Theory and Applications of Categories, Vol. 31, No. 38, 2016, pp. 1134–1175.

COSHEAFIFICATION

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Abstract. It is proved that for any small Grothendieck site X, there exists a coreflection (called *cosheafification*) from the category of precosheaves on X with values in a category $\mathbf{K}$, to the full subcategory of cosheaves, provided either $\mathbf{K}$ or $\mathbf{K}^{op}$ is locally presentable. If $\mathbf{K}$ is cocomplete, such a coreflection is built explicitly for the (pre)cosheaves with values in the category $\mathbf{Pro}\,(\mathbf{K})$ of pro-objects in $\mathbf{K}$. In the case of precosheaves on topological spaces, it is proved that any precosheaf with values in $\mathbf{Pro}\,(\mathbf{K})$ is *smooth*, i.e. is strongly locally isomorphic to a cosheaf. Constant cosheaves are constructed, and there are established connections with shape theory.

0. Introduction

A *presheaf* (*precosheaf*) on a topological space X with values in a category $\mathbf{K}$ is just a contravariant (covariant) functor from the category of open subsets of X to $\mathbf{K}$, while a *sheaf* (*cosheaf*) is such a functor satisfying some extra conditions. The category of (pre)cosheaves with values in $\mathbf{K}$ is dual to the category of (pre)sheaves with values in the opposite category $\mathbf{K}^{op}$.

While the theory of sheaves is well developed, and is covered by a plenty of publications, the theory of cosheaves is more poorly represented. The main reason for this is that *cofiltered limits* are *not* exact in the "usual" categories like sets, abelian groups, rings, or modules. On the contrary, *filtered colimits* are exact in the above categories, which allows to construct rather rich theories of sheaves with values in the "usual" categories. To sum up, the "usual" categories $\mathbf{K}$ are badly suited for cosheaf theory. Dually, the categories $\mathbf{K}^{op}$ are badly suited for sheaf theory.

The first step in building a suitable theory of cosheaves would be constructing a *cosheaf associated with a precosheaf* (simply: *cosheafification*). As is shown in this paper (see Theorem 3.1), it is possible in many situations, namely for precosheaves with values in an arbitrary *locally presentable category* (or a dual to such a category). The class of locally presentable categories is huge [Adámek and Rosický, 1994, Ch. 1, 4 and 5]. It includes all varieties and quasi-varieties of many-sorted algebras, and essentially algebraic categories [Adámek and Rosický, 1994, Theorem 3.36] of *partial algebras* like the category

The author is grateful for the referee's comments and suggestions that have helped to significantly improve the paper.

Received by the editors 2016-05-05 and, in final form, 2016-12-12.

Transmitted by Clemens Berger. Published on 2016-12-31.

2010 Mathematics Subject Classification: Primary 18F10, 18F20; Secondary 55P55, 55Q07, 14F20.

Key words and phrases: Cosheaves, smooth precosheaves, cosheafification, pro-category, cosheaf homology, locally presentable categories, accessible categories.

Cat of small categories, and the category **Pos** of posets. Even the class of locally *finitely* presentable categories is very large, and includes [Adámek and Rosický, 1994, Corollary 3.7 and Theorem 3.24] all varieties of many-sorted *finitary* algebras like **Set**, **Gr**, **Ab**, modules etc. and all quasi-varieties like the category **Gra** of graphs, the category of torsion-free abelian groups, or the category Σ-**Rel** of finitary relations.

However, our purpose is to prepare a foundation for future *homology* and *homotopy* theories of cosheaves (see Conjectures 0.3, 0.4 and 0.5 below). Therefore, we need a more or less *explicit* construction. Moreover, we need a construction satisfying *good exactness* properties. In [Funk, 1995] the cosheafification of precosheaves of sets on topological spaces is discussed. It is sketched there [Funk, 1995, Theorem 6.3] that on complete metric spaces, the cosheafification can be described explicitly by using the so-called "display space of a precosheaf". See Example 4.6 and 4.8. The construction of [Funk, 1995, Theorem 6.3] works there, but produces cosheaves that are hardly interesting for future applications. In [Woolf, 2009, Appendix B] it is claimed that the display space construction works for *any* topological space and *any* precosheaf of sets on it. However, his Lemma B.3 contains essential errors, see [Woolf, 2015]. Anyway, even the construction from [Funk, 1995] for complete metric spaces is *not an exact functor*, and therefore is *not* suitable for homology and homotopy theories of cosheaves.

In [Bredon, 1997] and [Bredon, 1968], it is assumed (correctly, in our opinion!) that a suitable cosheafification of a precosheaf should be *locally isomorphic* to the precosheaf. This notion is much stronger than a **K**-local isomorphism (Definition 2.20). We call a local isomorphism in the sense of Bredon a *strong local isomorphism* (Definition 2.24). Precosheaves that admit a "correct" cosheafification are called *smooth* (Definition 3.8, [Bredon, 1997, Corollary VI.3.2 and Definition VI.3.4], or [Bredon, 1968, Corollary 3.5 and Definition 3.7]). It is not clear whether one has enough smooth precosheaves for building a suitable theory of cosheaves (see Example 4.5, 4.6 and 4.8). In fact, Glen E. Bredon back in 1968 was rather pessimistic on the issue. See [Bredon, 1968], p. 2: "*The most basic concept in sheaf theory is that of a sheaf generated by a given presheaf. In categorical terminology this is the concept of a reflector from presheaves to sheaves. We believe that there is not much hope for the existence of a reflector from precosheaves to cosheaves*". It seems that he was still pessimistic in 1997: Chapter VI "Cosheaves and Čech homology" of his book [Bredon, 1997] is almost identical to [Bredon, 1968].

On the contrary, our approach seems to have solved the problem. If one allows (pre)cosheaves (defined on an *arbitrary* small Grothendieck site) to take values in a larger category, then the desired reflection (in fact, **co**reflection) can be constructed. It follows from our considerations in this paper, that the best candidate for such category is the pro-category $\mathbf{Pro}\,(\mathbf{K})$ (see Definition 1.19) for an *arbitrary cocomplete* category **K**. Our cosheafification is built like this (Definition 2.5):

$$\mathcal{A} \longmapsto \mathcal{A}_+ \longmapsto \mathcal{A}_{++} = \mathcal{A}_\#,$$

where $()_+$ is the operation dual to the well-known plus construction $()^+$ in sheaf theory. We have succeeded because of the niceness of the category $\mathbf{Pro}\,(\mathbf{K})$. For "usual" precosheaves (with values in $\mathbf{K}$) the above two-step process does not work. In [Prasolov, 2012], this approach was developed for precosheaves with values in $\mathbf{Pro}\left(\mathbf{Set}\right)$ and $\mathbf{Pro}\left(\mathbf{Ab}\right)$, and part (2) and (3) of Theorem 3.11 were proved. In this paper, the two statements are proved much easier, using a significantly more general part (1) of Theorem 3.11.

0.1. Remark. An interesting attempt is made in [Schneiders, 1987] where the author sketches a cosheaf theory on topological spaces with values in a category $\mathbf{L}$, dual to an "elementary" category $\mathbf{L}^{op}$. He proposes a candidate for such a category. Let $\alpha < \beta$ be two inaccessible cardinals. Then $\mathbf{L}$ is the category $\mathbf{Pro}_{\beta}\left(\mathbf{Ab}_{\alpha}\right)$ of abelian pro-groups $\left(G_j\right)_{j\in\mathbf{J}}$ such that $card\left(G_j\right) < \alpha$ and $card\left(Mor\left(\mathbf{J}\right)\right) < \beta$. However, our pro-category $\mathbf{Pro}\left(\mathbf{K}\right)$ cannot be used in the cosheaf theory from [Schneiders, 1987] because the category $\left(\mathbf{Pro}\left(\mathbf{K}\right)\right)^{op}$ is **not** elementary.

0.2. Remark. Another cosheaf theory on topological spaces was sketched in [Sugiki, 2001]: the (pre)cosheaves there take values in the category $\mathbf{Pro}\left(\mathbf{Mod}\left(k\right)\right)$ where k is a commutative quasi-noetherian [Prasolov, 2013, Definition 2.25] ring. Definition 2.2.7 of a cosheaf on a topological space X in [Sugiki, 2001] is equivalent to our definition of a cosheaf on the corresponding site $OPEN\left(X\right)$, see Example B.9 and Remark B.10. Theorem 2.2.8 in [Sugiki, 2001] states that the cosheafification exists. However, no proof of that theorem is given, and no explicit construction of such cosheafification is provided.

The cosheafification we have constructed guarantees that our precosheaves are always smooth (Corollary 3.9). Moreover, in Theorem 3.10, we give necessary and sufficient conditions for smoothness of a precosheaf with values in an "old" category $\mathbf{K}$: it is smooth iff our coreflection applied to that precosheaf produces a cosheaf which takes values in that old category.

Another difficulty in cosheaf theory is the lack of suitable *constant* cosheaves. In [Bredon, 1997] and [Bredon, 1968], such cosheaves are constructed only for locally connected spaces. See Examples 4.7 and 4.8. In Theorem 3.11, constant cosheaves are constructed. It turns out that they are closely connected to shape theory. Namely, the constant cosheaf $\left(G\right)_{\#}$ with values in $\mathbf{Pro}\left(\mathbf{K}\right)$ is isomorphic to the *pro-homotopy* (Definition 1.27) cosheaf $G\otimes_{\mathbf{Set}} pro\text{-}\pi_0$ (in particular $\left(\mathbf{pt}\right)_{\#}\simeq pro\text{-}\pi_0$), while the constant cosheaf $\left(A\right)_{\#}$ with values in $\mathbf{Pro}\left(\mathbf{Ab}\right)$ is isomorphic to the *pro-homology* (Definition 1.29) cosheaf $pro\text{-}H_0\left(\_,A\right)$.

In future papers, we are planning to develop homology of cosheaves, i.e. to study projective and flabby cosheaves, projective and flabby resolutions, and to construct the left satellites

$$H_n\left(X,\mathcal{A}\right) := L_n\Gamma\left(X,\mathcal{A}\right)$$

of the global sections functor

$$H_0\left(X,\mathcal{A}\right) := \Gamma\left(X,\mathcal{A}\right).$$

It is expected that deeper connections to shape theory will be discovered, as is stated in the two Conjectures below:

0.3. Conjecture. *On the site $NORM(X)$ (Example B.11), the left satellites of H_0 are naturally isomorphic to the pro-homology:*

$$H_n(X, A_{\#}) = H_n(X, \textit{pro-}H_0(\_, A)) \simeq \textit{pro-}H_n(X, A).$$

If X is Hausdorff paracompact, the above isomorphisms exist also for the standard site $OPEN(X)$ (Example B.9).

0.4. Conjecture. *On the site $NORM(X)$, the **non-abelian** left satellites of H_0 are naturally isomorphic to the pro-homotopy:*

$$\begin{aligned} H_n(X, S_{\#}) &= H_n(X, S \times \textit{pro-}\pi_0) \simeq S \times \textit{pro-}\pi_n(X), \\ H_n\left(X, (\mathbf{pt})_{\#}\right) &= H_n(X, \textit{pro-}\pi_0) \simeq \textit{pro-}\pi_n(X). \end{aligned}$$

If X is Hausdorff paracompact, the above isomorphisms exist also for the standard site $OPEN(X)$.

The main application (Theorem 3.11) deals with the case of topological spaces (i.e. the site $OPEN(X)$). Our constructions, however, are valid for general Grothendieck sites. The constructions in (strong) shape theory use essentially *normal* coverings instead of general coverings, therefore dealing with the site $NORM(X)$ instead of the site $OPEN(X)$. It seems that Theorem 3.11 is valid also for the site $NORM(X)$. Applying our machinery (from this paper and from future papers) to the site $FINITE(X)$ (Example B.12), we expect to obtain results on homology of the Stone-Čech compactification $\beta(X)$. To deal with the equivariant homology, one should apply the machinery to the equivariant site $OPEN_G(X)$ (Example B.13).

It is not yet clear how to generalize the above Conjectures to *strong shape theory*. However, we have some ideas how to do that.

Other possible applications could be in étale homotopy theory [Artin and Mazur, 1986] as is summarized in the following

0.5. Conjecture. *Let X^{et} be the site from Example B.14.*

1. *The left satellites of H_0 are naturally isomorphic to the étale pro-homology:*

$$H_n\left(X^{et}, A_{\#}\right) \simeq H_n^{et}(X, A).$$

2. *The non-abelian left satellites of H_0 are naturally isomorphic to the étale pro-homotopy:*

$$H_n\left(X^{et}, (\mathbf{pt})_{\#}\right) \simeq H_n\left(X^{et}, \pi_0^{et}\right) \simeq \pi_n^{et}(X).$$

0.6. Acknowledgement. *The author wishes to express his gratitude to Professor Carles Casacuberta. The idea of part (2) of Theorem 3.1 belongs to him. The long discussions with him helped the author to understand the importance of locally presentable and accessible categories.*

1. Preliminaries

Categories.

1.1. Notation.

1. We shall denote **limits** (inverse/projective limits) by $\varprojlim$, and **colimits** (direct/inductive limits) by $\varinjlim$.
2. If U is an object of a category $\mathbf{C}$, we shall usually write $U \in \mathbf{C}$ instead of $U \in Ob(\mathbf{C})$.

1.2. Definition. *A* ***diagram*** *in* $\mathbf{C}$ *is a functor*

$$\mathcal{D} : \mathbf{I} \longrightarrow \mathbf{C}$$

where $\mathbf{I}$ *is a* ***small*** *category. A* ***cone*** *(respectively* ***cocone****) of the diagram* $\mathcal{D}$ *is a pair* (U, φ) *where* $U \in \mathbf{C}$*, and* φ *is a morphism of functors* $\varphi : U^{const} \to \mathcal{D}$ *(respectively* $\mathcal{D} \to U^{const}$*). Here* U^{const} *is the constant functor:*

$$\begin{aligned} U^{const}(i) &= U, i \in \mathbf{I}, \\ U^{const}(i \to j) &= \mathbf{1}_U. \end{aligned}$$

1.3. Remark. We will also consider functors $\mathbf{C} \to \mathbf{D}$ where $\mathbf{C}$ is not small. However, such functors form a **quasi-category** $\mathbf{D}^{\mathbf{C}}$, because the morphisms $\mathbf{D}^{\mathbf{C}}(\mathcal{F}, \mathcal{G})$ form a **class**, but not in general a **set**.

1.4. Definition. *A category* $\mathbf{C}$ *is called* ***complete*** *if it admits small limits, and* ***cocomplete*** *if it admits small colimits.*

1.5. Remark. A complete category has a **terminal** object (a limit of an empty diagram). A cocomplete category has an **initial** object (a colimit of an empty diagram).

1.6. Definition. *A functor* $\mathcal{F} : \mathbf{C} \to \mathbf{D}$ *is called* ***left (right) exact*** *if it commutes with* ***finite*** *limits (colimits).* $\mathcal{F}$ *is called* ***exact*** *if it is both left and right exact.*

1.7. Definition. *A subcategory* $\mathbf{C} \subseteq \mathbf{D}$ *is called* ***reflective*** *(respectively* ***coreflective****) iff the inclusion* $\mathbf{C} \hookrightarrow \mathbf{D}$ *is a right (respectively left) adjoint. The left (respectively right) adjoint* $\mathbf{D} \to \mathbf{C}$ *is called a* ***reflection*** *(respectively* ***coreflection****).*

1.8. Definition. *Given* $U \in \mathbf{C}$*, let*

$$h_U : \mathbf{C}^{op} \longrightarrow \mathbf{Set}, \ h^U : \mathbf{C} \longrightarrow \mathbf{Set},$$

be the following functors:

$$\begin{aligned} &h_U(V) := Hom_{\mathbf{C}}(V, U), \ h^U(V) := Hom_{\mathbf{C}}(U, V), \\ &h_U(\alpha) := [(\gamma \in h_U(V) = Hom_{\mathbf{C}}(V, U)) \longmapsto (\gamma \circ \alpha \in Hom_{\mathbf{C}}(V', U) = h_U(V'))], \\ &h^U(\beta) := \left[\left(\gamma \in h^U(V) = Hom_{\mathbf{C}}(U, V)\right) \longmapsto \left(\beta \circ \gamma \in Hom_{\mathbf{C}}(U, V') = h^U(V')\right)\right], \end{aligned}$$

where

$$\begin{array}{rcl}(\alpha : V' \longrightarrow V) & \in & Hom_{\mathbf{C}}(V', V) = Hom_{\mathbf{C}^{op}}(V, V'), \\ (\beta : V \longrightarrow V') & \in & Hom_{\mathbf{C}}(V, V').\end{array}$$

1.9. Remark. The functors

$$h_{?} : \mathbf{C} \longrightarrow \mathbf{Set}^{\mathbf{C}^{op}}, \; h^{?} : \mathbf{C}^{op} \longrightarrow \mathbf{Set}^{\mathbf{C}},$$

are full embeddings, called the **Yoneda embeddings**.

