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STRICT ω-CATEGORIES ARE MONADIC OVER POLYGRAPHS

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ABSTRACT. We give a direct proof that the category of strict ω-categories is monadic over the category of polygraphs.

Introduction

This short note presents a proof of monadicity for the adjunction between the category $\mathbf{Cat}_\omega$ of strict ω-categories and the category $\mathbf{Pol}_\omega$ of polygraphs (or computads, as first introduced by Street in [Str76]). Here we follow the presentation and terminology of [Bur93, Mét03]. The reader may consult [Mét08] for a detailed description of the categories and functors referred to in this particular case, or [Bat98] for a broader perspective including generalized "A-computads" for a monad A on globular sets. The latter paper rightly asserts the monadicity theorem, but some parts of the proof rely on the fact that the category of A-computads is a presheaf category, which is precisely not true in the present case, where A is the monad of strict ω-categories [MZ08, Che13]. Since then, the status of monadicity for $\mathbf{Cat}_\omega$ has remained somewhat unclear (see e.g the entry "computad" on the nLab [nLa]). Our proof is based on the same ideas as developed in [Bat98], except that we avoid the presheaf argument and establish instead a lifting result (Lemma 2.1), possibly of independent interest.

As for notations, whenever a functor F is a right-adjoint, we denote its left-adjoint by F^*. Let us finally mention a small point about terminology. Given a functor $F : \mathbf{A} \to \mathbf{B}$, with left-adjoint F^*, and $T = FF^*$ the associated monad on $\mathbf{B}$, there is a comparison functor K from $\mathbf{A}$ to the category $\mathbf{B}^T$ of T-algebras: we call F *monadic* if K is an equivalence of categories, and *strictly monadic* if K is an isomorphism. We refer to [ML71, VI.7] for corresponding variants of Beck's monadicity criterion.

1. Three adjunctions

In this section, we briefly describe three pairs of adjoint functors between categories $\mathbf{Glob}_\omega$ of globular sets, $\mathbf{Cat}_\omega$ of strict ω-categories and $\mathbf{Pol}_\omega$ of polygraphs.

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As ω-categories are globular sets with extra structure, there is an obvious forgetful functor

$$U : \mathbf{Cat}_\omega \to \mathbf{Glob}_\omega$$

This functor U has a left-adjoint U^* taking a globular set X to the ω-category U^*X it generates. Moreover this adjunction is strictly monadic.

A second adjunction involves functors V, V^* between $\mathbf{Cat}_\omega$ and $\mathbf{Pol}_\omega$. Unlike U, the right adjoint V is not quite obvious. Thus, let C be an ω-category, the polygraph $P = V(C)$ is defined by induction, together with a morphism $\epsilon^C : V^*(P) \to C$:

- For $n = 0$, $P_0 = C_0$ and ϵ^C_0 is the identity.
- Suppose $n > 0$, and P, ϵ^C have been defined up to dimension $n-1$. The set of n-generators of P is then the set P_n of triples $p = (z, x, y)$ where $z \in C_n$, x, y are parallel cells in P^*_{n-1} and $z : \epsilon^C_{n-1}(x) \to \epsilon^C_{n-1}(y)$. The source and target of p in P^*_{n-1} are $x = s_{n-1}(p)$ and $y = t_{n-1}(p)$ respectively, and $\epsilon^C_n(p) = z$. By the universal property of polygraphs, ϵ^D_n extends uniquely to a map from P^*_n to C_n preserving compositions and identities. Functoriality of V is immediate and V is in fact right-adjoint to V^* (see [Bat98, Mét03]).

Note that

$$\epsilon^C : V^*V(C) \to C$$

is the counit of this adjunction and determines the standard polygraphic resolution of C.

We finally describe a functor

$$G : \mathbf{Pol}_\omega \to \mathbf{Glob}_\omega$$

Let P be a polygraph. Let us denote by $j_n : P_n \to P^*_n$ the canonical inclusion of the set of n-generators of P into the set of n-cells of $P^* = V^*(P)$. We define the globular set $X = G(P)$ dimensionwise, so that for each $n \in \mathbb{N}$, $X_n \subset P_n$:

- For $n = 0$, $X_0 = P_0$.
- Let $n > 0$ and suppose we have defined $X_k \subset P_k$ for all $k < n$, together with source and target maps building an $n-1$-globular set. Let $X_n \subset P_n$ be the set of n-generators a of P such that $s_{n-1}(a)$ and $t_{n-1}(a)$ belong to $j_{n-1}(X_{n-1})$ and define source and target maps $s^X_{n-1}, t^X_{n-1} : X_n \to X_{n-1}$ as the unique maps such that $j_{n-1}s^X_{n-1}(a) = s_{n-1}(a)$ and $j_{n-1}t^X_{n-1}(a) = t_{n-1}(a)$ for each $a \in X_n$. This extends X to an n-globular set.

