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THE EULER CHARACTERISTIC OF AN ENRICHED CATEGORY

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Abstract. We develop the homotopy theory of Euler characteristic (magnitude) of a category enriched in a monoidal model category. If a monoidal model category $\mathcal{V}$ is equipped with an Euler characteristic that is compatible with weak equivalences and fibrations in $\mathcal{V}$, then our Euler characteristic of $\mathcal{V}$-enriched categories is also compatible with weak equivalences and fibrations in the canonical model structure on the category of $\mathcal{V}$-enriched categories. In particular, we focus on the case of topological categories; i.e., categories enriched in the category of topological spaces. As its application, we obtain the ordinary Euler characteristic of a cellular stratified space X by computing the Euler characteristic of the face category $C(X)$.

1. Introduction

The Euler characteristic of a topological space is a classical homotopy invariant. However, the Euler characteristic is defined not only for topological spaces, but also for finite posets [Rot64], groupoids [BD01], and categories [Lei08], [BL08], [FLS11], [Nog11], [Nog13]. Moreover, Leinster defined an invariant for categories enriched in a monoidal category, called *magnitude* [Lei13]. Our work in this paper is based on magnitude, so we give a quick review of it here.

Let k be a rig (a ring without negatives). For finite sets I and J, an $I \times J$ *matrix* over k is a function $I \times J \to k$. For an $I \times J$ matrix ζ and a $J \times H$ matrix ξ, the $I \times H$ matrix $\zeta\xi$ is defined by $\zeta\xi(i,h) = \sum_j \zeta(i,j)\xi(j,h)$ for any i in I and h in H. An $I \times J$ matrix ζ has a $J \times I$ transpose ζ^*. Given a finite set I, we write $u_I : I \to k$ (or simply u) for the column vector with $u_I(i) = 1$ for all i in I. Let ζ be an $I \times J$ matrix over k. A *weighting* on ζ is a column vector $w : J \to k$ such that $\zeta w = u_I$. A *coweighting* on ζ is a row vector $v : I \to k$ such that $v\zeta = u_J^*$. The matrix ζ *has magnitude* if it has a weighting w and a coweighting v. Its *magnitude* is then

$$|\zeta| = \sum_j w(j) = \sum_i v(i) \in k.$$

This definition does not depend on the choice of a weighting and a coweighting.

Let $\mathcal{V}$ be a monoidal category, and let

$$|-| : (\mathrm{ob}(\mathcal{V})/\cong, \otimes, \mathbf{1}) \longrightarrow (k, \cdot, 1)$$

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be a monoid homomorphism where $\mathrm{ob}(\mathcal{V})$ is the collection of objects of $\mathcal{V}$ and $\mathrm{ob}(\mathcal{V})/\cong$ denotes the isomorphism classes of objects of $\mathcal{V}$. For a $\mathcal{V}$-category $\mathcal{A}$ having finitely many objects, the *similarity matrix* of $\mathcal{V}$ is the $\mathrm{ob}(\mathcal{A}) \times \mathrm{ob}(\mathcal{A})$ matrix $\xi_{\mathcal{A}}$ over k defined by $\xi_{\mathcal{A}}(a, b) = |\mathcal{A}(a, b)|$ for any objects a and b of $\mathcal{A}$. If $\xi_{\mathcal{A}}$ has magnitude, then $\mathcal{V}$-category $\mathcal{A}$ *has magnitude*; its magnitude is then $|\mathcal{A}| = |\xi_{\mathcal{A}}|$. In particular, Leinster studied the magnitude of finite metric spaces. A metric space can be regarded as a category enriched in the poset of non-negative extended real numbers $[0, \infty]$. See [Lei13] for more details.

In this paper, we consider the case in which a monoidal category is equipped with a model structure. A *model structure* on a category consists of three classes of morphisms, called *weak equivalences, fibrations, and cofibrations*, satisfying certain conditions, and this provides a framework to do homotopy theory [Qui67]. If a monoidal model category $\mathcal{V}$ satisfies certain conditions, a model structure is induced on the category $\mathcal{V}$-**Cat** of categories enriched in $\mathcal{V}$, called the *canonical model structure* [BM13]. Suppose that

$$|-| : (\mathrm{ob}(\mathcal{V})/\cong, \otimes, \mathbf{1}) \longrightarrow (k, \cdot, 1)$$

is a monoid homomorphism compatible with weak equivalences and fibrations in $\mathcal{V}$; i.e., $|-|$ is an invariant with respect to weak equivalences, and satisfies the product formula with respect to fibrations. This is a natural assumption for topologists when we regard $|-|$ as the standard topological Euler characteristic. The topological Euler characteristic χ is an invariant with respect to weak homotopy equivalences, and satisfies the product formula $\chi(E) = \chi(B)\chi(F)$ for a (Serre) fibration $E \to B$ with fiber F and connected base B. By following Leinster's work, we can define magnitude of $\mathcal{V}$-categories induced by the monoid homomorphism $|-|$. In this paper, we call it *Euler characteristic* when $\mathcal{V}$ is a monoidal model category, and a monoid homomorphism $|-|$ is compatible with weak equivalences and fibrations. Then, one can ask whether the induced invariant is compatible with weak equivalences and fibrations in $\mathcal{V}$-**Cat**. The following is a positive answer to the question:

1.1. Theorem. *Suppose that the category of $\mathcal{V}$-enriched categories admits the canonical model structure and $\chi(-)$ is the Euler characteristic of $\mathcal{V}$-enriched categories.*

1. *If $\mathcal{A}$ and $\mathcal{B}$ are weakly equivalent in $\mathcal{V}$-**Cat**, then the Euler characteristics of $\mathcal{A}$ and $\mathcal{B}$ are equal; i.e., $\chi(\mathcal{A}) = \chi(\mathcal{B})$.*
2. *If a $\mathcal{V}$-functor $p : \mathcal{E} \to \mathcal{B}$ is a fibration over a connected $\mathcal{V}$-category $\mathcal{B}$ in $\mathcal{V}$-**Cat** satisfying certain conditions, then χ satisfies the product formula.*

Computing the Euler characteristic of cellular stratified spaces is an application of the Euler characteristic of $\mathcal{V}$-categories. A *cellular stratified space* is a generalization of cell complexes, and it is introduced by Tamaki [Tama]. Although a cell complex is a space defined by attaching closed disks, a cellular stratified space is defined by attaching globular cells. A *globular n-cell* is a subspace of the closed disk D^n containing the interior of D^n. If X is a finite cell complex, the Euler characteristic $\chi(X)$ is obtained as an alternating sum of the cardinalities of the set of n-cells, however in the case of cellular stratified spaces this does not work. For example, the half-open interval $(0,1]$ is a cellular stratified space consisting of a 1-cell $(0,1]$ and a 0-cell $\{1\}$. We have $\chi((0,1]) = 1$ since $(0,1]$ is contractible. However, the alternating sum of the numbers of n-cells is $1-1=0$.

In [Tama], Tamaki shows that a nice cellular stratified space X is homotopy equivalent to the classifying space $BC(X)$ of the face category $C(X)$ enriched in the category of topological spaces. By the following theorem, we can show that the standard Euler characteristic $\chi(X)$ is equal to our Euler characteristic $\chi(C(X))$ of the topological category $C(X)$.

1.2. Theorem. *Let $\mathcal{A}$ be a finite measurable acyclic topological category (Definition 3.6 and 4.11). Then, the Euler characteristic $\chi(\mathcal{A})$ coincides with the topological Euler characteristic $\chi(B\mathcal{A})$ of the classifying space $B\mathcal{A}$ of $\mathcal{A}$.*

1.3. Corollary. *Let X be a finite locally polyhedral cellular stratified space (Definition 5.6) whose parameter spaces are finite CW-complexes. Then, we have $\chi(X) = \chi(C(X))$, where $C(X)$ is the cylindrical face category of X.*

This paper is organized as follows. In Section 2, we give a review of enriched categories and model categories including the canonical model structure on the category of enriched categories. In Section 3, we introduce the Euler characteristic of $\mathcal{V}$-enriched categories for a monoidal model category $\mathcal{V}$, and give a proof of Theorem 1.1. In Section 4, we focus on the case in which $\mathcal{V}$ is the category of topological spaces, and we prove Theorem 1.2 and Corollary 1.3.

2. Review of enriched categories and model categories

We first begin with a brief review of basic notions of model categories. It was originally introduced by Quillen in [Qui67] to do homotopy theory in general categories. See [Hov99] and [Hir03] as references about model categories.

2.1. Definition. *A* model structure *on a category M consists of three distinguished subcategories W, C, and F, called* weak equivalences*,* cofibrations*, and* fibrations*, and they satisfy the following properties:*

1. *If f and g are morphisms of M such that $g \circ f$ is defined, two of f, g, and $g \circ f$ are weak equivalences, then so is the third.*
2. *All W, C, and F are closed under retracts.*
3. *Every morphism in $W \cap C$ has the right lifting property for F, and every morphism in C has the right lifting property for $W \cap F$.*
4. *For any morphism f in M can be factored as $p \circ i$ by p in $F \cap W$ and i in C, and also f can be factored as $q \circ j$ by q in F and j in $C \cap W$.*

A morphism in $W \cap C$ is called a trivial cofibration*, and a morphism in $W \cap F$ is called a* trivial fibration*, respectively. A category M is a* model category *if it is equipped with a model structure on M and closed under all small limits and colimits.*

2.2. Definition. *Let M be a model category. The* homotopy category $\mathrm{Ho}(M)$ *is the category consisting of* $\mathrm{ob}(\mathrm{Ho}(M)) = \mathrm{ob}(M)$ *and* $\mathrm{Ho}(M)(X, Y) = M(QRX, QRY)/\simeq$*, where QR is the cofibrant and fibrant replacement and $\simeq$ is the homotopy relation.*

Generally, the homotopy category $\mathrm{Ho}(M)$ is defined as the localization

$$\ell : M \longrightarrow M[W^{-1}] = \mathrm{Ho}(M)$$

with respect to the class of weak equivalences W. It is determined uniquely up to equivalence of categories.

2.3. Remark. [Theorem 1.2.10 of [Hov99]] If X is cofibrant and Y is fibrant in a model category M, then there exists a natural isomorphism $\mathrm{Ho}(M)(X, Y) \cong M(X, Y)/\simeq$.

Next, we review of basic notions of enriched categories. Let $(\mathcal{V}, \otimes, \mathbf{1})$ denote a monoidal category throughout this paper. A $\mathcal{V}$-enriched category is a generalization of a category using hom-objects in $\mathcal{V}$ instead of hom-sets. For more details, see [Kel05].

