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CLOSURE OPERATORS IN ABELIAN CATEGORIES AND SPECTRAL SPACES

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Abstract. We give several new ways of constructing spectral spaces starting with objects in abelian categories satisfying certain conditions which apply, in particular, to Grothendieck categories. For this, we consider the spaces of invariants of closure operators acting on subobjects of a given object. The key to our results is a newly discovered criterion of Finocchiaro that uses ultrafilters to identify spectral spaces along with subbases of quasi-compact open sets.

1. Introduction

A topological space is said to be spectral if it is homeomorphic to the Zariski spectrum of a commutative ring. A major fact proved by Hochster in [14] is that such spaces can be characterized in purely topological fashion, i.e., a topological space X is spectral if and only if it satisfies the following conditions:

(1) X is quasi-compact.

(2) The quasi-compact open subsets of X are closed under finite intersections and form an open basis of X.

(3) Every nonempty irreducible closed subset of X has a unique generic point.

Indeed, the study of spectral spaces is a big topic and the literature on this subject is immense (see [17, Tag 08YF] for a brief introduction). Further, by using a newly discovered criterion of Finocchiaro [6, Corollary 3.3] for spectral spaces using ultrafilters, Finocchiaro, Fontana and Spirito [8] have recently given several new examples of spectral spaces arising in commutative algebra (see also [7], [9], [10]). For instance, the authors showed in [8] that the space of submodules of a given module M over a commutative ring R forms a spectral space. This immediately leads us to ask similar questions in algebraic geometry. For example, would a similar result hold in the category of quasi-coherent sheaves over a scheme X?

In this paper, our objective is to build spectral spaces starting from a general abelian category. We require that our abelian categories should satisfy the (AB5) axiom (see [13]) and also be well-powered, i.e., the collection of isomorphism classes of subobjects of any given object should be a set. In particular, this applies to all Grothendieck categories. For

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instance, if $\mathcal{A}$ is a sheaf of rings on a topological space X, then the categories $\mathcal{A}-Premod$ and $\mathcal{A}-Mod$ of presheaves and sheaves of $\mathcal{A}$-modules respectively are Grothendieck categories (see [2, § I.3, II.4]). If X is an arbitrary scheme, then Gabber has shown that the category $QCoh(X)$ of quasi-coherent sheaves on X is a Grothendieck category (see, for instance, [3, § 3]). As such, we feel that our results also show the power and versatility of Finocchiaro's new criterion [6] for spectral spaces.

We now describe the paper in greater detail. We begin in Section 2 by fixing an object M of the abelian category $\mathcal{C}$ along with a subobject $N \subseteq M$. We let $Fin(N, M)$ be the collection of all subobjects of M that may be obtained by adding to N a collection of finitely generated subobjects of M. If $\mathcal{C}$ is a category of modules or more generally, any category that is "locally finitely generated" (see, for instance, [1]), then $Fin(N, M)$ reduces to the collection of all subobjects of M that contain N. Then, our first main result is that $Fin(N, M)$ is a spectral space, for which we use the criterion of Finocchiaro [6] and adapt the methods of [8]. Thereafter, given any inclusion $\beta : N \hookrightarrow N'$ in the category $Sub(M)$ of subobjects of M, we construct a morphism

$$Fin(\beta) : Fin(N', M) \longrightarrow Fin(N, M) \tag{1}$$

It is important to emphasize that the maps in (1) are not just continuous, but also "spectral maps", i.e., the inverse of any quasi-compact open is still a quasi-compact and open set. The latter follows from carefully examining the explicit subbasis of quasi-compact opens that we construct in the process of showing that each $Fin(N, M)$ is a spectral space. Further, we consider the Grothendieck topology on $Sub(M)$ in which the coverings of any $N \in Sub(M)$ are given by filtered families $\{N_i \hookrightarrow N\}_{i \in I}$ in $Sub(M)$ satisfying $\sum_{i \in I} N_i = N$. Then, we show that the association $N \mapsto Fin(N, M)$ actually determines a sheaf on $Sub(M)$ taking values in the category of spectral spaces connected by spectral maps.

In Section 3, we work more generally with the space $Fin(N, M)$ equipped with a "closure operator" $c : Fin(N, M) \longrightarrow Fin(N, M)$. For this, we are also motivated by the methods of [8, § 3] with closure operators on submodules. Our idea of a closure operator in Definition 3.1 for abelian categories should be compared to closure operators for ideals and submodules, which provide a unifying framework for several closure notions in commutative algebra such as integral closure, radical closure and Frobenius closure (see, for instance, [4], [5], [8, § 3]). Given a closure operator, or more generally, an extensive and order-preserving operator c on $Fin(N, M)$, we show that the space $Fin^c(N, M)$ of fixed points of c is a spectral space when c is of finite type (see Definition 3.1). Further, when we have a family of such operators $c_N : Fin(N, M) \longrightarrow Fin(N, M)$, $N \in Sub(M)$ constructed by starting from a single operator $c = c_0 : Fin(0, M) \longrightarrow Fin(0, M)$, we show that the association $N \mapsto Fin^{c_N}(N, M)$ determines a sheaf on $Sub(M)$ taking values in the category of spectral spaces connected by spectral maps. We conclude Section 3 by giving several explicit examples of closure operators on quasi-coherent sheaves of ideals over a quasi-separated Noetherian scheme, thus producing spectral spaces.

We consider in Section 4 the general question of which subsets of $Fin(N, M)$ can arise as the collection of fixed points of a closure operator of finite type on $Fin(N, M)$.

We show that families of objects in $Fin(N,M)$ that are closed under intersections and filtered unions correspond to spaces of fixed points of closure operators of finite type on $Fin(N,M)$. In particular, they are spectral subspaces of $Fin(N,M)$. Finally, we consider the object $N+M^f \in Fin(N,M)$, where M^f is the sum of all finitely generated subobjects of M. Then, it is clear that $N+M^f$ contains every other object in $Fin(N,M)$. Hence, $N+M^f$ must be fixed by every closure operator on $Fin(N,M)$. As such, we consider the space $Fin^c(N,M)$ with the point $N+M^f$ removed and give some conditions for $Fin^c(N,M)\backslash\{N+M^f\}$ to be a spectral subspace. We conclude by giving a sufficient condition for $Fin^c(N,M)$ with the point $N+M^f$ removed to be such that every object is still contained in a maximal element.

2. Spectral spaces of subobjects

Let $\mathcal{C}$ be an abelian category. For each object $M \in \mathcal{C}$, we will denote by $Sub(M)$ the category of subobjects of M in $\mathcal{C}$. More explicitly, each object of $Sub(M)$ can be represented by a monomorphism $\alpha : N \hookrightarrow M$. Two monomorphisms $\alpha : N \hookrightarrow M$ and $\alpha' : N' \hookrightarrow M$ are said to be equivalent if there exists an isomorphism $\beta : N \longrightarrow N'$ such that $\alpha' \circ \beta = \alpha$. Then, the subobjects of M are equivalence classes of monomorphisms with target M. For the sake of simplicity, we will denote the object of $Sub(M)$ reprsented by $(\alpha : N \hookrightarrow M)$ simply by N and write $N \subseteq M$.

Further, a morphism β from an object $(\alpha : N \hookrightarrow M)$ in $Sub(M)$ to an object $(\alpha' : N' \hookrightarrow M)$ in $Sub(M)$ will be given by a morphism $\beta : N \longrightarrow N'$ in $\mathcal{C}$ such that $\alpha' \circ \beta = \alpha$. Since α is a monomorphism, it is immediate that there is at most one such morphism β and that β must also be a monomorphism. In this case, we will write $N \subseteq N'$.