1.10. Definition. *Let $U \in \mathbf{C}$. The **comma category** $\mathbf{C}_U$ is defined as follows:*

$$\begin{array}{l} Ob(\mathbf{C}_U) := \{(V \to U) \in Hom_{\mathbf{C}}(V, U)\}, \\ Hom_{\mathbf{C}_U}((\alpha_1 : V_1 \to U), (\alpha_2 : V_2 \to U)) := \{\beta : V_1 \to V_2 \mid \alpha_2 \circ \beta = \alpha_1\}. \end{array}$$

1.11. Definition. *Let $\mathcal{F} \in \mathbf{Set}^{\mathbf{C}^{op}}$. The **comma category** $\mathbf{C}_{\mathcal{F}}$ is defined as follows:*

$$\begin{array}{l} Ob(\mathbf{C}_{\mathcal{F}}) := \{(V, \alpha) \mid V \in \mathbf{C}, \alpha \in \mathcal{F}(V)\}, \\ Hom_{\mathbf{C}_U}((V_1, \alpha_1), (V_2, \alpha_2)) := \{\beta : V_1 \to V_2 \mid \mathcal{F}(\beta)(\alpha_2) = \alpha_1\}. \end{array}$$

1.12. Remark. The categories $\mathbf{C}_U$ and $\mathbf{C}_{h_U}$ are equivalent.

Locally presentable categories. The main reference here is [Adámek and Rosický, 1994, Chapter 1]. See [Adámek and Rosický, 1994, Definitions 1.1, 1.9, 1.13, and 1.17].

1.13. Notation. We denote by $\aleph_0$ the smallest infinite cardinal.

1.14. Definition. *Let λ be a regular cardinal, and $\mathbf{C}$ be a category.*

1. *A poset is called λ-**directed** provided that every subset of cardinality smaller than λ has an upper bound. A diagram $\mathcal{D} : \mathbf{I} \to \mathbf{C}$ where $\mathbf{I}$ is a λ-directed poset is called a λ-**directed diagram**. A poset or a diagram is called **directed** if it is $\aleph_0$-directed.*

2. *An object U of $\mathbf{C}$ is called λ-presentable provided that*

$$h^U = Hom_{\mathbf{C}}(U, \_) : \mathbf{C} \longrightarrow \mathbf{Set}$$

*preserves λ-directed colimits. U is called **finitely presentable** if it is $\aleph_0$-presentable.*

3. *$\mathbf{C}$ is called **locally** λ-**presentable** provided that it is cocomplete, and has a **set** A of λ-presentable objects such that every object is a λ-directed colimit of objects from A. $\mathbf{C}$ is called **locally presentable** if it is locally λ-presentable for some regular cardinal λ. $\mathbf{C}$ is called **locally finitely presentable** if it is locally $\aleph_0$-presentable.*

1.15. Remark. The notions above can be equivalently defined using more general λ-**filtered** diagrams: a small category $\mathbf{I}$ is called λ-filtered provided that each subcategory with less than λ morphisms has a cocone in $\mathbf{I}$. This means that:

1. $\mathbf{I}$ is non-empty.
2. For each collection $I_s, s \in S$, of less than λ objects of $\mathbf{I}$ there exists an object J and morphisms $f_s : I_s \to J, s \in S$, in $\mathbf{I}$.
3. For each collection $g_s : I_1 \to I_2, s \in S$, of less than λ morphisms in $\mathbf{I}$ there exists a morphism $f : I_2 \to J$ in $\mathbf{I}$ with $f \circ g_s$ independent of s.

A diagram $\mathcal{D} : \mathbf{I} \to \mathbf{C}$ is called λ-filtered if $\mathbf{I}$ is a λ-filtered category.

1.16. Remark. See [Adámek and Rosický, 1994, Remark 1.19]. A category is locally λ-presentable iff the following two conditions are satisfied:

1. Every object is a λ-directed (equivalently: λ-filtered) colimit of λ-presentable objects.
2. There exists, up to an isomorphism, only a set of λ-presentable objects.

By $Pres_\lambda \mathbf{C}$ we will denote a **set** of representatives for the isomorphism classes of λ-presentable objects of $\mathbf{C}$.

Pro-objects. The main reference is [Kashiwara and Schapira, 2006, Chapter 6].

The case of $\aleph_0$-(co)filtered categories and diagrams is of special interest in this subsection. They are simply called (co)filtered. See, e.g., [Mac Lane, 1998, Chapter IX.1] for filtered, and [Mardešić and Segal, 1982, Chapter I.1.4] for cofiltered categories.

1.17. Definition. *A category $\mathbf{I}$ is called **filtered** if $\mathbf{I}$ is $\aleph_0$-filtered. A category $\mathbf{I}$ is called **cofiltered** if $\mathbf{I}^{op}$ is filtered. A diagram $\mathcal{D} : \mathbf{I} \to \mathbf{K}$ is called (co)filtered if $\mathbf{I}$ is a (co)filtered category.*

1.18. Remark. In [Kashiwara and Schapira, 2006], such categories and diagrams are called **(co)filtrant**.

1.19. Definition. *Let $\mathbf{K}$ be a category. The pro-category $\mathbf{Pro}(\mathbf{K})$ (see [Kashiwara and Schapira, 2006, Definition 6.1.1], [Mardešić and Segal, 1982, Remark I.1.4], or [Artin and Mazur, 1986, Appendix]) is the category $\mathbf{L}^{op}$ where $\mathbf{L} \subseteq \mathbf{Set}^{\mathbf{K}}$ is the full subcategory of functors that are filtered colimits of representable functors, i.e. colimits of diagrams of the form*

$$\mathbf{I}^{op} \xrightarrow{\mathcal{X}^{op}} \mathbf{K}^{op} \xrightarrow{h^?} \mathbf{Set}^{\mathbf{K}}$$

where $\mathbf{I}$ is a cofiltered category, $\mathcal{X} : \mathbf{I} \to \mathbf{K}$ is a diagram, and $h^?$ is the second Yoneda embedding. We will simply denote such diagrams by $\mathcal{X} = (X_i)_{i \in \mathbf{I}}$.

Let two pro-objects be defined by the diagrams $\mathcal{X} = (X_i)_{i \in \mathbf{I}}$ and $\mathcal{Y} = (Y_j)_{j \in \mathbf{J}}$. Then

$$Hom_{\mathbf{Pro}(\mathbf{C})}(\mathcal{X}, \mathcal{Y}) = \varprojlim_{j \in \mathbf{J}} \varinjlim_{i \in \mathbf{I}} Hom_{\mathbf{C}}(X_i, Y_j).$$

1.20. Remark. $\mathbf{Pro}(\mathbf{K})$ is indeed a category even though $\mathbf{Set}^{\mathbf{K}}$ is a quasi-category: $Hom_{\mathbf{Pro}(\mathbf{K})}(\mathcal{X},\mathcal{Y})$ is a **set** for any $\mathcal{X}$ and $\mathcal{Y}$.

1.21. Remark. The category $\mathbf{K}$ is a full subcategory of $\mathbf{Pro}(\mathbf{K})$: any object $X \in \mathbf{K}$ gives rise to a **rudimentary** pro-object

$$(* \longmapsto X) \in \mathbf{Pro}(\mathbf{K}).$$

The proposition below allows us to recognize rudimentary pro-objects:

1.22. Proposition. *Let*

$$\mathcal{X} = (X_i)_{i\in\mathbf{I}} \in \mathbf{Pro}(\mathbf{C}),$$

and $Z \in \mathbf{C}$. Then $\mathcal{X} \simeq Z$ iff there exist an $i_0 \in \mathbf{I}$ and a morphism $\tau_0 : X_{i_0} \to Z$ satisfying the property: for any morphism $s : i \to i_0$, there exist a morphism $g : Z \to X_i$ and a morphism $t : j \to i$ satisfying

$$\begin{aligned} \tau_0 \circ X(s) \circ g &= \mathbf{1}_Z, \\ g \circ \tau_0 \circ X(s) \circ X(t) &= X(t). \end{aligned}$$

Proof. The statement is dual to [Kashiwara and Schapira, 2006, Proposition 6.2.1]. ∎

Pro-homotopy and pro-homology. Let $\mathbf{Top}$ be the category of topological spaces and continuous mappings. The following categories are closely related to $\mathbf{Top}$: the category $H(\mathbf{Top})$ of homotopy types, the category $\mathbf{Pro}(H(\mathbf{Top}))$ of pro-homotopy types, and the category $H(\mathbf{Pro}(\mathbf{Top}))$ of homotopy types of pro-spaces. The latter category is used in *strong shape theory*. It is finer than the former which is used in *shape theory*. The pointed versions $\mathbf{Pro}(H(\mathbf{Top}_*))$ and $H(\mathbf{Pro}(\mathbf{Top}_*))$ are defined similarly.

One of the most important tools in strong shape theory is a *strong expansion* (see [Mardešić, 2000], conditions (S1) and (S2) on p. 129). In this paper, it is sufficient to use a weaker notion: an $H(\mathbf{Top})$-*expansion* ([Mardešić and Segal, 1982, §I.4.1], conditions (E1) and (E2)). Those two conditions are equivalent to the following

1.23. Definition. *Let X be a topological space. A morphism $X \to (Y_j)_{j\in\mathbf{I}}$ in $\mathbf{Pro}(H(\mathbf{Top}))$ is called an $H(\mathbf{Top})$-expansion (or simply **expansion**) if for any polyhedron P the following mapping*

$$\varinjlim_j [Y_j, P] = \varinjlim_j Hom_{H(\mathbf{Top})}(Y_j, P) \longrightarrow Hom_{H(\mathbf{Top})}(X, P) = [X, P]$$

is bijective where $[Z, P]$ is the set of homotopy classes of continuous mappings from Z to P.

*An expansion is called **polyhedral** (or an $H(\mathbf{Pol})$-expansion) if all Y_j are polyhedra.*

1.24. Remark.

1. The pointed version of this notion (an $H(\mathbf{Pol}_*)$-expansion) is defined similarly.
2. For any (pointed) topological space X there exists an $H(\mathbf{Pol})$-expansion (an $H(\mathbf{Pol}_*)$-expansion), see [Mardešić and Segal, 1982, Theorem I.4.7 and I.4.10].
3. Any two $H(\mathbf{Pol})$-expansions ($H(\mathbf{Pol}_*)$-expansions) of a (pointed) topological space X are isomorphic in the category $\mathbf{Pro}(H(\mathbf{Pol}))$ ($\mathbf{Pro}(H(\mathbf{Pol}_*))$), see [Mardešić and Segal, 1982, Theorem I.2.6].

1.25. Definition. *An open covering is called* ***normal*** *[Mardešić and Segal, 1982, §I.6.2], iff there is a partition of unity subordinated to it.*

1.26. Remark. Theorem 8 from [Mardešić and Segal, 1982, App.1, §3.2], shows that an $H(\mathbf{Pol})$- or an $H(\mathbf{Pol}_*)$-expansion for X can be constructed using nerves of normal (see Definition 1.25) open coverings of X.

Pro-homotopy is defined in [Mardešić and Segal, 1982, p. 121]:

1.27. Definition. *For a (pointed) topological space X, define its pro-homotopy pro-sets*

$$pro\text{-}\pi_n(X) := (\pi_n(Y_j))_{j\in\mathbf{J}}$$

where $X \to (Y_j)_{j\in\mathbf{J}}$ is an $H(\mathbf{Pol})$-expansion if $n = 0$, and an $H(\mathbf{Pol}_*)$-expansion if $n \geq 1$.

1.28. Remark. Similar to the "usual" algebraic topology, $pro\text{-}\pi_0$ is a pro-set (an object of $\mathbf{Pro}(\mathbf{Set})$), $pro\text{-}\pi_1$ is a pro-group (an object of $\mathbf{Pro}(\mathbf{Gr})$), and $pro\text{-}\pi_n$ are abelian pro-groups (objects of $\mathbf{Pro}(\mathbf{Ab})$) for $n \geq 2$.

Pro-homology groups are defined in [Mardešić and Segal, 1982, §II.3.2], as follows:

1.29. Definition. *For a topological space X, and an abelian group A, define its pro-homology groups as*

$$pro\text{-}H_n(X, A) := (H_n(Y_j, A))_{j\in\mathbf{J}}$$

where $X \to (Y_j)_{j\in\mathbf{J}}$ is an $H(\mathbf{Pol})$-expansion.

2. (Pre)cosheaves

General sites. We fix a small Grothendieck site (see Definition B.3) $X = (\mathbf{C}_X, Cov(X))$, and a category $\mathbf{K}$.

2.1. Definition. *Assume that* $\mathbf{K}$ *is cocomplete.*

1. *A* ***precosheaf*** $\mathcal{A}$ *on* X *with values in* $\mathbf{K}$ *is a functor* $\mathcal{A} : \mathbf{C}_X \to \mathbf{K}$.
2. *A precosheaf* $\mathcal{A}$ *is* ***coseparated*** *provided*

$$\mathcal{A} \otimes_{\mathbf{Set}^{\mathbf{C}_X}} R \longrightarrow \mathcal{A} \otimes_{\mathbf{Set}^{\mathbf{C}_X}} h_U \simeq \mathcal{A}(U)$$

is an epimorphism for any $U \in \mathbf{C}_X$ *and for any covering sieve (Definition B.1 and B.3)* R *over* U. *The pairing* $\otimes_{\mathbf{Set}^{\mathbf{C}_X}}$ *is introduced in Definition A.6.*

3. *A precosheaf* $\mathcal{A}$ *is a* ***cosheaf*** *provided*

$$\mathcal{A} \otimes_{\mathbf{Set}^{\mathbf{C}_X}} R \longrightarrow \mathcal{A} \otimes_{\mathbf{Set}^{\mathbf{C}_X}} h_U \simeq \mathcal{A}(U)$$

is an isomorphism for any $U \in \mathbf{C}_X$ *and for any covering sieve* R *over* U.

2.2. Notation. Let $\mathbf{pCS}(X, \mathbf{K}) = \mathbf{K}^{\mathbf{C}_X}$ be the category of precosheaves on X with values in $\mathbf{K}$, and let $\mathbf{CS}(X, \mathbf{K})$ be the full subcategory of cosheaves.

2.3. Proposition. *Let* $G \in \mathbf{K}$, *let* $\mathcal{A} \in \mathbf{pCS}(X, \mathbf{K})$, *and let* $R \subseteq h_U$ *be a sieve. Then:*

1.

$$Hom_{\mathbf{K}}\left(\mathcal{A} \otimes_{\mathbf{Set}^{\mathbf{C}_X}} R, G\right) \simeq Hom_{\mathbf{Set}^{(\mathbf{C}_X)^{op}}}\left(R, Hom_{\mathbf{K}}(\mathcal{A}, G)\right) \simeq$$
$$\simeq \varprojlim_{(V \to U) \in \mathbf{C}_R} Hom_{\mathbf{K}}\left(\mathcal{A}(V), G\right) \simeq Hom_{\mathbf{K}}\left(\varinjlim_{(V \to U) \in \mathbf{C}_R} \mathcal{A}(V), G\right)$$

naturally in G, $\mathcal{A}$ *and* R. *The presheaf of* ***sets*** $Hom_{\mathbf{K}}(\mathcal{A}, G)$ *is introduced in Definition A.4.*

2.

$$\mathcal{A} \otimes_{\mathbf{Set}^{\mathbf{C}_X}} R \simeq \varinjlim_{(V \to U) \in \mathbf{C}_R} \mathcal{A}(V).$$

Proof.

1. Follows from Proposition A.7 and [Artin et al., 1972, Corollary I.3.5].
2. Let G in (1) runs over all objects of $\mathbf{K}$. It follows from (1) that $h^K \simeq h^L$ where

$$K = \mathcal{A} \otimes_{\mathbf{Set}^{\mathbf{C}_X}} R, \; L = \varinjlim_{(V \to U) \in \mathbf{C}_R} \mathcal{A}(V).$$

Due to Yoneda's lemma, $K \simeq L$.

∎

2.4. Proposition. *Let X be a small site with a pretopology (Definition B.7). Then for any sieve R generated by a cover $\{U_i \to U\}_{i\in I}$,*

$$\mathcal{A} \otimes_{\mathbf{Set}^{\mathbf{C}_X}} R \simeq \varinjlim_{(V\to U)\in \mathbf{C}_R} \mathcal{A}(V) \simeq coker \left(\coprod_{i,j\in I} \mathcal{A}(U_i \times_U U_j) \rightrightarrows \coprod_{i\in I} \mathcal{A}(U_i) \right).$$

Proof. Apply $Hom_{\mathbf{K}}(\_, G)$ when G runs over all objects of $\mathbf{K}$. Apply then [Artin et al., 1972, Proposition I.2.12], to the presheaf of **sets** $Hom_{\mathbf{K}}(\mathcal{A}, G)$. ∎

2.5. Definition. *Let X be a small site, and $\mathbf{K}$ be a category (cocomplete and closed under cofiltered limits). Let $\mathcal{A}$ be a precosheaf.*

1. *Define a precosheaf $(\mathcal{A})_+^{\mathbf{K}}$ (or simply $\mathcal{A}_+$) by the following:*

$$\mathcal{A}_+(U) = \varprojlim_{R} \varinjlim_{(V\to U)\in \mathbf{C}_R} \mathcal{A}(V) = \varprojlim_{R} \left(\mathcal{A} \otimes_{\mathbf{Set}^{\mathbf{C}_X}} R\right).$$

where $R \subseteq h^U$ runs over all covering sieves over U.