2.4. Definition. *A* $\mathcal{V}$-enriched category*, or simply, a* $\mathcal{V}$-category $\mathcal{A}$ *consists of a set of objects* $\mathrm{ob}(\mathcal{A})$ *and a* hom-object $\mathcal{A}(a, b)$ *of $\mathcal{V}$ for each pair of objects a and b of $\mathcal{A}$ with composition morphisms $\circ : \mathcal{A}(b, c) \otimes \mathcal{A}(a, b) \to \mathcal{A}(a, c)$, and the identity $\mathbf{1} \to \mathcal{A}(d, d)$ for each object a, b, c, d in $\mathcal{A}$, satisfying two coherence conditions with respect to associativity and unitality of composition. We call $\mathcal{A}$* finite *if* $\mathrm{ob}(\mathcal{A})$ *is finite.*

A $\mathcal{V}$-functor $f : \mathcal{A} \to \mathcal{B}$ *between two $\mathcal{V}$-categories $\mathcal{A}$ and $\mathcal{B}$ consists of a map f :* $\mathrm{ob}(\mathcal{A}) \to \mathrm{ob}(\mathcal{B})$ *on objects and a morphism $\mathcal{A}(a, b) \to \mathcal{B}(f(a), f(b))$ in $\mathcal{V}$ which preserves the composition and identities for each pair of objects a, b in* $\mathrm{ob}(\mathcal{A})$.

The following model structure on the category of small categories is called the folk model structure (see [JT91]). It is closely related to the model structure on the category of $\mathcal{V}$-categories.

2.5. Definition. *A functor $p : E \to B$ between small categories is called an* isofibration *if for any object e in E and any isomorphism $f : p(e) \to b$ in B, there exists an isomorphism $g : e \to e'$ in E such that $p(g) = f$.*

A functor $i : X \to Y$ is called an isocofibration *if it is injective on the set of objects.*

2.6. Theorem. [Folk model structure] *The category of small categories admits the following model structure:*

1. *The weak equivalences are equivalences of categories.*
2. *The fibrations are isofibrations.*
3. *The cofibrations are isocofibrations.*

When $\mathcal{V}$ is a nice monoidal model category [Hov99], the category $\mathcal{V}$-**Cat** of small $\mathcal{V}$-categories also admits a model structure called *canonical.* This is a mixture of the model structure on $\mathcal{V}$ and the above folk model structure on the category of small categories. In order to define the canonical model structure, we need the category of connected components $\pi_0\mathcal{A}$ of a $\mathcal{V}$-category $\mathcal{A}$.

2.7. Definition. *For an object X in a monoidal model category $\mathcal{V}$, the connected components $\pi_0 X$ of X is the set of morphisms $\mathrm{Ho}(\mathcal{V})(\mathbf{1}, X)$ in the homotopy category. Moreover, for a $\mathcal{V}$-category $\mathcal{A}$, the category of connected components $\pi_0\mathcal{A}$ of $\mathcal{A}$ is a small category whose set of objects is $\mathrm{ob}(\mathcal{A})$ and set of morphisms from an object a to an object b of $\mathcal{A}$ is $(\pi_0\mathcal{A})(a, b) = \pi_0(\mathcal{A}(a, b))$.*

We call the object X connected *if the set of connected components $\pi_0 X = \mathrm{Ho}(\mathcal{V})(\mathbf{1}, X)$ is a single point. A small category $\mathcal{C}$ is called* connected *when there exists a zigzag sequence of morphisms*

$$x \longrightarrow x_1 \longleftarrow x_2 \longrightarrow \cdots \longrightarrow x_{n-1} \longleftarrow y$$

starting at x and ending at y for any two objects x and y of $\mathcal{C}$. We call a $\mathcal{V}$-category $\mathcal{A}$ connected *if the category $\pi_0\mathcal{A}$ of connected components is connected. Moreover, we call $\mathcal{A}$* strongly connected *if $\mathcal{A}$ is connected and every hom-object $\mathcal{A}(x, y)$ is connected in $\mathcal{V}$.*

2.8. Definition. *Let $\mathcal{V}$ be a monoidal model category. A $\mathcal{V}$-functor $f : \mathcal{A} \to \mathcal{B}$ is called a* local weak equivalence *(resp.* local fibration, local trivial fibration*) if the morphism on hom-objects $\mathcal{A}(x, y) \to \mathcal{B}(f(x), f(y))$ is a weak equivalence (resp. fibration, trivial fibration) in $\mathcal{V}$ for each objects x, y of $\mathcal{A}$.*

1. *We define a $\mathcal{V}$-functor $f : \mathcal{A} \to \mathcal{B}$ to be a* DK-equivalence *if it is a local weak equivalence and $\pi_0 f : \pi_0\mathcal{A} \to \pi_0\mathcal{B}$ is an equivalence of categories.*
2. *We define a $\mathcal{V}$-functor $p : \mathcal{A} \to \mathcal{B}$ to be a* naive fibration *if it is a local fibration and $\pi_0 p : \pi_0\mathcal{A} \to \pi_0\mathcal{B}$ is an isofibration of categories.*
3. *We define a $\mathcal{V}$-category $\mathcal{A}$ to be* locally fibrant *if $\mathcal{A}(a, b)$ is fibrant in $\mathcal{V}$ for any objects a and b of $\mathcal{A}$.*

2.9. Definition. *Let $\mathcal{V}$ be a monoidal model category. A model structure on the category of small $\mathcal{V}$-categories is called* canonical *if it has the following properties:*

1. *A $\mathcal{V}$-category is fibrant if and only if it is locally fibrant.*
2. *A $\mathcal{V}$-functor $f : \mathcal{A} \to \mathcal{B}$ is a trivial fibration if and only if it is surjective on the set of objects and a local trivial fibration.*
3. *The weak equivalences are DK-equivalences.*
4. *The fibrations between fibrant objects are naive fibrations.*

The definition of the canonical model structure in [BM13] only requires conditions 1 and 2 stated above. Hence, our definition is narrower than [BM13]. However, in order to clarify the relation with DK-equivalences and naive fibrations, we use the above definition. Note that if there exists a model structure on $\mathcal{V}$ satisfying these conditions, it is determined uniquely.

2.10. Example. Here are some examples of canonical model structures.

1. An ordinary category is a category enriched in the category of sets $(\mathbf{Set}, \times, *)$. If $\mathbf{Set}$ is equipped with the trivial model structure, then the category of small categories $\mathbf{Cat}$ admits the canonical model structure which coincides with the folk model structure [JT91].

2. A topological category is a category enriched in the category of topological spaces $(\mathbf{Top}, \times, *)$. Let $\mathbf{CGWH}$ denote the full subcategory of compactly generated weak Hausdorff spaces. If $\mathbf{CGWH}$ is equipped with the classical model structure, then the category of $\mathbf{CGWH}$-categories admits the canonical model structure [Ili15].

3. A 2-category is a category enriched in the category of small categories $(\mathbf{Cat}, \times, *)$. If $\mathbf{Cat}$ is equipped with the folk model structure [JT91], then the category of 2-categories admits the canonical model structure [Lac02].

4. A simplicial category is a category enriched in the category of simplicial sets $(\mathbf{Set}^{\Delta^{\mathrm{op}}}, \times, *)$. If $\mathbf{Set}^{\Delta^{\mathrm{op}}}$ is equipped with the classical model structure, then the category of simplicial categories admits the canonical model structure [Ber07].

5. A DG-category over a ring R is a category enriched in the category of chain complexes $(\mathbf{Ch}_R, \otimes, R)$. If $\mathbf{Ch}_R$ is equipped with the projective model structure, then the category of DG-categories admits the canonical model structure [Tab05].

Berger and Moerdijk give a general condition on $\mathcal{V}$ for existence of the canonical model structure in Theorem 1.9 of [BM13].

3. The Euler characteristic of enriched categories

In this section, we focus on homotopical properties of the Euler characteristic.

3.1. The Euler characteristic of $\mathcal{V}$-categories. The Euler characteristic (magnitude) of $\mathcal{V}$-categories in [Lei13] is constructed by a monoid homomorphism

$$|-| : (\mathrm{ob}(\mathcal{V})/\cong, \otimes, \mathbf{1}) \to (k, \cdot, 1)$$

sending finite coproducts to finite sums from the set of isomorphism classes of objects to a rig k. We write it simply as $|-| : \mathcal{V} \to k$. Leinster introduced the following notions to define magnitude.

3.2. Definition. *Let $\mathcal{V}$ be a monoidal category with a monoid homomorphism $|-| : \mathcal{V} \to k$, and let $\mathcal{A}$ be a $\mathcal{V}$-category.*

1. *The* similarity matrix *of $\mathcal{A}$ is the function $\xi : \mathrm{ob}(\mathcal{A}) \times \mathrm{ob}(\mathcal{A}) \to k$ given by $\xi(a,b) = |\mathcal{A}(a,b)|$.*
2. *Let $u : \mathrm{ob}(\mathcal{A}) \to k$ denote the column vector with $u(a) = 1$ for any object a of $\mathcal{A}$. A* weighting *on $\mathcal{A}$ is a column vector $w : \mathrm{ob}(\mathcal{A}) \to k$ such that $\xi w = u$. Dually, a* coweighting *on $\mathcal{A}$ is a row vector $v : \mathrm{ob}(\mathcal{A}) \to k$ such that $v\xi = u^*$.*

Note that we have

$$\sum_{i \in \mathrm{ob}(\mathcal{A})} w(i) = u^* w = v \xi w = vu = \sum_{j \in \mathrm{ob}(\mathcal{A})} v(j)$$

if both a weighting and a coweighting exist. Moreover,

$$\sum_{i \in \mathrm{ob}(\mathcal{A})} w(i) = u^* w = v \xi w = v \xi w' = u^* w' = \sum_{i \in \mathrm{ob}(\mathcal{A})} w'(i)$$

for two (co)weightings w and w' on $\mathcal{A}$, and the equality guarantees the following definition.

3.3. Definition. [Definition 1.1.3 of [Lei13]] *Let $\mathcal{V}$ be a monoidal category with a monoid homomorphism, and let $\mathcal{A}$ be a $\mathcal{V}$-category. We say that $\mathcal{A}$ has magnitude if it has both a weighting w and a coweighting v on $\mathcal{A}$. Then, the magnitude of $\mathcal{A}$ is defined by*

$$|\mathcal{A}| = \sum_{i \in \mathrm{ob}(\mathcal{A})} w(i) = \sum_{j \in \mathrm{ob}(\mathcal{A})} v(j).$$

3.4. Remark. Let $\mathcal{V}$ be a monoidal category with a monoid homomorphism. For a $\mathcal{V}$-category $\mathcal{A}$, if the similarity matrix ξ of $\mathcal{A}$ has an inverse matrix $\xi^{-1} : \mathrm{ob}(\mathcal{A}) \times \mathrm{ob}(\mathcal{A}) \to k$, then there uniquely exist a weighting and a coweighting, which are given by $w(b) = \sum_{a \in \mathrm{ob}(\mathcal{A})} \xi^{-1}(a,b)$ and $v(a) = \sum_{b \in \mathrm{ob}(\mathcal{A})} \xi^{-1}(a,b)$. Hence, $\mathcal{A}$ has magnitude, and we have $|\mathcal{A}| = \sum_{a,b \in \mathrm{ob}(\mathcal{A})} \xi^{-1}(a,b)$.