Throughout this section and the rest of this paper, we will assume that the abelian category $\mathcal{C}$ is "well-powered" (see, for example, [11, § 3]), i.e., the category $Sub(M)$ is a small category for every object $M \in \mathcal{C}$. This holds, for instance, if the abelian category has a generator (see [11, Proposition 3.35]) and in particular every Grothendieck category is well-powered. We will also assume throughout that $\mathcal{C}$ satisfies (AB5), i.e., coproducts exist and filtered colimits commute with finite limits.

We recall that an object $M \in \mathcal{C}$ is said to be "finitely generated" if the co-representable functor $Hom_{\mathcal{C}}(M, \_\_) : \mathcal{C} \longrightarrow Set$ preserves filtered colimits of monomorphisms (see, for instance, [1] where such objects are referred to as "ω-generated"). For each object $M \in \mathcal{C}$, we denote by $fg(M)$ the collection of isomorphism classes of finitely generated subobjects of M and always set $M^f := \sum_{M' \in fg(M)} M'$.

For each pair (N,M) with $N \subseteq M$, we now set:

$$Fin(N,M) := \{T \mid N \subseteq T \subseteq M \text{ and } T = N + T^f = N + \sum_{T_0 \in fg(T)} T_0\} \tag{2}$$

Then, for each object $Q \in Fin(N,M)$, we consider the subsets:

$$V(Q,N,M) := \{T \in Fin(N,M) \mid Q \subseteq T\} \quad D(Q,N,M) := Fin(N,M)\backslash V(Q,N,M) \tag{3}$$

We can now define a topology on $Fin(N, M)$ whose subbasis is given by the collection $\mathcal{S}(N, M) := \{D(Q, N, M) \mid Q \in Fin(N, M)\}$. We show that this topology makes $Fin(N, M)$ into a spectral space, i.e., it is homeomorphic to the Zariski spectrum of some commutative ring. For this, we will use the characterization of spectral spaces via ultrafilters due to Finocchiaro [6]. We recall here (see, for instance, [6, § 1]) that a filter $\mathfrak{F}$ on a set X is a collection of subsets of X such that: (a) $\phi \notin \mathfrak{F}$, (b) $Y, Z \in \mathfrak{F} \Rightarrow Y \cap Z \in \mathfrak{F}$ and (c) $Y \subseteq Z \subseteq X$ and $Y \in \mathfrak{F}$ implies that $Z \in \mathfrak{F}$. Then, an ultrafilter $\mathfrak{F}$ of subsets of X is a maximal element in the collection of filters on X.

2.1. Proposition. *We fix an object $M \in \mathcal{C}$ and take an arbitrary subobject $N \subseteq M$. Then, the following holds for the topological space $Fin(N, M)$:*

(a) The subcollection

$$\mathcal{S}^f(N, M) := \{D(Q, N, M) \in \mathcal{S}(N, M) \mid \exists\, Q_0 \in fg(Q) \text{ such that } Q = N + Q_0\}$$

of $\mathcal{S}(N, M)$ also forms a subbasis for the topology on $Fin(N, M)$.

(b) $Fin(N, M)$ is a spectral space with $\mathcal{S}^f(N, M)$ as a subbasis of quasi-compact open sets.

Proof. (a) We consider some $Q \in Fin(N, M)$. From the definition in (2), it follows that $Q = N + \sum\limits_{Q_0 \in fg(Q)} Q_0$. Consequently, we must have:

$$D(Q, N, M) = \bigcup_{Q_0 \in fg(Q)} D(Q_0 + N, N, M) \tag{4}$$

From (4), it is clear that the topology on $Fin(N, M)$ given by the subbasis $\mathcal{S}^f(N, M)$ is identical to the topology given by the subbasis $\mathcal{S}(N, M)$.

(b) From the definitions in (3), it is immediate that $Fin(N, M)$ is a T_0-space. We now consider an ultrafilter $\mathfrak{F}$ of subsets of $Fin(N, M)$ and form the subobject:

$$Q_{\mathfrak{F}} := \sum_{V(Q,N,M) \in \mathfrak{F}} Q \tag{5}$$

Since $Fin(N, M)$ is closed under arbitrary sums, it is clear that $Q_{\mathfrak{F}} \in Fin(N, M)$. In particular, for $Q, Q' \in Fin(N, M)$, we must have $Q + Q' \in Fin(N, M)$ and we notice that $V(Q, N, M) \cap V(Q', N, M) = V(Q + Q', N, M)$. Since the ultrafilter $\mathfrak{F}$ is closed under finite intersections, it follows that the sum in (5) is filtered.

We now claim that for any set $D(Q, N, M) \in \mathcal{S}^f(N, M)$, the element $Q_{\mathfrak{F}} \in D(Q, N, M)$ if and only if $D(Q, N, M) \in \mathfrak{F}$. For this, we first suppose there exists some $D(Q, N, M) \in \mathcal{S}^f(N, M) \cap \mathfrak{F}$ but $Q_{\mathfrak{F}} \notin D(Q, N, M)$. Then, $Q_{\mathfrak{F}} \in V(Q, N, M)$, i.e., $Q \subseteq Q_{\mathfrak{F}}$. We can express Q as $Q = N + Q_0$ where $Q_0 \in fg(Q)$. Since Q_0 is finitely generated and $Q_{\mathfrak{F}}$ is expressed as a filtered sum in (5), it follows that we can take $Q' \in Fin(N, M)$ such that $V(Q', N, M) \in \mathfrak{F}$ and $Q_0 \subseteq Q'$. Thus, $Q = N + Q_0 \subseteq Q'$. This implies that $V(Q, N, M) \supseteq$

$V(Q', N, M)$ and hence $V(Q, N, M) \in \mathfrak{F}$. But then $\phi = D(Q, N, M) \cap V(Q, N, M) \in \mathfrak{F}$, which is a contradiction.

Conversely, suppose that we have $D(Q, N, M) \in \mathcal{S}^f(N, M)$ with $Q_{\mathfrak{F}} \in D(Q, N, M)$. Then, if $D(Q, N, M) \notin \mathfrak{F}$, it follows that its complement $V(Q, N, M)$ must lie in the ultrafilter $\mathfrak{F}$. But then the definition in (5) implies that $Q \subseteq Q_{\mathfrak{F}}$ and hence $Q_{\mathfrak{F}} \in V(Q, N, M)$, which contradicts the fact that $Q_{\mathfrak{F}} \in D(Q, N, M)$. It now follows from the criterion in [6, Corollary 3.3] that $Fin(N, M)$ is a spectral space. Further, it follows from [8, Corollary 1.2] that $\mathcal{S}^f(N, M)$ is a subbasis of quasi-compact open sets for $Fin(N, M)$. ∎

We now turn the association $N \mapsto Fin(N, M)$ in Proposition 2.1 into a functor on $Sub(M)$. For this, we recall (see [14]) that a map $f : X \longrightarrow X'$ of spectral spaces is said to be a spectral map if the inverse of any quasi-compact open in X' is open and quasi-compact in X.