$$\begin{array}{ccccc} X_n & \hookrightarrow & P_n & & \\ {\scriptstyle s^X_{n-1}}\downarrow\downarrow{\scriptstyle t^X_{n-1}} & & & \searrow\searrow^{t_{n-1}}_{s_{n-1}} & \\ X_{n-1} & \hookrightarrow & P_{n-1} & \xrightarrow[j_{n-1}]{} & P^*_{n-1} \end{array} \qquad (1)$$

The previous construction is clearly functorial and defines the required functor G. Remark that G admits a left adjoint $G^* : \mathbf{Glob}_\omega \to \mathbf{Pol}_\omega$ which takes the globular set X to a polygraph P such that $P_n = X_n$, in other words G^* defines a natural inclusion of $\mathbf{Glob}_\omega$ into $\mathbf{Pol}_\omega$.

Note that G forgets all generators of P that are not "hereditary globular", so that for instance $G(P)$ may have no cells at all beyond dimension 1. However, the following result shows that the functor G is not always trivial.

1.1. Lemma. *There is a natural isomorphism $\phi : GV \to U$, that is, the following diagram commutes up to a natural isomorphism*

$$\begin{array}{ccc} \mathbf{Cat}_\omega & \xrightarrow{V} & \mathbf{Pol}_\omega \\ {\scriptstyle U}\downarrow & \swarrow {\scriptstyle G} & \\ \mathbf{Glob}_\omega & & \end{array} \tag{2}$$

Proof. Let C be an ω-category, and $X = GV(C)$. For each $n \in \mathbb{N}$, let $\phi_n^C : X_n \to C_n$ be the composition of the following maps

$$X_n \hookrightarrow V(C)_n \xrightarrow{j_n} V^*V(C)_n \xrightarrow{\epsilon_n^C} C_n$$

As ϵ^C is an ω-morphism and (1) commutes, the family $(\phi_n^C)_{n\in\mathbb{N}}$ defines a globular morphism $\phi^C : GV(C) \to U(C)$, natural in C. Thus we get a natural transformation $\phi : GV \to U$.

Let us now define $\chi_n^C : C_n \to X_n$ by induction on n such that $\phi_n^C \circ \chi_n^C = 1_{C_n}$:

- For $n = 0$, $X_0 = C_0$ and $\phi_0^C = 1_{C_0} = 1_{X_0}$, so that $\chi_0^C : C_0 \to X_0$ is also $1_{C_0} = 1_{X_0}$.
- Suppose $n > 0$ and χ_k^C has been defined up to $k = n-1$, and let $z \in C_n$. Let $u = s_{n-1}(z)$ and $v = t_{n-1}(z)$ in C_{n-1}. By induction hypothesis, $\chi_{n-1}^C(u)$ and $\chi_{n-1}^C(v)$ belong to X_{n-1}. Let $x = j_{n-1}\chi_{n-1}^C(u)$, $y = j_{n-1}\chi_{n-1}^C(v)$ in $V^*V(C)_{n-1}$ and define $a = \chi_n^C(z) = (z, x, y)$. By construction $a \in X_n$ and $\phi_n^C(a) = z$.

It remains to prove that ϕ_n^C is injective. We reason again by induction on n:

- For $n = 0$, ϕ_0^C is an identity, hence injective.
- Suppose $n > 0$ and ϕ_{n-1}^C injective. Let $a_i = (z_i, x_i, y_i) \in X_n$ for $i = 0, 1$ such that $\phi_n^C(a_0) = \phi_n^C(a_1)$. Thus $z_0 = z_1$. Also

$$\begin{aligned} \phi_{n-1}^C(s_{n-1}^X(a_0)) &= s_{n-1}(\phi_n^C(a_0)) \\ &= s_{n-1}(\phi_n^C(a_1)) \\ &= \phi_{n-1}^C(s_{n-1}^X(a_1)) \end{aligned}$$

and because ϕ_{n-1}^C is injective,

$$s_{n-1}^X(a_0) = s_{n-1}^X(a_1)$$

Now

$$
\begin{aligned}
x_0 &= s_{n-1}(a_0) \\
&= j_{n-1}s^X_{n-1}(a_0) \\
&= j_{n-1}s^X_{n-1}(a_1) \\
&= s_{n-1}(a_1) \\
&= x_1
\end{aligned}
$$

Likewise $y_0 = y_1$, and we get $a_0 = a_1$. Hence ϕ^C_n is injective and we are done.

