilde{\epsilon}_s"] & (A,s) \arrow[loop below, "(\tilde{l}_s\tilde{r}_s)^{k_s^0}"] \arrow[r, "a_s"] & \cdots \arrow[r, "a_{t+1}"] & (A,t) \arrow[loop below, "(\tilde{l}_t\tilde{r}_t)^{k_t^1}"] \arrow[u, "\tilde{r}_t"'] &
\end{tikzcd}$$

implies the commutativity of said diagram in all cases.

Identity (9) Let $i-1>j$. The case $\mu(i-1)=\mu(j)$ is well known. Let $\mu(j)=s<t=\mu(i-1)$. The interchange law of the pasting diagram

$$\begin{tikzcd}
(B,t) \arrow[loop above, "(\tilde{r}_t\tilde{l}_t)^{k_t^1}"] \arrow[r, bend left, "1"] \arrow[r, bend right, "\tilde{r}_t\tilde{l}_t"'] \arrow[r, phantom, "\Downarrow\tilde{\eta}_t"] & (B,t) \arrow[loop above, "(\tilde{r}_t\tilde{l}_t)^{k_t^0}"] \arrow[r, "b_t"] & \dots \arrow[r, "b_{s+1}"] & (B,s) \arrow[loop above, "(\tilde{r}_s\tilde{l}_s)^{k_s^1}"] \arrow[d, "\tilde{l}_s"] & \\
 & & & (A,s) \arrow[r, bend left, "\tilde{l}_s\tilde{r}_s"] \arrow[r, bend right, "1"'] \arrow[r, phantom, "\Downarrow\tilde{\epsilon}_s"] & (A,s) \arrow[loop above, "(\tilde{l}_s\tilde{r}_s)^{k_s^0}"]
\end{tikzcd}$$

implies the commutativity of said diagram in all cases.

- Relations (11) and (12) are also satisfied in $\mathcal{ADJ}[n]$ because the interchange law holds. We leave it to the reader to write down the corresponding pasting diagram as above.

- The cosimplicial identity (10) is satisfied as a consequence of the triangle identities for the adjoints in $\mathcal{ADJ}[n]$.

- Relation (13) is a consequence of the commutativity of the diagram

$$\begin{tikzcd}
\tilde{b}_i \arrow[r, "\tilde{\eta}_{i-1}\tilde{b}_i"] \arrow[d, "\tilde{b}_i\tilde{\eta}_i"'] & \tilde{r}_{i-1}\tilde{l}_{i-1}\tilde{b}_i \arrow[d, "\tilde{r}_{i-1}\tilde{m}_i"] \\
\tilde{b}_i\tilde{r}_i\tilde{l}_i \arrow[r, equal] & \tilde{r}_{i-1}\tilde{a}_i\tilde{l}_i
\end{tikzcd}$$

and the interchange law. The previous diagram itself is commutative because of the following equality.

$$\begin{tikzcd}
(B,i) \arrow[r, "1"] \arrow[d, "\tilde{l}_i"'] \arrow[d, phantom, "\eta_i"] & \tilde{(B,i)} \arrow[r, "\tilde{b}_i"] & (B,i-1) \arrow[r, "1"] \arrow[d, "\tilde{l}_{i-1}"] \arrow[dl, phantom, "\tilde{\epsilon}_{i-1}"] \arrow[dr, phantom, "\tilde{\eta}_{i-1}"] & (B,i-1) \\
(A,i) \arrow[r, "\tilde{a}_i"'] \arrow[ur, "\tilde{r}_i"'] & (A,i-1) \arrow[r, "1"] \arrow[ur, "r_{i-1}"] & (A,i-1) \arrow[ur, "\tilde{r}_{i-1}"'] &
\end{tikzcd}
=
\begin{tikzcd}
(B,i) \arrow[r, "1"] \arrow[d, "\tilde{l}_i"'] \arrow[d, phantom, "\tilde{\eta}_i"] & (B,i) \arrow[r, "\tilde{b}_i"] & (B,i-1) \\
(A,i) \arrow[r, "\tilde{a}_i"'] \arrow[ur, "\tilde{r}_i"'] & (A,i-1) \arrow[ur, "\tilde{r}_{i-1}"] &
\end{tikzcd}$$

- Relation (14) is satisfied because the diagram

$$\begin{tikzcd}
\tilde{l}_{i-1}\tilde{r}_{i-1}\tilde{l}_{i-1}\tilde{b}_i \arrow[r, "\tilde{l}_{i-1}\tilde{r}_{i-1}\tilde{m}_i"] \arrow[d, "\tilde{\epsilon}_{i-1}\tilde{l}_{i-1}\tilde{b}_i"'] & \tilde{l}_{i-1}\tilde{r}_{i-1}\tilde{a}_i\tilde{l}_i \arrow[r, equal] & \tilde{l}_{i-1}\tilde{b}_i\tilde{r}_i\tilde{l}_i \arrow[r, "\tilde{m}_i\tilde{l}_i\tilde{r}_i"] & \tilde{a}_i\tilde{l}_i\tilde{r}_i\tilde{l}_i \arrow[d, "\tilde{a}_i\tilde{\epsilon}_i\tilde{l}_i"] \\
\tilde{l}_{i-1}\tilde{b}_i \arrow[rrr, "\tilde{m}_i"] & & & \tilde{a}_i\tilde{l}_i
\end{tikzcd}$$

commutes and the interchange law holds. The diagram is commutative since we have the following equality.

$$
\begin{tikzcd}
(B,i) \arrow[r,"1"] \arrow[d,"\tilde{l}_i"'] \arrow[d, phantom, shift left=3ex, "\tilde{\eta}_i"] & (B,i) \arrow[r,"\tilde{b}_i"] & (B,i-1) \arrow[d,"\tilde{l}_{i-1}"] \arrow[dl, phantom, "\tilde{\epsilon}_{i-1}"] \\
(A,i) \arrow[r,"\tilde{a}_i"] \arrow[ur,"\tilde{r}_i"'] \arrow[d,"\tilde{r}_i"] \arrow[dd, bend right=60, "1"'] \arrow[d, phantom, shift right=5ex, "\tilde{\epsilon}_i"] & (A,i-1) \arrow[r,"1"] \arrow[ur,"\tilde{r}_{i-1}"] & (A,i-1) \arrow[d,"\tilde{r}_{i-1}"] \\
(B,i) \arrow[r,"1"] \arrow[d,"\tilde{l}_i"'] \arrow[d, phantom, shift left=3ex, "\tilde{\eta}_i"] & (B,i) \arrow[r,"\tilde{b}_i"] & (B,i-1) \arrow[d,"\tilde{l}_{i-1}"] \arrow[dl, phantom, "\tilde{\epsilon}_{i-1}"] \\
(A,i) \arrow[r,"\tilde{a}_i"] \arrow[ur,"\tilde{r}_i"'] & (A,i-1) \arrow[r,"1"] \arrow[ur,"\tilde{r}_{i-1}"] & (A,i-1)
\end{tikzcd}
=
\begin{tikzcd}
(B,i) \arrow[r,"1"] \arrow[d,"\tilde{l}_i"'] \arrow[d, phantom, shift left=3ex, "\tilde{\eta}_i"] & (B,i) \arrow[r,"\tilde{b}_i"] & (B,i-1) \arrow[d,"\tilde{l}_{i-1}"] \arrow[dl, phantom, "\tilde{\epsilon}_{i-1}"] \\
(A,i) \arrow[r,"\tilde{a}_i"] \arrow[ur,"\tilde{r}_i"'] & (A,i-1) \arrow[r,"1"] \arrow[ur,"\tilde{r}_{i-1}"] & (A,i-1) \arrow[d,"\tilde{r}_{i-1}"] \arrow[dd, bend right=60, "1"'] \arrow[d, phantom, shift right=5ex, "\tilde{\epsilon}_{i-1}"] \\
 & & (A,i-1) \arrow[d,"\tilde{l}_{i-1}"] \\
 & & (B,i-1)
\end{tikzcd}
$$

Note the similarity with the proof of Proposition 2.2.4, except that the codomain is changed!