2.2. Proposition. *We fix an object $M \in \mathcal{C}$ and consider the category $Sub(M)$ of all subobjects of M. Then, the association $(\alpha : N \hookrightarrow M) \mapsto Fin(N, M)$ gives a contravariant functor from $Sub(M)$ to the category of spectral spaces connected by spectral maps.*

Proof. Given a morphism $\beta : N \hookrightarrow N'$ in $Sub(M)$, we define a map:

$$Fin(\beta) : Fin(N', M) \longrightarrow Fin(N, M) \qquad T' \mapsto N + T'^f = N + \sum_{T'_0 \in fg(T')} T'_0 \tag{6}$$

From Proposition 2.1, we know that $\mathcal{S}^f(N, M)$ is a subbasis of quasi-compact open sets of $Fin(N, M)$. As such, in order to show that $Fin(\beta)$ is a spectral map, it suffices to check that $Fin(\beta)^{-1}(D(Q, N, M)) \in \mathcal{S}^f(N', M)$ for each $D(Q, N, M) \in \mathcal{S}^f(N, M)$. Since $D(Q, N, M) \in \mathcal{S}^f(N, M)$, we can express $Q = N + Q_0$ for some $Q_0 \in fg(Q)$. We will now show that $Fin(\beta)^{-1}(D(Q, N, M)) = D(N' + Q_0, N', M) \in \mathcal{S}^f(N', M)$ which is equivalent to checking that:

$$Fin(\beta)^{-1}(V(Q, N, M)) = V(N' + Q_0, N', M) \tag{7}$$

We first take an arbitrary $T' \in V(N' + Q_0, N', M)$. Then, $Q_0 \in fg(T')$. From the definition in (6), it now follows that $Q = N + Q_0 \subseteq Fin(\beta)(T')$. Hence, we see that $T' \in Fin(\beta)^{-1}(V(Q, N, M))$. Conversely, let $T' \in Fin(N', M)$ be such that $T = Fin(\beta)(T') \in V(Q, N, M)$. Again, from (6), it follows that:

$$T = Fin(\beta)(T') = N + \sum_{T'_0 \in fg(T')} T'_0 \subseteq N' + \sum_{T'_0 \in fg(T')} T'_0 = T' \tag{8}$$

Since $T \subseteq T'$, we see that $Q_0 \subseteq Q \subseteq T \subseteq T'$. We already know that $T' \in Fin(N', M)$ contains N' and hence $T' \in V(N' + Q_0, N', M)$. This proves the result. ∎

Our next aim is to describe sheaf properties of the contravariant functor in Proposition 2.2. But, first we record the following simple results, which will help make the construction in (6) more transparent.

2.3. Lemma. *(a) Given* T_1, $T_2 \in Fin(N,M)$ *such that* $fg(T_1) = fg(T_2)$*, we must have* $T_1 = T_2$.

(b) Consider a morphism $\beta : N \hookrightarrow N'$ *in* $Sub(M)$ *and take an arbitrary* $T' \in Fin(N', M)$*. Let* $T = Fin(\beta)(T')$*. Then,* $T \subseteq T'$*. Moreover,* $fg(T) = fg(T')$*.*

Proof. (a) Since $T_1, T_2 \in Fin(N,M)$ and $fg(T_1) = fg(T_2)$, it follows from the definition in (2) that:

$$T_1 = N + \sum_{T_{10} \in fg(T_1)} T_{10} = N + \sum_{T_{20} \in fg(T_2)} T_{20} = T_2 \tag{9}$$

(b) We have $T = Fin(\beta)(T') \subseteq T'$ as in (8) in the proof of Proposition 2.2. Hence, $fg(T) \subseteq fg(T')$. On the other hand, from the definition in (6), we have:

$$T = Fin(\beta)(T') = N + \sum_{T'_0 \in fg(T')} T'_0 \tag{10}$$

It is clear from (10) that $fg(T') \subseteq fg(T)$. It follows that $fg(T') = fg(T)$.

∎

We will now say that a family $\{\alpha_i : N_i \hookrightarrow N\}_{i \in I}$ of morphisms in $Sub(M)$ indexed by a filtered set I is a 'covering family' for N if $colim_{i\in I} N_i = \sum_{i \in I} N_i = N$. It is clear that $\{1 : N \longrightarrow N\}$ is a covering family. Further, given a covering family $\{\alpha_i : N_i \hookrightarrow N\}_{i\in I}$ and a covering family $\{\alpha_{ij} : N_{ij} \hookrightarrow N_i\}_{j \in J_i}$ for each $i \in I$, we see that $\{N_{ij} \hookrightarrow N\}_{i\in I, j \in J_i}$ must be a covering family. Finally, since filtered colimits commute with finite limits in $\mathcal{C}$, covering families are stable under pullbacks. It follows that covering families give a Grothendieck topology on $Sub(M)$ in the sense of [15, Chapter III].

For a covering family $\{\alpha_i : N_i \hookrightarrow N\}_{i\in I}$, the inclusions $\alpha_{ij} : N_i \hookrightarrow N_j$ in the filtered system $\{N_i\}_{i\in I}$ give rise to an inverse system of spectral spaces $\{Fin(N_i, M)\}_{i \in I}$ connected by spectral maps. It follows (see [16, (A.1)]) that the limit of this system in the category of topological spaces is still a spectral space which is further equal to the limit of the system in the category of spectral spaces.

2.4. Proposition. *(a) Let* $\{\alpha_i : N_i \hookrightarrow N\}_{i\in I}$ *be a covering family in* $Sub(M)$*. Then, there is a homeomorphism of spectral spaces* $Fin(N,M) = \varprojlim_{i \in I} Fin(N_i, M)$*.*

(b) The association $(\alpha : N \hookrightarrow M) \mapsto Fin(N,M)$ *defines a sheaf on* $Sub(M)$ *taking values in the category of spectral spaces connected by spectral maps.*

Proof. First, we will show that $Fin(N,M) = \varprojlim_{i\in I} Fin(N_i, M)$ at the level of sets. For this we consider a collection $\{T_i\}_{i\in I}$ of objects with $T_i \in Fin(N_i, M)$ and $Fin(\alpha_{ij})(T_j) = T_i$ for any inclusion $\alpha_{ij} : N_i \hookrightarrow N_j$ in the filtered system $\{N_i\}_{i\in I}$ of subobjects of N. We now set $T := \sum_{i\in I} T_i$. Since each $T_i = N_i + \sum_{T_{i0} \in fg(T_i)} T_{i0}$ and $\sum_{i \in I} N_i = N$, it follows that $T \in Fin(N,M)$.

Given $i,\ j \in I$, we fix $k \in I$ such that $N_i, N_j \subseteq N_k$. Now since $T_i = Fin(\alpha_{ik})(T_k)$ and $T_j = Fin(\alpha_{jk})(T_k)$, it follows from Lemma 2.3(b) that $fg(T_i) = fg(T_k) = fg(T_j)$. Hence, the collection $fg(T_i)$ is identical for each $i \in I$. Further, since I is filtered, any finitely generated $T_0 \in fg(T) = fg(\sum_{i\in I} T_i)$ must lie in $fg(T_{i_0})$ for some $i_0 \in I$ and hence in every $fg(T_i)$, $i \in I$. Thus, $fg(T) = fg(T_i)$ for each $i \in I$. It follows that $T_i = N_i + \sum_{T_{i0}\in fg(T_i)} T_{i0} = N_i + \sum_{T_0\in fg(T)} T_0 = Fin(\alpha_i)(T)$ for each $i \in I$. Finally, if $S \in Fin(N, M)$ is another object such that $Fin(\alpha_i)(S) = T_i$ for each $i \in I$, we must have $fg(S) = fg(T_i) = fg(T)$. Since $S,\ T \in Fin(N, M)$, it now follows from Lemma 2.3(a) that $S = T$.

By construction, the maps $Fin(\alpha_i) : Fin(N, M) \longrightarrow Fin(N_i, M)$ are already continuous. Consider an element $D(Q, N, M)$ of the subbasis $\mathcal{S}^f(N, M)$ of $Fin(N, M)$. Then, there exists $Q_0 \in fg(Q)$ such that $Q = N + Q_0$. We fix $i \in I$. From the proof of Proposition 2.2, we know that $Fin(\alpha_i)^{-1}(D(N_i + Q_0, N_i, M)) = D(N + Q_0, N, M) = D(Q, N, M)$. Hence, the topology on $Fin(N, M)$ given by the subbasis $\mathcal{S}^f(N, M)$ is the coarsest topology for which the maps $Fin(\alpha_i) : Fin(N, M) \longrightarrow Fin(N_i, M)$ are all continuous. This proves (a). The result of (b) follows from (a) and the definition of covering families in $Sub(M)$. ∎

3. Spectral spaces determined by closure operators

Let R be a commutative ring. In commutative algebra, a closure operator (see, for instance, [4, § 2.1], [5]) associates to each ideal I in R an ideal $\bar{I}$ such that $I \subseteq \bar{I}$, $\bar{\bar{I}} = \bar{I}$ and $\bar{I} \subseteq \bar{J}$ whenever $I \subseteq J$. We begin this section by adapting this idea to abelian categories.