3.5. Definition. *Let $\mathcal{V}$ be a monoidal model category, and let $\mathcal{W}$ be a full subcategory of $\mathcal{V}$ that is closed under finite coproducts and direct summands. A* measure *of $\mathcal{V}$ is a monoid homomorphism sending finite coproducts to finite sums*

$$|-| : (\mathrm{ob}(\mathcal{W})/\!\sim, \otimes, \mathbf{1}) \longrightarrow (k, \cdot, 1)$$

from the set of weak equivalence classes of $\mathrm{ob}(\mathcal{W})$ to a rig k. We write it simply as $|-| : \mathcal{W} \to k$.

3.6. Definition. *For a monoidal model category $\mathcal{V}$ with a measure $|-| : \mathcal{W} \to k$, we call a $\mathcal{V}$-category $\mathcal{A}$ measurable on $\mathcal{W}$ if $\mathcal{A}(a,b)$ belongs to $\mathrm{ob}(\mathcal{W})$ for any objects a and b of $\mathcal{A}$.*

3.7. Definition. *Let $\mathcal{V}$ be a monoidal model category and $|-| : \mathcal{W} \to k$ be a measure of $\mathcal{V}$ from a full subcategory $\mathcal{W}$ of $\mathcal{V}$ to a rig k. A $\mathcal{V}$-category $\mathcal{A}$* has Euler characteristic *if $\mathcal{A}$ is measurable on $\mathcal{W}$ and has magnitude for the measure. Then, we call the magnitude* Euler characteristic *of $\mathcal{A}$.*

Proposition 1.4.3 and 1.4.4 of [Lei13] show the following properties of Euler characteristic with respect to products and coproducts.

3.8. Proposition. [Lei13] *Let $\mathcal{V}$ be a monoidal model category with a measure. Suppose that $\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n$ are $\mathcal{V}$-categories having Euler characteristics. Then,*

1. *$\chi(\otimes_i \mathcal{A}_i) = \prod_i \mathcal{A}_i$ when $\mathcal{V}$ is a symmetric monoidal model category. Furthermore, the unit $\mathcal{V}$-category $\mathcal{K}$ consisting of only a single point and the unit $\mathbf{1}$ in $\mathcal{V}$ has Euler characteristic one.*
2. *$\chi(\coprod_i \mathcal{A}_i) = \sum_i \chi(\mathcal{A}_i)$. Furthermore, the initial $\mathcal{V}$-category $\emptyset$ has Euler characteristic zero.*

3.9. The canonical model structure and the Euler characteristic. We investigate the relation between the Euler characteristics of $\mathcal{V}$-categories and the canonical model structure on the category of $\mathcal{V}$-categories.

We first show invariance of the Euler characteristic of $\mathcal{V}$-categories with respect to DK-equivalences. If $\mathcal{V}$ is a monoidal model category, the homotopy category $\mathrm{Ho}(\mathcal{V})$ admits a monoidal structure by the total derived functor of the tensor product of $\mathcal{V}$ (see Section 4.3 in [Hov99]). The localizing functor $\ell : \mathcal{V} \to \mathrm{Ho}(\mathcal{V})$ is a monoidal functor sending weak equivalences to isomorphisms. Thus, it induces a 2-functor $\ell_* : \mathcal{V}\text{-}\mathbf{Cat} \to \mathrm{Ho}(\mathcal{V})\text{-}\mathbf{Cat}$. A $\mathcal{V}$-functor $f : \mathcal{A} \to \mathcal{B}$ is a DK-equivalence if and only if $\ell_*(f) : \ell_*(\mathcal{A}) \to \ell_*(B)$ is an equivalence of $\mathrm{Ho}(\mathcal{V})$-categories. Hence, the following theorem follows from Proposition 1.4.1 of [Lei13].

3.10. Theorem. *Let $\mathcal{V}$ be a monoidal model category with a measure. Suppose that two finite measurable $\mathcal{V}$-categories $\mathcal{A}$ and $\mathcal{B}$ are DK-equivalent. Then, their Euler characteristics are equal $\chi(\mathcal{A}) = \chi(\mathcal{B})$ if $\mathcal{A}$ or $\mathcal{B}$ has Euler characteristic.*

Another important homotopical property of the Euler characteristic is the product formula with respect to fibrations. In the category of topological spaces, a fibration $p : E \to B$ over a connected base B with fiber F yields under suitable hypotheses the equality $\chi(E) = \chi(B)\chi(F)$. When B has connected components B_i, the above equality is generally extended as the following form:

$$\chi(E) = \sum_{i \in \pi_0 B} \chi(B_i)\chi(F_i),$$

where F_i is the fiber over a point of B_i. We focus on the relation between the Euler characteristic of $\mathcal{V}$-categories and naive fibrations.

3.11. Definition. *Let $p : E \to B$ be a morphism in a monoidal category $\mathcal{V}$. The* fiber *F_b of p over $b : \mathbf{1} \to B$ is defined by the pullback*

$$\begin{tikzcd} F_b \arrow[r] \arrow[d] & E \arrow[d, "p"] \\ \mathbf{1} \arrow[r, "b"'] & B \end{tikzcd}$$

in $\mathcal{V}$. The fiber *$\mathcal{F}_b$ of a $\mathcal{V}$-functor $q : \mathcal{E} \to \mathcal{B}$ over an object b of $\mathcal{B}$ can be defined in $\mathcal{V}$-**Cat** similarly to the above case. Note that choosing an object b of $\mathcal{B}$ gives a $\mathcal{V}$-functor $\mathcal{K} \to \mathcal{B}$ from the unit $\mathcal{V}$-category $\mathcal{K}$ consisting of a single object and the unit $\mathbf{1}$ as the hom-object. The fiber $\mathcal{F}_b$ is a $\mathcal{V}$-category consisting of the inverse image $q^{-1}(b)$ as objects and the fiber of $q : \mathcal{E}(x, y) \to \mathcal{B}(b, b)$ over the identity $1_b : \mathbf{1} \to \mathcal{B}(b, b)$ as the hom-object $\mathcal{F}_b(x, y)$.*

3.12. Lemma. *Let $\mathcal{V}$ be a right proper monoidal model category with cofibrant unit. Suppose that $p : E \to B$ is a fibration between fibrant objects in $\mathcal{V}$, and B is connected. Then, any two fibers F_b and $F_{b'}$ are weakly equivalent.*

Proof. The morphisms $b, b' : \mathbf{1} \to B$ are homotopic to each other. Hence, there exist a cylinder object C with a factorization $\mathbf{1} \coprod \mathbf{1} \overset{j_0 \coprod j_1}{\to} C \to \mathbf{1}$ and a morphism $h : C \to B$ such that $h \circ j_0 = b$ and $h \circ j_1 = b'$. We have the following commutative diagram:

$$\begin{tikzcd} \mathbf{1} \arrow[r, "b"] \arrow[d] & B \arrow[d, equal] & E \arrow[l, "p"'] \arrow[d, equal] \\ C \arrow[r, "h"] & B & E \arrow[l, "p"'] \\ \mathbf{1} \arrow[r, "b'"] \arrow[u] & B \arrow[u, equal] & E \arrow[l, "p"'] \arrow[u, equal] \end{tikzcd}$$

in which each vertical morphism is a weak equivalence. By Proposition 13.3.9 of [Hir03], we conclude that F_b and $F_{b'}$ are weakly equivalent. ∎

3.13. Definition. *Let $\mathcal{V}$ be a right proper monoidal model category with cofibrant unit. We say that a measure $|-| : \mathcal{W} \to k$ of $\mathcal{V}$* preserves fibrations *if $\mathcal{W}$ consists of fibrant objects, and for any fibration $p : E \to B$ such that E, B, and any fiber belong to $\mathcal{W}$ and the base B is decomposed as a finite coproduct $\coprod_i B_i$ of connected objects B_i, we have*

$$|E| = \sum_i |B_i| \cdot |F_i|,$$

where F_i is the fiber of p over any morphism $\mathbf{1} \to B_i$ (by Lemma 3.12, $|F_i|$ does not depend on the choice of a morphism $\mathbf{1} \to B_i$).

Unfortunately, the fibers of a naive fibration are not DK-equivalent to each other in general even if the base $\mathcal{V}$-category is connected. For example, a finite left G-set A for a finite non-trivial group G gives an ordinary category A_G with two objects $0,1$, and hom-sets $A_G(0,0)=G$, $A_G(0,1)=A$, $A_G(1,0)=\emptyset$, and $A_G(1,1)=*$. The projection from A_G to the category $0\to 1$ is an isofibration. However, the fibers $F_0=G$ and $F_1=*$ are not DK-equivalent. Thus, we consider the following condition on objects introduced in [BM13].

Let us first introduce the notion of $\mathcal{V}$-interval objects according to [BM13]. A natural model of intervals in $\mathcal{V}$-**Cat** can be considered as $\mathcal{I}$ consisting of $\mathrm{ob}(\mathcal{I})=\{0,1\}$ and $\mathcal{I}(i,j)=\mathbf{1}$ for $0\leqq i,j\leqq 1$. A $\mathcal{V}$-interval object is defined as a cofibrant $\mathcal{V}$-category that has two objects $\{0,1\}$ and is weakly equivalent to $\mathcal{I}$ in $\mathcal{V}\text{-}\mathbf{Cat}_{\{0,1\}}$. Here, $\mathcal{V}\text{-}\mathbf{Cat}_{\{0,1\}}$ is the category of $\mathcal{V}$-categories which have two objects $\{0,1\}$, and $\mathcal{V}$-functors whose maps on objects are identities. More generally, for a set S, we can define $\mathcal{V}\text{-}\mathbf{Cat}_S$ as the category of $\mathcal{V}$-categories which have the set of objects S. The *transferred model structure* on $\mathcal{V}\text{-}\mathbf{Cat}_S$ consists of local weak equivalences and local fibrations.

3.14. Definition. [Definition 1.11 of [BM13]] *Let $\mathcal{V}$ be a monoidal model category such that $\mathcal{V}\text{-}\mathbf{Cat}_{\{0,1\}}$ admits the transferred model structure. A* $\mathcal{V}$-interval object *is a cofibrant object in the transferred model structure on $\mathcal{V}\text{-}\mathbf{Cat}_{\{0,1\}}$ which is weakly equivalent to $\mathcal{I}$.*

3.15. Assumption. *Assume that our monoidal model category $\mathcal{V}$ satisfies the following conditions:*

1. *$\mathcal{V}$ is adequate (Definition 1.1 in [BM13]), and right proper.*
2. *There exists a generating set of $\mathcal{V}$-intervals (Definition 1.11 in [BM13]).*
3. *$\mathcal{V}$ is cartesian with cofibrant unit.*
4. *The set of connected components $\pi_0(X)$ is empty if and only if X is the initial object $\emptyset$ in $\mathcal{V}$.*

By Proposition 2.5 in [BM13], the first condition guarantees that $\mathcal{V}\text{-}\mathbf{Cat}_{\{0,1\}}$ admits the transferred model structure. Regarding the third condition, a cartesian monoidal category is a monoidal category whose tensor product $\otimes$ coincides with the categorical product $\times$. Hence, the unit object $\mathbf{1}$ is a terminal object. In particular, $\mathcal{V}$ is symmetric.