A consequence of the existence of $S_{(X,x),(Y,y)}$ is that $T_{(X,x),(Y,y)}$ is full. To show that $T_{(X,x),(Y,y)}$ is faithful, it is enough to check that $S_{(X,x),(Y,y)}$ is full. Remark that by construction, generators of $\mathcal{ADJ}[n]((X,x),(Y,y))$ are given by (equivalence classes) of diagrams

$$
\begin{tikzcd}
(X,x) \arrow[r,"f"] & (Z,z) \arrow[r, bend left, "g"] \arrow[r, bend right, "h"'] \arrow[r, phantom, "\alpha\Downarrow"] & (W,w) \arrow[r,"k"] & (Y,y)
\end{tikzcd}
$$

, where f,k are morphisms generated by the 1-cells, and $\alpha : g \Rightarrow h$ is a generating 2-cell, that is, a unit or a counit. First, from 3.3.8 we conclude that by construction, all morphisms of the previous form with $\alpha = \tilde{m}_i$ for $1 \le i \le n$ are in the image of $S_{(X,x),(Y,y)}$. Proposition 3.3.6 shows that the generator

$$
\begin{tikzcd}
(X,x) \arrow[r,"f"] & (A,t) \arrow[r, bend left, "\tilde{l}_t\tilde{r}_t"] \arrow[r, bend right, "1"'] \arrow[r, phantom, "\tilde{\epsilon}_t\Downarrow"] & (A,t) \arrow[r,"k"] & (Y,y)
\end{tikzcd}
$$

is in the image of $S_{(X,x),(Y,y)}$ when $f = \tilde{l}_t \cdot f'$. If $f = \tilde{a}_{t+1}f'$, remark that the generator is actually equal to the composite

$$
\begin{tikzcd}
(X,x) \arrow[r,"f'"] & (A,t+1) \arrow[r,"\tilde{r}_{t+1}"] & (B,t+1) \arrow[r, bend left, "\tilde{l}_t\tilde{b}_{t+1}"] \arrow[r, bend right, "\tilde{a}_{t+1}\tilde{l}_{t+1}"'] \arrow[r, phantom, "\tilde{m}_{t+1}\Downarrow"] & (A,t) \arrow[r,"k"] & (Y,y)
\end{tikzcd}
$$

$$\circ$$

$$
\begin{tikzcd}
(X,x) \arrow[r,"f'"] & (A,t+1) \arrow[r, bend left, "\tilde{l}_{t+1}\tilde{r}_{t+1}"] \arrow[r, bend right, "1"'] \arrow[r, phantom, "\tilde{\epsilon}_{t+1}\Downarrow"] & (A,t+1) \arrow[r,"\tilde{a}_{t+1}"] & (A,t) \arrow[r,"k"] & (Y,y)
\end{tikzcd}
$$

Since Proposition 3.3.6 also permits to conclude the case ϵ_n, by induction all generators with α a counit are in the image of $S_{(X,x),(Y,y)}$. Proposition 3.3.7 shows that the generator

$$
\begin{tikzcd}
(X,x) \arrow[r,"f"] & (B,t) \arrow[r, bend left, "1"] \arrow[r, bend right, "\tilde{r}_t\tilde{l}_t"'] \arrow[r, phantom, "\tilde{\eta}_t\Downarrow"] & (B,t) \arrow[r,"k"] & (Y,y)
\end{tikzcd}
$$

is in the image of $S_{(X,x),(Y,y)}$. ■

4. Some homotopically interesting observations

In this part, we provide some evidence that the hom-categories of $\mathbf{Adj}[n]$ can be of interest to the homotopy theorist. We start by showing that they are Reedy categories.

4.1. The hom-categories of $\mathbf{Adj}[n]$ are Reedy. Let $\mathscr{D} = \mathbf{Adj}[n]((X,x),(Y,y))$ with $x \geq y$. We show here that $\mathscr{D}$ is a Reedy category (in the sense of Hovey, [8, p.124]). Let $\mathscr{D}_-$ be the subcategory containing all objects and maps

$$\begin{tikzcd} \mathbf{m} \arrow[r, "f"] \arrow[d, "\mu"'] & \mathbf{m}' \arrow[dl, "\mu'"] \\ {[y,x]} \arrow[ur, phantom, "\Rightarrow"] & \end{tikzcd}$$

where f is surjective, and $\mathscr{D}_+$ the subcategory containing all objects and strict injective maps. That $(\mathscr{D}_-, \mathscr{D}_+)$ is a factorization system follows from the combination of the factorization system $(\mathrm{Mor}\ i_{X,Y} \downarrow^l_{ic} [y,x], \mathrm{Mor}\ i_{X,Y} \downarrow^l_{s} [y,x])$ (see factorization (5)) with the more classical factorization system $(\{\text{surjective strict map}\}, \{\text{injective strict map}\})$ of $i_{X,Y} \downarrow^l_s [y,x]$. This can also be recovered from the *double factorization system* (as defined in [13]), also called *ternary factorization system*, consisting of the following three sets of maps

$$(\mathrm{Mor}\ i_{X,Y} \downarrow^l_{ic} [y,x], \{\text{strict surjective maps}\}, \{\text{strict injective maps}\}).$$

For each n, the poset of morphisms $\mathcal{P}_{\mathbf{n}} = \{\mathbf{n} \to [y,x]\}$ is finite. It is a classical result that one can extend any poset to a total order using some finite choices. We apply this to the dual of $\mathcal{P}_{\mathbf{n}}$ to get a functor $\lambda_n : \mathcal{P}_{\mathbf{n}}^{\mathrm{op}} \to |\mathcal{P}_{\mathbf{n}}|$. More explicitly, if $\mathcal{P}_{\mathbf{n}}$ has N elements, λ_n is an order-reversing bijection $\mathcal{P}_{\mathbf{n}} \to \{0, \ldots, N-1\}$. We define the degree function $d : |\mathscr{D}| \to \mathbb{N} \times \mathbb{N}$ by

$$d(\nu : \mathbf{n} \to [y,x]) = (n, \lambda_n(\nu))$$

where the product is endowed with the lexicographical order.

Let $f : (\mathbf{n}, \nu) \to (\mathbf{m}, \mu)$ be a strict injective map. Then, either $n < m$ or $n = m$. The first case readily implies that $d(\mathbf{n}, \nu) < d(\mathbf{m}, \mu)$, while in the second case, the strictness of f implies that $\nu = \mu$ and thus f is the identity. This shows that non-identity morphisms in $\mathscr{D}_+$ raise the degree.

Let $g : (\mathbf{n}, \nu) \to (\mathbf{m}, \mu)$ be a surjective map. Then, either $n > m$ or $n = m$. In the first case, $n > m$ already implies $d(\mathbf{n}, \nu) > d(\mathbf{m}, \mu)$. In the second case, either $\nu < \mu$ or g is an an identity morphism. But then $\lambda_n(\nu) > \lambda_n(\mu)$, and thus $d(\mathbf{n}, \nu) > d(\mathbf{m}, \mu)$, as expected. We thus have proven the desired result.

4.1.1. Proposition. *For all $X, Y \in \{A, B\}$, $n \in \mathbb{N}$ and $n \geq x \geq y \geq 0$, the hom-category $\mathbf{Adj}[n]((X,x),(Y,y))$ is a Reedy category.*

4.2. The extended bar construction. Let $(\Delta \times \Delta)_+$ be the full subcategory of $\Delta_+ \times \Delta_+$ containing all objects but $(\mathbf{0}, \mathbf{0})$. A biaugmented bisimplicial set is a functor $(\Delta \times \Delta)_+^{\mathrm{op}} \to \mathbf{Set}$. We use standard notation for bisimplicial sets, which we recall below.