■

2. Lifting lemma

The forgetful functor $U : \mathbf{Cat}_\omega \to \mathbf{Glob}_\omega$ is faithful, but clearly not full. However, globular morphisms lift to ω-morphisms in the sense of the following result:

2.1. LEMMA. *Let C, D be ω-categories and $\alpha : U(C) \to U(D)$ be a globular morphism. Then there is a unique morphism $\overline{\alpha} : V(C) \to V(D)$ in $\mathbf{Pol}_\omega$ such that the following square commutes:*

$$
\begin{array}{ccc}
UV^*V(C) & \xrightarrow{UV^*(\overline{\alpha})} & UV^*V(D) \\
\downarrow{\scriptstyle U(\epsilon^C)} & & \downarrow{\scriptstyle U(\epsilon^D)} \\
U(C) & \xrightarrow[\alpha]{} & U(D)
\end{array}
\qquad (3)
$$

PROOF. We build the required morphism $\overline{\alpha} : V(C) \to V(D)$ by induction on the dimension. Note that diagram (3) yields a diagram in **Sets** at any given dimension n. We may therefore drop the letter U in the following computations. Also $\overline{\alpha}^*$ is short for $V^*(\overline{\alpha})$.

- For $n = 0$, we have $V(C)_0 = C_0$, $V(D)_0 = D_0$; also ϵ^C_0 and ϵ^D_0 are identities, so that $\overline{\alpha}_0 = \alpha_0$ is the unique solution.

- Suppose $n > 0$ and we have defined $\overline{\alpha}$ satisfying the commutation condition, up to dimension $n-1$. Let $p = (z, x, y)$ be an n-generator of $V(C)$. Suppose $\overline{\alpha}(p) = (z', x', y')$: the commutation condition implies $z' = \alpha(z)$, $x' = \overline{\alpha}^*_{n-1}(x)$ and $y' = \overline{\alpha}^*_{n-1}(x)$, so that $\overline{\alpha}$ extends in at most one way to dimension n, and uniqueness holds. As for the existence, x, y are parallel $(n-1)$-cells in $V^*V(C)_{n-1}$; by induction hypothesis, their images $x' = \overline{\alpha}^*_{n-1}(x)$ and $y' = \overline{\alpha}^*_{n-1}(x)$ are $(n-1)$-parallel cells in $V^*V(D)$. Again, by induction hypothesis, (3) commutes in dimension $n-1$; also α

is a globular map, hence

$$\begin{aligned} s_{n-1}(z') &= s_{n-1}(\alpha_n(z)) \\ &= \alpha_{n-1}(s_{n-1}(z)) \\ &= \alpha_{n-1}(\epsilon^C_{n-1}(x)) \\ &= \epsilon^D_{n-1}(\overline{\alpha}^*_{n-1}(x)) \\ &= \epsilon^D_{n-1}(x') \end{aligned}$$

and likewise

$$t_{n-1}(z') = \epsilon^D_{n-1}(y')$$

Therefore $p' = (z', x', y')$ is an n-generator of $V(D)$. Also $s_{n-1}(p') = x' = \overline{\alpha}^*_{n-1}(x) = \overline{\alpha}^*_{n-1}(s_{n-1}(p))$ and $t_{n-1}(p') = y' = \overline{\alpha}^*_{n-1}(y) = \overline{\alpha}^*_{n-1}(t_{n-1}(p))$, so that $\overline{\alpha}$ extends to a morphism in $\mathbf{Pol}_\omega$ up to dimension n. Finally the diagram (3) commutes in dimension n : it is sufficient to check this on generators, but

$$\begin{aligned} \epsilon^D_n \overline{\alpha}^*_n(p) &= \epsilon^D_n(p') \\ &= z' \\ &= \alpha_n(z) \\ &= \alpha_n \epsilon^C_n(p) \end{aligned}$$

and we are done.

∎

3. Monadicity

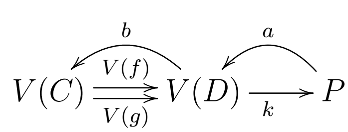
We now turn to the main result.

3.1. Theorem. *The functor* $V : \mathbf{Cat}_\omega \to \mathbf{Pol}_\omega$ *is monadic.*

Proof. Recall that monadicity means here that $\mathbf{Cat}_\omega$ is *equivalent* to the category of algebras of the monad VV^* on $\mathbf{Pol}_\omega$. By using the corresponding version of Beck's criterion, this amounts to show that (i) V reflects isomorphisms and (ii) if f, g is a parallel pair of ω-morphisms such that the pair $V(f)$, $V(g)$ has a split coequalizer in $\mathbf{Pol}_\omega$, then f, g has a coequalizer in $\mathbf{Cat}_\omega$, and V preserves coequalizers of such pairs (see for instance [ML71, VI.7, exercises 3 and 6]).