3.16. Definition. [Section 2.2 in [BM13]] *A $\mathcal{V}$-functor is said to be* path-lifting *if it has the right lifting property with respect to $\{i\}\to\mathcal{H}$, $i=0,1$, for any $\mathcal{V}$-interval object.*

3.17. Lemma. *Let $\mathcal{V}$ be an adequate monoidal model category with cofibrant unit. For a path-lifting $\mathcal{V}$-functor $p:\mathcal{E}\to\mathcal{B}$ between locally fibrant $\mathcal{V}$-categories $\mathcal{E}$ and $\mathcal{B}$, the induced functor $\pi_0(p):\pi_0(\mathcal{E})\to\pi_0(\mathcal{B})$ is an isofibration.*

Proof. Suppose that e is an object of $\mathcal{E}$ and an object b of $\mathcal{B}$ is homotopy equivalent to $p(e)$. Let $\mathcal{J}$ denote the $\mathcal{V}$-category on $\{0,1\}$ representing a single directed morphism, i.e. $\mathcal{J}(0,0) = \mathcal{J}(0,1) = \mathcal{J}(1,1) = \mathbf{1}$, and $\mathcal{J}(1,0) = \emptyset$. For an isomorphism $\varphi : p(e) \to b$ in $\pi_0(\mathcal{B})$, a corresponding morphism $\mathbf{1} \to \mathcal{B}(p(e), b)$ yields the representing functor $r : \mathcal{J} \to \mathcal{B}$. Proposition 2.24 of [BM13] guarantees the coherence axiom (Definition 2.18 of [BM13]), i.e. there exists a cofibration $k : \mathcal{J} \to \mathcal{H}$ into a $\mathcal{V}$-interval object $\mathcal{H}$ and an extension $h : \mathcal{H} \to \mathcal{B}$ such that $h \circ k = r$. The path-lifting property extends h to a $\mathcal{V}$-functor $\tilde{h} : \mathcal{H} \to \mathcal{E}$ satisfying $p \circ \tilde{h} = h$, $\tilde{h}(0) = e$. The functor $\pi_0(\tilde{h})$ yields an isomorphism $\tilde{\varphi} : e \to \tilde{h}(1)$ in $\pi_0(\mathcal{E})$ such that $\pi_0(p)(\tilde{\varphi}) = \varphi$. ∎

3.18. Corollary. *Let $\mathcal{V}$ be a right proper and adequate monoidal model category with cofibrant unit and a generating set of $\mathcal{V}$-intervals. Then, the canonical model structure on $\mathcal{V}$-**Cat** exists and is right proper.*

Proof. Theorem 1.10 in [BM13] states that $\mathcal{V}$-**Cat** admits a right proper model structure whose weak equivalences are DK-equivalences, and fibrations are path-lifting local fibrations. The fibrations between locally fibrant $\mathcal{V}$-categories are naive fibrations by Lemma 3.17. This model structure is canonical in our sense. ∎

3.19. Lemma. *Let $\mathcal{V}$ be a monoidal model category satisfying Assumption 3.15. The unit $\mathbf{1}$ of $\mathcal{V}$ is connected.*

Proof. Since $\mathbf{1}$ is a terminal object, this is fibrant and the set of morphisms $\mathcal{V}(\mathbf{1},\mathbf{1})$ consists of a single point. Then we have $\pi_0(\mathbf{1}) = \mathrm{Ho}(\mathcal{V})(\mathbf{1},\mathbf{1}) \cong \mathcal{V}(\mathbf{1},\mathbf{1})/\simeq\, = *$. ∎

3.20. Lemma. *Let $\mathcal{V}$ be a monoidal model category satisfying Assumption 3.15. Any $\mathcal{V}$-category can be decomposed as $\mathcal{A} = \coprod_i \mathcal{A}_i$ for connected $\mathcal{V}$-categories $\mathcal{A}_i$.*

Proof. The category of connected components $\pi_0\mathcal{A}$ can be decomposed as $\pi_0\mathcal{A} = \coprod_i \mathcal{B}_i$ for connected subcategories $\mathcal{B}_i$ of $\pi_0\mathcal{A}$. Consider the full subcategory $\mathcal{A}_i$ of $\mathcal{A}$ having the same objects as $\mathcal{B}_i$. Then $\mathcal{A}_i$ is connected and $\mathcal{A} = \coprod_i \mathcal{A}_i$ since $\mathcal{A}(a,b) = \emptyset$ if and only if $\pi_0\mathcal{A}(a,b) = \emptyset$ by Assumption 3.15. ∎

3.21. Definition. *Let $\mathcal{V}$ be a monoidal model category satisfying Assumption 3.15, and let a and b be two objects in a $\mathcal{V}$-category $\mathcal{A}$. We say that a and b are;*

1. disjoint *if both $\mathcal{A}(a,b)$ and $\mathcal{A}(b,a)$ are initial objects in $\mathcal{V}$,*
2. equivalent *if there exists a $\mathcal{V}$-interval object $\mathcal{H}$ and a $\mathcal{V}$-functor $h : \mathcal{H} \to \mathcal{A}$ such that $h(0) = a$ and $h(1) = b$,*
3. homotopy equivalent *if a and b are isomorphic in $\pi_0(\mathcal{A})$.*

If two objects a and b of $\mathcal{A}$ are equivalent, then these are homotopy equivalent by applying π_0 to the functor $h : \mathcal{H} \to \mathcal{A}$. Furthermore, we call that $\mathcal{A}$ has (homotopy) coherent objects *if any two objects of $\mathcal{A}$ are disjoint or (homotopy) equivalent.*

3.22. Lemma. *Let $\mathcal{V}$ be a monoidal model category satisfying Assumption 3.15. Suppose that $p : \mathcal{E} \to \mathcal{B}$ is a naive fibration between fibrant $\mathcal{V}$-categories, and $\mathcal{B}$ is connected and has coherent objects. Then, any two fibers $\mathcal{F}_b$ and $\mathcal{F}_{b'}$ are DK-equivalent.*

Proof. Since $\mathcal{B}$ is connected and has coherent objects, for any two objects b and b' in $\mathcal{B}$, there exist a $\mathcal{V}$-interval object $\mathcal{H}$ and $h : \mathcal{H} \to \mathcal{B}$ such that $h(0) = b$ and $h(1) = b'$. Note that every $\mathcal{V}$-interval object is DK-equivalent to the unit $\mathcal{V}$-category $\mathcal{K}$. We have the following commutative diagram:

$$\begin{tikzcd}
\mathcal{K} \arrow[r, "b"] \arrow[d] & \mathcal{B} \arrow[d, equal] & \mathcal{E} \arrow[l, "p"'] \arrow[d, equal] \\
\mathcal{H} \arrow[r, "h"] & \mathcal{B} \arrow[d, equal] & \mathcal{E} \arrow[l, "p"'] \arrow[d, equal] \\
\mathcal{K} \arrow[r, "b'"] \arrow[u] & \mathcal{B} & \mathcal{E} \arrow[l, "p"']
\end{tikzcd}$$

in which each vertical functor is a DK-equivalence. By Proposition 13.3.9 of [Hir03], we conclude that $\mathcal{F}_b$ and $\mathcal{F}_{b'}$ are DK-equivalent. ∎

The following lemma is shown as Lemma 2.12 of [BM13].

3.23. Lemma. *Let $\mathcal{V}$ be a monoidal model category. For two homotopy equivalent objects a and b, and any object c of a $\mathcal{V}$-category $\mathcal{A}$, the two hom-objects $\mathcal{A}(a, c)$ and $\mathcal{A}(b, c)$ (resp. $\mathcal{A}(c, a)$ and $\mathcal{A}(c, b)$) are weakly equivalent to each other in $\mathcal{V}$.*

In the rest of this section, we assume that a rig k of the range of a measure has a multiplicative inverse for every non-zero element.

3.24. Lemma. *Let $\mathcal{V}$ be a monoidal model category with a measure satisfying Assumption 3.15. Suppose that $\mathcal{B}$ is a non-empty $\mathcal{V}$-category having Euler characteristic, connected, and has homotopy coherent objects. Then, we have $\chi(\mathcal{B}) = |\mathcal{B}(b, b)|^{-1}$ for any object b of $\mathcal{B}$.*

Proof. Fix an object b of $\mathcal{B}$. Lemma 3.23 shows that $|\mathcal{B}(x, y)| = |\mathcal{B}(b, b)|$ for any objects x and y of $\mathcal{B}$ since any two objects are homotopy equivalent. The similarity matrix $\xi_{\mathcal{B}}$ is $\xi_{\mathcal{B}}(x, y) = |\mathcal{B}(b, b)|$ for any objects x and y of $\mathcal{B}$. A weighting w on $\mathcal{B}$ can be defined as $w(x) = (\mathrm{ob}(\mathcal{B})^\sharp \cdot |\mathcal{B}(b, b)|)^{-1}$, where $\mathrm{ob}(\mathcal{B})^\sharp$ is the cardinality of $\mathrm{ob}(\mathcal{B})$. We obtain the Euler characteristic $\chi(\mathcal{B}) = \sum_{x \in \mathrm{ob}(\mathcal{B})} w(x) = |\mathcal{B}(b, b)|^{-1}$. ∎

3.25. Theorem. *Suppose that $\mathcal{V}$ is a monoidal model category satisfying Assumption 3.15, and it is equipped with a measure that preserves fibrations. Moreover, suppose that $p : \mathcal{E} \to \mathcal{B}$ is a naive fibration between fibrant $\mathcal{V}$-categories, and $\mathcal{E}$, $\mathcal{B}$, and any fiber have Euler characteristics. If both $\mathcal{E}$ and $\mathcal{B}$ have coherent objects, and $\mathcal{B}$ is strongly connected, we have*

$$\chi(\mathcal{E}) = \chi(\mathcal{B})\chi(\mathcal{F}),$$

where $\mathcal{F}$ is the fiber of p over an object of $\mathcal{B}$.