4.2.1. Notation.

- If X is a biaugmented bisimplicial set, $X_{n,m} := X(\mathbf{n+1}, \mathbf{m+1})$.
- If X is a biaugmented bisimplicial set, $d_i^h := X(d^i \times 1) : X_{n_0+1,n_1} \to X_{n_0,n_1}$ and $d_i^v = X(1 \times d^i) : X_{n_0,n_1+1} \to X_{n_0,n_1}$. Likewise, $s_j^h := X(s^j \times 1) : X_{n_0,n_1} \to X_{n_0+1,n_1}$ and $s_j^v = X(1 \times s^j) : X_{n_0,n_1} \to X_{n_0,n_1+1}$.
- We denote by $\Delta[n,m]$ the functor $(\Delta \times \Delta)_+^{\mathrm{op}} \to \mathbf{Set}$ represented by $(\mathbf{n+1}, \mathbf{m+1})$.

A biaugmented simplicial set looks like

$$\begin{tikzcd}
\vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \iddots \\
X_{-1,2} \arrow[d] & X_{0,2} \arrow[l] \arrow[d] & X_{1,2} \arrow[l] \arrow[d] & X_{2,2} \arrow[l] \arrow[d] & \cdots \arrow[l] \\
X_{-1,1} \arrow[d] & X_{0,1} \arrow[l] \arrow[d] & X_{1,1} \arrow[l] \arrow[d] & X_{2,1} \arrow[l] \arrow[d] & \cdots \arrow[l] \\
X_{-1,0} & X_{0,0} \arrow[l] \arrow[d] & X_{1,0} \arrow[l] \arrow[d] & X_{2,0} \arrow[l] \arrow[d] & \cdots \arrow[l] \\
 & X_{0,-1} & X_{1,-1} \arrow[l] & X_{2,-1} \arrow[l] & \cdots . \arrow[l]
\end{tikzcd}$$

Let us consider the category $\Delta_+ \downarrow^l \mathbf{2} = \mathbf{Mnd}[1]((B,1),(B,0))$. There is faithful functor $j : \Delta_+ \times \Delta_+ \to \Delta_+ \downarrow^l \mathbf{2}$ that is bijective on objects and given by the ordinal sum $j(\mathbf{n}, \mathbf{m}) = (\mathbf{n} \to \mathbf{1}) + (\mathbf{m} \to \mathbf{1})$ for all $\mathbf{n}, \mathbf{m} \in \Delta_+$. It is similarly defined on maps. The image of this functor is $\Delta_+ \downarrow_s^l \mathbf{2}$, the subcategory of strict maps. The functor j restricts to a functor

$$j : (\Delta \times \Delta)_+ \to \Delta \downarrow^l \mathbf{2}$$

whose image is still the subcategory of strict maps. We thus have a category isomorphism $j : (\Delta \times \Delta)_+ \to \Delta \downarrow_s^l \mathbf{2}$. For convenience, we will write $(\mathbf{n}, \mathbf{m})$ to denote $j(\mathbf{n}, \mathbf{m})$ and use the notation for bisimplicial sets for vertical and horizontal faces and degeneracies.

4.2.2. Proposition. *A functor $\left(\Delta \downarrow^l \mathbf{2}\right)^{\mathrm{op}} \to \mathbf{Set}$ is a biaugmented bisimplicial set X together with maps $\tau^* : X_{k,l+1} \to X_{k+1,l}$ such that the following diagram are commutative*

- *For $0 \le i \le k$*

$$\begin{tikzcd}
X_{k,l} \arrow[r, "\tau^*"] \arrow[d, "d_i^h"] & X_{k+1,l-1} \arrow[d, "d_i^h"] \\
X_{k-1,l} \arrow[r, "\tau^*"] & X_{k,l-1}
\end{tikzcd}
\qquad
\begin{tikzcd}
X_{k,l} \arrow[r, "\tau^*"] \arrow[d, "s_i^h"] & X_{k+1,l-1} \arrow[d, "s_i^h"] \\
X_{k+1,l} \arrow[r, "\tau^*"] & X_{k+2,l-1}
\end{tikzcd}$$

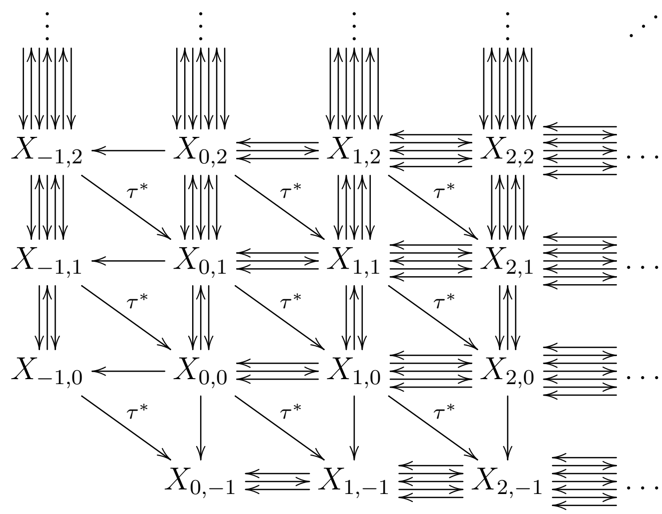


Figure 1: Picture of a functor $\left(\Delta \downarrow^l \mathbf{2}\right)^{\mathrm{op}} \to \mathbf{Set}$

- $$\begin{tikzcd} X_{k,l} \arrow[r, "\tau^*"] \arrow[dr, "d_0^v"'] & X_{k+1,l-1} \arrow[d, "d_{k+1}^h"] \\ & X_{k,l-1} \end{tikzcd} \qquad \begin{tikzcd} X_{k,l} \arrow[rr, "\tau^*"] \arrow[d, "s_0^v"] & & X_{k+1,l-1} \arrow[d, "s_{k+1}^h"] \\ X_{k,l+1} \arrow[r, "\tau^*"] & X_{k+1,l} \arrow[r, "\tau^*"] & X_{k+2,l-1} \end{tikzcd}$$
- *For* $0 \leq i \leq l-1$ $$\begin{tikzcd} X_{k,l} \arrow[r, "\tau^*"] \arrow[d, "d_{i+1}^v"] & X_{k+1,l-1} \arrow[d, "d_i^v"] \\ X_{k,l-1} \arrow[r, "\tau^*"] & X_{k+1,l-2} \end{tikzcd} \qquad \begin{tikzcd} X_{k,l} \arrow[r, "\tau^*"] \arrow[d, "s_{i+1}^v"] & X_{k+1,l-1} \arrow[d, "s_i^v"] \\ X_{k,l+1} \arrow[r, "\tau^*"] & X_{k+1,l} \end{tikzcd}$$

Proof. Theorem 3.3.2 provides us with a presentation of the category $\Delta \downarrow^l \mathbf{2}$ which corresponds exactly with the statement of the proposition after identification of $\mu \to \mathbf{2}$ with $(\mu^{-1}(0), \mu^{-1}(1))$ and dualization. ∎

Thus, representing only the images of the generators, a functor $\left(\Delta \downarrow^l \mathbf{2}\right)^{\mathrm{op}} \to \mathscr{C}$ can be pictured as in Figure 1. We will now provide explicitly a functor indexed by $(\Delta \downarrow^l \mathbf{2})^{\mathrm{op}}$ associated to a morphism of adjunctions.