3.1. Definition. *Let $M \in \mathcal{C}$ and fix a subobject $N \subseteq M$. We define the following notions for an operator $c : Fin(N, M) \longrightarrow Fin(N, M)$:*

(a) Extensive: for each $T \in Fin(N, M)$, we have $T \subseteq c(T)$.

(b) Idempotent: for each $T \in Fin(N, M)$, we have $c(c(T)) = c(T)$.

(c) Order-preserving: for any $T_1,\ T_2 \in Fin(N, M)$ with $T_1 \subseteq T_2$, we have $c(T_1) \subseteq c(T_2)$.

(d) Finite type: for any $T \in Fin(N, M)$, we have $c(T) = \sum_{T_0\in fg(T)} c(N + T_0)$.

In particular, an operator $c : Fin(N, M) \longrightarrow Fin(N, M)$ that is extensive, idempotent and order-preserving will be referred to as a closure operator.

We urge the reader to compare Definition 3.1 above to the definition of closure in categories of modules in [4, Definition 7.0.1].

3.2. Lemma. *Let $c : Fin(N, M) \longrightarrow Fin(N, M)$ be an operator that is extensive, order-preserving and of finite type. For any $T \in Fin(N, M)$, set $c^\infty(T) := \sum_{k=1}^{\infty} c^k(T)$. Then, the object $c^\infty(T)$ is fixed by the operator c, i.e., $c(c^\infty(T)) = c^\infty(T)$.*

Proof. We take an arbitrary $T_0 \in fg(c^\infty(T))$. Since T_0 is finitely generated, we can take $k_0 \in \mathbb{N}$ such that $T_0 \subseteq c^{k_0}(T)$. Further since $N \subseteq T \subseteq c^{k_0}(T)$, it follows that $N+T_0 \subseteq c^{k_0}(T)$. This gives $c(N+T_0) \subseteq c^{k_0+1}(T) \subseteq c^\infty(T)$ for each $T_0 \in fg(c^\infty(T))$. The operator c being of finite type, we now have $c(c^\infty(T)) = \sum_{T_0 \in fg(c^\infty(T))} c(N+T_0) \subseteq c^\infty(T)$. This proves the result.

■

3.3. Proposition. *Let $c : Fin(N,M) \longrightarrow Fin(N,M)$ be an operator that is extensive, order-preserving and of finite type. Then, the collection $Fin^c(N,M)$ of all objects in $Fin(N,M)$ fixed by c is a spectral space. Further, the inclusion $Fin^c(N,M) \hookrightarrow Fin(N,M)$ is a spectral map.*

Proof. We maintain the notation from Section 2. From Proposition 2.1 it follows that the collection $D(Q,N,M) \cap Fin^c(N,M)$ forms a subbasis for the topology on $Fin^c(N,M)$, where $D(Q,N,M) \in \mathcal{S}^f(N,M)$. Now, let $\mathfrak{F}$ be an ultrafilter of subsets of $Fin^c(N,M)$ and set:

$$T_{\mathfrak{F}} := \sum_{T \in Fin^c(N,M), V(T,N,M) \cap Fin^c(N,M) \in \mathfrak{F}} T \tag{11}$$

Now suppose that T_1, $T_2 \in Fin^c(N,M)$ are such that $V(T_1,N,M) \cap Fin^c(N,M)$ and $V(T_2,N,M) \cap Fin^c(N,M)$ lie in $\mathfrak{F}$. Using Lemma 3.2, we know that $c^\infty(T_1+T_2) \in Fin^c(N,M)$. Further, we notice that $V(c^\infty(T_1+T_2),N,M) \cap Fin^c(N,M) = V(T_1+T_2,N,M) \cap Fin^c(N,M) = V(T_1,N,M) \cap V(T_2,N,M) \cap Fin^c(N,M) \in \mathfrak{F}$. Hence, the sum in (11) is filtered.

We claim that $T_{\mathfrak{F}} \in Fin^c(N,M)$. For this, we take an arbitrary $T_0 \in fg(T_{\mathfrak{F}})$. Since the sum in (11) is filtered, it follows that we can find some $T \in Fin^c(N,M)$ with $V(T,N,M) \cap Fin^c(N,M) \in \mathfrak{F}$ such that $T_0 \subseteq T$. Then, $N + T_0 \subseteq T$ and since $T \in Fin^c(N,M)$, we get $c(N+T_0) \subseteq c(T) = T \subseteq T_{\mathfrak{F}}$. Since c is of finite type, we now have $c(T_{\mathfrak{F}}) = \sum_{T_0 \in fg(T_{\mathfrak{F}})} c(N+T_0) \subseteq T_{\mathfrak{F}}$ and hence $T_{\mathfrak{F}} \in Fin^c(N,M)$.

Finally, we consider some $Q \in Fin(N,M)$ such that $Q = N+Q_0$ for some $Q_0 \in fg(Q)$. We will show that

$$T_{\mathfrak{F}} \in D(Q,N,M) \cap Fin^c(N,M) \qquad \Leftrightarrow \qquad D(Q,N,M) \cap Fin^c(N,M) \in \mathfrak{F} \tag{12}$$

First, we suppose that $D(Q,N,M) \cap Fin^c(N,M) \notin \mathfrak{F}$. Since $\mathfrak{F}$ is an ultrafilter, it follows that the complement $V(Q,N,M) \cap Fin^c(N,M) \in \mathfrak{F}$. We now notice that $V(Q,N,M) \cap Fin^c(N,M) = V(c^\infty(Q),N,M) \cap Fin^c(N,M)$. From Lemma 3.2, we know that $c^\infty(Q) \in Fin^c(N,M)$ and the definition in (11) implies that $c^\infty(Q) \subseteq T_{\mathfrak{F}}$. Hence, $Q \subseteq c^\infty(Q) \subseteq T_{\mathfrak{F}}$ which shows that $T_{\mathfrak{F}} \notin D(Q,N,M) \cap Fin^c(N,M)$.

Conversely, suppose that $T_{\mathfrak{F}} \notin D(Q,N,M) \cap Fin^c(N,M)$. Then, $T_{\mathfrak{F}} \in V(Q,N,M) \cap Fin^c(N,M)$, i.e., $Q \subseteq T_{\mathfrak{F}}$. By assumption, we can express $Q = N+Q_0$, where $Q_0 \in fg(Q)$. Since the sum in (11) is filtered, it follows that we can find $T' \in Fin^c(N,M)$ with $V(T',N,M) \cap Fin^c(N,M) \in \mathfrak{F}$ such that $Q_0 \subseteq T'$. Then, $Q = N + Q_0 \subseteq T'$ and we get $V(T',N,M) \cap Fin^c(N,M) \subseteq V(Q,N,M) \cap Fin^c(N,M)$. Combining with the fact that

$V(T',N,M)\cap Fin^c(N,M)\in\mathfrak{F}$, we see that $V(Q,N,M)\cap Fin^c(N,M)\in\mathfrak{F}$. Hence, the complement $D(Q,N,M)\cap Fin^c(N,M)\notin\mathfrak{F}$.