PROOF. The $\mathcal{V}$-category $\mathcal{E}$ can be decomposed as the finite coproduct $\mathcal{E} = \coprod_i \mathcal{E}_i$ for connected subcategories $\mathcal{E}_i$. Consider the following two pullback diagrams for an arbitrary object b in $\mathcal{B}$:

$$\begin{tikzcd}
\mathcal{F}_i \arrow[r] \arrow[d] & \mathcal{F}_b \arrow[r] \arrow[d] & \mathcal{K} \arrow[d, "b"] \\
\mathcal{E}_i \arrow[r] & \mathcal{E} \arrow[r, "p"'] & \mathcal{B},
\end{tikzcd}$$

where the $\mathcal{V}$-category $\mathcal{F}_i$ is the full subcategory of $\mathcal{F}_b$ having $p^{-1}(b) \cap \mathrm{ob}(\mathcal{E}_i)$ as objects. We can choose an object x_i of $\mathcal{F}_i$ for all i since $\pi_0(p)$ is an isofibration, and $\mathcal{B}$ is connected and has coherent objects. We have the coproduct decomposition of $\mathcal{F}_b$ by $\mathcal{F}_b = \coprod_i \mathcal{F}_i$ since $\mathcal{F}(x, y) = \emptyset$ if $\mathcal{E}(x, y) = \emptyset$. Since $\mathcal{E}$ has coherent objects, $\mathcal{E}_i$ also does. Moreover, $\mathcal{F}_i$ also does by the pullback diagram. The Euler characteristic of $\mathcal{F}_b$ is

$$\chi(\mathcal{F}_b) = \sum_i \chi(\mathcal{F}_i) = \sum_i (|\mathcal{F}_i(x_i, x_i)|)^{-1}$$

from Lemma 3.24. Since p is a naive fibration, the morphism

$$p : \mathcal{E}(x_i, x_i) \longrightarrow \mathcal{B}(p(x_i), p(x_i))$$

is a fibration in $\mathcal{V}$, and we have

$$|\mathcal{E}(x_i, x_i)| = |\mathcal{B}(p(x_i), p(x_i))| \cdot |\mathcal{F}_i(x_i, x_i)|$$

since $\mathcal{B}(p(x_i), p(x_i))$ is connected. The following calculation shows the result:

$$\begin{aligned}
\chi(\mathcal{E}) &= \sum_i \chi(\mathcal{E}_i) \\
&= \sum_i (|\mathcal{E}(x_i, x_i)|)^{-1} \\
&= \sum_i (|\mathcal{B}(p(x_i), p(x_i))| \cdot |\mathcal{F}_i(x_i, x_i)|)^{-1} \\
&= |\mathcal{B}(b, b)|^{-1} \sum_i (|\mathcal{F}_i(x_i, x_i)|)^{-1} \\
&= \chi(\mathcal{B})\chi(\mathcal{F}).
\end{aligned}$$

■

Let us consider the case in which $\mathcal{B}$ is not strongly connected. Since there is no guarantee that an object X in $\mathcal{V}$ can be decomposed by connected objects, we need the following assumption.

3.26. Assumption. *We assume that our monoidal model category $\mathcal{V}$ with a measure $|-| : \mathcal{W} \to k$ satisfies Assumption 3.15 and the following properties:*

1. *The product $\times$ is compatible with the coproducts; i.e., the canonical comparison map $\coprod_{i,j} (X_i \times Y_j) \to (\coprod_i X) \times \left(\coprod_j Y_j\right)$ is invertible.*
2. *Any object X in $\mathcal{W}$ is fibrant, and has finitely many connected components.*
3. *Any object X in $\mathcal{W}$ is decomposed as a finite coproduct $X = \coprod_{i\in\pi_0 X} X_i$ for connected objects X_i, and the following is a pullback diagram:*

$$\begin{tikzcd}
X_i \arrow[r] \arrow[d] & X = \coprod_i X_i \arrow[d] \\
\mathbf{1} \arrow[r, "i"'] & \coprod_i \mathbf{1}.
\end{tikzcd}$$

By the pullback diagram above, these connected objects X_i are determined uniquely up to isomorphism for an object X. A morphism $\mu : X \times Y \to Z$ in $\mathcal{V}$ induces a map on connected components

$$\pi_0 X \times \pi_0 Y \longrightarrow \pi_0 Z, \quad (i, j) \mapsto ij.$$

When the three objects X, Y, and Z are decomposed as $X = \coprod_{i\in\pi_0 X} X_i$, $Y = \coprod_{j\in\pi_0 Y} Y_j$, and $Z = \coprod_{k\in\pi_0 Z} Z_k$ respectively, we have the following diagram:

$$\begin{tikzcd}
 & Z_{ij} \arrow[rr] \arrow[dd] & & Z \arrow[dd] \\
X_i \times Y_j \arrow[ur, dotted, "\mu_{ij}"] \arrow[rr] \arrow[dd] & & X \times Y \cong \coprod_{i,j} (X_i \times Y_j) \arrow[ur, "\mu"] \arrow[dd] & \\
 & \mathbf{1} \arrow[rr, "ij"'] & & \coprod_k \mathbf{1} \\
\mathbf{1} \arrow[ur, equals] \arrow[rr, "{(i,j)}"'] & & \coprod_{i,j} \mathbf{1}, \arrow[ur] &
\end{tikzcd}$$

where the front and the back diagrams are pullbacks. The diagram induces a morphism $\mu_{ij} : X_i \times Y_j \to Z_{ij}$ that makes the diagram above commute.

3.27. Lemma. *Let $\mathcal{V}$ be a monoidal model category with a measure satisfying Assumption 3.26. A monoid object X in $\mathcal{W}$ induces a monoid structure on $\pi_0 X$. Let X be decomposed as the coproduct of the connected objects*

$$X = \coprod_{i\in\pi_0 X} X_i$$

from Assumption 3.26. If $\pi_0 X$ is a group, all connected objects X_i are weakly equivalent to each other.

Proof. Denote the unit of $\pi_0 X$ by e. For an element i of $\pi_0 X$, the connected objects $\pi_0(X_{i^{-1}}) = \mathrm{Ho}(\mathcal{V})(1, X_{i^{-1}})$ consists of a single point. Consider the following composition:

$$X_i \cong X_i \times \mathbf{1} \xrightarrow{1\times i^{-1}} X_i \times X_{i^{-1}} \xrightarrow{\mu_{ii^{-1}}} X_e.$$

The inverse map in $\mathrm{Ho}(\mathcal{V})$ is

$$X_e \cong X_e \times \mathbf{1} \xrightarrow{1\times i} X_e \times X_i \xrightarrow{\mu_{ei}} X_i$$

since the multiplication μ on X satisfies the associativity and unitality conditions. Thus, any connected objects X_i is weakly equivalent to X_e. ∎

3.28. Theorem. *Suppose that $\mathcal{V}$ is a monoidal model category satisfying Assumption 3.26, and it is equipped with a measure that preserves fibrations. Moreover, suppose that $p : \mathcal{E} \to \mathcal{B}$ is a naive fibration between $\mathcal{V}$-categories, and $\mathcal{E}$, $\mathcal{B}$, and any fiber have Euler characteristics. If both $\mathcal{E}$ and $\mathcal{B}$ have coherent objects, and $\mathcal{B}$ is connected, we have*

$$\chi(\mathcal{E}) = \chi(\mathcal{B})\chi(\mathcal{F}),$$

where $\mathcal{F}$ is the fiber of p over an object of $\mathcal{B}$.

Proof. This is shown similarly to Theorem 3.25. Note that both $\mathcal{E}$ and $\mathcal{B}$ are fibrant $\mathcal{V}$-categories since the hom-objects of them belong to $\mathcal{W}$, and $\mathcal{W}$ consists of fibrant objects by Assumption 3.26. We have the finite coproduct decomposition $\mathcal{E} = \coprod_i \mathcal{E}_i$ for connected $\mathcal{V}$-categories $\mathcal{E}_i$ and $\mathcal{B}(b,b) = \coprod_j \mathcal{B}(b,b)_j$ for connected objects $\mathcal{B}(b,b)_j$ for an arbitrary object b in $\mathcal{B}$. Choose an object x_i in $\mathcal{E}_i$ such that $p(x_i) = b$ for each i. Since the measure preserves fibrations, we have

$$|\mathcal{E}(x_i, x_i)| = \sum_j |\mathcal{B}(p(x_i), p(x_i))_j| \cdot |F_j|,$$

where F_j is the fiber of $p : \mathcal{E}(x_i, x_i) \to \mathcal{B}(p(x_i), p(x_i))$ over the morphism

$$\mathbf{1} \longrightarrow \mathcal{B}(p(x_i), p(x_i))$$

picking out a point in the connected component indexed by j. If e of $\pi_0\mathcal{B}(p(x_i), p(x_i))$ denotes the unit, then F_e is weakly equivalent to $\mathcal{F}(x_i, x_i)$. Consider the following pullback diagram:

$$\begin{tikzcd}
F_j \arrow[r] \arrow[d] & \mathcal{E}(x_i, x_i)_j \arrow[r] \arrow[d] & \mathcal{E}(x_i, x_i) \arrow[d, "p"] \\
\mathbf{1} \arrow[r] & \mathcal{B}(p(x_i), p(x_i))_j \arrow[r] & \mathcal{B}(p(x_i), p(x_i)).
\end{tikzcd}$$

Similarly to Lemma 3.12 and Lemma 3.22, F_j is weakly equivalent to F_e and $\mathcal{F}(x_i, x_i)$ since $B(p(x_i), p(x_i))_j$ and $B(p(x_i), p(x_i))_e$ are weakly equivalent to each other by Lemma

3.27. By Lemma 3.24, we have

$$\begin{aligned}
\chi(\mathcal{E}) &= \sum_i \chi(\mathcal{E}_i) \\
&= \sum_i (|\mathcal{E}(x_i, x_i)|)^{-1} \\
&= \sum_i \sum_j (|\mathcal{B}(p(x_i), p(x_i))_j| \cdot |F_j|)^{-1} \\
&= \sum_i \left(\pi_0\mathcal{B}(p(x_i), p(x_i))^{\sharp} \cdot |\mathcal{B}(p(x_i), p(x_i))_e| \cdot |\mathcal{F}(x_i, x_i)|\right)^{-1} \\
&= \sum_i (|\mathcal{B}(p(x_i), p(x_i))| \cdot |\mathcal{F}(x_i, x_i)|)^{-1} \\
&= \chi(\mathcal{B})\chi(\mathcal{F}).
\end{aligned}$$

Hence, the result follows. ■

3.29. Corollary. *Suppose that $\mathcal{V}$ is a monoidal model category satisfying Assumption 3.26, and it is equipped with a measure that preserves fibrations. Moreover, suppose that $p : \mathcal{E} \to \mathcal{B}$ is a naive fibration between $\mathcal{V}$-categories, and $\mathcal{E}$, $\mathcal{B}$, and any fiber have Euler characteristics. If both $\mathcal{E}$ and $\mathcal{B}$ have coherent objects, and $\mathcal{B}$ is a finite coproduct $\coprod_i \mathcal{B}_i$ for connected $\mathcal{V}$-categories $\mathcal{B}_i$, we have*

$$\chi(\mathcal{E}) = \sum_i \chi(\mathcal{B}_i)\chi(\mathcal{F}_i),$$

where $\mathcal{F}_i$ is the fiber of p over an object of $\mathcal{B}_i$.