Remark that by its universal property, $\mathbf{Adj}[n]^{\mathrm{coop}} \cong \mathbf{Adj}[n]$. This isomorphism interchanges the objects (A, x) and $(B, n-x)$ and thus determines an isomorphism

$$\mathbf{Adj}[n]((A,x),(B,y))^{\mathrm{op}} \cong \mathbf{Adj}[n]((A,n-y),(B,n-x)).$$

Observe that $\mathbf{Adj}[n]((A,n-y),(B,n-x)) \cong \mathbf{Adj}[n]((A,x),(B,y))$, and thus it also determines an isomorphism

$$\mathbf{Adj}[n]((A,x),(B,y))^{\mathrm{op}} \cong \mathbf{Adj}[n]((A,x),(B,y)).$$

As a particular case of Theorem 3.3.9, given an adjunction morphism in **Cat**,

$$
\begin{tikzcd}
\mathscr{B} \arrow[r, bend left, "L"] \arrow[r, phantom, "\perp"] \arrow[d, "F"'] & \mathscr{A} \arrow[l, bend left, "R"] \arrow[d, "G"] \\
\mathscr{B}' \arrow[r, bend left, "L'"] \arrow[r, phantom, "\perp"] & \mathscr{A}' \arrow[l, bend left, "R'"]
\end{tikzcd}
$$

there is a functor $\mathbf{Adj}[1]((A,1),(B,0)) \to \mathbf{Cat}(\mathscr{A},\mathscr{B}')$. If we fix an object $X \in |\mathscr{A}|$, we can further compose this functor, so as to get a functor

$$\mathbf{Adj}[n]((B,1),(B,0))^{\mathrm{op}} \xrightarrow{r_1^*} \mathbf{Adj}[n]((A,1),(B,0))^{\mathrm{op}} \xrightarrow{\cong} \mathbf{Adj}[n]((A,1),(B,0)) \longrightarrow \mathscr{B}'.$$

If we write $T = RL$ and $T' = R'L'$ for the monads associated to the adjunctions, the diagram is given as follows.

$$
\begin{tikzcd}
\vdots \arrow[d, shift left=2] \arrow[d] \arrow[d, shift right=2] \arrow[d, <-, shift left=1] \arrow[d, <-, shift right=1] & \vdots \arrow[d, shift left=2] \arrow[d] \arrow[d, shift right=2] \arrow[d, <-, shift left=1] \arrow[d, <-, shift right=1] & \vdots \arrow[d, shift left=2] \arrow[d] \arrow[d, shift right=2] \arrow[d, <-, shift left=1] \arrow[d, <-, shift right=1] & \vdots \arrow[d, shift left=2] \arrow[d] \arrow[d, shift right=2] \arrow[d, <-, shift left=1] \arrow[d, <-, shift right=1] & \iddots \\
T'^3F(RX) \arrow[d, shift left=2] \arrow[d] \arrow[d, shift right=2] \arrow[d, <-, shift left=1] \arrow[d, <-, shift right=1] \arrow[dr, "\tau^*"] & T'^3FT(RX) \arrow[l] \arrow[l, shift left=2] \arrow[r, shift left=1] \arrow[d, shift left=2] \arrow[d] \arrow[d, shift right=2] \arrow[d, <-, shift left=1] \arrow[d, <-, shift right=1] \arrow[dr, "\tau^*"] & T'^3FT^2(RX) \arrow[l, shift left=2] \arrow[l, shift right=1] \arrow[r, shift left=1] \arrow[d, shift left=2] \arrow[d] \arrow[d, shift right=2] \arrow[d, <-, shift left=1] \arrow[d, <-, shift right=1] \arrow[dr, "\tau^*"] & T'^3FT^3(RX) \arrow[l, shift left=2] \arrow[l] \arrow[l, shift right=2] \arrow[r, shift left=1] \arrow[r, shift right=1] \arrow[d, shift left=2] \arrow[d] \arrow[d, shift right=2] \arrow[d, <-, shift left=1] \arrow[d, <-, shift right=1] & \cdots \arrow[l, shift left=2] \arrow[l] \arrow[l, shift right=2] \\
T'^2F(RX) \arrow[d, shift left=2] \arrow[d, shift right=2] \arrow[d, <-] \arrow[dr, "\tau^*"] & T'^2FT(RX) \arrow[l] \arrow[l, shift left=2] \arrow[r, shift left=1] \arrow[d, shift left=2] \arrow[d, shift right=2] \arrow[d, <-] \arrow[dr, "\tau^*"] & T'^2FT^2(RX) \arrow[l, shift left=2] \arrow[l, shift right=1] \arrow[r, shift left=1] \arrow[d, shift left=2] \arrow[d, shift right=2] \arrow[d, <-] \arrow[dr, "\tau^*"] & T'^2FT^3(RX) \arrow[l, shift left=2] \arrow[l] \arrow[l, shift right=2] \arrow[r, shift left=1] \arrow[r, shift right=1] \arrow[d, shift left=2] \arrow[d, shift right=2] \arrow[d, <-] & \cdots \arrow[l, shift left=2] \arrow[l] \arrow[l, shift right=2] \\
T'F(RX) \arrow[d, dashed] \arrow[dr, "\tau^*"] & T'FT(RX) \arrow[l] \arrow[l, shift left=2] \arrow[r, shift left=1] \arrow[d] \arrow[dr, "\tau^*"] & T'FT^2(RX) \arrow[l, shift left=2] \arrow[l, shift right=1] \arrow[r, shift left=1] \arrow[d] \arrow[dr, "\tau^*"] & T'FT^3(RX) \arrow[l, shift left=2] \arrow[l] \arrow[l, shift right=2] \arrow[r, shift left=1] \arrow[r, shift right=1] \arrow[d] & \cdots \arrow[l, shift left=2] \arrow[l] \arrow[l, shift right=2] \\
FRX & FT(RX) \arrow[l, dashed] \arrow[l, shift left=2] \arrow[r, shift left=1] & FT^2(RX) \arrow[l, shift left=2] \arrow[l, shift right=1] \arrow[r, shift left=1] & FT^3(RX) \arrow[l, shift left=2] \arrow[l] \arrow[l, shift right=2] \arrow[r, shift left=1] \arrow[r, shift right=1] & \cdots \arrow[l, shift left=2] \arrow[l] \arrow[l, shift right=2]
\end{tikzcd}
$$

The i-th row is obtained by applying $(T')^{i+1}F$ to the bar construction of the T-algebra $(RX, R\epsilon_X)$. The codiagonal maps are instances of the natural transformation which is part of the monad morphism induced by the adjunction morphism. The j-th column is the bar construction of the T'-algebra obtained from the T-algebra $T^{j+1}RX$ through the monad morphism. An easy observation (and consequence of 4.3.4 below) is that the diagonal of this biaugmented bisimplicial object is also a simplicial resolution of FRX as a T'-algebra which is weakly equivalent to the classical bar construction of FRX.

4.3. The extended Artin-Mazur codiagonal. The composite

$$(\Delta \times \Delta)_+ \xrightarrow{j} \Delta \downarrow^l \mathbf{2} \xrightarrow{p} \Delta$$

is ordinal sum, and thus there is a diagram

$$
\begin{tikzcd}
\mathbf{Set}^{\Delta^{\mathrm{op}}} \arrow[r, shift left, "p^*"] \arrow[r, phantom, "\perp"] & \mathbf{Set}^{(\Delta\downarrow^l \mathbf{2})^{\mathrm{op}}} \arrow[l, shift left, "\mathrm{Ran}_p"] \arrow[r, shift left, "j^*"] \arrow[r, phantom, "\perp"] & \mathbf{Set}^{(\Delta\times\Delta)_+^{\mathrm{op}}} \arrow[l, shift left, "\mathrm{Ran}_j"]
\end{tikzcd}
$$

in which the composition $\mathrm{Ran}_p \circ \mathrm{Ran}_j$ is isomorphic to the Artin-Mazur codiagonal Tot, which is by definition Ran_+. Let us call Ran_j the *extended Artin-Mazur codiagonal* and write $\overline{\mathrm{Tot}} := \mathrm{Ran}_j$. We describe in more detail the right Kan extensions Ran_p and $\overline{\mathrm{Tot}}$. The first one is quite easy to describe. Indeed, if $\Delta[n] := \Delta(-, \mathbf{n+1}) : \Delta^{\mathrm{op}} \to \mathbf{Set}$ is the simplicial set represented by $[n] = \mathbf{n+1}$, and $(\mathbf{m_0+1}, \mathbf{m_1+1})$ is an object of $\Delta \downarrow^l \mathbf{2}$, $p^*\Delta[n]_{m_0,m_1} = \Delta(\mathbf{m_0+m_1+2}, \mathbf{n+1})$. Remark that

$$p^*\Delta[n]_{m_0,m_1} \subseteq \left(\Delta \downarrow^l \mathbf{2}\right)((\mathbf{m_0+1}, \mathbf{m_1+1}), (\mathbf{0}, \mathbf{n+1})).$$