It now follows from the criterion in [6, Corollary 3.3] that $Fin^c(N,M)$ is a spectral space. Further, it follows from [8, Corollary 1.2] that the collection $D(Q,N,M)\cap Fin^c(N,M)$ forms a subbasis of quasi-compact opens for $Fin^c(N,M)$, where $D(Q,N,M)$ varies over all elements of $\mathcal{S}^f(N,M)$. From Proposition 2.1, we know that $\mathcal{S}^f(N,M)$ is a subbasis of quasi-compact opens for $Fin(N,M)$ and it follows that the inclusion $Fin^c(N,M)\hookrightarrow Fin(N,M)$ is a spectral map. ∎

The next result will give a method for constructing closure operators on the sets $Fin(N,M)$ and also for modifying them into closure operators of finite type. Before that, we notice that the sets $Fin(N,M)$ are not necessarily closed under ordinary intersections. Accordingly, the role of intersections in $Fin(N,M)$ is played by the following construction: given a family $\{T_i\}_{i\in I}$ of objects in $Fin(N,M)$, we set:

$$\bigcap_{i\in I}^{f} T_i := N+(\textstyle\bigcap_{i\in I}T_i)^f\in Fin(N,M) \tag{13}$$

In particular, we note that if $T\in Fin(N,M)$ is such that $T\subseteq\bigcap_{i\in I}T_i$, then $T\subseteq\bigcap\limits_{i\in I}^{f} T_i$.

3.4. Proposition. *(a) Let $c: Fin(N,M)\longrightarrow Fin(N,M)$ be an operator that is extensive and order-preserving. Then, there exists a closure operator $cl: Fin(N,M)\longrightarrow Fin(N,M)$ such that $cl(T)=c(T)$ whenever $T=c(T)$ for some $T\in Fin(N,M)$.*

(b) Let $cl: Fin(N,M)\longrightarrow Fin(N,M)$ be a closure operator. Then, there exists a closure operator $cl^f: Fin(N,M)\longrightarrow Fin(N,M)$ of finite type such that $cl^f(T)=cl(T)$ whenever $T=N+T_0$ for some finitely generated object T_0.

Proof. (a) We notice that the object $N+M^f\in Fin(N,M)$ contains every object in $Fin(N,M)$. Since c is extensive, it follows that $N+M^f=c(N+M^f)$. This shows that the set $r(T):=\{T'\in Fin(N,M)\mid T'\supseteq T,\ T'=c(T')\}$ is non-empty for each $T\in Fin(N,M)$. We now define $cl: Fin(N,M)\longrightarrow Fin(N,M)$ by setting for each $T\in Fin(N,M)$:

$$cl(T):=\bigcap_{T'\in r(T)}^{f} T' \tag{14}$$

We now claim that $c(cl(T))=cl(T)$. For this, we notice that $cl(T)\subseteq T'$ for each $T'\in r(T)$ and hence $c(cl(T))\subseteq c(T')=T'$ for each $T'\in r(T)$, i.e., $c(cl(T))\subseteq\bigcap_{T'\in r(T)}T'$. But then, we must have $c(cl(T))\subseteq cl(T)$ from the definition in (14). Since c is extensive, we get $c(cl(T))=cl(T)$. Hence, $cl(T)\in r(cl(T))$ and it is now clear from the definition in (14) that the operation cl is idempotent, i.e., $cl(cl(T))=cl(T)$ for each $T\in Fin(N,M)$. It is also clear from (14) that cl satisfies the other properties for being a closure operator on $Fin(N,M)$.

(b) Given a closure operator cl on $Fin(N,M)$, we set:

$$cl^f: Fin(N,M)\longrightarrow Fin(N,M)\qquad cl^f(T):=\sum_{T_0\in fg(T)}cl(N+T_0) \tag{15}$$

In order to see that cl^f is idempotent, we consider some finitely generated $G \subseteq cl^f(T)$. Since the sum in (15) is filtered, we can take $T_0 \in fg(T)$ such that $G \subseteq cl(N + T_0)$. But then, $N + G \subseteq cl(N + T_0)$ and hence $cl(N + G) \subseteq cl(cl(N + T_0)) = cl(N + T_0) \subseteq cl^f(T)$. The definition in (15) now shows that $cl^f(cl^f(T)) = cl^f(T)$. On the other hand, it is also clear from (15) that cl^f is extensive, order-preserving and of finite type. ∎

We will now show how to construct closure operators on $Fin(N, M)$ starting from operators on ideals in ordinary rings. Since $\mathcal{C}$ is an abelian category, $End_{\mathcal{C}}(M)$ is a unital ring (not necessarily commutative) for each object $M \in \mathcal{C}$. The multiplication on $End_{\mathcal{C}}(M)$ is taken to be $f \cdot g := (g \circ f)$ for any $f, g \in End_{\mathcal{C}}(M)$. Then, given a subobject $N \subseteq M$, $Hom_{\mathcal{C}}(M, N)$ may be viewed as a left ideal in $End_{\mathcal{C}}(M)$. Suppose now that we are given an operator that associates each left ideal $I \subseteq End_{\mathcal{C}}(M)$ to another left ideal $\bar{I} \subseteq End_{\mathcal{C}}(M)$. We now define:

$$c : Fin(N, M) \longrightarrow Fin(N, M) \quad T \mapsto T + \sum_{g \in \overline{Hom_{\mathcal{C}}(M,T)} \subseteq End_{\mathcal{C}}(M)} Im(g)^f \tag{16}$$

If we assume that the association $I \mapsto \bar{I}$ satisfies $\bar{I} \subseteq \bar{J}$ for any left ideals $I \subseteq J$ in $End_{\mathcal{C}}(M)$, it follows that the operator c in (16) is extensive and order-preserving. Using Proposition 3.4, c can be used to construct a closure operator on $Fin(N, M)$ and further modified into a closure operator of finite type.

We will now study the functorial properties of the spectral spaces $Fin^c(N, M)$. For this, we fix in the rest of this section a closure operator $c : Fin(0, M) \longrightarrow Fin(0, M)$ of finite type. For any $N \subseteq M$, this induces an operator:

$$c_N : Fin(N, M) \longrightarrow Fin(N, M) \qquad T \mapsto N + c(T^f) = N + \sum_{T_0 \in fg(T)} c(T_0) \tag{17}$$

It is clear from (17) that each induced operator $c_N : Fin(N, M) \longrightarrow Fin(N, M)$ is extensive, order preserving and of finite type. We let $Fin^{c_N}(N, M)$ denote the collection of objects in $Fin(N, M)$ that are fixed by the operator c_N.

3.5. Proposition. *The association of each subobject N of M to $Fin^{c_N}(N, M)$ determines a contravariant functor from $Sub(M)$ to the category of spectral spaces connected by spectral maps.*

Proof. Let $\gamma : N \hookrightarrow N'$ be a morphism in $Sub(M)$ and consider the induced spectral map $Fin(\gamma) : Fin(N', M) \longrightarrow Fin(N, M)$ as defined in the proof of Proposition 2.2. We claim that $Fin(\gamma)$ restricts to a map $Fin^c(\gamma) : Fin^{c_{N'}}(N', M) \longrightarrow Fin^{c_N}(N, M)$. For this, we consider some $T \in Fin(N', M)$ with $T = c_{N'}(T)$. From (17), this means that $T = N' + T^f = N' + c(T^f)$.

On the other hand, by definition, $Fin(\gamma)(T) = N + T^f$. Since $N + T^f \subseteq T$, it follows that $(N + T^f)^f \subseteq T^f$ and hence $(N + T^f)^f = T^f$. Applying the definition in (17), we get $c_N(Fin(\gamma)(T)) = c_N(N + T^f) = N + c(T^f)$.

We now claim that $Fin(\gamma)(T) = N + T^f = N + c(T^f) = c_N(Fin(\gamma)(T))$. For this, we note that $fg(N + c(T^f)) \subseteq fg(N' + c(T^f)) = fg(T) = fg(T^f) \subseteq fg(N + T^f)$. Hence, $fg(N + c(T^f)) = fg(N + T^f)$ and it follows from Lemma 2.3(a) that $N + c(T^f) = N + T^f \in Fin(N, M)$. Finally, since $Fin^c(\gamma) : Fin^{c_{N'}}(N', M) \longrightarrow Fin^{c_N}(N, M)$ is obtained by restricting $Fin(\gamma)$ between the subspaces $Fin^{c_{N'}}(N', M)$ and $Fin^{c_N}(N, M)$, it follows directly from (7) that $Fin^c(\gamma)$ is still a spectral map.