Proof. Let $\mathcal{E}_i$ be the full subcategory of $\mathcal{E}$ whose set of objects is the inverse image $p^{-1}(\mathrm{ob}(\mathcal{B}_i))$. Then $\mathcal{E}$ is decomposed as the coproduct $\coprod_i \mathcal{E}_i$ and

$$\begin{tikzcd}
\mathcal{E}_i \arrow[r] \arrow[d] & \mathcal{E} \arrow[d, "p"] \\
\mathcal{B}_i \arrow[r] & \mathcal{B}
\end{tikzcd}$$

is a pullback diagram. The left vertical morphism $\mathcal{E}_i \to \mathcal{B}_i$ is a fibration over the connected base $\mathcal{B}_i$ with the fiber $\mathcal{F}_i$. Theorem 3.28 shows $\chi(\mathcal{E}_i) = \chi(\mathcal{B}_i)\chi(\mathcal{F}_i)$, and Proposition 3.8 completes this proof. ■

3.30. Examples.

1. Let $(\mathbf{Set}, \times, *)$ be the category of sets equipped with the trivial model structure. Denote the full subcategory of finite sets by $\mathbf{set}$, and define a measure $\sharp : \mathbf{set} \to \mathbb{Z} \subset \mathbb{Q}$ by the cardinality of sets. A category enriched in $\mathbf{Set}$ is a small category. It gives the Euler characteristic of finite categories that coincides with the one defined in

[Lei08]. The category of small categories admits the canonical model structure that coincides with the folk model structure. Corollary 3.29 implies that an isofibration between finite groupoids satisfies the product formula.

2. Let $(\mathbf{Cat}, \times, *)$ be the category of small categories equipped with the folk model structure. Denote the full subcategory of finite categories having Leinster's Euler characteristic [Lei08] by **cat**, and define a measure $\chi : \mathbf{cat} \to \mathbb{Q}$ as the Euler characteristic of finite categories. This gives the Euler characteristic of 2-categories. Corollary 3.29 implies that a fibration of 2-categories [Lac02] between finite 2-groupoids satisfies the product formula. A *2-groupoid* G is a 2-category whose 1-morphisms and 2-morphisms are invertible. For an object x of G, define $\pi_1(G, x)$ as $\pi_0 G(x, x)$ and $\pi_2(G, f)$ as the set of 2-morphisms from a 1-morphism f to itself. Lemma 3.24 implies
$$\chi(G) = \sum_{[x]\in\pi_0 G} \left(\sum_{[f]\in\pi_1(G,x)} (\pi_2(G, f)^\sharp)^{-1} \right)^{-1}.$$
Furthermore, Lemma 3.27 shows that $\pi_2(G, f)^\sharp = \pi_2(G, 1_x)^\sharp$ for any element $[f]$ of $\pi_1(G, x)$. Hence,
$$\chi(G) = \sum_{[x]\in\pi_0 G} \frac{\pi_2(G, 1_x)^\sharp}{\pi_1(G, x)^\sharp}.$$

3. Let $(\mathbf{CGWH}, \times, *)$ be the category of compactly generated weak Hausdorff spaces equipped with the classical model structure. Denote the full subcategory of spaces having the homotopy type of a finite CW-complex by **cw**, and define a measure $\chi : \mathbf{cw} \to \mathbb{Z} \subset \mathbb{Q}$ by the topological Euler characteristic. This gives the Euler characteristic of categories enriched in **cw**. We will investigate this case deeply in the next section.

4. The classifying spaces of topological categories

4.1. The geometric realization of simplicial spaces. The classifying space of a topological category was introduced in [Seg68]. It is defined as the geometric realization of a simplicial space called the nerve.

4.2. Definition. *The category* Δ *consists of totally ordered sets*
$$[n] = \{0 < 1 < 2 < \cdots < n\}$$
for $n \geqq 0$ *as objects and order preserving maps between them as morphisms. A* simplicial space *is a functor from the opposite category* Δ^{op} *of* Δ *to the category* **Top** *of spaces*
$$\Delta^{\mathrm{op}} \longrightarrow \mathbf{Top}.$$

For a simplicial space X and an order preserving map $\varphi : [n] \to [m]$, let φ_* denote $X(\varphi) : X_m \to X_n$. The category $\mathbf{Top}^{\Delta^{\mathrm{op}}}$ of simplicial spaces is defined as the functor category from Δ^{op} to **Top**. *Let Δ_+ denote the subcategory of Δ having the same objects as Δ and injective order preserving maps as morphisms. If $n > m$, there is no morphism $[n] \to [m]$ in Δ_+. A* Δ-space *is a functor*

$$\Delta_+^{\mathrm{op}} \longrightarrow \mathbf{Top}.$$

Let $\mathbf{Top}^{\Delta_+^{\mathrm{op}}}$ denote the category of Δ-spaces. The canonical inclusion functor $\Delta_+ \to \Delta$ induces the forgetful functor

$$\flat : \mathbf{Top}^{\Delta^{\mathrm{op}}} \longrightarrow \mathbf{Top}^{\Delta_+^{\mathrm{op}}}.$$

A simplicial space can be described as a sequence of spaces X_n equipped with face maps $d_j : X_n \to X_{n-1}$ and degeneracy maps $s_i : X_{n-1} \to X_n$ satisfying the simplicial identities (see [May92]). Similarly, a Δ-space is a sequence of spaces equipped with only face maps. The above functor $\flat$ makes simplicial spaces forget their degeneracy maps.

4.3. Definition. *A* cosimplicial space *is a functor $\Delta \to$* **Top**. *The* standard cosimplicial space *is a functor taking $[n]$ to*

$$\Delta^n = \left\{ (t_0, \cdots, t_n) \in \mathbb{R}^{n+1} \;\middle|\; t_i \geqq 0, \sum_{i=0}^{n} t_i = 1 \right\}$$

and taking a map $\varphi : [n] \to [m]$ to $\varphi^* : \Delta^n \to \Delta^m$ which is the linear extension of the map $v_i \mapsto v_{\varphi(i)}$ on vertices of standard simplices.

4.4. Definition. *Let X be a simplicial space. The* geometric realization $|X|$ *of X is the space defined by*

$$|X| = \left(\coprod_{n \geqq 0} \Delta^n \times X_n \right) / (t, \varphi_*(x)) \sim (\varphi^*(t), x)$$

for all order preserving maps $\varphi : [n] \to [m]$ and points x in X_m and t in Δ^n. Similarly, the geometric realization of a Δ-space Y is defined by

$$||Y|| = \left(\coprod_{n \geqq 0} \Delta^n \times Y_n \right) / (t, \varphi_*(y)) \sim (\varphi^*(t), y)$$

for all injective order preserving maps $\varphi : [n] \to [m]$ and points y in Y_m and t in Δ^n. The fat realization $||X||$ *of a simplicial space X is defined as the geometric realization $||X^\flat||$ of the Δ-space $X^\flat$. We obtain the canonical projection $||X|| \to |X|$ by taking the quotient of the degenerate part.*

4.5. Definition. *A* characteristic map *of a space X is a map $\varphi : D^n \to X$ from the closed n-disk D^n whose restriction $\varphi|_{\mathrm{Int}(D^n)} : \mathrm{Int}(D^n) \to X$ to the interior of D^n is an embedding. The image $\varphi(\mathrm{Int}(D^n))$ is denoted by e, and called an n-*cell *of X. We say that n is the* dimension *of e, and denote it $\dim e$. A Hausdorff space X is a* cell complex *if there exists a family of characteristic maps $\{\varphi_\lambda\}_{\lambda\in\Lambda}$ of X, and it satisfies the following conditions:*

1. *The space X is the disjoint union of $\{e_\lambda\}_{\lambda\in\Lambda}$ as a set, i.e. $X = \bigcup_{\lambda\in\Lambda} e_\lambda$ and $e_\lambda \cap e_\mu = \emptyset$ whenever $\lambda \neq \mu$.*
2. *Let $X^{(n)}$ denote the subspace $\bigcup_{\dim e_\lambda \leqq n} e_\lambda$ of X. Each characteristic map $\varphi_\lambda : D^n \to X$ satisfies $\varphi_\lambda(\partial D^n) \subset X^{(n-1)}$.*

A subspace A of a cell complex $X = \bigcup_{\lambda\in\Lambda} e_\lambda$ is called a subcomplex *of X if there exists $\Lambda' \subset \Lambda$ such that $A = \bigcup_{\mu\in\Lambda'} e_\mu$, and $\overline{e_\alpha} \subset A$ for each α in Λ'. A cell complex is called a* CW-complex *when it satisfies the two conditions; closure-finiteness and weak topology conditions (see [Hat02]).*

The fat realization was introduced by Segal in [Seg74]. Compared with the normal geometric realization, the fat realization is easy to treat in homotopy theory. The following two properties do not hold in general for the ordinary geometric realization.

4.6. Proposition. [Proposition A.1 of [Seg74]]

1. *For a simplicial space X, if each X_n has the homotopy type of a CW-complex, then so does $||X||$.*
2. *If $X \to Y$ is a map of simplicial spaces such that $X_n \to Y_n$ is a homotopy equivalence for each n, then the induced map $||X|| \to ||Y||$ is a homotopy equivalence.*

For a simplicial space X and a non-negative integer m, denote

$$||X||^{(m)} = \left(\coprod_{0\leqq n\leqq m} \Delta^n \times X_n \right) /(t, \varphi_*(x)) \sim (\varphi^*(t), x).$$

We have the following pushout diagram:

$$\begin{tikzcd} \partial\Delta^n \times X_n \arrow[r] \arrow[d] & ||X||^{(n-1)} \arrow[d] \\ \Delta^n \times X_n \arrow[r] & ||X||^{(n)}, \end{tikzcd}$$

and the fat realization $||X||$ is the following sequential colimit:

$$||X||^{(0)} \subset ||X||^{(1)} \subset \cdots \subset ||X||^{(n)} \subset \cdots .$$

This construction is a key point of the proof of the above proposition in [Seg74]. Since the geometric realization of a Δ-space has the same construction, it has the same properties as the fat realization.

4.7. Corollary.

1. *For a Δ-space X, if each X_n has the homotopy type of a CW-complex, then so does $||X||$.*

2. *If $X \to Y$ is a map of Δ-spaces such that $X_n \to Y_n$ is a homotopy equivalence for each n, then the induced map $||X|| \to ||Y||$ is a homotopy equivalence.*

Regarding the second property above, we observe more explicit cell structure of $||X||$. We regard the standard simplex Δ^n as a natural CW-complex, and denote the n-face $\mathrm{Int}(\Delta^n)$ by τ^n.