As a consequence, if $X \in \mathbf{Set}^{(\Delta\downarrow^l \mathbf{2})^{\mathrm{op}}}$, a natural transformation $\phi : p^*\Delta[n] \to X$ is uniquely determined by $\phi_{-1,n}(\mathrm{id}_{\mathbf{n+1}}) \in X_{-1,n}$, and moreover this correspondence is bijective. This shows that $\mathrm{Ran}_p(X)_\bullet = X_{-1,\bullet}$. As a consequence, $\overline{\mathrm{Tot}}_{-1,\bullet} \cong \mathrm{Tot}$. In order to compute $\overline{\mathrm{Tot}}$, we generalize slightly a lemma by Cegarra and Remedios [4, (12)]. Denote by $(\Delta \downarrow^l \mathbf{2})[n_0, n_1]$ the functor $(\Delta \downarrow^l \mathbf{2})^{\mathrm{op}} \to \mathbf{Set}$ represented by $(\mathbf{n_0+1}, \mathbf{n_1+1}) = ([n_0], [n_1])$.

4.3.1. Lemma. *Let $[n] = [n_0] + [n_1]$ with $n_1 \geq 0$. There is a coequalizer in $\mathbf{Set}^{(\Delta\times\Delta)_+^{\mathrm{op}}}$*

$$\coprod_{p=n_0+1}^{n-1} \Delta[p, n-p-1] \underset{1\times d_0}{\overset{d^{p+1}\times 1}{\rightrightarrows}} \coprod_{p=n_0+1}^{n} \Delta[p, n-p] \xrightarrow{\sum_p x_p} j^*(\Delta \downarrow^l \mathbf{2})[n_0, n_1].$$

Proof. Let $[m] = [m_0] + [m_1]$ and $f \in j^*(\Delta \downarrow^l \mathbf{2})[n_0, n_1]_{m_0,m_1}$. The map $f : [m] \to [n]$ is a non-decreasing map with the extra condition (when $m_1 \neq -1$) that $f(m_0+1) \geq n_0 + 1$.d Define $\tilde{f} : [m] \to [n+1]$ by $\tilde{f}(x) = \begin{cases} f(x) & x \leq m_0 \\ df(x)+1 & x > m_0 d \end{cases}$ and let $p = f(m_0+1)$ (or $p = n$ if $m_1 = -1$). Remark that $\tilde{f} : (\mathbf{m_0+1}, \mathbf{m_1+1}) \to (\mathbf{p+1}, \mathbf{n-p+1})$ is a strict map, and thus belongs to $(\Delta \times \Delta)_+$.Since $p = f(m_0+1) \geq n_0 + 1$,

$$s^p : [n+1] \to [n] \in j^*(\Delta \downarrow^l \mathbf{2})[n_0, n_1]_{p,n-p}.$$

Observe that $s^p\tilde{f} = f$. Let the natural transformation $x_p : \Delta[p, n-p] \to j^*(\Delta \downarrow^l \mathbf{2})[n_0, n_1]$ be represented by s^p. The previous argument shows that

$$\sum_p x_p : \coprod_{p=n_0+1}^{n+1} \Delta[p, n-p] \to j^*(\Delta \downarrow^l \mathbf{2})[n_0, n_1]$$

is surjective. Suppose now that $f = x_{p_1}(f_1)$ and $f = x_{p_2}(f_2)$ and observe that

$$f(m_0) \leq p_1, p_2 \leq f(m_0+1).$$

- If $p = p_1 = p_2$, $f_1(x) = f_2(x)$ for all $x \notin f^{-1}(p)$. Let $x \in f^{-1}(p)$. If $x \leq m_0$ then $f(m_0) = p$ and thus $p \leq f_i(x) \leq f_i(m_0) = p$, since f_i is strict. Similarly, if $x \geq m_0 + 1$, then $p + 1 \leq f_i(x) \leq p + 1$. This shows that $f_1 = f_2$.

- If $p_2 = p_1 + 1$, then $f(m_0) \neq f(m_1)$ and $p_1 < f(m_0 + 1)$. Up to modifying the splitting of the codomain, f is a strict map

$$f : (\mathbf{m_0+1}, \mathbf{m_1+1}) \to (\mathbf{p_1+1}, \mathbf{n-p_1})$$

Post-composing by d^{p_1+1} yields two maps

$$(\mathbf{m_0+1}, \mathbf{m_1+1}) \xrightarrow{f} (\mathbf{p_1+1}, \mathbf{n-p_1}) \xrightarrow{d^{p_1+1}=1\times d^0} (\mathbf{p_1+1}, \mathbf{n-p_1+1})$$

$$(\mathbf{m_0+1}, \mathbf{m_1+1}) \xrightarrow{f} (\mathbf{p_1+1}, \mathbf{n-p_1}) \xrightarrow{d^{p_1+1}=d^{p_1+1}\times 1} (\mathbf{p_1+2}, \mathbf{n-p_1}).$$

The first point implies that f_1 is the first composite and f_2 the second one.

- If $p_1 < p_2$, we can iterate the previous point to finish the proof.

■

4.3.2. Lemma. *For every n_0, $j^*(\Delta \downarrow^l \mathbf{2})[n_0, -1] \cong \Delta[n_0, -1]$.*

Proof. Since $(\mathbf{n_0+1}, \mathbf{0})$ corresponds to the minimal map $\mathbf{n_0+1} \to \mathbf{2}$, all maps to it must be strict. ■

4.3.3. Corollary. *Let $X \in \mathbf{Set}^{(\Delta\times\Delta)^{\mathrm{op}}_+}$. Its extended Artin-Mazur codiagonal is given explicitly as follows.*

- *For $n_0, n_1 \geq 0$*

$$\overline{\mathrm{Tot}}(X)_{n_0-1,n_1} = \left\{ (x_{n_0}, \dots x_{n_0+n_1}) \in \prod_{p=n_0}^{n_0+n_1} X_{p,n_0+n_1-p} : d^h_{p+1}(x_{p+1}) = d^v_0(x_p) \right\}.$$

- $\overline{\mathrm{Tot}}(X)_{n_0,-1} = X_{n_0,-1}$.

- *For $x = (x_{n_0}, \dots x_{n_0+n_1}) \in \overline{\mathrm{Tot}}(X)_{n_0-1,n_1}$,*

$$\begin{aligned}
d^v_j(x) &= (d^v_j(x_{n_0}), d^v_{j-1}(x_{n_0+1}), \dots, d^v_1(x_{n_0+j-1}), d^h_{j+n_0}(x_{n_0+j+1}), \dots d^h_{j+n_0}(x_{n_0+n_1})),\\
s^v_j(x) &= (s^v_j(x_{n_0}), s^v_{j-1}(x_{n_0+1}), \dots, s^v_0(x_{n_0+j}), s^h_{j+n_0}(x_{n_0+j}), \dots s^h_{j+n_0}(x_{n_0+n_1})),\\
d^h_j(x) &= (d^h_j(x_{n_0}), \dots, d^h_j(x_{n_0+n_1})),\\
s^h_j(x) &= (s^h_j(x_{n_0}), \dots, s^h_j(x_{n_0+n_1})),\\
\tau^*(x) &= (x_{n_0+1}, \dots, x_{n_0+n_1}).
\end{aligned}$$

Proof. Remark that Lemma 4.3.1 together with the description of $\mathrm{Dec}\Delta[n]$ given in Cegarra and Remedios's lemma [4, (12)] implies that $\overline{\mathrm{Tot}}_{n_0-1,n_1} = \mathrm{Tot}(X_{n_0+\bullet,\bullet})$. One can also derive the result directly from Lemma 4.3.1 using Yoneda's lemma. ■

4.3.4. Proposition. *Let $X \in \mathbf{Set}^{(\Delta\downarrow^l \mathbf{2})^{\mathrm{op}}}$. Then, for all $n \geq 0$ the augmentation $d_0^h : X_{\bullet,n} \to X_{-1,n}$ of the n-th row admits a vertical extra degeneracy given by*

$$s_{k+1} := \tau^* \circ s_0^v : X_{k,n} \to X_{k+1,n}.$$

Proof. We should check that three identities are verified.