2. *Let*

$$\lambda(U) = \lambda_{U,R'} \circ \lambda_{R'} : \mathcal{A}_+(U) \longrightarrow \mathcal{A}(U)$$

be the composition of canonical morphisms (not depending on R')

$$\mathcal{A}_+(U) = \varprojlim_{R} \varinjlim_{(V\to U)\in \mathbf{C}_R} \mathcal{A}(V) \xrightarrow{\lambda_{R'}} \varinjlim_{(V\to U)\in \mathbf{C}_{R'}} \mathcal{A}(V) \xrightarrow{\lambda_{U,R'}} \mathcal{A}(U).$$

The family $(\lambda(U))_{U\in \mathbf{C}_X}$ defines the morphism of functors

$$\lambda : ()_+ \longrightarrow \mathbf{1}_{\mathbf{pS}(X,\mathbf{K})} : \mathbf{pS}(X,\mathbf{K}) \longrightarrow \mathbf{pS}(X,\mathbf{K}).$$

2.6. Proposition. *If the topology is induced by a pretopology (Definition B.7), then*

$$\mathcal{A}_+(U) = \varprojlim_{\{U_i\to U\}} coker \left(\coprod_{i,j\in I} \mathcal{A}(U_i \times_U U_j) \rightrightarrows \coprod_{i\in I} \mathcal{A}(U_i) \right).$$

Proof. Follows from Proposition 2.4. ∎

2.7. Remark. Our plus construction $()_+$ is dual to the plus construction $()^+$ for presheaves (Definition B.18).

2.8. Lemma. *Let*

$$f : \mathcal{X} = (X_i)_{i\in \mathbf{I}} \longrightarrow \mathcal{Y} = (Y_j)_{j\in \mathbf{J}}$$

be a morphism in $\mathbf{Pro}(\mathbf{K})$. Then f is an epimorphism iff

$$Hom_{\mathbf{Pro}(\mathbf{K})}(\mathcal{Y}, G) \longrightarrow Hom_{\mathbf{Pro}(\mathbf{K})}(\mathcal{X}, G)$$

*is injective for any **rudimentary** (Remark 1.21) object*

$$G \in \mathbf{K} \subseteq \mathbf{Pro}(\mathbf{K}).$$

Proof. Let

$$\mathcal{Z} = (Z_s)_{s\in\mathbf{S}} \in \mathbf{Pro}\,(\mathbf{K})\,.$$

For any $s \in \mathbf{S}$, the mapping

$$Hom_{\mathbf{Pro}(\mathbf{K})}\,(\mathcal{Y}, Z_s) \longrightarrow Hom_{\mathbf{Pro}(\mathbf{K})}\,(\mathcal{X}, Z_s)$$

is injective. It follows that

$$Hom_{\mathbf{Pro}(\mathbf{K})}\,(\mathcal{Y}, \mathcal{Z}) = \varprojlim_{s\in\mathbf{S}} Hom_{\mathbf{Pro}(\mathbf{K})}\,(\mathcal{Y}, Z_s) \longrightarrow \varprojlim_{s\in\mathbf{S}} Hom_{\mathbf{Pro}(\mathbf{K})}\,(\mathcal{X}, Z_s) = Hom_{\mathbf{Pro}(\mathbf{K})}\,(\mathcal{X}, \mathcal{Z})$$

is injective as well, therefore f is an epimorphism.

Conversely, if f is an epimorphism, then

$$Hom_{\mathbf{Pro}(\mathbf{K})}\,(\mathcal{Y}, \mathcal{Z}) \longrightarrow Hom_{\mathbf{Pro}(\mathbf{K})}\,(\mathcal{X}, \mathcal{Z})$$

is injective for **any** $\mathcal{Z} \in \mathbf{Pro}\,(\mathbf{K})$. Since $G \in \mathbf{Pro}\,(\mathbf{K})$, the mapping

$$Hom_{\mathbf{Pro}(\mathbf{K})}\,(\mathcal{Y}, G) \longrightarrow Hom_{\mathbf{Pro}(\mathbf{K})}\,(\mathcal{X}, G)$$

is injective as well. ∎

2.9. Corollary. *Assume* $\mathbf{K}$ *is cocomplete.*

1. *A morphism* $f : G \to H$ *in* $\mathbf{K}$ *is an epimorphism iff it is an epimorphism in* $\mathbf{Pro}\,(\mathbf{K})$*.*

2. *Let* $\mathcal{A} \in \mathbf{pCS}\,(X, \mathbf{K})$*. Then* $\mathcal{A}$ *is coseparated (a cosheaf) iff it is coseparated (a cosheaf) when considered as a precosheaf with values in* $\mathbf{Pro}\,(\mathbf{K})$*.*

Proof. The full inclusion $\mathbf{K} \subseteq \mathbf{Pro}\,(\mathbf{K})$ commutes with colimits [Kashiwara and Schapira, 2006, dual to Corollary 6.1.17]. ∎

2.10. Proposition. *Let* $\mathbf{K}$ *be a cocomplete category, let* $\mathbf{L}$ *be either* $\mathbf{K}$ *or* $\mathbf{Pro}\,(\mathbf{K})$*, and let* $\mathcal{A} \in \mathbf{pCS}\,(X, \mathbf{L})$ *be a precosheaf. If* $G \in \mathbf{K}$*, consider the presheaf of sets*

$$Hom_{\mathbf{L}}\,(\mathcal{A}, G) \in \mathbf{pS}\,(X, \mathbf{Set})$$

(Definition A.4). Then:

1. $\mathcal{A}$ *is coseparated iff* $Hom_{\mathbf{L}}\,(\mathcal{A}, G)$ *is separated (Definition B.15) for any* $G \in \mathbf{K}$*.*

2. $\mathcal{A}$ *is a cosheaf iff* $Hom_{\mathbf{L}}\,(\mathcal{A}, G)$ *is a sheaf (Definition B.15) for any* $G \in \mathbf{K}$*.*

PROOF. It follows from Proposition A.7, that for any sieve $R \subseteq h_U$,

$$Hom_{\mathbf{L}}\left(\mathcal{A} \otimes_{\mathbf{Set}^{\mathbf{C}_X}} R, G\right) \simeq Hom_{\mathbf{Set}(\mathbf{C}_X)^{op}}\left(R, Hom_{\mathbf{L}}\left(\mathcal{A}, G\right)\right).$$

Consider the diagrams

$$\mathcal{A} \otimes_{\mathbf{Set}^{\mathbf{C}_X}} R \xrightarrow{\varphi} \mathcal{A} \otimes_{\mathbf{Set}^{\mathbf{C}_X}} h_U \simeq \mathcal{A}\left(U\right)$$

and

$$\begin{array}{ccc} Hom_{\mathbf{L}}\left(\mathcal{A}\left(U\right), G\right) & \xrightarrow[\varphi_G]{} & Hom_{\mathbf{L}}\left(\mathcal{A} \otimes_{\mathbf{Set}^{\mathbf{C}_X}} R, G\right) \\ \simeq \downarrow & & \downarrow \simeq \\ Hom_{\mathbf{L}}\left(\mathcal{A}, G\right)\left(U\right) & \xrightarrow{\psi_G} & Hom_{\mathbf{Set}(\mathbf{C}_X)^{op}}\left(R, Hom_{\mathbf{L}}\left(\mathcal{A}, G\right)\right) \end{array}$$

where U runs over objects of $\mathbf{C}_X$, and R runs over covering sieves.

1. If $\mathbf{L} = \mathbf{K}$, then φ is an epimorphism $\iff$ φ_G is a monomorphism for any $G \in \mathbf{K}$ $\iff$ ψ_G is a monomorphism for any $G \in \mathbf{K}$ $\iff$ $Hom_{\mathbf{L}}\left(\mathcal{A}, G\right)$ is a separated presheaf of sets for any $G \in \mathbf{K}$.

   If $\mathbf{L} = \mathbf{Pro}\left(\mathbf{K}\right)$, then, due to Lemma 2.8, φ is an epimorphism $\iff$ φ_G is a monomorphism for any $G \in \mathbf{K}$ $\iff$ $Hom_{\mathbf{L}}\left(\mathcal{A}, G\right)$ is a separated presheaf for any $G \in \mathbf{K}$.

2. If $\mathbf{L} = \mathbf{K}$, then φ is an isomorphism $\iff$ φ_G is an isomorphism for any $G \in \mathbf{K}$ $\iff$ ψ_G is an isomorphism for any $G \in \mathbf{K}$ $\iff$ $Hom_{\mathbf{L}}\left(\mathcal{A}, G\right)$ is a sheaf of sets for any $G \in \mathbf{K}$.

   If $\mathbf{L} = \mathbf{Pro}\left(\mathbf{K}\right)$, then, since $\mathbf{Pro}\left(\mathbf{K}\right)^{op}$ is a full subcategory of $\mathbf{Set}^{\mathbf{K}}$, φ is an isomorphism $\iff$ φ_G is an isomorphism for any $G \in \mathbf{K}$ $\iff$ $Hom_{\mathbf{L}}\left(\mathcal{A}, G\right)$ is a sheaf for any $G \in \mathbf{K}$.

■

2.11. PROPOSITION. *Assume that $\mathbf{K}$ is cocomplete. Let $\mathcal{A} \in \mathbf{pCS}\left(X, \mathbf{Pro}\left(\mathbf{K}\right)\right)$, and $G \in \mathbf{K}$. Then there is a natural (in $\mathcal{A}$ and G) isomorphism*

$$Hom_{\mathbf{Pro}(\mathbf{K})}\left(\left(\mathcal{A}\right)_+^{\mathbf{Pro}(\mathbf{K})}, G\right) \simeq \left(Hom_{\mathbf{Pro}(\mathbf{K})}\left(\mathcal{A}, G\right)\right)_{\mathbf{Set}}^+,$$

where $()^+$ is the plus construction for sheaves (Definition B.18).

PROOF. The functor

$$Hom_{\mathbf{Pro}(\mathbf{K})}\left(\_, G\right) : \mathbf{Pro}\left(\mathbf{K}\right) \longrightarrow \mathbf{Set}^{op}$$

commutes with small colimits [Kashiwara and Schapira, 2006, dual to Corollary 6.1.17], and cofiltered limits [Kashiwara and Schapira, 2006, dual to Theorem 6.1.8]. ■

2.12. THEOREM. *Assume* $\mathbf{K}$ *is cocomplete. Let*

$$\lambda(\mathcal{A}) : \mathcal{A}_+ = (\mathcal{A})_+^{\mathbf{Pro}(\mathbf{K})} \longrightarrow \mathcal{A}$$

be the canonical morphism of functors

$$()_+ \longrightarrow \mathbf{1}_{\mathbf{pCS}(X,\mathbf{Pro}(\mathbf{K}))} : \mathbf{pCS}(X, \mathbf{Pro}(\mathbf{K})) \longrightarrow \mathbf{pCS}(X, \mathbf{Pro}(\mathbf{K}))$$

from Definition 2.5. Then:

1. *The functor* $()_+$ *is right exact.*
2. *For any* $\mathcal{A}$, $\mathcal{A}_+$ *is a coseparated precosheaf.*
3. *A presheaf* $\mathcal{A}$ *is coseparated iff* $\lambda(\mathcal{A})$ *is an epimorphism. In that case* $\mathcal{A}_+$ *is a cosheaf.*
4. *The following conditions are equivalent:*
   (a) $\lambda(\mathcal{A})$ *is an isomorphism.*
   (b) $\mathcal{A}$ *is a cosheaf.*
5. *The functor* $()_\#^{\mathbf{Pro}(\mathbf{K})} = ()_{++}^{\mathbf{Pro}(\mathbf{K})}$ *is right adjoint to the inclusion*

$$i_{X,\mathbf{Pro}(\mathbf{K})} : \mathbf{pCS}(X, \mathbf{Pro}(\mathbf{K})) \hookrightarrow \mathbf{pCS}(X, \mathbf{Pro}(\mathbf{K})).$$

PROOF. Let G run over objects of $\mathbf{K}$ (not of $\mathbf{Pro}(\mathbf{K})$).

1. The functor $\mathcal{A} \mapsto \mathcal{A}_+$ is the composition of a colimit $\varinjlim_{(V\to U)\in\mathbf{C}_R} \mathcal{A}(V)$ which commutes with arbitrary colimits, and a codirected limit $\varprojlim_{R\subseteq h_U}$ which commutes with **finite** colimits [Kashiwara and Schapira, 2006, dual to Proposition 6.1.19]. Therefore, $()_+$ is right exact (commutes with finite colimits).

2. Due to Proposition 2.11,

$$Hom_{\mathbf{Pro}(\mathbf{K})}(\mathcal{A}_+, G) \simeq \left(Hom_{\mathbf{Pro}(\mathbf{K})}(\mathcal{A}, G)\right)^+.$$

   Due to [Artin et al., 1972, Proposition II.3.2], $Hom_{\mathbf{K}}(\mathcal{A}_+, G)$ is separated for any $G \in \mathbf{K}$. Apply Proposition 2.10.

3. Due to [Artin et al., 1972, Proposition II.3.2],

$$Hom_{\mathbf{Pro}(\mathbf{K})}(\mathcal{A}, G) \longrightarrow Hom_{\mathbf{Pro}(\mathbf{K})}(\mathcal{A}_+, G)$$

   is a monomorphism iff $Hom_{\mathbf{Pro}(\mathbf{K})}(\mathcal{A}, G)$ is separated. In that case $Hom_{\mathbf{Pro}(\mathbf{K})}(\mathcal{A}_+, G)$ is a sheaf. Apply Proposition 2.10.

4. Due to [Artin et al., 1972, Proposition II.3.2],
$$Hom_{\mathbf{Pro(K)}}\left(\mathcal{A},G\right)\longrightarrow Hom_{\mathbf{Pro(K)}}\left(\mathcal{A}_{+},G\right)$$
is an isomorphism iff $Hom_{\mathbf{Pro(K)}}\left(\mathcal{A},G\right)$ is a sheaf for any $G\in\mathbf{K}$. Apply Proposition 2.10.

5. We need to prove that for any cosheaf $\mathcal{B}$, any morphism $\mathcal{B}\rightarrow\mathcal{A}$ has a unique decomposition
$$\mathcal{B}\longrightarrow\mathcal{A}_{++}\longrightarrow\mathcal{A}.$$
The existence is easy: since $\mathcal{B}_{++}\rightarrow\mathcal{B}$ is an isomorphism, take the decomposition
$$\mathcal{B}\simeq\mathcal{B}_{++}\longrightarrow\mathcal{A}_{++}\longrightarrow\mathcal{A}.$$
To prove uniqueness, consider two decompositions
$$\begin{tikzcd}\mathcal{B} \arrow[r, shift left, "\alpha"] \arrow[r, shift right, "\beta"'] & \mathcal{A}_{++} \arrow[r] & \mathcal{A}\end{tikzcd}$$
and apply $Hom_{\mathbf{Pro(K)}}\left(\_,G\right)$:
$$\begin{tikzcd}Hom_{\mathbf{Pro(K)}}\left(\mathcal{A},G\right) \arrow[r] & Hom_{\mathbf{Pro(K)}}\left(\mathcal{A},G\right)^{++} \arrow[r, shift left, "Hom_{\mathbf{Pro(K)}}(\alpha,G)"] \arrow[r, shift right, "Hom_{\mathbf{Pro(K)}}(\beta,G)"'] & Hom_{\mathbf{Pro(K)}}\left(\mathcal{B},G\right).\end{tikzcd}$$
It follows that $Hom_{\mathbf{Pro(K)}}\left(\alpha,G\right)=Hom_{\mathbf{Pro(K)}}\left(\beta,G\right)$ for any $G\in\mathbf{K}$, therefore $\alpha=\beta$, because $\mathbf{Pro}\left(\mathbf{K}\right)^{op}$ is a full subcategory of $\mathbf{Set}^{\mathbf{K}}$.

■

Topological spaces. Throughout this subsection, X is a topological space considered as the site $OPEN\left(X\right)$ (see Example B.9 and Remark B.10), and $\mathbf{K}$ is a cocomplete category.

2.13. Proposition. *Let $\mathcal{A}$ be a cosheaf with values in $\mathbf{K}$. Then $\mathcal{A}\left(\varnothing\right)$ is an initial object in $\mathbf{K}$.*

Proof. Let $\left\{U_i\rightarrow\varnothing\right\}_{i\in I}$ be the empty covering, i.e. the set of indices I is empty. It is clear that
$$Y=\coprod_{i\in I}\mathcal{A}\left(U_i\right)$$
is an initial object in $\mathbf{C}$.

If $\mathcal{A}$ is a cosheaf, then
$$\mathcal{A}\left(\varnothing\right)=coker\left(\coprod_{i\in\varnothing}\mathcal{A}\left(U_i\right)\rightrightarrows\coprod_{(i,j)\in\varnothing}\mathcal{A}\left(U_i\cap U_j\right)\right)=coker\left(Y\rightrightarrows Y\right)\simeq Y.$$

■

2.14. Corollary. *If, in the conditions of Proposition 2.13, $\mathbf{K}$ is $\mathbf{Set}$ or $\mathbf{Pro}(\mathbf{Set})$, then $\mathcal{A}(\varnothing)=\varnothing$. If $\mathbf{K}$ is $\mathbf{Ab}$ or $\mathbf{Pro}(\mathbf{Ab})$, then $\mathcal{A}(\varnothing)=0$.*

2.15. Corollary. *A cosheaf with values in $\mathbf{Ab}$ or $\mathbf{Pro}(\mathbf{Ab})$ **is never a cosheaf** when considered as a precosheaf with values in $\mathbf{Set}$ or $\mathbf{Pro}(\mathbf{Set})$.*

2.16. Definition. *Let $G\in\mathbf{K}$. We denote by the same letter G the following **constant** precosheaf on X with values in $\mathbf{K}$ or $\mathbf{Pro}(\mathbf{K})$: $G(U):=G$ for all open subsets U.*

To introduce local isomorphisms, one needs the notion of a *costalk*, which is dual to the notion of a stalk (Definition B.23) in sheaf theory.