■

As in Section 2, we now consider a covering family for an object $N \in Sub(M)$, i.e., a family $\{\alpha_i : N_i \longrightarrow N\}_{i\in I}$ of morphisms in $Sub(M)$ indexed by a filtered set I satisfying $colim_{i\in I} N_i = \sum_{i\in I} N_i = N$. The following result now gives the sheaf properties of the functor in Proposition 3.5.

3.6. Proposition. *(a) Let $\{\alpha_i : N_i \hookrightarrow N\}_{i\in I}$ be a covering family of morphisms in $Sub(M)$. Then, there is a homeomorphism of spectral spaces*

$$Fin^{c_N}(N, M) = \varprojlim_{i\in I} Fin^{c_{N_i}}(N_i, M)$$

(b) The association $(\alpha : N \hookrightarrow M) \mapsto Fin^{c_N}(N, M)$ defines a sheaf on $Sub(M)$ taking values in the category of spectral spaces connected by spectral maps.

Proof. We consider a family $\{T_i\}_{i\in I}$ of objects with each $T_i \in Fin^{c_{N_i}}(N_i, M)$ and $Fin^c(\alpha_{ij})(T_j) = T_i$ for each inclusion $\alpha_{ij} : N_i \hookrightarrow N_j$ in the filtered system $\{N_i\}_{i\in I}$ of subobjects of N. We already know from the proof of Proposition 2.4 that $T = \sum_{i\in I} T_i$ is the unique object in $Fin(N, M)$ such that $Fin(\alpha_i)(T) = T_i$ for each $i \in I$.

We claim that $T \in Fin^{c_N}(N, M)$. We fix $i_0 \in I$. Since $T_{i_0} \in Fin^{c_{N_{i_0}}}(N_{i_0}, M)$, we know that $c(T^f_{i_0}) \subseteq N_{i_0} + c(T^f_{i_0}) = c_{N_{i_0}}(T_{i_0}) = T_{i_0} \subseteq T$. From the proof of Proposition 2.4, we also know that $fg(T) = fg(T_i)$ for each $i \in I$ and hence $c(T^f) = c(T^f_{i_0})$. We now obtain $c_N(T) = N + c(T^f) = N + c(T^f_{i_0}) \subseteq T$. Hence, $T \in Fin^{c_N}(N, M)$ and we see that $Fin^{c_N}(N, M) = \varprojlim_{i\in I} Fin^{c_{N_i}}(N_i, M)$ at the level of sets.

Finally, we note that the inverse limit topology on $Fin^{c_N}(N, M)$ is the coarsest topology for which the maps $Fin^c(\alpha_i) : Fin^{c_N}(N, M) \longrightarrow Fin^{c_{N_i}}(N_i, M)$ are all continuous. Since the maps $Fin^c(\alpha_i)$ are obtained by restricting the $Fin(\alpha_i)$, it now follows from similar considerations as in the proof of Proposition 2.4 that $Fin^{c_N}(N, M) = \varprojlim_{i\in I} Fin^{c_{N_i}}(N_i, M)$ as spectral spaces. This proves (a). Again, as in the proof of Proposition 2.4, the result of (b) follows from (a) and the definition of covering families in $Sub(M)$.

■

We conclude this section by listing some explicit consequences of the results. Suppose that $\mathcal{R}$ is some full subcategory of the category of commutative rings with identity. A family of closure operators persistent with respect to $\mathcal{R}$ (see [4, Definition 4.3.1]) is given by a collection $\mathbf{c} = \{c_R\}_{R\in\mathcal{R}}$ with c_R a closure operator on (not necessarily proper) ideals of R for each $R \in \mathcal{R}$. Further, the operators $\{c_R\}_{R\in\mathcal{R}}$ are compatible in the following sense:

for any morphism $\phi : R \longrightarrow S$ in $\mathcal{R}$ and any ideal $I \subseteq R$, we have $\phi(c_R(I))S \subseteq c_S(\phi(I)S)$. When the morphism $\phi : R \longrightarrow S$ is flat, this reduces to $c_R(I) \otimes_R S \subseteq c_S(I \otimes_R S)$.

The family $\mathbf{c} = \{c_R\}_{R\in\mathcal{R}}$ is said to commute with localization in $\mathcal{R}$ (see [4, § 4.6]) if for any morphism $R \longrightarrow T$ in $\mathcal{R}$ such that T is a localization of R with respect to some multiplicatively closed subset, we have the equality $c_R(I) \otimes_R T = c_T(I \otimes_R T)$.

3.7. Proposition. *Let $\mathcal{R}$ be a full subcategory of the category of commutative rings with identity. Let $\mathbf{c} = \{c_R\}_{R\in\mathcal{R}}$ be a family of closure operators persistent with respect to $\mathcal{R}$ and which commutes with localization in $\mathcal{R}$. Let $\phi : R \longrightarrow S$ be a morphism in $\mathcal{R}$ such that the induced morphism $Spec(S) \longrightarrow Spec(R)$ is a Zariski open immersion of affine schemes. Suppose that for any multiplicatively closed set $W \subseteq R$, the localization $R[W^{-1}]$ lies in $\mathcal{R}$. Then, for any ideal $I \subseteq R$, we have an equality $c_R(I) \otimes_R S = c_S(I \otimes_R S)$ of ideals in S.*

Proof. Since $Spec(S) \longrightarrow Spec(R)$ is a Zariski immersion, the morphism $\phi : R \longrightarrow S$ must be flat. Then, for every ideal $I \subseteq R$, the persistence condition on $\mathbf{c} = \{c_R\}_{R\in\mathcal{R}}$ gives us an inclusion $c_R(I) \otimes_R S \subseteq c_S(I \otimes_R S)$ of ideals in S.

Further, the affine open $Spec(S) \subseteq Spec(R)$ can be given a Zariski cover of the form $\{Spec(T_k) \longrightarrow Spec(S)\}_{k\in K}$, where each $T_k = R_{t_k}$ is a localization of R with respect to some element $t_k \in R$. For every $k \in K$, this induces an inclusion $c_R(I) \otimes_R S \otimes_S T_k \subseteq c_S(I \otimes_R S) \otimes_S T_k$ of ideals in T_k. Applying the persistence condition on $\mathbf{c} = \{c_R\}_{R\in\mathcal{R}}$ to the flat morphism $S \longrightarrow T_k$ and the ideal $I \otimes_R S \subseteq S$, we obtain

$$c_R(I)\otimes_R T_k = c_R(I)\otimes_R S\otimes_S T_k \subseteq c_S(I\otimes_R S)\otimes_S T_k \subseteq c_{T_k}(I\otimes_R S\otimes_S T_k) = c_{T_k}(I\otimes_R T_k) \quad (18)$$

However, since the family $\mathbf{c} = \{c_R\}_{R\in\mathcal{R}}$ commutes with localization, we must have $c_R(I) \otimes_R T_k = c_{T_k}(I \otimes_R T_k)$. From (18), it now follows that each of the induced inclusions $c_R(I) \otimes_R S \otimes_S T_k \subseteq c_S(I \otimes_R S) \otimes_S T_k$ of ideals in T_k is an identity. Since $\{Spec(T_k) \longrightarrow Spec(S)\}_{k\in K}$ is a Zariski cover of $Spec(S)$, it follows that we must have $c_R(I) \otimes_R S = c_S(I \otimes_R S)$ for each ideal $I \subseteq R$. ∎