(i)

$$\begin{aligned} d_{k+1}^h \circ s_{k+1} &= d_{k+1}^h \circ \tau^* \circ s_0^v \\ &= d_0^v \circ s_0^v \\ &= \mathrm{id}. \end{aligned}$$

(ii) For $0 \leq j \leq k$

$$\begin{aligned} d_j^h \circ s_{k+1} &= d_j^h \circ \tau^* \circ s_0^v \\ &= \tau^* d_j^h \circ s_0^v \\ &= s_k \circ d_j^h. \end{aligned}$$

(iii) For $0 \leq j \leq k$

$$\begin{aligned} s_j^h \circ s_{k+1} &= s_j^h \circ \tau^* \circ s_0^v \\ &= \tau^* s_j^h \circ s_0^v \\ &= s_{k+2} \circ s_j^h. \end{aligned}$$

and when $j = k+1$,

$$\begin{aligned} s_{k+1}^h \circ s_{k+1} &= s_{k+1}^h \circ \tau^* \circ s_0^v \\ &= (\tau^*)^2 \circ s_0^v \circ s_0^v \\ &= (\tau^*)^2 \circ s_1^v \circ s_0^v \\ &= \tau^* \circ s_0^v \circ \tau^* \circ s_0^v \\ &= s_{k+2} \circ s_{k+1}. \end{aligned}$$

■

4.3.5. Corollary. *Let $X \in \mathbf{Set}^{(\Delta\downarrow^l \mathbf{2})^{\mathrm{op}}}$. There is a weak equivalence* $\mathrm{Tot}(j^*X) \simeq X_{-1,\bullet}$.

Proof. By Proposition 4.3.4, all rows of j^*X are homotopically discrete with $\pi_0(X_{\bullet,n}) = X_{-1,n}$. Let $\Delta^h_{X_{-1,\bullet}}$ denote the bisimplicial set that is constant in the horizontal direction with value $X_{-1,\bullet}$ in the vertical direction. The bisimplicial set map $d_0 : j^*X \to \Delta^h_{X_{-1,\bullet}}$ is row-wise a weak equivalence. As a consequence, its diagonal and thus its Artin-Mazur codiagonal are weak equivalences as well, since the diagonal and the Artin-Mazur codiagonal are weakly equivalent ([3, Theorem 1.1]). Since $X_{-1,\bullet}$ is constant in each row, $\mathrm{Tot}(X_{-1,\bullet}) = X_{-1,\bullet}$. Thus, our map is a weak equivalence. ■

As a result, we get several new homotopical models for the diagonal of a bisimplicial set. For instance $d\overline{\mathrm{Tot}}(X) \simeq \mathrm{Tot}\overline{\mathrm{Tot}}(X) \simeq \overline{\mathrm{Tot}}(X)_{-1,\bullet} = \mathrm{Tot}(X)$. One can also keep iterating the extended codiagonal.

A. Presentation of a 2-category by a computad

We want to construct an analogue of a free category on a graph, but for 2-categories. Computads will play the role of graphs. Since 2-categories are categories enriched in categories, computads will be graphs enriched in graphs. The material of this Appendix is taken from [21].

A.1. Definition. *A* computad $\mathscr{G}$ *is a graph* $\mathrm{Gr}\mathscr{G}$ *together with, for each pair of vertices* A, B *of* $\mathrm{Gr}\mathscr{G}$*, another graph* $\mathscr{G}(A,B)$ *whose vertex set is a subset of the set of paths in the graph* $\mathrm{Gr}\mathscr{G}$ *from* A *to* B*.*

Any small 2-category $\mathscr{C}$ can be seen as a computad $U(\mathscr{C})$ by defining $\mathrm{Gr}U(\mathscr{C})$ to be the underlying graph of the underlying category of $\mathscr{C}$ and with graphs $U(\mathscr{C})(A,B)$ given by $U(\mathscr{C})(A,B)\big((f_1,\dots,f_n),(g_1,\dots,g_m)\big) = U\mathscr{C}(A,B)(f_1\circ\dots\circ f_n, g_1\circ\dots\circ g_m)$. This underlying functor also has a left adjoint, which is given in the following definition.

A.2. Definition. *For a computad* $\mathscr{G}$*, the 2-category* $\mathcal{F}\mathscr{G}$ *is constructed as follows.*

- *The set of objects is the vertex set of* $\mathrm{Gr}\mathscr{G}$*.*
- *For* A, B *two vertices, we define a pair of graphs* $D^2\mathscr{G}(A,B), D^1\mathscr{G}(A,B)$ *as follows. The vertex sets of both graphs are the set of paths in* $\mathrm{Gr}\mathscr{G}$ *which start at* A *and end at* B*. A diagram* $\begin{tikzcd} A \arrow[r, "f"] & W \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\alpha\Downarrow"] & X \arrow[r, "g"] & Y \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\beta\Downarrow"] & Z \arrow[r, "h"] & B \end{tikzcd}$*, where* f, g, h *are paths in* $|\mathscr{G}|$ *and* α*,* β *are arrows in* $\mathscr{G}(W,X)$ *and* $\mathscr{G}(Y,Z)$ *respectively, is an arrow of* $D^2\mathscr{G}(A,B)$*, from the top path to the bottom one. On the other hand, a diagram* $\begin{tikzcd} A \arrow[r, "f"] & W \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\alpha\Downarrow"] & X \arrow[r, "g"] & B \end{tikzcd}$ *of a similar form is an arrow in* $D^1\mathscr{G}(A,B)$ *from the top path to the bottom one.*

  The category $\mathcal{F}\mathscr{G}(A,B)$ *is the coequalizer in* **Cat** *of the diagram*

$$\mathcal{F}(D^2\mathscr{G}(A,B)) \rightrightarrows \mathcal{F}(D^1\mathscr{G}(A,B)) \longrightarrow \mathcal{F}\mathscr{G}(A,B)$$

  where the two arrows correspond to decomposing a diagram

$$\begin{tikzcd} A \arrow[r, "f"] & W \arrow[r, bend left, "a_0"] \arrow[r, bend right, "a_1"'] \arrow[r, phantom, "\alpha\Downarrow"] & X \arrow[r, "g"] & Y \arrow[r, bend left, "b_0"] \arrow[r, bend right, "b_1"'] \arrow[r, phantom, "\beta\Downarrow"] & Z \arrow[r, "h"] & B \end{tikzcd}$$

either as the composite

$$
\begin{tikzcd}
A \arrow[r, "f"] & W \arrow[r, bend left, "a_0"] \arrow[r, bend right, "a_1"'] \arrow[r, phantom, "\alpha\Downarrow"] & X \arrow[r, "g"] & Y \arrow[r, "b_0"] & Z \arrow[r, "h"] & B
\end{tikzcd}
$$

$$\circ$$

$$
\begin{tikzcd}
A \arrow[r, "f"] & W \arrow[r, "a_1"] & X \arrow[r, "g"] & Y \arrow[r, bend left, "b_0"] \arrow[r, bend right, "b_1"'] \arrow[r, phantom, "\beta\Downarrow"] & Z \arrow[r, "h"] & B
\end{tikzcd}
$$

or

$$
\begin{tikzcd}
A \arrow[r, "f"] & W \arrow[r, "a_0"] & X \arrow[r, "g"] & Y \arrow[r, bend left, "b_0"] \arrow[r, bend right, "b_1"'] \arrow[r, phantom, "\beta\Downarrow"] & Z \arrow[r, "h"] & B
\end{tikzcd}
$$

$$\circ$$

$$
\begin{tikzcd}
A \arrow[r, "f"] & W \arrow[r, bend left, "a_0"] \arrow[r, bend right, "a_1"'] \arrow[r, phantom, "\alpha\Downarrow"] & X \arrow[r, "g"] & Y \arrow[r, "b_1"] & Z \arrow[r, "h"] & B.
\end{tikzcd}
$$