2.17. Definition. *Assume that a category $\mathbf{K}$ admits cofiltered limits. Let $\mathcal{A}$ be a precosheaf with values in $\mathbf{K}$, and let $x\in X$. The **costalk** of $\mathcal{A}$ at x is*

$$\mathcal{A}^{x}:=\varprojlim_{U\in J(x)}\mathcal{A}(U)$$

where $J(x)$ is the family of open neighborhoods of x.

2.18. Remark. In a situation when $\mathbf{K}\subseteq\mathbf{L}$ is a subcategory, and $\mathcal{A}\in\mathbf{pS}(X,\mathbf{K})$ we will use notations $(\mathcal{A})_{\mathbf{K}}^{x}$ and $(\mathcal{A})_{\mathbf{L}}^{x}$ depending on whether the limit is taken in the category $\mathbf{K}$ or in the category $\mathbf{L}$.

2.19. Example. Let

$$\begin{array}{rcl}\mathbf{K} & \subseteq & \mathbf{Pro}(\mathbf{K}),\\ \mathcal{A} & \in & \mathbf{pCS}(X,\mathbf{K})\subseteq\mathbf{pCS}(X,\mathbf{Pro}(\mathbf{K})).\end{array}$$

Then $(\mathcal{A})_{\mathbf{K}}^{x}$ is just the limit

$$(\mathcal{A})_{\mathbf{K}}^{x}=\varprojlim_{U\in J(x)}\mathcal{A}(U),$$

while $(\mathcal{A})_{\mathbf{Pro}(\mathbf{K})}^{x}$ is the pro-object represented by the cofiltered diagram

$$(\mathcal{A})_{\mathbf{Pro}(\mathbf{K})}^{x}=(\mathcal{A}(U))_{U\in J(x)}.$$

2.20. Definition. *Let $\mathbf{K}$ admit cofiltered limits, and let $f:\mathcal{A}\to\mathcal{B}$ be a morphism in the category of precosheaves $\mathbf{pCS}(X,\mathbf{K})$. We say that f is a **local isomorphism** iff $f^{x}:\mathcal{A}^{x}\to\mathcal{B}^{x}$ is an isomorphism for any $x\in X$. In a situation when $\mathbf{K}\subseteq\mathbf{L}$, and*

$$\mathcal{A},\mathcal{B}\in\mathbf{pCS}(X,\mathbf{K})\subseteq\mathbf{pCS}(X,\mathbf{L}),$$

we will say that f is $\mathbf{K}$-local (respectively $\mathbf{L}$-local) isomorphism iff

$$(f)_{\mathbf{K}}^{x}:(\mathcal{A})_{\mathbf{K}}^{x}\to(\mathcal{B})_{\mathbf{K}}^{x}$$

(respectively $(f)_{\mathbf{L}}^{x}:(\mathcal{A})_{\mathbf{L}}^{x}\to(\mathcal{B})_{\mathbf{L}}^{x}$) is an isomorphism for any $x\in X$.

2.21. Proposition. *Let* $\mathbf{K}$ *be a cocomplete category admitting cofiltered limits. Assume that* $\mathbf{CS}(X,\mathbf{K}) \subseteq \mathbf{pCS}(X,\mathbf{K})$ *is coreflective, and the coreflection is given by the functor*

$$()_{\#} : \mathbf{pCS}(X,\mathbf{K}) \longrightarrow \mathbf{CS}(X,\mathbf{K}).$$

Then for any precosheaf $\mathcal{A}$*, the natural morphism* $\mathcal{A}_{\#} \to \mathcal{A}$ *is a local isomorphism.*

Proof. Let $x \in X$, and $G \in \mathbf{K}$. Denote by $\mathcal{P}_{x,G}$ the following **pointed** precosheaf: $\mathcal{P}_{x,G}(U)$ is an initial object J when $x \notin U$, and $\mathcal{P}_{x,G}(U) = G$ when $x \in U$. It is easy to check that $\mathcal{P}_{x,G}$ is in fact a **cosheaf**, and that for any precosheaf $\mathcal{C}$,

$$Hom_{\mathbf{pCS}(X,\mathbf{K})}(\mathcal{P}_{x,G},\mathcal{C}) \simeq \varprojlim_{U\in J(x)} Hom_{\mathbf{K}}(G,\mathcal{C}(U)) \simeq Hom_{\mathbf{K}}(G,\mathcal{C}^{x}),$$

naturally in G and $\mathcal{C}$. Using the adjointness isomorphism, one gets

$$Hom_{\mathbf{K}}(G,\mathcal{A}^{x}) \simeq Hom_{\mathbf{pCS}(X,\mathbf{K})}(\mathcal{P}_{x,G},\mathcal{A}) \simeq Hom_{\mathbf{CS}(X,\mathbf{K})}(\mathcal{P}_{x,G},\mathcal{A}_{\#}) \simeq Hom_{\mathbf{K}}(G,(\mathcal{A}_{\#})^{x}),$$

for any $G \in \mathbf{K}$. Therefore, $\mathcal{A}^{x} \simeq (\mathcal{A}_{\#})^{x}$, as desired. ∎

2.22. Example. Let $\mathcal{A}$ be a precosheaf of abelian groups on X. According to [Bredon, 1968], Section 2, or [Bredon, 1997], Definition V.12.1, $\mathcal{A}$ is called *locally zero* iff for any $x \in X$ and any open neighborhood U of x there exists another open neighborhood V, $x \in V \subseteq U$, such that $\mathcal{A}(V) \to \mathcal{A}(U)$ is zero. If we consider, however, the precosheaf $\mathcal{A}$ as a precosheaf of abelian pro-groups, then it follows from Proposition 1.22 that $\mathcal{A}$ is locally zero iff for any $x \in X$, $\mathcal{A}^{x}$ is the zero object in the category $\mathbf{Pro}(\mathbf{Ab})$.

2.23. Definition. *A precosheaf* $\mathcal{A}$ *of abelian (pro-)groups on* X *is called* ***locally zero*** *if* $(\mathcal{A})^{x}_{\mathbf{Pro}(\mathbf{Ab})} = 0$ *for any* $x \in X$.

2.24. Definition. *Let* $\mathcal{A} \to \mathcal{B}$ *be a morphism of precosheaves (with values in* $\mathbf{K}$ *or* $\mathbf{Pro}(\mathbf{K})$*) on* X*. It is called a* ***local isomorphism in the sense of Bredon*** *(shorty:* ***strong local isomorphism****) iff* $(\mathcal{A})^{x}_{\mathbf{Pro}(\mathbf{K})} \to (\mathcal{B})^{x}_{\mathbf{Pro}(\mathbf{K})}$ *is an isomorphism for any* $x \in X$.

2.25. Remark. Strong local isomorphisms are local isomorphisms. Indeed, it follows from [Kashiwara and Schapira, 2006, dual to Proposition 6.3.1], that if $(A_i)_{i\in\mathbf{I}} \to (B_j)_{J\in\mathbf{J}}$ is an isomorphism in $\mathbf{Pro}(\mathbf{K})$, then

$$\varprojlim_{i\in\mathbf{I}} A_i \longrightarrow \varprojlim_{j\in\mathbf{J}} B_j$$

is an isomorphism in $\mathbf{K}$.

2.26. Proposition. *Let* $f : \mathcal{A} \to \mathcal{B}$ *be a morphism of precosheaves on* X *with values in* $\mathbf{Ab}$ *or* $\mathbf{Pro}(\mathbf{Ab})$*. Then* f *is a strong local isomorphism iff both* $\ker(f)$ *and* $coker(f)$ *are locally zero.*

Proof. Since cofiltered limits are exact in $\mathbf{Pro}(\mathbf{Ab})$ [Kashiwara and Schapira, 2006, dual to Proposition 6.1.19], the sequence

$$(\ker(f))^{x}_{\mathbf{Pro}(\mathbf{Ab})} \longrightarrow (\mathcal{A})^{x}_{\mathbf{Pro}(\mathbf{Ab})} \xrightarrow{f^{x}} (\mathcal{B})^{x}_{\mathbf{Pro}(\mathbf{Ab})} \longrightarrow (coker(f))^{x}_{\mathbf{Pro}(\mathbf{Ab})}$$

is exact. Since $\mathbf{Pro}(\mathbf{Ab})$ is an abelian category [Kashiwara and Schapira, 2006, Chapter 8.6], f^{x} is an isomorphism iff both $(\ker(f))^{x}_{\mathbf{Pro}(\mathbf{Ab})}$ and $(coker(f))^{x}_{\mathbf{Pro}(\mathbf{Ab})}$ are zero. ∎

2.27. Remark. It follows from Proposition 2.26 that a morphism $f : \mathcal{A} \to \mathcal{B}$ of precosheaves of abelian groups is a local isomorphism in the sense of [Bredon, 1968, Section 3], or [Bredon, 1997, Definition V.12.2], iff it is a strong local isomorphism in our sense.

2.28. Proposition. *Let* $\mathbf{K}$ *be cocomplete, and let*

$$(f : \mathcal{A} \longrightarrow \mathcal{B}) \in Hom_{\mathbf{pCS}(X,\mathbf{Pro(K)})}(\mathcal{A}, \mathcal{B}) .$$

Then f *is a strong local isomorphism iff*

$$Hom_{\mathbf{Pro(K)}}(\mathcal{B}, G) \longrightarrow Hom_{\mathbf{Pro(K)}}(\mathcal{A}, G)$$

is a local isomorphism of $\mathbf{Set}$*-valued presheaves for all* $G \in \mathbf{K}$*.*

Proof.

$$Hom_{\mathbf{Pro(K)}}(\_, G) : \mathbf{Pro}(\mathbf{K}) \longrightarrow \mathbf{Set}$$

converts cofiltered limits into filtered colimits [Kashiwara and Schapira, 2006, dual to Corollary 6.1.17]. ■

3. Main results

General sites. Let $X = (\mathbf{C}_X, Cov(X))$ be a small site (Definition B.3), and let $\mathbf{K}$ be a category. Let $\mathbf{Pro}(\mathbf{K})$ be the corresponding pro-category (Definition 1.19).

Let $\mathbf{pCS}(X, \mathbf{K}) = \mathbf{K}^{\mathbf{C}_X}$ be the category of precosheaves on X with values in $\mathbf{K}$, and let $\mathbf{CS}(X, \mathbf{K})$ (if $\mathbf{K}$ is cocomplete) be the full subcategory of cosheaves (Notation 2.2).

3.1. Theorem.

1. *If* $\mathbf{K}$ *is locally presentable (Definition 1.14), then* $\mathbf{CS}(X, \mathbf{K}) \subseteq \mathbf{pCS}(X, \mathbf{K})$ *is a coreflective subcategory.*
2. *If* $\mathbf{K}^{op}$ *is locally presentable, then* $\mathbf{CS}(X, \mathbf{K}) \subseteq \mathbf{pCS}(X, \mathbf{K})$ *is a coreflective subcategory.*
3. *If* $\mathbf{K}^{op}$ *is locally finitely presentable, then the coreflection* $\mathbf{pCS}(X, \mathbf{K}) \to \mathbf{CS}(X, \mathbf{K})$ *is given by*
$$\mathcal{A} \longmapsto (\mathcal{A})^{\mathbf{K}}_{\#} = (\mathcal{A})^{\mathbf{K}}_{++}$$
(see Definition 2.5).
4. *Assume* $\mathbf{K}$ *is cocomplete, and* $\mathcal{A} \in \mathbf{pCS}(X, \mathbf{K})$*. Then:*
   (a) $\mathcal{A}$ *is coseparated (a cosheaf) iff it is coseparated (a cosheaf) when considered as a precosheaf with values in* $\mathbf{Pro}(\mathbf{K})$*.*

(b)

$$\mathbf{CS}\left(X,\mathbf{Pro}\left(\mathbf{K}\right)\right)\subseteq\mathbf{pCS}\left(X,\mathbf{Pro}\left(\mathbf{K}\right)\right)$$

is coreflective, and the coreflection

$$\mathbf{pCS}\left(X,\mathbf{Pro}\left(\mathbf{K}\right)\right)\longrightarrow\mathbf{CS}\left(X,\mathbf{Pro}\left(\mathbf{K}\right)\right)$$

is given by

$$\mathcal{A}\longmapsto\left(\mathcal{A}\right)_{\#}^{\mathbf{Pro}(\mathbf{K})}=\left(\mathcal{A}\right)_{++}^{\mathbf{Pro}(\mathbf{K})}.$$

3.2. Corollary. *Assume that either* $\mathbf{K}$ *or* $\mathbf{K}^{op}$ *is locally presentable. Then*

$$\mathbf{CS}\left(X,\mathbf{Pro}\left(\mathbf{K}\right)\right)\subseteq\mathbf{pCS}\left(X,\mathbf{Pro}\left(\mathbf{K}\right)\right)$$

is coreflective, and the coreflection

$$\mathbf{pCS}\left(X,\mathbf{Pro}\left(\mathbf{K}\right)\right)\longrightarrow\mathbf{CS}\left(X,\mathbf{Pro}\left(\mathbf{K}\right)\right)$$

is given by

$$\mathcal{A}\longmapsto\left(\mathcal{A}\right)_{\#}^{\mathbf{Pro}(\mathbf{K})}=\left(\mathcal{A}\right)_{++}^{\mathbf{Pro}(\mathbf{K})}.$$

Proof. If $\mathbf{K}$ is locally presentable, it is both complete and cocomplete [Adámek and Rosický, 1994, Corollary 2.47]. If $\mathbf{K}^{op}$ is locally presentable, then again $\mathbf{K}$ is both complete and cocomplete. The statement follows from Theorem 3.1 (4). ■

3.3. Proof of Theorem 3.1 (1).

Proof. The proof goes through the following three steps:

1. $\mathbf{CS}\left(X,\mathbf{K}\right)$ is the $\mathbf{K}$-valued model $\mathbf{Mod}(\mathfrak{S},\mathbf{K})$ [Adámek and Rosický, 1994, Definition 2.55 and 2.60], of the following $\varinjlim$-sketch

   $$\mathfrak{S}=\left(\mathbf{C}_X,\mathfrak{L}=\varnothing,\mathfrak{C},\mathbf{K},\sigma\right).$$

   $\mathfrak{C}$ is the family of diagrams

   $$\mathbf{C}_R\subseteq\mathbf{C}_U\longrightarrow\mathbf{C}_X$$

   in $\mathbf{C}_X$ (R runs over covering sieves over U), where $\sigma\left(R\right)$ is the corresponding cocone (Definition 1.2)

   $$\sigma\left(R\right)=\left(\mathbf{C}_R\hookrightarrow\mathbf{C}_U\right).$$

   A precosheaf

   $$\mathcal{A}\in\mathbf{pCS}\left(X,\mathbf{K}\right)=\mathbf{K}^{\mathbf{C}_X}$$

   is a cosheaf iff

   $$\left(\varinjlim_{(V\to U)\in\mathbf{C}_R}\mathcal{A}\left(V\right)\right)\longrightarrow\mathcal{A}\left(U\right)$$

   is an isomorphism for all $U\in\mathbf{C}_X$ and for all sieves $R\in Cov\left(U\right)$. Therefore, $\mathcal{A}$ is a cosheaf iff $\mathcal{A}$ maps any cocone $\sigma\left(R\right)$ into a $\varinjlim$-cocone in $\mathbf{K}$, i.e. $\mathbf{CS}\left(X,\mathbf{K}\right)$ is indeed the model $\mathbf{Mod}(\mathfrak{S},\mathbf{K})$.

2. Due to [Adámek and Rosický, 1994, Theorem 2.60], the category $\mathbf{Mod}(\mathfrak{S}, \mathbf{K})$ is accessible. Since $\mathfrak{S}$ is a $\varinjlim$-sketch ($\mathfrak{L} = \varnothing$), the category is cocomplete, therefore locally presentable [Adámek and Rosický, 1994, Corollary 2.47]. See also [Adámek and Rosický, 1994, Remark 2.63].

3. Due to [Adámek and Rosický, 1994, Theorem 1.58 and Theorem 1.20], the category $\mathbf{CS}\,(X, \mathbf{K})$, being locally presentable, is co-wellpowered, and has a generator. The inclusion
$$i_{X,\mathbf{K}} : \mathbf{CS}\,(X, \mathbf{K}) \hookrightarrow \mathbf{pCS}\,(X, \mathbf{K})$$
clearly preserves direct limits, therefore, due to the dual to [Adámek and Rosický, 1994, Freyd's special adjoint functor theorem, Ch. 0.7], $i_{X,\mathbf{K}}$ is a left adjoint.