For the remainder of this section, let X be a quasi-separated Noetherian scheme and let $\mathcal{C} := QCoh(X)$, the category of quasi-coherent sheaves on X. It is well known (see, for instance, [12, Proposition 7]) that $QCoh(X)$ is a locally finitely generated Grothendieck category. Further, any quasi-coherent sheaf of ideals $\mathcal{I} \subseteq \mathcal{O}_X$ inside the structure sheaf $\mathcal{O}_X$ is finitely generated in $QCoh(X)$. In particular, any closure operator on $Sub(\mathcal{O}_X) = Fin(0, \mathcal{O}_X)$ must be of finite type. We now consider the following:

(1) Let $\mathcal{R}$ be the category of all commutative rings with identity and let $\mathbf{c} = \{c_R\}_{R\in\mathcal{R}}$ be a family of closure operators persistent with respect to $\mathcal{R}$ and which commutes with localization in $\mathcal{R}$. For example, the radical closure and the integral closure satisfy this property (see [4, § 4.3, § 4.6]). We recall here that the integral closure of an ideal $I \subseteq R$ is the collection of all $r \in R$ such that there exists $n \in \mathbb{N}$ and $a_i \in I^i$ for $1 \leq i \leq n$ such that:

$$r^n + \sum_{i=1}^{n} a_i r^{n-i} = 0 \quad (19)$$

We can now use $\mathbf{c} = \{c_R\}_{R\in\mathcal{R}}$ to construct a closure operator on $Sub(\mathcal{O}_X)$ as follows : given any quasi-coherent sheaf of ideals $\mathcal{I} \subseteq \mathcal{O}_X$, we define its closure $c_X(\mathcal{I})$ by setting $c_X(\mathcal{I})(U) = c_{\mathcal{O}_X(U)}(\mathcal{I}(U))$ for any affine open $U \subseteq X$. From Proposition 3.7, it is clear that $c_X(\mathcal{I})$ is a quasi-coherent sheaf of ideals. It now follows that the collection of quasi-coherent sheaves of ideals fixed by the operator c_X forms a spectral space.

(2) Let $\mathcal{R}$ be the full subcategory of all commutative rings of characteristic p and let $\mathbf{c} = \{c_R\}_{R\in\mathcal{R}}$ be a family of closure operators persistent with respect to $\mathcal{R}$ and which commutes with localization in $\mathcal{R}$. If we suppose that X is a scheme over $Spec(\mathbb{F}_p)$, we can now use $\mathbf{c} = \{c_R\}_{R\in\mathcal{R}}$ to construct a closure operator c_X on quasi-coherent sheaves of ideals over X as in (1). Again, it follows from the results above that the collection of quasi-coherent sheaves of ideals fixed by this closure operator forms a spectral space.

For example, the Frobenius closure (see [4, § 4.3, § 4.6]) is persistent with respect to characteristic p rings and commutes with localization. We recall that the Frobenius closure (see, for instance, [4, § 2]) is defined as follows: for $R \in \mathcal{R}$, an ideal $I \subseteq R$ and any $n \in \mathbb{N}$, we denote by $I^{[p^n]}$ the ideal generated by the p^n-th powers of elements of I. Then, an element $x \in R$ lies in the Frobenius closure I^F of the ideal I if there exists some $n \in \mathbb{N}$ such that $x^{p^n} \in I^{[p^n]}$.

(3) Let $\mathcal{R}$ be the full subcategory of all integral domains and let $\mathbf{c} = \{c_R\}_{R\in\mathcal{R}}$ be a family of closure operators persistent with respect to $\mathcal{R}$ and which commutes with localization in $\mathcal{R}$. The plus closure, for example, satisfies this property (see [4, § 4.3, § 4.6]). For $R \in \mathcal{R}$, we recall that an element $x \in R$ lies in the plus closure I^+ of an ideal $I \subseteq R$ if there exists an injective map of integral domains $R \longrightarrow S$ making S a finite R-module with $x \in IS$. If the scheme X is taken to be integral, we now see that the fixed points of the closure operator c_X on quasi-coherent sheaves of ideals constructed as in (1) give a spectral space.

4. Families of subobjects and fixed points of closure operators

We fix an object $M \in \mathcal{C}$ and a subobject $N \subseteq M$. From the previous section, we know that if $c : Fin(N,M) \longrightarrow Fin(N,M)$ is a closure operator of finite type, the collection $Fin^c(N,M)$ of the fixed points of the operator c forms a spectral space. In this final section, we give a complete description for the subsets of $Fin(N,M)$ that can arise as fixed points of a closure operator of finite type. We start with a special case.

4.1. Proposition. *Suppose that M^f is Noetherian, i.e., every subobject of M^f is a finitely generated object. Let $\mathcal{T} = \{T_i\}_{i\in I}$ be a collection of objects in $Fin(N,M)$ such that $N + M^f \in \mathcal{T}$ and for any non-empty subset $J \subseteq I$, we must have $\bigcap^f_{j\in J} T_j \in \mathcal{T}$. Then $\mathcal{T}$ is the set of fixed points of a closure operator of finite type on $Fin(N,M)$. In particular, $\mathcal{T}$ is a spectral subspace of $Fin(N,M)$.*

Proof. We define an operator $c : Fin(N,M) \longrightarrow Fin(N,M)$ as follows: for any $U \in Fin(N,M)$ we set

$$c(U) := \bigcap^f_{T\in\mathcal{T},T\supseteq U} T = N + (\bigcap_{T\in\mathcal{T},T\supseteq U} T)^f \tag{20}$$

It is clear that c is a closure operator. Further, since M^f is Noetherian, the object $U^f \subseteq M^f$ is finitely generated. As $U \in Fin(N,M)$ may be expressed as $U = N + U^f$, it follows that c is also of finite type.

By the assumption on the family $\mathcal{T}$, we notice that $c(U) := \bigcap^f_{T\in\mathcal{T},T\supseteq U} T$ lies in $\mathcal{T}$ for any $U \in Fin(N,M)$. Hence, if $U \in Fin(N,M)$ is such that $c(U) = U$, we must have $U \in \mathcal{T}$. The converse is also clear.

■

4.2. Proposition. *Let $\mathcal{T} = \{T_i\}_{i\in I}$ be a collection of objects in $Fin(N,M)$ such that $N + M^f \in \mathcal{T}$. Then, the following are equivalent:*

(1) The collection $\mathcal{T}$ satisfies the following two conditions:

(a) For any non-empty subset $J \subseteq I$, the object $\bigcap_{j\in J}{}^f T_j \in \mathcal{T}$.

(b) Let $\{T_k\}_{k\in K}$ be a collection of elements of $\mathcal{T}$ indexed by a filtered set K. Then, $\sum_{k\in K} T_k$ lies in $\mathcal{T}$.

(2) There exists a closure operator c of finite type on $Fin(N,M)$ such that $\mathcal{T}$ is the set of fixed points of c. In particular, $\mathcal{T}$ is a spectral subspace of $Fin(N,M)$.

Proof. (1) $\Rightarrow$ (2) : Take an arbitrary object $U \in Fin(N,M)$. Now, for each $U_0 \in fg(U)$, set:

$$c(N + U_0) := \bigcap^f_{T\supseteq U_0,T\in\mathcal{T}} T \tag{21}$$

We now define the operator $c : Fin(N,M) \longrightarrow Fin(N,M)$ by setting:

$$c(U) := \sum_{U_0\in fg(U)} c(N + U_0) \tag{22}$$

It is clear that c is extensive and order-preserving. To show that c is idempotent, we consider some object $U_{00} \in fg(c(U))$. Since the sum in (22) is filtered, we must have some $U_0 \in fg(U)$ such that $U_{00} \subseteq c(N+U_0)$. It follows from (21) that $c(N+U_{00}) \subseteq c(N+U_0)$. It now follows from (22) that $c(c(U)) = \sum_{U_{00}\in fg(c(U))} c(N + U_{00}) \subseteq c(U)$. Hence, c is a closure operator of finite type.