First, if $f : C \to D$ is an ω-morphism such that $V(f)$ is an isomorphism, then $GV(f)$ is an isomorphism in $\mathbf{Glob}_\omega$ and by Lemma 1.1, $U(f)$ is an isomorphism. Now, U reflects isomorphisms, hence f is an isomorphism. Therefore V reflects isomorphisms as required.

Now, let $f, g : C \to D$ be a pair of ω-morphisms and suppose



(4)

is a split coequalizer in $\mathbf{Pol}_\omega$ where $k \circ a = 1_P$, $V(f) \circ b = 1_{V(D)}$ and $V(g) \circ b = a \circ k$. By applying the functor G to (4), we get a split coequalizer in $\mathbf{Glob}_\omega$:

$$\begin{tikzcd} GV(C) \arrow[r, shift left, "GV(f)"] \arrow[r, shift right, "GV(g)"'] & GV(D) \arrow[l, bend right, "G(b)"'] \arrow[r, "G(k)"'] & G(P) \arrow[l, bend right, "G(a)"'] \end{tikzcd} \tag{5}$$

Then, by using the natural isomorphism ϕ of Lemma 1.1, we obtain the following diagram

$$\begin{tikzcd} GV(C) \arrow[r, shift left, "GV(f)"] \arrow[r, shift right, "GV(g)"'] \arrow[d, "\phi^C"'] & GV(D) \arrow[l, bend right, "G(b)"'] \arrow[d, "\phi^D"'] \arrow[dr, "G(k)"] & \\ U(C) \arrow[r, shift left, "U(f)"] \arrow[r, shift right, "U(g)"'] & U(D) \arrow[l, bend left, "\beta"] \arrow[r, "l"] & G(P) \arrow[l, bend left, "\alpha"] \arrow[ul, bend right, "G(a)"'] \end{tikzcd} \tag{6}$$

where $\alpha = \phi^D \circ G(a)$, $l = G(k) \circ (\phi^D)^{-1}$ and $\beta = \phi^C \circ G(b) \circ (\phi^D)^{-1}$. Therefore $l \circ \alpha = 1_{G(P)}$, $U(f) \circ \beta = 1_{U(D)}$ and

$$\begin{aligned} U(g) \circ \beta &= U(g) \circ \phi^C \circ G(b) \circ (\phi^D)^{-1} \\ &= \phi^D \circ GV(g) \circ G(b) \circ (\phi^D)^{-1} \\ &= \phi^D \circ G(a) \circ G(k) \circ (\phi^D)^{-1} \\ &= \alpha \circ l \end{aligned}$$

and the bottom line of (6) is a split coequalizer diagram in $\mathbf{Glob}_\omega$. Now the functor U is strictly monadic, so that there is a unique ω-morphism $h : D \to E$ such that $U(E) = G(P)$ and $U(h) = l$ and moreover this unique morphism makes

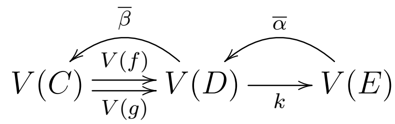
$$\begin{tikzcd} C \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & D \arrow[r, "h"] & E \end{tikzcd} \tag{7}$$

a coequalizer diagram in $\mathbf{Cat}_\omega$. Note that, by construction, $U(E) = G(P)$.

It remains to show that $V(h) : V(D) \to V(E)$ is a coequalizer of the pair $V(f)$, $V(g)$ in $\mathbf{Pol}_\omega$. By applying Lemma 2.1 to $\alpha : U(E) \to U(D)$ and to $\beta : U(D) \to U(C)$, we get unique morphisms $\overline{\alpha} : V(E) \to V(D)$ and $\overline{\beta} : V(D) \to V(C)$ satisfying the required commutation condition. Consider the following diagram:

$$\begin{tikzcd} UV^*V(E) \arrow[r, "UV^*(\overline{\alpha})"] \arrow[d, "U(\epsilon^E)"'] & UV^*V(D) \arrow[r, "UV^*V(h)"] \arrow[d, "U(\epsilon^D)"] & UV^*V(E) \arrow[d, "U(\epsilon^E)"] \\ U(E) \arrow[r, "\alpha"'] & U(D) \arrow[r, "U(h)"'] & U(E) \end{tikzcd} \tag{8}$$

The left-hand square commutes by hypothesis, and the right-hand square commutes by the naturality of ϵ, whence the outer square also commutes. As $U(h) \circ \alpha = 1_{U(E)}$, the uniqueness of the lifting in Lemma 2.1 implies that $V(h) \circ \overline{\alpha} = 1_{V(E)}$. By the same uniqueness argument, we get $V(f) \circ \overline{\beta} = 1_{V(D)}$ and $V(g) \circ \overline{\beta} = \overline{\alpha} \circ V(h)$. Therefore the following diagram is a split coequalizer in $\mathbf{Pol}_\omega$



and we are done. ■

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