∎

3.4. Proof of Theorem 3.1 (2).

Proof. Let $\mathbf{L} = \mathbf{K}^{op}$. Since
$$\mathbf{pCS}\,(X, \mathbf{K})^{op} \simeq \mathbf{pS}\,(X, \mathbf{K}^{op}) \simeq \mathbf{pS}\,(X, \mathbf{L})$$
and
$$\mathbf{CS}\,(X, \mathbf{K})^{op} \simeq \mathbf{S}\,(X, \mathbf{K}^{op}) \simeq \mathbf{S}\,(X, \mathbf{L})\,,$$
it is enough to apply Theorem B.22: $\mathbf{S}\,(X, \mathbf{L}) \subseteq \mathbf{pS}\,(X, \mathbf{L})$ is a reflective subcategory. ∎

3.5. Proof of Theorem 3.1 (3).

Proof. Let again $\mathbf{L} = \mathbf{K}^{op}$. Since
$$\mathbf{pCS}\,(X, \mathbf{K})^{op} \simeq \mathbf{pS}\,(X, \mathbf{K}^{op}) \simeq \mathbf{pS}\,(X, \mathbf{L})$$
and
$$\mathbf{CS}\,(X, \mathbf{K})^{op} \simeq \mathbf{S}\,(X, \mathbf{K}^{op}) \simeq \mathbf{S}\,(X, \mathbf{L})\,,$$
it is enough to apply Theorem B.21: $\mathbf{S}\,(X, \mathbf{L}) \subseteq \mathbf{pS}\,(X, \mathbf{L})$ is a reflective subcategory, and a reflection is given by
$$\mathcal{A} \longmapsto (\mathcal{A})^{++}_{\mathbf{K}^{op}}\,.$$
Therefore,
$$\mathbf{CS}\,(X, \mathbf{L}) \subseteq \mathbf{pCS}\,(X, \mathbf{L})$$
is a coreflective subcategory, and a coreflection is given by
$$\mathcal{A} \longmapsto (\mathcal{A})^{++}_{\mathbf{K}^{op}} = (\mathcal{A})^{\mathbf{K}}_{++}\,.$$

∎

3.6. PROOF OF THEOREM 3.1 (4).

PROOF. **(a)** Follows from Corollary 2.9.

(b) Follows from Theorem 2.12. ∎

TOPOLOGICAL SPACES. Throughout this subsection, X is a topological space considered as the site $OPEN\,(X)$ (see Example B.9 and Remark B.10).

3.7. THEOREM. *Let* $\mathbf{K}$ *be a cocomplete category.*

1. *For any precosheaf* $\mathcal{A}$ *on* X *with values in* $\mathbf{Pro}\,(\mathbf{K})$*, the counit adjunction morphism* $(\mathcal{A})_{\#}^{\mathbf{Pro}(\mathbf{K})} \to \mathcal{A}$ *is a strong local isomorphism (Definition 2.24).*
2. *Any strong local isomorphism* $\mathcal{A} \to \mathcal{B}$ *between cosheaves on* X *with values in* $\mathbf{Pro}\,(\mathbf{K})$*, is an isomorphism.*
3. *If* $\mathcal{B} \to \mathcal{A}$ *is a strong local isomorphism, and* $\mathcal{B}$ *is a cosheaf, then the natural morphism* $\mathcal{B} \to (\mathcal{A})_{\#}^{\mathbf{Pro}(\mathbf{K})}$ *is an isomorphism.*

PROOF.

1. Apply Proposition 2.21 to the category $\mathbf{Pro}\,(\mathbf{K})$.
2. Let
$$(f : \mathcal{A} \to \mathcal{B}) \in Hom_{\mathbf{CS}(X,\mathbf{Pro}(\mathbf{K}))}\,(\mathcal{A},\mathcal{B})$$
be a strong local isomorphism between cosheaves, and G run over objects of $\mathbf{K}$. Due to Proposition 2.28 and 2.10,
$$Hom_{\mathbf{Pro}(\mathbf{K})}\,(f,G) : Hom_{\mathbf{Pro}(\mathbf{K})}\,(\mathcal{B},G) \longrightarrow Hom_{\mathbf{Pro}(\mathbf{K})}\,(\mathcal{A},G)$$
is a local isomorphism between sheaves of sets. It is well-known (see, for example, [Bredon, 1997, Ch. I.1]) that a local isomorphism between sheaves of sets is an isomorphism, therefore $Hom_{\mathbf{Pro}(\mathbf{K})}\,(f,G)$ is an isomorphism for any $G \in \mathbf{K}$. f is then an isomorphism because $(\mathbf{Pro}\,(\mathbf{K}))^{op}$ is a full subcategory of $\mathbf{Set}^{\mathbf{K}}$.
3. It is assumed that the composition
$$\mathcal{B} \longrightarrow (\mathcal{A})_{\#}^{\mathbf{Pro}(\mathbf{K})} \longrightarrow \mathcal{A}$$
of two morphisms is a strong local isomorphism. The second morphism is a strong local isomorphism, too. Therefore the first morphism is a strong local isomorphism between cosheaves, thus an isomorphism.

∎

3.8. Definition. *A precosheaf $\mathcal{A}$ is called* ***smooth*** *([Bredon, 1997, Corollary VI.3.2 and Definition VI.3.4], or [Bredon, 1968, Corollary 3.5 and Definition 3.7]), iff there exist precosheaves $\mathcal{B}$ and $\mathcal{B}'$, a cosheaf $\mathcal{C}$, and strong local isomorphisms $\mathcal{A} \to \mathcal{B} \leftarrow \mathcal{C}$, or, equivalently, strong local isomorphisms $\mathcal{A} \leftarrow \mathcal{B}' \to \mathcal{C}$.*

3.9. Corollary. *Let $\mathbf{K}$ be a cocomplete category. Any precosheaf with values in $\mathbf{Pro}\,(\mathbf{K})$ is smooth.*

Proof. Consider the diagram

$$\mathcal{A} \xrightarrow{\mathbf{1}_{\mathcal{A}}} \mathcal{A} \longleftarrow (\mathcal{A})_{\#}^{\mathbf{Pro}(\mathbf{K})},$$

or the diagram

$$\mathcal{A} \longleftarrow (\mathcal{A})_{\#}^{\mathbf{Pro}(\mathbf{K})} \xrightarrow{\mathbf{1}_{\mathcal{A}}} (\mathcal{A})_{\#}^{\mathbf{Pro}(\mathbf{K})}.$$

■

The results on cosheaves and precosheaves with values in $\mathbf{Pro}\,(\mathbf{K})$ can be applied to "usual" ones (like in [Bredon, 1968] and [Bredon, 1997, Chapter VI]), with values in $\mathbf{K}$, because $\mathbf{K}$ is a full subcategory of $\mathbf{Pro}\,(\mathbf{K})$ (see Remark 1.21).

The connection between the two types of (pre)cosheaves can be summarized in the following

3.10. Theorem. *Let $\mathbf{K}$ be cocomplete, and let $\mathcal{A}$ be a precosheaf on a topological space X with values in $\mathbf{K}$.*

1. *$\mathcal{A}$ is coseparated (a cosheaf) iff it is coseparated (a cosheaf) when considered as a precosheaf with values in $\mathbf{Pro}\,(\mathbf{K})$.*

2. *$\mathcal{A}$ is smooth iff $(\mathcal{A})_{\#}^{\mathbf{Pro}(\mathbf{K})}$ takes values in $\mathbf{K}$, i.e. $(\mathcal{A})_{\#}^{\mathbf{Pro}(\mathbf{K})}(U) \in \mathbf{K}$ (in other words, $(\mathcal{A})_{\#}^{\mathbf{Pro}(\mathbf{K})}(U)$ is a* rudimentary *pro-object, see Remark 1.21) for any open subset $U \subseteq X$.*

Proof.

1. Follows from Theorem 3.1 (4a).

2. If $(\mathcal{A})_{\#}^{\mathbf{Pro}(\mathbf{K})}$ takes values in $\mathbf{K}$, consider the diagram

$$\mathcal{A} \xrightarrow{\mathbf{1}_{\mathcal{A}}} \mathcal{A} \longleftarrow (\mathcal{A})_{\#}^{\mathbf{Pro}(\mathbf{K})}$$

of strong local isomorphisms in $\mathbf{pCS}\,(X, \mathbf{K})$. The diagram guarantees that $\mathcal{A}$ is smooth.

Conversely, assume that $\mathcal{A}$ is smooth. There exists either a diagram

$$\mathcal{A} \longrightarrow \mathcal{B} \longleftarrow \mathcal{C}$$

or a diagram

$$\mathcal{A} \longleftarrow \mathcal{B}' \longrightarrow \mathcal{C}$$

of strong local isomorphisms with a **cosheaf** $\mathcal{C} \in \mathbf{CS}(X, \mathbf{K})$. In the first case, the diagram

$$(\mathcal{A})_{\#}^{\mathbf{Pro}(\mathbf{K})} \longrightarrow (\mathcal{B})_{\#}^{\mathbf{Pro}(\mathbf{K})} \longleftarrow \mathcal{C}$$

consists of strong local isomorphisms (therefore isomorphisms, due to Theorem 3.7) between cosheaves. It follows that $(\mathcal{A})_{\#}^{\mathbf{Pro}(\mathbf{K})}$ takes values in $\mathbf{K}$, since $\mathcal{C}$ does. In the second case, the diagram

$$(\mathcal{A})_{\#}^{\mathbf{Pro}(\mathbf{K})} \longleftarrow (\mathcal{B})_{\#}^{\mathbf{Pro}(\mathbf{K})} \longrightarrow (\mathcal{C})_{\#}^{\mathbf{Pro}(\mathbf{K})} \longrightarrow \mathcal{C}$$

consists of strong local isomorphisms between cosheaves. It follows again that $(\mathcal{A})_{\#}^{\mathbf{Pro}(\mathbf{K})}$ takes values in $\mathbf{K}$, since $\mathcal{C}$ does.

■

We are now able to construct *constant* cosheaves, and to establish connections to shape theory.

3.11. THEOREM. *Let* $\mathbf{K}$ *be a cocomplete category, and let* $G \in \mathbf{K}$*.*

1. *The precosheaf*

$$\mathcal{P}(U) := G \otimes_{\mathbf{Set}} \textit{pro-}\pi_0(U)$$

(Definition A.3), where pro-π_0 *is the pro-homotopy functor from Definition 1.27 (see also [Mardešić and Segal, 1982, p. 121]), is a cosheaf. Let* G *be the constant precosheaf corresponding to* G *(Definition 2.16) on* X *with values in* $\mathbf{Pro}(\mathbf{K})$*. Then* $(G)_{\#}$ *is naturally isomorphic to* $\mathcal{P}$*.*

2. *Let* $\mathbf{K} = \mathbf{Set}$*. The precosheaf*

$$\mathcal{Q}(U) := G \times \textit{pro-}\pi_0(U)$$

is a cosheaf. Let G *be the constant precosheaf corresponding to* G *on* X *with values in* $\mathbf{Pro}(\mathbf{Set})$*. Then* $(G)_{\#}$ *is naturally isomorphic to* $\mathcal{Q}$*.*

3. *Let* $\mathbf{K} = \mathbf{Ab}$*. The precosheaf*

$$\mathcal{H}(U) := \textit{pro-}H_0(U, G)$$

where pro-H_0 *is the pro-homology functor from Definition 1.29 (see also [Mardešić and Segal, 1982, §II.3.2]), is a cosheaf. Let* G *be the constant precosheaf corresponding to* G *(Definition 2.16) on* X *with values in* $\mathbf{Pro}(\mathbf{Ab})$*. Then* $(G)_{\#}$ *is naturally isomorphic to* $\mathcal{H}$*.*

3.12. COROLLARY.

1. *pro-*π_0 *is a cosheaf.*
2. $(\mathbf{pt})_{\#} \simeq \textit{pro-}\pi_0$ *where* $\mathbf{pt}$ *is the one-point constant precosheaf.*

PROOF. Put $G = \mathbf{pt}$ in Theorem 3.11 (2). ■

3.13. Proposition. *Let* $\mathbf{K}$ *be a cocomplete category. For any* $G, H \in \mathbf{K}$ *and any topological space* U*, the set*

$$Hom_{\mathbf{Pro(K)}}\left(G \otimes_{\mathbf{Set}} pro\text{-}\pi_0\left(U\right), H\right)$$

is naturally (in G*,* H *and* U*) isomorphic to the set* $Hom_{\mathbf{K}}\left(G, H\right)^U$ *of continuous functions* $U \to Hom_{\mathbf{K}}\left(G, H\right)$ *where* $Hom_{\mathbf{K}}\left(G, H\right)$ *is supplied with the discrete topology.*

Proof. Let $U \to \left(Y_j\right)_{j\in\mathbf{J}}$ be a polyhedral expansion. Then

$$G \otimes_{\mathbf{Set}} pro\text{-}\pi_0\left(U\right) = \left(G \otimes_{\mathbf{Set}} \pi_0\left(Y_j\right)\right)_{j\in\mathbf{J}}.$$

Therefore,

$$\begin{aligned}
&Hom_{\mathbf{Pro(K)}}\left(G \otimes_{\mathbf{Set}} pro\text{-}\pi_0\left(U\right), H\right) \simeq \varinjlim_{j\in\mathbf{J}} Hom_{\mathbf{K}}\left(G \otimes_{\mathbf{Set}} \pi_0\left(Y_j\right), H\right) \simeq \\
&\simeq \varinjlim_{j\in\mathbf{J}} Hom_{\mathbf{Set}}\left(\pi_0\left(Y_j\right), Hom_{\mathbf{K}}\left(G, H\right)\right) \simeq \varinjlim_{j\in\mathbf{J}} Hom_{\mathbf{Top}}\left(Y_j, Hom_{\mathbf{K}}\left(G, H\right)\right) \simeq \\
&\simeq \varinjlim_{j\in\mathbf{J}} Hom_{H(\mathbf{Top})}\left(Y_j, Hom_{\mathbf{K}}\left(G, H\right)\right) \simeq Hom_{H(\mathbf{Top})}\left(U, Hom_{\mathbf{K}}\left(G, H\right)\right) \simeq \\
&\simeq Hom_{\mathbf{Top}}\left(U, Hom_{\mathbf{K}}\left(G, H\right)\right) \simeq Hom_{\mathbf{K}}\left(G, H\right)^U.
\end{aligned}$$

The bijections

$$\begin{aligned}
&Hom_{\mathbf{Top}}\left(Y_j, Hom_{\mathbf{K}}\left(G, H\right)\right) \simeq Hom_{H(\mathbf{Top})}\left(Y_j, Hom_{\mathbf{K}}\left(G, H\right)\right), \\
&Hom_{H(\mathbf{Top})}\left(U, Hom_{\mathbf{K}}\left(G, H\right)\right) \simeq Hom_{\mathbf{Top}}\left(U, Hom_{\mathbf{K}}\left(G, H\right)\right),
\end{aligned}$$

above are due to the fact that $Hom_{\mathbf{K}}\left(G, H\right)$ is discrete, therefore each homotopy class of mappings consists of a single mapping. The bijection

$$\varinjlim_{j\in\mathbf{J}} Hom_{H(\mathbf{Top})}\left(Y_j, Hom_{\mathbf{K}}\left(G, H\right)\right) \simeq Hom_{H(\mathbf{Top})}\left(U, Hom_{\mathbf{K}}\left(G, H\right)\right)$$

follows from the definition of an expansion. Since the spaces Y_j, being polyhedra, are locally connected, and $Hom_{\mathbf{K}}\left(G, H\right)$ is discrete, the bijections

$$Hom_{\mathbf{Set}}\left(\pi_0\left(Y_j\right), Hom_{\mathbf{K}}\left(G, H\right)\right) \simeq Hom_{\mathbf{Top}}\left(Y_j, Hom_{\mathbf{K}}\left(G, H\right)\right)$$

follow easily. ■

3.14. Proof of Theorem 3.11:

Proof.

1. Due to Proposition 2.10 and 2.11, it is enough to prove that, for any $H \in \mathbf{K}$, the presheaf of sets
$$\mathcal{B} := Hom_{\mathbf{Pro(K)}}\left(G \otimes_{\mathbf{Set}} pro\text{-}\pi_0, H\right)$$
is a sheaf, and that $\mathcal{C}^{\#} \simeq \mathcal{B}$ for the constant presheaf of sets
$$\mathcal{C} := Hom_{\mathbf{Pro(K)}}\left(G, H\right) = Hom_{\mathbf{K}}\left(G, H\right).$$

Due to Proposition 3.13, for any open subset U of X,

$$\mathcal{B}(U) = Hom_{\mathbf{Pro}(\mathbf{K})}(G \otimes_{\mathbf{Set}} pro\text{-}\pi_0, H) \simeq Hom_{\mathbf{K}}(G,H)^U .$$

For any open covering $\{U_i \to U\}_{i\in I}$ the space U is isomorphic in the category **Top** to the cokernel

$$coker\left(\coprod_{i,j\in I} U_i \cap U_j \rightrightarrows \coprod_{i\in I} U_i\right),$$

therefore

$$\begin{aligned}\mathcal{B}(U) &= Hom_{\mathbf{K}}(G,H)^U \simeq \ker\left(\prod_{i\in I} Hom_{\mathbf{K}}(G,H)^{U_i} \rightrightarrows \prod_{i,j\in I} Hom_{\mathbf{K}}(G,H)^{U_i\cap U_j}\right) \\ &\simeq \ker\left(\prod_{i\in I}\mathcal{B}(U_i) \rightrightarrows \prod_{i,j\in I}\mathcal{B}(U_i\cap U_j)\right).\end{aligned}$$

Therefore, $\mathcal{B}$ is a sheaf. To prove that $\mathcal{C}^{\#} \simeq \mathcal{B}$, it is enough, due to Theorem 3.7 and Proposition 2.28, to prove that $\mathcal{C} \to \mathcal{B}$ is a **Set**-local isomorphism of presheaves. The stalks $\mathcal{C}_x = Hom_{\mathbf{K}}(G,H)$ are constant. Let $x \in X$, and let $J(x)$ be the set of open neighborhoods of x. Since

$$\mathcal{B}_x = \varinjlim_{U\in J(x)}\mathcal{B}(U) \simeq \varinjlim_{U\in J(x)} Hom_{\mathbf{K}}(G,H)^U \simeq Hom_{\mathbf{K}}(G,H) \simeq \mathcal{C}_x,$$

the morphism $\mathcal{C} \to \mathcal{B}$ is indeed a local isomorphism.