4.8. Definition. *Let X be a cell complex. The set of cells of X is denoted by $P(X)$. We give a partial order on $P(X)$ defined by $e_\lambda \leqq e_\mu$ if $e_\lambda \subset \overline{e_\mu}$ for any e_λ and e_μ of $P(X)$. We call $P(X)$ the face poset of X.*

4.9. Proposition. *Let X be a Δ-space which is degreewise of the homotopy type of a CW-complex. Then, $||X||$ has the homotopy type of a CW-complex consisting of cells $\tau^n \times \sigma$ for σ in $P(X_n)$ and $n \geqq 0$.*

Proof. We prove the claim by induction on m for $||X||^{(m)}$. When $m = 0$, the space $||X||^{(0)} = \Delta^0 \times X_0$ is a CW-complex consisting of $\tau^0 \times \sigma$ for σ in $P(X_0)$. Assume that $||X||^{(m-1)}$ is a CW-complex consisting of $\tau^n \times \sigma$ for σ in $P(X_n)$ and $0 \leqq n \leqq m-1$. Consider the following pushout diagram:

$$\begin{tikzcd}
\partial\Delta^m \times X_m \arrow[r, "f"] \arrow[d] & {||X||^{(m-1)}} \arrow[d] \\
\Delta^m \times X_m \arrow[r] & {||X||^{(m)}}.
\end{tikzcd}$$

We may assume f to be a cellular map by the cellular approximation theorem. Since $\Delta^m \times X_m$ is the product CW-complex and $\partial\Delta^m \times X_m$ is a subcomplex, the above pushout diagram implies that $||X||^m$ is a CW-complex consisting of $\tau^n \times \sigma$ for an element σ of $P(X_n)$ and $0 \leqq n \leqq m$. ∎

4.10. The Euler characteristic of acyclic categories. In this subsection, we show that the Euler characteristic of an acyclic topological category $\mathcal{A}$ coincides with that of the classifying space $B\mathcal{A}$ of $\mathcal{A}$. Recall that the category of topological spaces is equipped with the monoidal model structure and the measure $\chi : \mathbf{cw} \to \mathbb{Z}$ of Example 3.30. We use this measure in the rest of this paper.

4.11. Definition. *A topological category $\mathcal{T}$ is* acyclic *if it satisfies the following two properties:*

1. *The space $\mathcal{T}(x,x)$ consists of a single point for any object x of $\mathcal{T}$.*

2. *For objects y and z of $\mathcal{T}$, if $y \neq z$ and $\mathcal{T}(y,z)$ is not empty, then $\mathcal{T}(z,y)$ is empty.*

4.12. Definition. *Let $\mathcal{T}$ be a topological category. The nerve $N\mathcal{T}$ is a simplicial space defined by*

$$N_n\mathcal{T} = \coprod_{x_i \in \mathrm{ob}(\mathcal{T})} \mathcal{T}(x_{n-1}, x_n) \times \mathcal{T}(x_{n-2}, x_{n-1}) \times \cdots \times \mathcal{T}(x_0, x_1).$$

The face map $d_j : N_n\mathcal{T} \to N_{n-1}\mathcal{T}$ is given by composing or removing morphisms; that is,

$$d_j(f_n, \ldots, f_1) = \begin{cases} (f_n, \ldots, f_2) & \text{if } j = 0, \\ (f_n, \ldots, f_{j+1} \circ f_j, \ldots, f_1) & \text{if } 0 < j < n, \\ (f_{n-1}, \ldots, f_1) & \text{if } j = n, \end{cases}$$

and the degeneracy map $s_i : N_n\mathcal{T} \to N_{n+1}\mathcal{T}$ is given by inserting an identity morphism; that is,

$$s_i(f_n, \ldots, f_1) = (f_n, \ldots, f_i, 1, f_{i-1}, \ldots f_1).$$

The classifying space *$B\mathcal{T}$ of $\mathcal{T}$ is defined as the geometric realization $|N\mathcal{T}|$ of the nerve $N\mathcal{T}$.*

4.13. Definition. *Let $\mathcal{A}$ be an acyclic topological category. The* non-degenerate nerve *$\overline{N}\mathcal{A}$ is a Δ-space defined by*

$$\overline{N}_n\mathcal{A} = \coprod_{x_i \neq x_{i-1}} \mathcal{A}(x_{n-1}, x_n) \times \mathcal{A}(x_{n-2}, x_{n-1}) \times \cdots \times \mathcal{A}(x_0, x_1).$$

The face map $d_j : \overline{N}_n\mathcal{A} \to \overline{N}_{n-1}\mathcal{A}$ is given as for the ordinary nerve $N\mathcal{A}$. Since $\mathcal{A}$ is acyclic, the composition of non-identity morphisms in $\mathcal{A}$ is also non-identity; therefore, the maps are well-defined.

4.14. Theorem. [Lemma B.13 in [Tama]] *If $\mathcal{A}$ is a finite acyclic topological category, then the classifying space $B\mathcal{A}$ is homeomorphic to $||\overline{N}\mathcal{A}||$.*

Let $\mathcal{A}$ be a finite measurable acyclic category. We give a partial order on the set of objects $\mathrm{ob}(\mathcal{A})$ such that $a \leqq b$ if $\mathcal{A}(a, b)$ is not empty. For simplicity, let $\chi(a, b)$ denote the Euler characteristic of $\mathcal{A}(a, b)$. If $a \nleqq b$, we have $\chi(a, b) = 0$ since $\mathcal{A}(a, b)$ is empty.

The following lemma is a generalization of Corollary 1.5 of [Lei08] and Hall's theorem for posets (Proposition 3.8.5 of [Sta12]). Leinster deals with finite skeletal categories (isomorphic objects must be equal) in which the only endomorphisms are identities, however as he mentioned in the part before Proposition 2.11 in [Lei08], this condition is equivalent to acyclicity in Definition 4.11.

4.15. Proposition. *The similarity matrix of a finite measurable acyclic category $\mathcal{A}$ has an inverse matrix μ, and it is given by*

$$\mu(a, b) = \sum_{j=0}^{\infty} \sum_{a = a_0 < \cdots < a_{j-1} < a_j = b} (-1)^j \chi(a_{j-1}, b) \cdots \chi(a, a_1),$$

where we regard the inner sum as one if $j = 0$.

Proof. Consider the product of μ and $\xi_{\mathcal{A}}$;

$$\begin{aligned}\sum_{b\in\mathrm{ob}(\mathcal{A})}\mu(a,b)\xi_{\mathcal{A}}(b,c) &= \sum_{b\in\mathrm{ob}(\mathcal{A})}\sum_{j=0}^{\infty}\sum_{a=a_0<\cdots<a_{j-1}<a_j=b}(-1)^j\chi(b,c)\chi(a_{j-1},b)\cdots\chi(a,a_1)\\ &= \sum_{j=0}^{\infty}\sum_{a=a_0<\cdots<a_{j-1}<a_j\leqq c}(-1)^j\chi(a_j,c)\chi(a_{j-1},a_j)\cdots\chi(a,a_1).\end{aligned}$$

When $a=c$, any object a_k in a sequence $a=a_0<\cdots<a_{j-1}<a_j\leqq c$ must be a since $\mathcal{A}$ is acyclic. Then, the right-hand side is equal to one. If $a\neq c$, then the right-hand side is

$$\chi(a,c)+(-1)\sum_{a<a_0\leqq c}\chi(a_1,c)\chi(a,a_1)+(-1)^2\sum_{a<a_1<a_2\leqq c}\chi(a_2,c)\chi(a_1,a_2)\chi(a,a_1)+\cdots.$$

The i-th term with $a_{i-1}\neq c$ and the $(i+1)$-th term with $a_i=c$ are canceled; therefore, the alternating sum collapses to zero. Hence, μ is an inverse matrix. ∎

4.16. Corollary. *The Euler characteristic of a finite measurable acyclic category $\mathcal{A}$ is*

$$\chi(\mathcal{A})=\sum_{j=0}^{\infty}\sum_{a_0<\cdots<a_j}(-1)^j\chi(a_{j-1},a_j)\cdots\chi(a_0,a_1).$$

Proof. By Remark 3.4, the Euler characteristic is

$$\chi(\mathcal{A})=\sum_{a,b\in\mathrm{ob}(\mathcal{A})}\mu(a,b)=\sum_{j=0}^{\infty}\sum_{a_0<\cdots<a_j}(-1)^j\chi(a_{j-1},a_j)\cdots\chi(a_0,a_1).$$

∎

4.17. Theorem. *The Euler characteristic $\chi(\mathcal{A})$ of a finite measurable acyclic category $\mathcal{A}$ coincides with the Euler characteristic $\chi(B\mathcal{A})$ of the classifying space $B\mathcal{A}$.*

Proof. By Theorem 4.14 and Proposition 4.9, the classifying space $B\mathcal{A}$ has the homotopy type of a finite CW-complex whose n-cell corresponds to $\tau^j\times\sigma^{k_j}\times\cdots\times\sigma^{k_1}$ for $\tau^j=\mathrm{Int}(\Delta^j)$ and a k_i-cell σ^{k_i} of $\mathcal{A}(a_i,a_{i+1})$ such that $n=j+\sum_{i=1}^{j}k_i$ and $a_i\neq a_{i+1}$ for any i. Let $\mathcal{A}_{a,b}^{(k)}$ denote the number of k-cells of $\mathcal{A}(a,b)$. The Euler characteristic $\chi(B\mathcal{A})$ can be computed as the alternating sum of the numbers of cells as follows:

$$\begin{aligned}
\chi(B\mathcal{A}) &= \sum_{n=0}^{\infty}(-1)^n \,(\text{the number of } n\text{-cells of } B\mathcal{A}\ ) \\
&= \sum_{n=0}^{\infty}(-1)^n \sum_{j=0}^{\infty} \sum_{j+k_1+\cdots+k_j=n} \sum_{a_0<\cdots<a_j} \mathcal{A}^{(k_j)}_{a_{j-1},a_j} \cdots \mathcal{A}^{(k_1)}_{a_0,a_1} \\
&= \sum_{n=0}^{\infty}\sum_{j=0}^{\infty} \sum_{j+k_1+\cdots+k_j=n} \sum_{a_0<\cdots<a_j} (-1)^j(-1)^{k_j}\mathcal{A}^{(k_j)}_{a_{j-1},a_j} \cdots (-1)^{k_1}\mathcal{A}^{(k_1)}_{a_0,a_1} \\
&= \sum_{j=0}^{\infty}\sum_{1\leq \ell\leq j} \sum_{k_\ell=0}^{\infty} \sum_{a_0<\cdots<a_j} (-1)^j(-1)^{k_j}\mathcal{A}^{(k_j)}_{a_{j-1},a_j} \cdots (-1)^{k_1}\mathcal{A}^{(k_1)}_{a_0,a_1} \\
&= \sum_{j=0}^{\infty} \sum_{a_0<\cdots<a_j} (-1)^j \chi(a_{j-1},a_j)\cdots\chi(a_0,a_1).
\end{aligned}$$

Consequently we have $\chi(B\mathcal{A}) = \chi(\mathcal{A})$ by Corollary 4.16. ∎

5. The Euler characteristic of a cellular stratified space

A cellular stratified space is a generalization of a cell complex introduced by Tamaki in [Tama]. He introduces many examples of cellular stratified spaces, for instance, regular cell complexes, complements of complexified hyperplane arrangements, and configuration spaces of graphs and spheres [Tama], [Tamb], [Tamc].