More generally, the 2-cells in $F\mathscr{G}$ can be thought of as being built up from the ones in $\mathscr{G}$, by the operation of pasting (see [21]). For instance, the diagram

$$
\begin{tikzcd}
& A_2 \arrow[r, bend left, "f_2"] \arrow[r, bend right, "f_3"'] \arrow[r, phantom, "\Downarrow\alpha_1"] \arrow[d, phantom, "\Downarrow\alpha_2"] & A_3 \arrow[r, "f_6"] \arrow[d, phantom, "\Downarrow\alpha_3"] & A_4 \arrow[dr, "f_7"] & \\
A_1 \arrow[ur, "f_1"] \arrow[urr, bend right, "f_4"'] \arrow[rrrr, "f_5"'] & & {} & & A_5
\end{tikzcd}
$$

represents the composite

$$
\begin{tikzcd}
A_1 \arrow[r, "f_1"] & A_2 \arrow[r, bend left, "f_2"] \arrow[r, bend right, "f_3"'] \arrow[r, phantom, "\Downarrow\alpha_1"] & A_3 \arrow[r, "f_7 f_6"] & A_5
\end{tikzcd}
$$

$$\circ$$

$$
\begin{tikzcd}
A_1 \arrow[r, "\emptyset"] & A_1 \arrow[r, bend left, "f_3 f_1"] \arrow[r, bend right, "f_4"'] \arrow[r, phantom, "\Downarrow\alpha_2"] & A_3 \arrow[r, "f_7 f_6"] & A_5
\end{tikzcd}
$$

$$\circ$$

$$
\begin{tikzcd}
A_1 \arrow[r, "\emptyset"] & A_1 \arrow[r, bend left, "f_7 f_6 f_4"] \arrow[r, bend right, "f_5"'] \arrow[r, phantom, "\Downarrow\alpha_3"] & A_5 \arrow[r, "\emptyset"] & A_5.
\end{tikzcd}
$$

The category of $\mathscr{V}$-categories is cocomplete as long as $\mathscr{V}$ is cocomplete, as first established in [23]. This implies in particular that 2-**Cat** is cocomplete. We describe now a very particular kind of coequalizer in this category.

A.3. Definition. *A* presentation of a 2-category $\mathscr{C}$ by computads *is a pair of computads $\mathscr{G}, \mathscr{H}$ with same object set and a coequalizer* $\mathcal{F}\mathscr{G} \overset{F}{\underset{G}{\rightrightarrows}} \mathcal{F}\mathscr{H} \longrightarrow \mathscr{C}$ *in* 2-**Cat***, where F, G are identities on objects.*

Remark that in the conditions of the definition above, $\mathscr{C}(X,Y)$ is always generated by the image of $\mathcal{F}\mathscr{H}(X,Y)$.

B. The lax comma category

Let us recall some definitions that go back at least to John Gray [7].

B.1. Definition. *Let $F, G : \mathscr{C} \to \mathscr{D}$ be two 2-functors. A* lax natural transformation *$(\alpha, a) : F \to G$ is a collection $\alpha_C : F(C) \to G(C)$ of 1-cells of $\mathscr{D}$, indexed by objects $C \in |\mathscr{C}|$, together with a collection $a_f : G(f)\alpha_C \Rightarrow \alpha_{C'}F(f)$ of 2-cells of $\mathscr{D}$, indexed by 1-cells $f : C \to C' \in \mathscr{C}$, as pictured in the following diagram.*

$$
\begin{tikzcd}
FC \arrow[r, "Ff"] \arrow[d, "\alpha_C"'] & FC' \arrow[d, "\alpha_{C'}"] \\
GC \arrow[r, "Gf"'] \arrow[ur, phantom, "\overset{a_f}{\Nearrow}"] & GC'.
\end{tikzcd}
$$

This data is subject to the following axioms.

Naturality of a: *For all C, C', a is a natural transformation*

$$
\begin{tikzcd}
 & \mathscr{D}(FC,FC') \arrow[dr, "(\alpha_{C'})_*"] & \\
\mathscr{C}(C,C') \arrow[ur, "F_{CC'}"] \arrow[dr, "G_{CC'}"'] & \Uparrow a & \mathscr{D}(FC,GC'). \\
 & \mathscr{D}(GC,GC') \arrow[ur, "(\alpha_C)^*"'] &
\end{tikzcd}
$$

More explicitly, for $\phi : f \Rightarrow f'$ a 2-cell in $\mathscr{C}(C,C')$, the following diagram of 2-cells commutes.

$$
\begin{tikzcd}[row sep=huge]
FC \arrow[r, bend left, "Ff"] \arrow[r, bend right, "Ff'"'] \arrow[r, phantom, "F\phi\Downarrow"] \arrow[d, "\alpha_C"'] & FC' \arrow[d, "\alpha_{C'}"] \\
GC \arrow[r, bend left, "Gf"] \arrow[r, bend right, "Gf'"'] \arrow[r, phantom, "G\phi\Downarrow"] \arrow[ur, phantom, "\overset{a_f}{\Nearrow}\ \underset{a_{f'}}{\Nearrow}"] & GC'.
\end{tikzcd}
$$

Composition: *If $C \xrightarrow{f} C' \xrightarrow{f'} C''$ is a diagram of 1-cells in $\mathscr{C}$, there is an equality of 2-cells*

$$\begin{tikzcd} FC \arrow[r, "Ff"] \arrow[d, "\alpha_C"'] \arrow[dr, phantom, "\nearrow a_f"] & FC' \arrow[r, "Ff'"] \arrow[d, "\alpha_{C'}"] \arrow[dr, phantom, "\nearrow a_{f'}"] & FC'' \arrow[d, "\alpha_{C''}"] \\ GC \arrow[r, "Gf"'] & GC' \arrow[r, "Gf'"'] & GC'' \end{tikzcd} \quad = \quad \begin{tikzcd} FC \arrow[r, "F(f'f)"] \arrow[d, "\alpha_C"'] \arrow[dr, phantom, "\nearrow a_{f'f}"] & FC'' \arrow[d, "\alpha_{C''}"] \\ GC \arrow[r, "G(f'f)"'] & GC''. \end{tikzcd}$$

Unit: $a_{1_C} = 1_{\alpha_C}$.