2. If $\mathbf{K} = \mathbf{Set}$, and $G \in \mathbf{Set}$, then $G \otimes_{\mathbf{Set}} pro\text{-}\pi_0 \simeq G \times pro\text{-}\pi_0$.

3. If $\mathbf{K} = \mathbf{Ab}$, and $G \in \mathbf{Ab}$, then $G \otimes_{\mathbf{Set}} pro\text{-}\pi_0 \simeq pro\text{-}H_0(\_, G)$. Indeed, let $U \to (Y_j)_{j\in\mathbf{J}}$ be a polyhedral expansion. Since the polyhedra Y_j are locally connected,

$$H_0(Y_j, G) \simeq Hom_{\mathbf{Set}}(\pi_0 Y_j, G) \simeq \coprod_{\pi_0 Y_j} G \simeq G \otimes_{\mathbf{Set}} \pi_0(Y_j).$$

■

4. Examples

Below is a series of examples of (pre)cosheaves with values in various categories.

COSHEAVES.

4.1. Example. Let A be an abelian group, and let $\Sigma_n(\_, A)$ be a precosheaf that assigns to U the colimit of the following sequence:

$$S_n(U,A) \xrightarrow{\mathbf{ba}} S_n(U,A) \xrightarrow{\mathbf{ba}} S_n(U,A) \xrightarrow{\mathbf{ba}} \ldots$$

where $S_n(U,A)$ is the group of singular A-valued n-chains on U, and **ba** is the barycentric subdivision. It is proved in [Bredon, 1968], Section 10, and [Bredon, 1997], Proposition VI.12.1, that $\Sigma_n(\_, A)$ is a cosheaf of abelian groups (and of abelian pro-groups, due to Theorem 3.10).

4.2. Example. Let π_0 be a precosheaf of sets that assigns to U the set $\pi_0(U)$ of path-connected components of U. Then π_0 is a cosheaf of sets (and of pro-sets, due to Theorem 3.10). This cosheaf is constant if X is locally path-connected, and is not constant in general. Indeed, π_0 is clearly coseparated. Let $\{U_i \to U\}_{i\in I}$ be an open covering, and let $P \in U_s$ and $Q \in U_t$ be two points lying in the same path-connected component. Therefore, there exists a continuous path $g:[0,1] \longrightarrow U$ with $g(0) = P$ and $g(1) = Q$. Using Lebesgue's Number Lemma, one proves that P and Q define equal elements of the cokernel below. Therefore, the mapping

$$\mathit{coker}\left(\coprod_{i,j}\pi_0(U_i\cap U_j) \rightrightarrows \coprod_i \pi_0(U_i)\right) \longrightarrow \pi_0(U)$$

is injective, thus bijective, and π_0 is a cosheaf.

4.3. Example. Let A be an abelian group, and let $H_0^S(\_, A)$ be the precosheaf of abelian groups that assigns to U the zeroth singular homology group $H_0^S(X,A)$. Then $H_0^S(\_, A)$ is a cosheaf. Indeed,

$$H_0^S = \mathit{coker}\left(\Sigma_1(\_, A) \longrightarrow \Sigma_0(\_, A)\right)$$

where $\Sigma_n(\_, A)$ is the cosheaf from Example 4.1. The embedding

$$\mathbf{CS}(X, \mathbf{Pro}(\mathbf{Ab})) \longrightarrow \mathbf{pCS}(X, \mathbf{Pro}(\mathbf{Ab})),$$

being left adjoint to $()_{\#}$, commutes with colimits. Therefore, $H_0^S(\_, A)$ is a cosheaf because $\Sigma_1(\_, A)$ and $\Sigma_0(\_, A)$ are cosheaves. $H_0^S(\_, A)$ is constant if X is locally path-connected. However, $H_0^S(\_, A)$ is not constant in general, see Example 4.8.

4.4. Example. Let **Gpd** be the category of small groupoids. Consider the following precosheaf $\Pi_1 \in \mathbf{pCS}(X, \mathbf{Gpd})$: for an open subset $U \subseteq X$ let $\Pi_1(U)$ be the fundamental groupoid of U. Then, due to the main theorem in [Brown and Salleh, 1984], for any open covering $(U_i)_{i\in I}$ of U, the morphism

$$\mathit{coker}\left(\coprod_{i,j\in I}\Pi_1(U_i\cap U_j) \rightrightarrows \coprod_{i\in I} \Pi_1(U_i)\right) \longrightarrow \Pi_1(U)$$

is an isomorphism of groupoids. Therefore, Π_1 is a cosheaf of groupoids.

Precosheaves.

4.5. Example. Let X be the closed interval $[0,1]$, and let $\mathcal{A}$ assign to U the group $S_1(U,\mathbb{Z})$ of singular 1-chains on U. It is proved in [Bredon, 1968, Remark 5.9], and [Bredon, 1997, Example VI.5.9], that this precosheaf of abelian groups is not smooth.

4.6. Example. Fix $n\geq 1$. Let again $X=[0,1]$, and let $\mathcal{A}$ assign to U the **set** $Simp_n(U)$ of singular n-**simplices** on U, i.e.

$$\mathcal{A}(U):=Simp_n(U)=U^{\Delta^n}=Hom_{\mathbf{Top}}(\Delta^n,U).$$

Then $\mathcal{A}$ is not smooth as a precosheaf of sets. Indeed, let $\mathcal{B}=(\mathcal{A})_+^{\mathbf{Pro(Set)}}$. For an open $U\subseteq X$,

$$\mathcal{B}(U)=\left(B_{\{U_i\}}\right)_{\{U_i\}}$$

where $\{U_i\}$ runs over open covers of U, and

$$B_{\{U_i\}}=\{\sigma:\Delta^n\longrightarrow U\mid \exists i\,(\sigma(\Delta^n)\subseteq U_i)\}.$$

It can be checked that:

1. $\mathcal{B}$ is a cosheaf of pro-sets.
2. For any $U\neq\varnothing$, the pro-object $\mathcal{B}(U)$ is **not** rudimentary (see Remark 1.21 and Proposition 1.22).

It follows that

$$(\mathcal{A})_\#^{\mathbf{Pro(Set)}}\simeq(\mathcal{A})_{++}^{\mathbf{Pro(Set)}}\simeq(\mathcal{A})_+^{\mathbf{Pro(Set)}}\simeq\mathcal{B},$$

and this cosheaf does not take values in **Set**. Therefore, due to Theorem 3.10, $\mathcal{A}$ is **not smooth**. However, since **Set** is locally presentable (even locally finitely presentable), there exists, due to Theorem 3.1(1), a cosheafification

$$()_\#^{\mathbf{Set}}:\mathbf{pCS}(X,\mathbf{Set})\longrightarrow\mathbf{CS}(X,\mathbf{Set}).$$

It can be checked that $(\mathcal{A})_\#^{\mathbf{Set}}$ is **rather trivial**: $(\mathcal{A})_\#^{\mathbf{Set}}(U)=U$, i.e. the result is as if our space X were a **discrete** space. The natural morphism $(\mathcal{A})_\#^{\mathbf{Set}}\to\mathcal{A}$ sends any point

$$a\in(\mathcal{A})_\#^{\mathbf{Set}}(U)=U$$

to the **constant** $(\sigma(t)\equiv a)$ singular simplex

$$\sigma\in Simp_n(U)=\mathcal{A}(U).$$

Let us calculate the costalks:

$$\begin{aligned}(\mathcal{A})^x_{\mathbf{Set}}&=\bigcap_{U\in J(x)}Simp_n(U)=\mathbf{pt},\\ \left((\mathcal{A})_\#^{\mathbf{Set}}\right)^x_{\mathbf{Set}}&=\bigcap_{U\in J(x)}U=\mathbf{pt},\end{aligned}$$

while

$$(\mathcal{A})^x_{\mathbf{Pro(Set)}} \simeq (\mathcal{B})^x_{\mathbf{Pro(Set)}}$$

are non-rudimentary pro-sets. It is clear that $(\mathcal{A})^{\mathbf{Set}}_{\#} \to \mathcal{A}$ is a **Set**-local isomorphism, but **not** a strong local isomorphism (because $\mathcal{A}$ is not smooth).

4.7. Example. Let π be a precosheaf of sets that assigns to U the set $\pi(U)$ of connected components of U. This precosheaf is coseparated. If X is **locally connected**, then, for any open subset $U \subseteq X$, the pro-homotopy set $pro\text{-}\pi_0(U)$ is isomorphic to the **rudimentary** (Remark 1.21) pro-set $\pi(U)$. It follows from Theorem 3.10, that $\pi \simeq (\mathbf{pt})_{\#}$ where **pt** is the one-point constant precosheaf. Therefore, **pt** is smooth, and π a constant cosheaf (compare to [Bredon, 1968, Remark 5.11]).

In general, if X is **not** locally connected, π is **not a cosheaf**. Indeed, let

$$X = Y \cup Z \subseteq \mathbb{R}^2,$$

where Y is the line segment between the points $(0,1)$ and $(0,-1)$, and Y is the graph of $y = \sin\left(\frac{1}{x}\right)$ for $0 < x \leq 2\pi$. Let further

$$\begin{aligned} X &= U = U_1 \cup U_2, \\ U_1 &= \left\{(x,y) \in X \mid y > -\frac{1}{2}\right\}, \\ U_2 &= \left\{(x,y) \in X \mid y < \frac{1}{2}\right\}. \end{aligned}$$

X is a connected (not locally connected!) compact metric space. Take $P = (0,1) \in U_1$ and $Q = \left(\frac{3\pi}{2}, -1\right) \in U_2$. Since $U = X$ is connected, these two points are mapped to the same element of $\pi(U)$ under the canonical mapping $U_1 \sqcup U_2 \longrightarrow U$. However, these two points define **different** elements of the colimit

$$coker\left(\pi(U_1 \cap U_2) \rightrightarrows \pi(U_1) \sqcup \pi(U_2)\right).$$

Therefore,

$$coker\left(\pi(U_1 \cap U_2) \rightrightarrows \pi(U_1) \sqcup \pi(U_2)\right) \longrightarrow \pi(U) = \pi(X)$$

is not injective, and π is not a cosheaf.

See also Example 4.8.

4.8. Example. Let X be the following sequence converging to zero (together with the limit):

$$X = \{0\} \cup \left\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots\right\} \subseteq \mathbb{R}.$$

The precosheaves π and π_0 from Examples 4.7 and 4.2 coincide on X. Therefore, $\pi = \pi_0$ is a cosheaf. However, it is **not** constant. To see this, just compare the costalks at different points $x \in X$: $(\pi)^x_{\mathbf{Pro(Set)}} = \{\mathbf{pt}\}$ if $x \neq 0$, while $(\pi)^0_{\mathbf{Pro(Set)}}$ is a **non-rudimentary**

(Remark 1.21) pro-set. Consider the constant precosheaf $\mathbf{pt}$. Due to Corollary 3.12, $(\mathbf{pt})_{\#}^{\mathbf{Pro(Set)}} \simeq pro\text{-}\pi_0$. The latter cosheaf does **not** take values in **Set**, therefore, due to Theorem 3.10, the precosheaf **pt** is **not** smooth. Similarly, it can be proved, that the cosheaf $H_0^S(\_, A)$ from Example 4.3 is not constant on X, while the constant precosheaf A is not smooth, because $(A)_{\#} \simeq pro\text{-}H_0(\_, A)$ does not take values in **Ab**.

It appears that the cosheaf $(\mathbf{pt})_{\#}^{\mathbf{Set}}$ is **rather trivial**. Similarly to Example 4.6, it can be proved that $(\mathbf{pt})_{\#}^{\mathbf{Set}}(U) = U$, i.e. the result is as if our space X were a **discrete** space.

A. Pairings

A.1. Definition. *Let* $\mathbf{K}$ *be a category. Assume that* $\mathbf{K}$ *is complete in (2) below, and cocomplete in (3) below. Given*

$$G, H \in \mathbf{K},\ Z \in \mathbf{Set},$$

define

1.

$$Hom_{\mathbf{K}}(G, H) \in \mathbf{Set};$$

2.

$$Hom_{\mathbf{Set}}(Z, G) := \prod_{Z} G \in \mathbf{K};$$

3.

$$G \otimes_{\mathbf{Set}} Z = Z \otimes_{\mathbf{Set}} G := \coprod_{Z} G \in \mathbf{K}.$$

A.2. Remark. The first two assignments are contravariant in the first argument and covariant in the second argument:

$$\begin{aligned} Hom_{\mathbf{K}}(\_, \_) &: \mathbf{K}^{op} \times \mathbf{K} \longrightarrow \mathbf{Set}, \\ Hom_{\mathbf{Set}}(\_, \_) &: \mathbf{Set}^{op} \times \mathbf{K} \longrightarrow \mathbf{K}, \end{aligned}$$

while the third assignment is covariant in both arguments:

$$\begin{aligned} \_ \otimes_{\mathbf{Set}} \_ &: \mathbf{K} \times \mathbf{Set} \longrightarrow \mathbf{K}, \\ \_ \otimes_{\mathbf{Set}} \_ &: \mathbf{Set} \times \mathbf{K} \longrightarrow \mathbf{K}. \end{aligned}$$

A.3. Definition. *Let* $\mathcal{X} = (X_i)_{i \in \mathbf{I}} \in \mathbf{Pro}(\mathbf{Set})$*, and* $Y \in \mathbf{K}$*. Define*

$$\mathcal{X} \otimes_{\mathbf{Set}} Y = Y \otimes_{\mathbf{Set}} \mathcal{X} \in \mathbf{Pro}(\mathbf{K})$$

by

$$\mathcal{X} \otimes_{\mathbf{Set}} Y = (X_i \otimes_{\mathbf{Set}} Y)_{i \in \mathbf{I}}.$$

A.4. Definition. *Let* $\mathbf{K}$ *be a complete and cocomplete category,* $\mathbf{C}$ *be a small category,*

$$\mathcal{A} : \mathbf{C} \longrightarrow \mathbf{K},\ \mathcal{B} : \mathbf{C} \longrightarrow \mathbf{Set},$$

be functors, and let

$$G \in \mathbf{K},\ Z \in \mathbf{Set}.$$

Then define the following functors:

1.
$$Hom_{\mathbf{K}}(G, \mathcal{A}) : \mathbf{C} \longrightarrow \mathbf{Set},$$
$$Hom_{\mathbf{K}}(G, \mathcal{A})(U) := Hom_{\mathbf{K}}(G, \mathcal{A}(U));$$

2.
$$Hom_{\mathbf{K}}(\mathcal{A}, G) : \mathbf{C}^{op} \longrightarrow \mathbf{Set},$$
$$Hom_{\mathbf{K}}(\mathcal{A}, G)(U) := Hom_{\mathbf{K}}(\mathcal{A}(U), G);$$

3.
$$Hom_{\mathbf{Set}}(\mathcal{B}, G) : \mathbf{C}^{op} \longrightarrow \mathbf{K},$$
$$Hom_{\mathbf{Set}}(\mathcal{B}, G) := Hom_{\mathbf{Set}}(\mathcal{B}(U), G);$$

4.
$$\mathcal{A} \otimes_{\mathbf{Set}} Z = Z \otimes_{\mathbf{Set}} \mathcal{A} : \mathbf{C} \longrightarrow \mathbf{K},$$
$$(\mathcal{A} \otimes_{\mathbf{Set}} Z)(U) = (Z \otimes_{\mathbf{Set}} \mathcal{A})(U) := Z \otimes_{\mathbf{Set}} (\mathcal{A}(U));$$

5.
$$\mathcal{B} \otimes_{\mathbf{Set}} G = G \otimes_{\mathbf{Set}} \mathcal{B} : \mathbf{C} \longrightarrow \mathbf{K},$$
$$(\mathcal{B} \otimes_{\mathbf{Set}} G)(U) = (G \otimes_{\mathbf{Set}} \mathcal{B})(U) := G \otimes_{\mathbf{Set}} (\mathcal{B}(U)).$$

A.5. Remark. The assignments $Hom_{\mathbf{K}}$ and $Hom_{\mathbf{Set}}$ are contravariant in the first argument and covariant in the second argument:

$$\begin{aligned} Hom_{\mathbf{K}}(\_,\_) &: \mathbf{K}^{op} \times \mathbf{K}^{\mathbf{C}} \longrightarrow \mathbf{Set}^{\mathbf{C}}, \\ Hom_{\mathbf{K}}(\_,\_) &: \left(\mathbf{K}^{\mathbf{C}}\right)^{op} \times \mathbf{K} \longrightarrow \mathbf{Set}^{\mathbf{C}^{op}}, \\ Hom_{\mathbf{Set}}(\_,\_) &: \left(\mathbf{Set}^{\mathbf{C}}\right)^{op} \times \mathbf{K} \longrightarrow \mathbf{K}^{\mathbf{C}^{op}}, \end{aligned}$$

while the assignments $\otimes_{\mathbf{Set}}$ are covariant in both arguments:

$$\begin{aligned} \_ \otimes_{\mathbf{Set}} \_ &: \mathbf{K}^{\mathbf{C}} \times \mathbf{Set} \longrightarrow \mathbf{K}^{\mathbf{C}}, \\ \_ \otimes_{\mathbf{Set}} \_ &: \mathbf{Set} \times \mathbf{K}^{\mathbf{C}} \longrightarrow \mathbf{K}^{\mathbf{C}}, \\ \_ \otimes_{\mathbf{Set}} \_ &: \mathbf{Set}^{\mathbf{C}} \times \mathbf{K} \longrightarrow \mathbf{K}^{\mathbf{C}}, \\ \_ \otimes_{\mathbf{Set}} \_ &: \mathbf{K} \times \mathbf{Set}^{\mathbf{C}} \longrightarrow \mathbf{K}^{\mathbf{C}}. \end{aligned}$$