From (21), (22) and the conditions (a) and (b) on $\mathcal{T}$, it is clear that the right hand side of (22) always lies in $\mathcal{T}$ for any object $U \in Fin(N,M)$. Hence, if $U \in Fin(N,M)$ is such that $c(U) = U$, we must have $U \in \mathcal{T}$. Conversely, take an arbitrary $T \in \mathcal{T}$ and consider $T_0 \in fg(T)$. Then, by (21), we know that $c(N + T_0) \subseteq N + T^f$. Applying (22), we get $c(T) = \sum_{T_0\in fg(T)} c(N + T_0) \subseteq N + T^f = T$. Since c is extensive, this implies $c(T) = T$.

(2) $\Rightarrow$ (1) : Let $c : Fin(N,M) \longrightarrow Fin(N,M)$ be a closure operator of finite type and let $\mathcal{T} := \{T_i\}_{i\in I} = Fin^c(N,M)$. In order to show that $\mathcal{T}$ satisfies condition (a), we take

a non-empty $J \subseteq I$ and consider $T := \bigcap_{j\in J}^{f} T_j$. Then, $c(T) \subseteq \bigcap_{j\in J} c(T_j) = \bigcap_{j\in J} T_j$ and hence $c(T) \subseteq \bigcap_{j\in J}^{f} T_j = T$. This shows that $c(T) = T$ and hence $\mathcal{T}$ satisfies condition (a).

On the other hand, let $\{T_k\}_{k\in K}$ be a collection of elements from $\mathcal{T}$ indexed by a filtered set K. Set $T := \sum_{k\in K} T_k$. Then, for any $T_0 \in fg(T)$, we can find $k_0 \in K$ such that $T_0 \subseteq T_{k_0}$. Then, $c(N + T_0) \subseteq c(T_{k_0}) = T_{k_0} \subseteq T$. Since c is of finite type, we obtain $c(T) = \sum_{T_0 \in fg(T)} c(N + T_0) \subseteq T$. Hence, $c(T) = T$, which proves condition (b).

■

We have noted before that the object $N + M^f$ is the unique object in $Fin(N, M)$ that contains every other object in $Fin(N, M)$. Since any closure operator $c : Fin(N, M) \longrightarrow Fin(N, M)$ must be extensive, it follows that $N + M^f$ must be fixed by c. In the last two results, we will consider the set $Fin^c(N, M)$ with the point $N + M^f$ removed. First, we give conditions for $Fin^c(N, M)\backslash\{N + M^f\}$ to be a spectral subspace of $Fin(N, M)$.

4.3. Proposition. *Let $c : Fin(N, M) \longrightarrow Fin(N, M)$ be a closure operator of finite type. Then, we have the following:*

(a) If there exists $M_0 \in fg(M)$ such that $N + M^f = N + M_0$, then $Fin^c(N, M)\backslash\{N + M^f\}$ is a spectral subspace of $Fin(N, M)$.

(b) Suppose that $c : Fin(N, M) \longrightarrow Fin(N, M)$ satisfies the additional property that for any $M_0' \in fg(M)$, the object $c(N + M_0')$ is given by $c(N + M_0') = N + M_0' + M_0''$ for some finitely generated object $M_0'' \in fg(M)$. Then, if $Fin^c(N, M)\backslash\{N + M^f\}$ is a spectral subspace of $Fin(N, M)$, there exists $M_0 \in fg(M)$ such that $N + M^f = N + M_0$.

Proof. (a) For the sake of convenience, we set $X := Fin^c(N, M)\backslash\{N + M^f\}$. The sets $D(Q, N, M) \cap X$ with $D(Q, N, M) \in \mathcal{S}^f(N, M)$ form a subbasis for the topology on X. Then, we consider an ultrafilter $\mathfrak{F}$ of subsets of X and set:

$$T_{\mathfrak{F}} = \sum_{T \in Fin^c(N,M), V(T,N,M)\cap X \in \mathfrak{F}} T \tag{23}$$

Now if T_1, $T_2 \in Fin^c(N, M)$ are such that $V(T_1, N, M) \cap X$ and $V(T_2, N, M) \cap X$ both lie in $\mathfrak{F}$, we note that $c(T_1 + T_2) \in Fin^c(N, M)$ and $V(c(T_1 + T_2), N, M) \cap X = V(T_1 + T_2, N, M) \cap X = V(T_1, N, M) \cap V(T_2, N, M) \cap X \in \mathfrak{F}$. This shows that the sum in (23) is filtered. We can also check as in the proof of Proposition 3.3 that $T_{\mathfrak{F}} \in Fin^c(N, M)$.

We now claim that $T_{\mathfrak{F}} \neq N + M^f$. Indeed, if $T_{\mathfrak{F}} = N + M^f = N + M_0$, there must be some term in the filtered sum (23) above that contains the finitely generated object M_0 and hence that term must be equal to $N + M^f = N + M_0$. But then $V(N + M^f, N, M) \cap X = \phi \in \mathfrak{F}$ which is a contradiction. Hence, $T_{\mathfrak{F}} \in X$. It now follows as in the proof of Proposition 3.3 that X is spectral.

(b) We take an arbitrary $T \in X = Fin^c(N, M)\backslash\{N + M^f\}$. Since $T \neq N + M^f$, we can find some finitely generated $Q_0 \in fg(M)$ such that $T \not\supseteq Q_0$. Then, $T \in D(N + Q_0, N, M) \cap X$. Thus, the collection $D(N + Q_0, N, M) \cap X$ as Q_0 varies over all objects in $fg(M)$ gives a cover of X. Since X is spectral and hence quasi-compact, this gives us a finite collection $\{Q_1, ..., Q_n\} \subseteq fg(M)$ such that $X = \bigcup_{i=1}^{n} D(N + Q_i, N, M) \cap X$. We now set $Q := \sum_{i=1}^{n} Q_i$ and consider the object $c(N + Q) \in Fin^c(N, M)$. It is clear from the definition of Q that $c(N + Q) \notin D(N + Q_i, N, M)$ for each $1 \leq i \leq n$. Hence, $c(N + Q) = N + M^f$. Finally, by assumption on c, we can find some $Q' \in fg(M)$ such that $c(N + Q) = N + Q + Q'$. Hence, $N + M^f = c(N + Q) = N + Q + Q'$, which proves the result.

∎

We conclude by giving a sufficient condition so that the set $Fin^c(N, M)$ with the point $N + M^f$ removed still contains maximal elements.

4.4. Proposition. *Let $c : Fin(N, M) \longrightarrow Fin(N, M)$ be a closure operator of finite type. Suppose there exists $M_0 \in fg(M)$ such that $N + M^f = N + M_0$. Then, every element in the set $Fin^c(N, M)\backslash\{N + M^f\}$ is contained in a maximal element.*

Proof. We consider a totally ordered collection $\{T_i\}_{i\in I}$ of objects in $Fin^c(N, M)\backslash\{N + M^f\}$ and set $T := \sum_{i\in I} T_i$. We claim that $T \in Fin^c(N, M)$. Indeed, for each $T_0 \in fg(T)$ we can find $i_0 \in I$ with $T_0 \subseteq T_{i_0}$ and hence $c(N + T_0) \subseteq c(T_{i_0}) = T_{i_0} \subseteq T$. Since c is of finite type, it follows that $c(T) = \sum_{T_0 \in fg(T)} c(N + T_0) \subseteq T$.

We now claim that $T \neq N + M^f = N + M_0$. Otherwise, since $M_0 \in fg(M)$, we have some $i_0' \in I$ with $M_0 \subseteq T_{i_0'}$. Then, $N + M^f = N + M_0 \subseteq T_{i_0'}$ and $N + M^f$ being the maximal element of $Fin(N, M)$, we get $N + M^f = T_{i_0'}$ which is a contradiction. The result follows from Zorn's lemma.

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