B.2. Definition. *Given two lax natural transformations* $\begin{tikzcd} \mathscr{C} \arrow[r, bend left, "F"] \arrow[r, bend right, "G"'] \arrow[r, phantom, "(\alpha,a)\Downarrow\ \Downarrow(\beta,b)"] & \mathscr{D} \end{tikzcd}$ *from the* 2-*functor* F *to the* 2-*functor* G*, a* modification $(\alpha, a) \overset{M}{\Rrightarrow} (\beta, b)$ *is a collection of* 2-*cells of* $\mathscr{D}$*,* $M_C : \alpha_C \to \beta_C$ *for all* $C \in |\mathscr{C}|$*, such that the following diagram of* 2-*cells commutes for all* $f : C \to C' \in \mathscr{C}$*.*

$$\begin{tikzcd}[column sep=large] FC \arrow[r] \arrow[d, bend right=50, "\alpha_C"'] \arrow[d, bend left=50, "\beta_C"] \arrow[d, phantom, "\overset{M_C}{\Rrightarrow}"] \arrow[dr, phantom, "\nearrow a_f"] & FC' \arrow[d, bend right=50, "\alpha_{C'}"'] \arrow[d, bend left=50, "\beta_{C'}"] \arrow[d, phantom, "\overset{M_{C'}}{\Rrightarrow}"] \\ GC \arrow[r] \arrow[ur, phantom, "\nearrow b_f"'] & GC' \end{tikzcd}$$

B.3. Definition. *We define the* 2-*category* $\mathbf{Lax}(\mathscr{C}, \mathscr{D})$ *by letting the objects be the* 2-*functors* $\mathscr{C} \to \mathscr{D}$*, the* 1-*cells be the lax natural transformations between* 2-*functors and the* 2-*cells be the modifications between such lax natural transformations.*

We introduce now lax comma 2-categories, which Gray called 2-comma categories. Since one can form strict, pseudo and oplax versions of these constructions, we prefer to add the adjective lax.

B.4. Definition. *Given two* 2-*functors* $\mathscr{A} \xrightarrow{F} \mathscr{C} \xleftarrow{G} \mathscr{B}$ *, we define the* lax comma 2-category $F \downarrow^l G$ *as follows.*

- *An object is a pair of objects* $A \in |\mathscr{A}|$ *and* $B \in |\mathscr{B}|$ *together with a* 1-*cell* $f : FA \longrightarrow GB$*.*

- *A morphism from* $f : FA \to GB$ *to* $f' : FA' \to GB'$ *is a pair of 1-cells* $a : A \to A'$*,* $b : B \to B'$ *together with a* 2-*cell* $\theta : G(b)f \Rightarrow f'F(a)$*;*

$$\begin{tikzcd} FA \arrow[r, "Fa"] \arrow[d, "f"'] \arrow[dr, phantom, "\nearrow\theta"] & FA' \arrow[d, "f'"] \\ GB \arrow[r, "Gb"'] & GB' \end{tikzcd}$$

- *A* 2*-cell* $(a, b, \theta) \Rightarrow (a', b', \theta')$ *is a pair of* 2*-cells* $\alpha : a \Rightarrow a'$, $\beta : b \Rightarrow b'$ *such that the following diagram of* 2*-cells is commutative.*

$$\begin{tikzcd}[column sep=large, row sep=huge]
FA \arrow[r, bend left, "Fa"] \arrow[r, bend right, "Fa'"'] \arrow[r, phantom, "F(\alpha)\Downarrow"] \arrow[d, "f"'] \arrow[dr, phantom, "\overset{\theta}{\Rightarrow}\quad\overset{\theta'}{\Rightarrow}"] & FA' \arrow[d, "f'"] \\
GB \arrow[r, bend left, "Gb"] \arrow[r, bend right, "Gb'"'] \arrow[r, phantom, "G(\beta)\Downarrow"] & GB'.
\end{tikzcd}$$

That is, $f'F(\alpha) \circ \theta = \theta' \circ G(\beta)f$.

- *Composition is given by pasting such diagrams.*

These 2-categories satisfy a 2-universal property. This 2-universal property was already known by Gray, [7, Proposition I.5.1, page 102].

B.5. Proposition. *Let* $\mathscr{X}$ *be a* 2*-category and* $\mathscr{A} \xrightarrow{F} \mathscr{C} \xleftarrow{G} \mathscr{B}$ *a diagram of* 2*-functors, and consider*

$$\mathbf{Lax}(\mathscr{X}, \mathscr{A}) \xrightarrow{F_*} \mathbf{Lax}(\mathscr{X}, \mathscr{C}) \xleftarrow{G_*} \mathbf{Lax}(\mathscr{X}, \mathscr{B})$$

There is an isomorphism of 2*-categories*

$$\mathbf{Lax}(\mathscr{X}, F \downarrow^l G) \cong F_* \downarrow^l G_*$$

which is natural in $\mathscr{X}$*.*

References

[1] Ricardo Andrade. *From manifolds to invariants of E_n-algebras.* PhD thesis, M.I.T., 2010.

[2] C. Auderset. Adjonctions et monades au niveau des 2-catégories. *Cahiers Topologie Géom. Différentielle*, 15:3–20, 1974.

[3] A. M. Cegarra and Josué Remedios. The relationship between the diagonal and the bar constructions on a bisimplicial set. *Topology Appl.*, 153(1):21–51, 2005.

[4] Antonio M. Cegarra and Josué Remedios. The behaviour of the $\overline{W}$-construction on the homotopy theory of bisimplicial sets. *Manuscripta Math.*, 124(4):427–457, 2007.

[5] Jean-Marc Cordier and Timothy Porter. Vogt's theorem on categories of homotopy coherent diagrams. *Math. Proc. Cambridge Philos. Soc.*, 100(1):65–90, 1986.

[6] P. Gabriel and M. Zisman. *Calculus of fractions and homotopy theory.* Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 35. Springer-Verlag New York, Inc., New York, 1967.

[7] John W. Gray. *Formal category theory: adjointness for* 2*-categories.* Lecture Notes in Mathematics, Vol. 391. Springer-Verlag, Berlin, 1974.

[8] Mark Hovey. *Model categories*, volume 63 of *Mathematical Surveys and Monographs.* American Mathematical Society, Providence, RI, 1999.

[9] G. M. Kelly. Basic concepts of enriched category theory. *Repr. Theory Appl. Categ.*, (10):vi+137, 2005. Reprint of the 1982 original [Cambridge Univ. Press, Cambridge; MR0651714].

[10] G. M. Kelly and Ross Street. Review of the elements of 2-categories. In *Category Seminar (Proc. Sem., Sydney, 1972/1973)*, pages 75–103. Lecture Notes in Math., Vol. 420. Springer, Berlin, 1974.

[11] Stephen Lack and Ross Street. The formal theory of monads. II. *J. Pure Appl. Algebra*, 175(1-3):243–265, 2002. Special volume celebrating the 70th birthday of Professor Max Kelly.

[12] Saunders MacLane. *Categories for the working mathematician.* Springer-Verlag, New York-Berlin, 1971. Graduate Texts in Mathematics, Vol. 5.

[13] Aleš Pultr and Walter Tholen. Free Quillen factorization systems. *Georgian Math. J.*, 9(4):807–820, 2002. Dedicated to Professor Hvedri Inassaridze on the occasion of his 70th birthday.

[14] Emily Riehl and Dominic Verity. The 2-category theory of quasi-categories. *Adv. Math.*, 280:549–642, 2015.

[15] Emily Riehl and Dominic Verity. Completeness results for quasi-categories of algebras, homotopy limits, and related general constructions. *Homology Homotopy Appl.*, 17(1):1–33, 2015.

[16] Emily Riehl and Dominic Verity. Homotopy coherent adjunctions and the formal theory of monads. *Adv. Math.*, 286:802–888, 2016.

[17] Emily Riehl and Dominic Verity. Fibrations and Yoneda's lemma in an ∞-cosmos. *J. Pure Appl. Algebra*, 221(3):499–564, 2017.

[18] Emily Riehl and Dominic Verity. Kan extensions and the calculus of modules for ∞–categories. *Algebr. Geom. Topol.*, 17(1):189–271, 2017.

[19] Stephen Schanuel and Ross Street. The free adjunction. *Cahiers Topologie Géom. Différentielle Catég.*, 27(1):81–83, 1986.

[20] Ross Street. The formal theory of monads. *J. Pure Appl. Algebra*, 2(2):149–168, 1972.

[21] Ross Street. Limits indexed by category-valued 2-functors. *J. Pure Appl. Algebra*, 8(2):149–181, 1976.

[22] Rainer M. Vogt. Homotopy limits and colimits. *Math. Z.*, 134:11–52, 1973.

[23] Harvey Wolff. V-cat and V-graph. *J. Pure Appl. Algebra*, 4:123–135, 1974.

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