A.6. Definition. *Let* $\mathbf{K}$ *be a complete and cocomplete category, let* $\mathbf{C}$ *be a small category, and let*

$$\mathcal{A} : \mathbf{C} \longrightarrow \mathbf{K},\ \mathcal{B} : \mathbf{C} \longrightarrow \mathbf{Set},\ \mathcal{F} : \mathbf{C}^{op} \longrightarrow \mathbf{Set},$$

be functors. Then define the following objects:

1. $Hom_{\mathbf{Set}^{\mathbf{C}}}(\mathcal{B}, \mathcal{A}) \in \mathbf{K}$ *is the* ***end*** *[Mac Lane, 1998, Chapter IX.5], of the bifunctor* $(U, V) \mapsto Hom_{\mathbf{Set}}(\mathcal{B}(U), \mathcal{A}(V))$*, i.e.*

$$Hom_{\mathbf{Set}^{\mathbf{C}}}(\mathcal{B}, \mathcal{A}) := \ker\left(\prod_{U} Hom_{\mathbf{Set}}(\mathcal{B}(U), \mathcal{A}(U)) \rightrightarrows \prod_{U\to V} Hom_{\mathbf{Set}}(\mathcal{B}(U), \mathcal{A}(V))\right).$$

2. $\mathcal{A}\otimes_{\mathbf{Set}^{\mathbf{C}}}\mathcal{F} = \mathcal{F}\otimes_{\mathbf{Set}^{\mathbf{C}^{op}}}\mathcal{A} \in \mathbf{K}$ *is the* ***coend*** *[Mac Lane, 1998, Chapter IX.6], of the bifunctor* $(U, V) \mapsto \mathcal{A}(U) \otimes_{\mathbf{Set}} \mathcal{F}(V)$*, i.e.*

$$\mathcal{A}\otimes_{\mathbf{Set}^{\mathbf{C}}}\mathcal{F} := coker\left(\coprod_{U\to V} \mathcal{A}(U) \otimes_{\mathbf{Set}} \mathcal{F}(V) \rightrightarrows \coprod_{U} \mathcal{A}(U) \otimes_{\mathbf{Set}} \mathcal{F}(U)\right).$$

A.7. Proposition. *Let* $G \in \mathbf{K}$*,* $\mathcal{A} \in \mathbf{K}^{\mathbf{C}}$*, and* $\mathcal{B} \in \mathbf{Set}^{\mathbf{C}}$*. Then*

$$Hom_{\mathbf{K}^{\mathbf{C}}}(\mathcal{B} \otimes_{\mathbf{Set}} G, \mathcal{A}) \simeq Hom_{\mathbf{K}}(G, Hom_{\mathbf{Set}^{\mathbf{C}}}(\mathcal{B}, \mathcal{A})) \simeq Hom_{\mathbf{Set}^{\mathbf{C}}}(\mathcal{B}, Hom_{\mathbf{K}}(G, \mathcal{A}))$$

naturally in G*,* $\mathcal{A}$*, and* $\mathcal{B}$*.*

Proof.

$$\begin{aligned}
& Hom_{\mathbf{K}^{\mathbf{C}}}(\mathcal{B} \otimes_{\mathbf{Set}} G, \mathcal{A}) \simeq \\
& \simeq \ker\left(\prod_{U} Hom_{\mathbf{K}}((\mathcal{B} \otimes_{\mathbf{Set}} G)(U), \mathcal{A}(U)) \rightrightarrows \prod_{U\to V} Hom_{\mathbf{K}}((\mathcal{B} \otimes_{\mathbf{Set}} G)(U), \mathcal{A}(V))\right) \\
& \simeq \ker\left(\prod_{U}\prod_{\mathcal{B}(U)} Hom_{\mathbf{K}}(G, \mathcal{A}(U)) \rightrightarrows \prod_{U\to V}\prod_{\mathcal{B}(U)} Hom_{\mathbf{K}}(G, \mathcal{A}(V))\right) \\
& \simeq Hom_{\mathbf{K}}\left(G, \ker\left(\prod_{U}\prod_{\mathcal{B}(U)} \mathcal{A}(U) \rightrightarrows \prod_{U\to V}\prod_{\mathcal{B}(U)} \mathcal{A}(V)\right)\right) \\
& \simeq Hom_{\mathbf{K}}\left(G, \ker\left(\prod_{U} Hom_{\mathbf{Set}}(\mathcal{B}(U), \mathcal{A}(U)) \rightrightarrows \prod_{U\to V} Hom_{\mathbf{Set}}(\mathcal{B}(U), \mathcal{A}(V))\right)\right) \\
& \simeq Hom_{\mathbf{K}}(G, Hom_{\mathbf{Set}^{\mathbf{C}}}(\mathcal{B}, \mathcal{A})).
\end{aligned}$$

Similarly,

$$
\begin{aligned}
& Hom_{\mathbf{K}^{\mathbf{C}}}(\mathcal{B} \otimes_{\mathbf{Set}} G, \mathcal{A}) \simeq \\
& \simeq \ker \left(\prod_{U} \prod_{\mathcal{B}(U)} Hom_{\mathbf{K}}(G, \mathcal{A}(U)) \rightrightarrows \prod_{U \to V} \prod_{\mathcal{B}(U)} Hom_{\mathbf{K}}(G, \mathcal{A}(V)) \right) \\
& \simeq \ker \left(\prod_{U} Hom_{\mathbf{Set}}(\mathcal{B}(U), Hom_{\mathbf{K}}(G, \mathcal{A}(U))) \rightrightarrows \prod_{U \to V} Hom_{\mathbf{Set}}(\mathcal{B}(U), Hom_{\mathbf{K}}(G, \mathcal{A}(V))) \right) \\
& \simeq Hom_{\mathbf{Set}^{\mathbf{C}}}(\mathcal{B}, Hom_{\mathbf{K}}(G, \mathcal{A})).
\end{aligned}
$$

■

Proposition below is a variant of Yoneda's Lemma:

A.8. Proposition. *Let $G \in \mathbf{K}$, $U \in \mathbf{C}$, and $\mathcal{A} \in \mathbf{K}^{\mathbf{C}^{op}}$. Then*

$$
Hom_{\mathbf{K}^{\mathbf{C}^{op}}}(h_U \otimes_{\mathbf{Set}} G, \mathcal{A}) \simeq Hom_{\mathbf{K}}(G, \mathcal{A}(U)) = (Hom_{\mathbf{K}}(G, \mathcal{A}))(U)
$$

naturally in G, U, and $\mathcal{A}$.

Proof. Using the Yoneda isomorphism $Hom_{\mathbf{Set}^{\mathbf{C}^{op}}}(h_U, \mathcal{A}) \simeq \mathcal{A}(U)$, and Proposition A.7, one gets

$$
Hom_{\mathbf{K}^{\mathbf{C}^{op}}}(h_U \otimes_{\mathbf{Set}} G, \mathcal{A}) \simeq Hom_{\mathbf{K}}(G, Hom_{\mathbf{Set}^{\mathbf{C}^{op}}}(h_U, \mathcal{A})) \simeq Hom_{\mathbf{K}}(G, \mathcal{A}(U)).
$$

■

B. Grothendieck topologies and (pre)sheaves

B.1. Definition. *Let $\mathbf{C}$ be a category. A **sieve** R over $U \in \mathbf{C}$ is a subfunctor $R \subseteq h_U$ of*

$$
h_U = Hom_{\mathbf{C}}(\_, U) : \mathbf{C}^{op} \longrightarrow \mathbf{Set}.
$$

B.2. Remark. Compare with [Kashiwara and Schapira, 2006, Definition 16.1.1].

B.3. Definition. *A Grothendieck site (or simply a **site**) X is a pair $(\mathbf{C}_X, Cov(X))$ where $\mathbf{C}_X$ is a category, and*

$$
Cov(X) = \bigcup_{U \in \mathbf{C}_X} Cov(U),
$$

*where $Cov(U)$ are the sets of **covering sieves** over U, satisfying the axioms GT1-GT4 from [Kashiwara and Schapira, 2006, Definition 16.1.2], or, equivalently, the axioms T1-T3 from [Artin et al., 1972, Definition II.1.1]. The site is called **small** iff $\mathbf{C}_X$ is a small category.*

B.4. Remark. The class (or a set, if X is small) $Cov\,(X)$ is called the **topology** on X.

B.5. Notation. Given $U \in \mathbf{C}_X$, and $R \in Cov\,(X)$, denote simply

$$\mathbf{C}_U := (\mathbf{C}_X)_U\,,\ \mathbf{C}_R := (\mathbf{C}_X)_R\,,$$

where $(\mathbf{C}_X)_U$ and $(\mathbf{C}_X)_R$ are the comma-categories defined earlier in Definition 1.10 and Definition 1.11.

B.6. Proposition. *Let $G \in \mathbf{K}$, and let $R \subseteq h_U$ be a sieve. Then*

1.

$$Hom_{\mathbf{K}^{\mathbf{C}^{op}}}\left(G \otimes_{\mathbf{Set}} R, \mathcal{A}\right) \simeq Hom_{\mathbf{K}}\left(G, Hom_{\mathbf{Set}^{\mathbf{C}^{op}}}\left(R, \mathcal{A}\right)\right) \simeq Hom_{\mathbf{K}}\left(G, \varprojlim_{(V\to U)\in\mathbf{C}_R} \mathcal{A}\left(V\right)\right).$$

2.

$$Hom_{\mathbf{Set}^{\mathbf{C}^{op}}}\left(R, \mathcal{A}\right) \simeq \varprojlim_{(V\to U)\in\mathbf{C}_R} \mathcal{A}\left(V\right).$$

Proof.

1. It follows from Proposition A.7 and [Artin et al., 1972, Corollary I.3.5], that, naturally in $G \in \mathbf{K}$,

$$\begin{aligned}
&Hom_{\mathbf{K}^{\mathbf{C}^{op}}}\left(G \otimes_{\mathbf{Set}} R, \mathcal{A}\right) \simeq Hom_{\mathbf{K}}\left(G, Hom_{\mathbf{Set}^{\mathbf{C}^{op}}}\left(R, \mathcal{A}\right)\right) \\
&\simeq Hom_{\mathbf{Set}^{\mathbf{C}^{op}}}\left(R, Hom_{\mathbf{K}}\left(G, \mathcal{A}\right)\right) \simeq \varprojlim_{(V\to U)\in\mathbf{C}_R} Hom_{\mathbf{K}}\left(G, \mathcal{A}\right)\left(V\right) \\
&\simeq \varprojlim_{(V\to U)\in\mathbf{C}_R} Hom_{\mathbf{K}}\left(G, \mathcal{A}\left(V\right)\right) \simeq Hom_{\mathbf{K}}\left(G, \varprojlim_{(V\to U)\in\mathbf{C}_R} \mathcal{A}\left(V\right)\right).
\end{aligned}$$

2. Let

$$K = Hom_{\mathbf{Set}^{\mathbf{C}^{op}}}\left(R, \mathcal{A}\right) \in \mathbf{K},\ L = \varprojlim_{(V\to U)\in\mathbf{C}_R} \mathcal{A}\left(V\right) \in \mathbf{K}.$$

   We have just proved that $h_K \simeq h_L \in \mathbf{Set}^{\mathbf{K}^{op}}$. It follows from Yoneda's Lemma that $K \simeq L$.

∎

B.7. Definition. *We say that the topology on a small site X is induced by a* ***pretopology*** *if each object $U \in \mathbf{C}_X$ is supplied with* ***covers*** *$\{U_i \to U\}_{i\in I}$, satisfying [Artin et al., 1972, Definition II.1.3] (compare to [Kashiwara and Schapira, 2006, Definition 16.1.5]), and the covering sieves $R \in Cov\,(X)$ are* ***generated*** *by covers:*

$$R = R_{\{U_i \to U\}} \subseteq h_U,$$

where $R_{\{U_i \to U\}}\,(V)$ consists of morphisms $(V \to U) \in h_U\,(V)$ admitting a decomposition

$$(V \to U) = (V \to U_i \to U)\,.$$

B.8. Remark. We use the word **covers** for general sites, and reserve the word **coverings** for open coverings of topological spaces.

B.9. Example. Let X be a topological space. We will call the site $OPEN\,(X)$ below the **standard site** for X:

$$OPEN\,(X) = \left(\mathbf{C}_{OPEN(X)}, Cov\,(OPEN\,(X))\right).$$

$\mathbf{C}_{OPEN(X)}$ has open subsets of X as objects and inclusions $U \subseteq V$ as morphisms. The pretopology on $OPEN\,(X)$ consists of open coverings

$$\{U_i \subseteq U\}_{i\in I} \in \mathbf{C}_{OPEN(X)}.$$

The corresponding topology consists of sieves $R_{\{U_i\subseteq U\}} \subseteq h_U$ where

$$(V \subseteq U) \in R_{\{U_i\subseteq U\}}\,(U) \iff \exists i \in I\ \ (V \subseteq U_i)\,.$$

B.10. Remark. We will often denote the standard site simply by $X = (\mathbf{C}_X, Cov\,(X))$.

B.11. Example. Let again X be a topological space. Consider the site

$$NORM\,(X) = \left(\mathbf{C}_{NORM(X)}, Cov\,(NORM\,(X))\right)$$

where $\mathbf{C}_{NORM(X)} = \mathbf{C}_X$, while the pretopology on $NORM\,(X)$ consists of **normal** (Definition 1.25) coverings $\{U_i \subseteq U\}$.

B.12. Example. Let again X be a topological space. Consider the site

$$FINITE\,(X) = \left(\mathbf{C}_{FINITE(X)}, Cov\,(FINITE\,(X))\right)$$

where $\mathbf{C}_{FINITE(X)} = \mathbf{C}_X$, while the pretopology on $FINITE\,(X)$ consists of **finite** normal coverings $\{U_i \subseteq U\}$.

B.13. Example. Let G be a topological group, and X be a G-space. The corresponding site $OPEN_G\,(X)$ has G-invariant open subsets of X as objects of $\mathbf{C}_{OPEN_G(X)}$ and the pretopology consisting of G-invariant open coverings (compare to [Artin, 1962, Example 1.1.4], or [Tamme, 1994, Example (1.3.2)]).

B.14. Example. Let X be a noetherian scheme, and define the site X^{et} by: $\mathbf{C}_{X^{et}}$ is the category of schemes Y/X étale, finite type, while the pretopology on X^{et} consists of finite surjective families of maps. See [Artin, 1962, Example 1.1.6], or [Tamme, 1994, II.1.2].

B.15. Definition. *Let $X = (\mathbf{C}_X, Cov(X))$ be a small site, and let $\mathbf{K}$ be a complete category.*

1. *A **presheaf** $\mathcal{A}$ on X with values in $\mathbf{K}$ is a functor $\mathcal{A} : (\mathbf{C}_X)^{op} \to \mathbf{K}$.*
2. *A presheaf $\mathcal{A}$ is **separated** provided that for any $U \in \mathbf{C}_X$*
$$\mathcal{A}(U) = Hom_{\mathbf{Set}^{(\mathbf{C}_X)^{op}}}(h_U, \mathcal{A}) \longrightarrow Hom_{\mathbf{Set}^{(\mathbf{C}_X)^{op}}}(R, \mathcal{A})$$
is a monomorphism for any covering sieve R over U.
3. *A presheaf $\mathcal{A}$ is a **sheaf** provided that for any $U \in \mathbf{C}_X$*
$$\mathcal{A}(U) = Hom_{\mathbf{Set}^{(\mathbf{C}_X)^{op}}}(h_U, \mathcal{A}) \longrightarrow Hom_{\mathbf{Set}^{(\mathbf{C}_X)^{op}}}(R, \mathcal{A})$$
is an isomorphism for any covering sieve R over U.

The pairing $Hom_{\mathbf{Set}^{(\mathbf{C}_X)^{op}}}$ is introduced in Definition A.4.

B.16. Notation. Let $\mathbf{pS}(X, \mathbf{K}) = \mathbf{K}^{(\mathbf{C}_X)^{op}}$ be the category of presheaves on X with values in $\mathbf{K}$, and let $\mathbf{S}(X, \mathbf{K})$ be the full subcategory of sheaves.

B.17. Proposition. *Let X be a small site with a pretopology (Definition B.7). Then for any presheaf $\mathcal{A}$ with values in a complete category $\mathbf{K}$, and for any sieve R generated by a cover $\{U_i \to U\}_{i \in I}$,*
$$Hom_{\mathbf{Set}^{(\mathbf{C}_X)^{op}}}(R, \mathcal{A}) \simeq \varprojlim_{(V \to U) \in \mathbf{C}_R} \mathcal{A}(V) \simeq \ker\left(\prod_{i \in I} \mathcal{A}(U_i) \rightrightarrows \prod_{i,j \in I} \mathcal{A}(U_i \times_U U_j) \right).$$

Proof. Apply [Artin et al., 1972, Proposition I.2.12] to the presheaves of **sets** $Hom_{\mathbf{K}}(G, \mathcal{A})$, where G runs over objects of $\mathbf{K}$. ∎

B.18. Definition. *Let X be a small site, and $\mathbf{K}$ be a category (complete and closed under filtered limits). Let $\mathcal{A}$ be a presheaf.*

1. *Define a presheaf $(\mathcal{A})^+_{\mathbf{K}}$ (or simply $\mathcal{A}^+$) by the following:*
$$\mathcal{A}^+(U) = \varinjlim_{R} \varprojlim_{(V \to U) \in \mathbf{C}_R} \mathcal{A}(V) = \varinjlim_{R} Hom_{\mathbf{Set}^{(\mathbf{C}_X)^{op}}}(R, \mathcal{A}),$$
where $R \subseteq h^U$ runs over all covering sieves over U.

2. *Let*

$$\lambda(U) = \lambda_{R'} \circ \lambda_{U,R'} : \mathcal{A}(U) \longrightarrow \mathcal{A}^{+}(U)$$

be the composition of canonical morphisms (not depending on R')

$$\mathcal{A}(U) \xrightarrow{\lambda_{U,R'}} \varprojlim_{(V\to U)\in\mathbf{C}_{R'}} \mathcal{A}(V) \xrightarrow{\lambda_{R'}} \varinjlim_{R} \varprojlim_{(V\to U)\in\mathbf{C}_{R}} \mathcal{A}(V) = \mathcal{A}^{+}(U).$$

The family $(\lambda(U))_{U\in\mathbf{C}_X}$ defines the morphism of functors

$$\lambda : \mathbf{1}_{\mathbf{pS}(X,\mathbf{K})} \longrightarrow ()^{+} : \mathbf{pS}(X,\mathbf{K}) \longrightarrow \mathbf{pS}(X,\mathbf{K}).$$

B.19. Proposition. *If the topology is induced by a pretopology (Definition B.7), then*

$$\mathcal{A}^{+}(U) = \varinjlim_{\{U_i\to U\}} \ker\left(\prod_{i}\mathcal{A}(U_i) \rightrightarrows \prod_{i,j}\mathcal{A}(U_i\times_U U_j)\right)$$

where $\{U_i\to U\}$ runs over the covers of U.

Proof. Follows from Proposition B.17. ■

B.20. Proposition.

1. *If $G\in\mathbf{K}$ is a finitely presentable object, then there is a natural isomorphism*

$$Hom_{\mathbf{K}}\left(G,\mathcal{A}^{+}\right) \simeq Hom_{\mathbf{K}}(G,\mathcal{A})^{+}.$$

2. *If $\mathfrak{G}\subseteq\mathbf{K}$ is a* ***strong generator*** *[Adámek and Rosický, 1994, Definition 0.6], then:*

   (a) *$\mathcal{A}\in\mathbf{pS}(X,\mathbf{K})$ is separated iff $Hom_{\mathbf{K}}(G,\mathcal{A})\in\mathbf{pS}(X,\mathbf{Set})$ is separated for any $G\in\mathfrak{G}$.*

   (b) *$\mathcal{A}\in\mathbf{pS}(X,\mathbf{K})$ is a sheaf iff $Hom_{\mathbf{K}}(G,\mathcal{A})\in\mathbf{pS}(X,\mathbf{Set})$ is a sheaf for any $G\in\mathfrak{G}$.*

Proof.

1. If $G\in Pres_{\aleph_0}\mathbf{K}$ (see Remark 1.16), then the functor $Hom_{\mathbf{K}}(G,\_)$ commutes with directed colimits and arbitrary limits. Therefore, $Hom_{\mathbf{K}}(G,\mathcal{A})^{+}\simeq Hom_{\mathbf{K}}(G,\mathcal{A}^{+})$.

2. $Hom_{\mathbf{K}}(G,\_)$ commutes with arbitrary limits. Therefore, for any covering sieve $R\subseteq h_U$,

$$Hom_{\mathbf{K}}\left(G, \varprojlim_{(V\to U)\in\mathbf{C}_R}\mathcal{A}(V)\right) \simeq \varprojlim_{(V\to U)\in\mathbf{C}_R} Hom_{\mathbf{K}}(G,\mathcal{A}(V)).$$

The morphism

$$\mathcal{A}(U) \longrightarrow \varprojlim_{(V \to U) \in \mathbf{C}_R} \mathcal{A}(V)$$

is a monomorphism (respectively, an isomorphism) iff

$$Hom_{\mathbf{K}}(G, \mathcal{A}(U)) \longrightarrow \varprojlim_{(V \to U) \in \mathbf{C}_R} Hom_{\mathbf{K}}(G, \mathcal{A}(V))$$

is a monomorphism (respectively, an isomorphism) for any $G \in \mathfrak{G}$.

■

B.21. THEOREM. *Assume that* $\mathbf{K}$ *is a finitely presentable category (Definition 1.14). Let*

$$(\lambda(\mathcal{A}) : \mathcal{A} \to \mathcal{A}^{+})_{\mathcal{A} \in \mathbf{pS}(X, \mathbf{K})}$$

be the canonical morphism of functors

$$\lambda : \mathbf{1}_{\mathbf{pS}(X,\mathbf{K})} \longrightarrow ()^{+} : \mathbf{pS}(X, \mathbf{K}) \longrightarrow \mathbf{pS}(X, \mathbf{K})$$

from Definition B.18. Then:

1. *The functor* $()^{+}$ *is left exact.*
2. *For any presheaf* $\mathcal{A}$, $\mathcal{A}^{+}$ *is a separated presheaf.*
3. *A presheaf* $\mathcal{A}$ *is separated iff* $\lambda(\mathcal{A})$ *is a monomorphism. In that case* $\mathcal{A}^{+}$ *is a sheaf.*
4. *The following conditions are equivalent:*
   (a) $\lambda(\mathcal{A})$ *is an isomorphism.*
   (b) $\mathcal{A}$ *is a sheaf.*
5. *The functor* $()^{\#}_{\mathbf{K}} = ()^{++}_{\mathbf{K}}$ *is left adjoint to the inclusion*

$$i_{X,\mathbf{K}} : \mathbf{S}(X, \mathbf{K}) \hookrightarrow \mathbf{pS}(X, \mathbf{K}).$$

PROOF. Let $Pres_{\aleph_0}\mathbf{K}$ be a set of representatives for the isomorphism classes of finitely presentable objects of $\mathbf{K}$ (see Remark 1.16). This set forms a **strong generator** [Adámek and Rosický, 1994, Theorem 1.20], for $\mathbf{K}$.

1. The functor $\mathcal{A} \mapsto \mathcal{A}^{+}$ is the composition of a limit $\varprojlim_{(V \to U) \in \mathbf{C}_R} \mathcal{A}(V)$ which commutes with arbitrary limits, and a directed colimit $\varinjlim_{R \subseteq h_U}$ which commutes with **finite** limits [Adámek and Rosický, 1994, Proposition 1.59]. Therefore, $()^{+}$ is left exact (commutes with finite limits).

2. Due to [Artin et al., 1972, Proposition II.3.2], $Hom_{\mathbf{K}}(G, \mathcal{A}^+)$ is separated for any $G \in Pres_{\aleph_0}\mathbf{K}$. Apply Proposition B.20.

3. Due to [Artin et al., 1972, Proposition II.3.2],

$$Hom_{\mathbf{K}}(G, \mathcal{A}) \longrightarrow Hom_{\mathbf{K}}\left(G, \mathcal{A}^+\right)$$

is a monomorphism iff $Hom_{\mathbf{K}}(G, \mathcal{A})$ is separated for any $G \in Pres_{\aleph_0}\mathbf{K}$. In that case $Hom_{\mathbf{K}}(G, \mathcal{A}^+)$ is a sheaf. Apply Proposition B.20.

4. Due to [Artin et al., 1972, Proposition II.3.2],

$$Hom_{\mathbf{K}}(G, \mathcal{A}) \longrightarrow Hom_{\mathbf{K}}\left(G, \mathcal{A}^+\right)$$

is an isomorphism iff $Hom_{\mathbf{K}}(G, \mathcal{A})$ is a sheaf for any $G \in Pres_{\aleph_0}\mathbf{K}$. Apply Proposition B.20.

5. We need to prove that for any sheaf $\mathcal{B}$, any morphism $\mathcal{A} \to \mathcal{B}$ has a unique decomposition

$$\mathcal{A} \longrightarrow \mathcal{A}^{++} \longrightarrow \mathcal{B}.$$

The existence is easy: since $\mathcal{B} \longrightarrow \mathcal{B}^{++}$ is an isomorphism, take the decomposition

$$\mathcal{A} \longrightarrow \mathcal{A}^{++} \longrightarrow \mathcal{B}^{++} \simeq \mathcal{B}.$$

To prove uniqueness, consider two decompositions

$$\begin{tikzcd} \mathcal{A} \arrow[r] & \mathcal{A}^{++} \arrow[r, shift left, "\alpha"] \arrow[r, shift right, "\beta"'] & \mathcal{B} \end{tikzcd}$$

and apply $Hom_{\mathbf{K}}(G, \_)$:

$$\begin{tikzcd} Hom_{\mathbf{K}}(G, \mathcal{A}) \arrow[r] & Hom_{\mathbf{K}}(G, \mathcal{A})^{++} \arrow[r, shift left, "Hom_{\mathbf{K}}(G,\alpha)"] \arrow[r, shift right, "Hom_{\mathbf{K}}(G,\beta)"'] & Hom_{\mathbf{K}}(G, \mathcal{B}) . \end{tikzcd}$$

It follows that $Hom_{\mathbf{K}}(G, \alpha) = Hom_{\mathbf{K}}(G, \beta)$ for any $G \in Pres_{\aleph_0}\mathbf{K}$, therefore $\alpha = \beta$.

∎

B.22. Theorem. *Let X be a small site, and $\mathbf{K}$ be a locally λ-presentable category. Then*

$$\mathbf{S}(X,\mathbf{K}) \subseteq \mathbf{pS}(X,\mathbf{K})$$

is a reflective subcategory.

Proof. Due to [Adámek and Rosický, 1994, Corollary 1.54], $\mathbf{pS}(X,\mathbf{K}) = \mathbf{K}^{\mathbf{C}_X}$ is a locally λ-presentable category. For each covering sieve $R \subseteq h_U$ and each $G \in \mathbf{K}$, let

$$g_{R,G} : G \otimes_{\mathbf{Set}} R \longrightarrow G \otimes_{\mathbf{Set}} h_U$$

be the corresponding morphism in $\mathbf{pS}(X,\mathbf{K})$. For a presheaf $\mathcal{A}$, apply $Hom_{\mathbf{pS}(X,\mathbf{K})}(\_,\mathcal{A})$:

$$\begin{aligned}
&Hom_{\mathbf{pS}(X,\mathbf{K})}(G \otimes_{\mathbf{Set}} R, \mathcal{A}) \simeq Hom_{\mathbf{pS}(X,\mathbf{Set})}(R, Hom_{\mathbf{Set}}(G,\mathcal{A})),\\
&Hom_{\mathbf{pS}(X,\mathbf{K})}(G \otimes_{\mathbf{Set}} h_U, \mathcal{A}) \simeq Hom_{\mathbf{K}}(G,\mathcal{A}(U)) = Hom_{\mathbf{K}}(G,\mathcal{A})(U),\\
&Hom_{\mathbf{pS}(X,\mathbf{K})}(g_{R,G},\mathcal{A}) \simeq \left(Hom_{\mathbf{K}}(G,\mathcal{A})(U) \longrightarrow Hom_{\mathbf{pS}(X,\mathbf{Set})}(R, Hom_{\mathbf{K}}(G,\mathcal{A}))\right).
\end{aligned}$$

Assume that G runs over $Pres_{\lambda}\mathbf{K}$. Then the following conditions are equivalent:

1. $Hom_{\mathbf{pS}(X,\mathbf{K})}(g_{R,G},\mathcal{A})$ is a bijection for all $g_{R,G}$.
2. $Hom_{\mathbf{K}}(G,\mathcal{A})$ is a sheaf of sets for all $G \in Pres_{\lambda}(\mathbf{K})$.
3. $\mathcal{A}$ is a sheaf.

Choose a regular cardinal $\mu \geq \lambda$ such that for all G both $G \otimes_{\mathbf{Set}} R$ and $G \otimes_{\mathbf{Set}} h_U$ are μ-presentable. It follows that $\mathbf{S}(X,\mathbf{K}) \subseteq \mathbf{pS}(X,\mathbf{K})$ is the μ-orthogonality class $\{g_{R,G}\}^{\perp}$ [Adámek and Rosický, 1994, Definition 1.35], in $\mathbf{pS}(X,\mathbf{K})$, and therefore [Adámek and Rosický, 1994, Theorem 1.39], $\mathbf{S}(X,\mathbf{K})$ is a reflective subcategory of $\mathbf{pS}(X,\mathbf{K})$. ■

(Pre)sheaves on topological spaces. Throughout this Subsection, X is a topological space considered as the site $OPEN(X)$ (see Example B.9 and Remark B.10).

B.23. Definition. *Assume that a category $\mathbf{K}$ admits filtered colimits. Let $\mathcal{A}$ be a presheaf with values in $\mathbf{K}$, and let $x \in X$. The **stalk** of $\mathcal{A}$ at x is*

$$\mathcal{A}_x := \varinjlim_{U \in J(x)} \mathcal{A}(U)$$

where $J(x)$ is the family of open neighborhoods of x.

B.24. Remark. In a situation when $K \subseteq L$ is a subcategory, and $\mathcal{A} \in \mathbf{pS}(X,\mathbf{K})$, we will use notations $(\mathcal{A})_x^{\mathbf{K}}$ and $(\mathcal{A})_x^{\mathbf{L}}$ depending on whether the colimit is taken in the category $\mathbf{K}$ or in the category $\mathbf{L}$.

B.25. Definition. *Let* $\mathbf{K}$ *admit filtered colimits, and let* $f : \mathcal{A} \to \mathcal{B}$ *be a morphism in the category of presheaves* $\mathbf{pS}(X, \mathbf{K})$*. We say that* f *is a* ***local isomorphism*** *iff* $f_x : \mathcal{A}_x \to \mathcal{B}_x$ *is an isomorphism for any* $x \in X$*. In a situation when* $\mathbf{K} \subseteq \mathbf{L}$*, and*

$$\mathcal{A}, \mathcal{B} \in \mathbf{pS}(X, \mathbf{K}) \subseteq \mathbf{pS}(X, \mathbf{L}),$$

we will say that f *is* $\mathbf{K}$*-local (respectively* $\mathbf{L}$*-local) isomorphism iff*

$$(f)_x^{\mathbf{K}} : (\mathcal{A})_x^{\mathbf{K}} \longrightarrow (\mathcal{B})_x^{\mathbf{K}}$$

(respectively $(f)_x^{\mathbf{L}} : (\mathcal{A})_x^{\mathbf{L}} \longrightarrow (\mathcal{B})_x^{\mathbf{L}}$*) is an isomorphism for any* $x \in X$*.*

B.26. Proposition. *Let* $\mathbf{K}$ *be a complete category admitting filtered colimits. Assume that* $\mathbf{S}(X, \mathbf{K}) \subseteq \mathbf{pS}(X, \mathbf{K})$ *is reflective, and the reflection is given by the functor*

$$()^{\#} : \mathbf{pS}(X, \mathbf{K}) \longrightarrow \mathbf{S}(X, \mathbf{K}).$$

Then for any presheaf $\mathcal{A}$*, the natural morphism* $\mathcal{A} \to \mathcal{A}^{\#}$ *is a local isomorphism.*

Proof. Let $x \in X$, and $G \in \mathbf{K}$. Denote by $\mathcal{P}_{x,G}$ the following **pointed** presheaf: $\mathcal{P}_{x,G}(U)$ is a terminal object T when $x \notin U$, and $\mathcal{P}_{x,G}(U) = G$ when $x \in U$. It is easy to check that $\mathcal{P}_{x,G}$ is in fact a **sheaf**, and that for any presheaf $\mathcal{C}$,

$$Hom_{\mathbf{pS}(X,\mathbf{K})}(\mathcal{C}, \mathcal{P}_{x,G}) \simeq \varprojlim_{U \in J(x)} Hom_{\mathbf{K}}(\mathcal{C}(U), G) \simeq Hom_{\mathbf{K}}(\mathcal{C}_x, G),$$

naturally in G and $\mathcal{C}$. Using the adjointness isomorphism, one gets

$$Hom_{\mathbf{K}}(\mathcal{A}_x, G) \simeq Hom_{\mathbf{pS}(X,\mathbf{K})}(\mathcal{A}, \mathcal{P}_{x,G}) \simeq Hom_{\mathbf{S}(X,\mathbf{K})}\left(\mathcal{A}^{\#}, \mathcal{P}_{x,G}\right) \simeq Hom_{\mathbf{K}}\left(\left(\mathcal{A}^{\#}\right)_x, G\right),$$

for any $G \in \mathbf{K}$. Therefore, $\mathcal{A}_x \simeq \left(\mathcal{A}^{\#}\right)_x$, as desired. ∎

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