5.1. Definition. *A* stratification *on a Hausdorff space X indexed by a poset Λ is a map $\pi : X \to \Lambda$ satisfying the following properties:*

1. *For each λ in $\Lambda, \pi^{-1}(\lambda)$ is connected and open in $\overline{\pi^{-1}(\lambda)}$.*
2. *For λ and μ in Λ, $\pi^{-1}(\lambda) \subset \overline{\pi^{-1}(\mu)}$ if and only if $\lambda \leqq \mu$.*

For simplicity, we denote $e_\lambda = \pi^{-1}(\lambda)$ and call it a face*. The indexing poset Λ is called the* face poset *of X and denoted by $P(X)$. A* stratified space *(X,π) is a pair of a space X and its stratification π. Sometimes we denote it X for simplicity. When the face poset is finite, we call the stratified space* finite*. We say a face e_λ is* normal *if $e_\mu \subset \overline{e_\lambda}$ whenever $e_\mu \cap \overline{e_\lambda} \neq \emptyset$. When all faces are normal, the stratification is said to be* normal.

Let (X,π_X) and (Y,π_Y) be stratified spaces. A morphism of stratified spaces $(f,\overline{f})$ is a pair of a continuous map $f : X \longrightarrow Y$ and a map of posets $\overline{f} : P(X) \longrightarrow P(Y)$ making the following diagram commutative:

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "\pi_X"'] & Y \arrow[d, "\pi_Y"] \\
P(X) \arrow[r, "\overline{f}"'] & P(Y).
\end{tikzcd}$$

When f is a homeomorphism, the map $(f, \overline{f})$ is called a subdivision*.*

5.2. Definition.

1. *A* globular n-cell *is a subspace D of the disk D^n containing the interior* $\mathrm{Int}(D^n)$ *of D^n.*

2. *For a space X and a subspace e of X, an n-*cell structure *on e, or simply an n-*cell *e is a quotient map $\varphi : D \to \overline{e}$ from a globular n-cell D whose restriction $\varphi|_{\mathrm{Int}(D^n)} : \mathrm{Int}(D^n) \to e$ is a homeomorphism. We say that n is the* dimension *of e, and denote it $\dim e$. We say that an n-cell is* closed *when D is the closed disk D^n.*

3. *A* cellular stratified space *is a stratified space whose faces are equipped with cell structures such that*
$$\varphi(\partial D_\lambda) \subset \bigcup_{\dim e_\mu < \dim e_\lambda} e_\mu,$$
where ∂D_λ is the boundary of D_λ. When all cells are closed (called closed cellular stratified space*), this means a cell complex in Def 4.5. Furthermore, a cellular stratified space is called a* CW-stratified space *if it satisfies the closure finite and weak topology conditions (see Definition 2.19 in [Tama]).*

   Let (X, π) be a stratified space and A be a subspace of X. If the restriction $\pi|_A$ is a stratification on A, the pair $(A, \pi|_A)$ is called a stratified subspace *of (X, π).*

5.3. Definition. [Definition 3.21 of [Tama]] *A* cylindrical structure *on a normal cellular stratified space X consists of*

- *a normal stratification on ∂D^n containing ∂D_λ as a stratified subspace for each n-cell $\varphi_\lambda : D_\lambda \to \overline{e_\lambda}$,*
- *a stratified space $P_{\mu,\lambda}$ called the* parameter space*, and a morphism of stratified spaces*
$$b_{\mu,\lambda} : P_{\mu,\lambda} \times D_\mu \longrightarrow \partial D_\lambda$$
for each pair of cells $e_\mu \subset \partial e_\lambda$, and
- *a morphism of stratified spaces*
$$c_{\lambda_0,\lambda_1,\lambda_2} : P_{\lambda_1,\lambda_2} \times P_{\lambda_0,\lambda_1} \longrightarrow P_{\lambda_0,\lambda_2}$$
for each sequence $\overline{e_{\lambda_0}} \subset \overline{e_{\lambda_1}} \subset \overline{e_{\lambda_2}}$

satisfying the following conditions:

1. *The restriction of $b_{\mu,\lambda}$ to $P_{\mu,\lambda} \times \mathrm{Int}(D_\mu)$ is a homeomorphism onto its image.*

2. The following three types of diagrams are commutative:

$$\begin{tikzcd} D_\lambda \arrow[r, "\varphi_\lambda"] & X \\ P_{\mu,\lambda} \times D_\mu \arrow[u, "b_{\mu,\lambda}"] \arrow[r, "\mathrm{pr}_2"] & D_\mu \arrow[u, "\varphi_\mu"'] \end{tikzcd}$$

$$\begin{tikzcd} P_{\lambda_1,\lambda_2} \times P_{\lambda_0,\lambda_1} \times D_{\lambda_0} \arrow[r, "1\times b_{\lambda_0,\lambda_1}"] \arrow[d, "c_{\lambda_0,\lambda_1,\lambda_2}\times 1"'] & P_{\lambda_1,\lambda_2} \times D_{\lambda_1} \arrow[d, "b_{\lambda_1,\lambda_2}"] \\ P_{\lambda_0,\lambda_2} \times D_{\lambda_0} \arrow[r, "b_{\lambda_0,\lambda_2}"'] & D_{\lambda_2} \end{tikzcd}$$

$$\begin{tikzcd} P_{\lambda_2,\lambda_3} \times P_{\lambda_1,\lambda_2} \times P_{\lambda_0,\lambda_1} \arrow[r, "c\times 1"] \arrow[d, "1\times c"'] & P_{\lambda_1,\lambda_3} \times P_{\lambda_0,\lambda_1} \arrow[d, "c"] \\ P_{\lambda_2,\lambda_3} \times P_{\lambda_0,\lambda_2} \arrow[r, "c"'] & P_{\lambda_0,\lambda_3}. \end{tikzcd}$$

3. We have

$$\partial D_\lambda = \bigcup_{e_\mu \subset \partial e_\lambda} b_{\mu,\lambda}(P_{\mu,\lambda} \times \mathrm{Int}(D_\mu))$$

as a stratified space.

A normal cellular stratified space equipped with a cylindrical structure is called a cylindrically normal cellular stratified space.

5.4. Definition. *For a cylindrically normal cellular stratified space X, define a topological category $C(X)$ as follows. The set of objects* $\mathrm{ob}(C(X))$ *is the set $P(X)$. The space of morphisms is defined by*

$$C(X)(\mu, \lambda) = P_{\mu,\lambda}$$

for each $e_\mu \subset \overline{e_\lambda}$ in $P(X)$, and the composition is defined by the map $c_{\lambda_0,\lambda_1,\lambda_2}$.

Note that $C(X)$ is an acyclic topological category.

Tamaki constructs an embedding $BC(X) \hookrightarrow X$ from the classifying space of $C(X)$ to the original cylindrically normal cellular stratified space X. Note that he considers the more general "stellar" stratified spaces rather than cellular stratified spaces in his paper [Tama].

5.5. Theorem. [Theorem 4.15 in [Tama]] *There exists an embedding $BC(X) \hookrightarrow X$ for a cylindrically normal cellular stratified space X. Furthermore, when all cells are closed, the above embedding is a homeomorphism.*

In order to show that the above embedding is a homotopy equivalence for a general cylindrical cellular stratified space, Tamaki considers the following condition. See section 3.3 in [Tama] for details.

5.6. Definition. *A* locally polyhedral cellular stratified space *consists of*

- *a cylindrically normal CW-stratified space X,*
- *a family of Euclidean polyhedral complexes $\tilde{F}_\lambda$ indexed by λ in $P(X)$ and*
- *a family of homeomorphisms $\alpha_\lambda : \tilde{F}_\lambda \to D^{\dim e_\lambda}$ indexed by λ in $P(X)$,*

satisfying the following conditions:

1. *For each cell e_λ, the homeomorphism $\alpha : \tilde{F}_\lambda \to D^{\dim e_\lambda}$ is a subdivision of the stratified space, where the stratification of $D^{\dim e_\lambda}$ is defined by the cylindrical structure.*
2. *For each pair $e_\mu < e_\lambda$, the parameter space $P_{\mu,\lambda}$ is a locally cone-like space and the composition*
$$P_{\mu,\lambda} \times F_\mu \xrightarrow{1\times\alpha_\mu} P_{\mu,\lambda} \times D_\mu \xrightarrow{b_{\mu,\lambda}} D_\lambda \xrightarrow{\alpha_\lambda^{-1}} F_\lambda$$
is a PL map, where $F_\lambda = \alpha^{-1}(D_\lambda)$.

Each α_λ is called a polyhedral replacement *of the cell structure map of e_λ.*

5.7. Theorem. [Theorem 4.16 in [Tama]] *If X is a locally polyhedral cellular stratified space, then there exists an embedding $BC(X) \hookrightarrow X$ whose image is a deformation retract of X.*

5.8. Corollary. *If X is a locally polyhedral cellular stratified space, then the classifying space $BC(X)$ of the cylindrical face category $C(X)$ of X is homotopy equivalent to X.*

Now, we obtain the following main theorem. This is a generalization of a result on the Euler characteristic of ordinary cell complexes (Proposition 3.8.8 in [Sta12]).

5.9. Theorem. *Let X be a finite locally polyhedral cellular stratified space. If each parameter space $P_{\lambda,\mu}$ belongs to* **cw** *for $\lambda < \mu$ in $P(X)$, then the Euler characteristic $\chi(X)$ of X is equal to $\chi(C(X))$ of the cylindrical face category.*

Proof. It follows from Theorem 4.17 and Corollary 5.8 that

$$\chi(X) = \chi(BC(X)) = \chi(C(X)).$$

■

5.10. Example. In Example 4.7 in [Tama], it is shown that the complex projective space with the minimal decomposition $\mathbb{C}\mathbf{P}^n = e^0 \cup e^2 \cup \cdots \cup e^{2n}$ is a closed cylindrical cellular stratified space whose face category $C(\mathbb{C}\mathbf{P}^n)$ has the form

$$\bullet \xrightarrow{S^1} \bullet \xrightarrow{S^1} \bullet \xrightarrow{S^1} \cdots \xrightarrow{S^1} \bullet.$$

Since $\chi(S^1) = 0$, the similarity matrix ξ of $C(\mathbb{C}\mathbf{P}^n)$ is the unit matrix of dimension $n+1$. We can take a weighting w as $w(e^{2i}) = 1$ for all $0 \leqq i \leqq n$. Hence, $\chi(\mathbb{C}\mathbf{P}^n) = \chi\left(C\left(\mathbb{C}\mathbf{P}^n\right)\right) = n+1$.

However, for obtaining $\chi(\mathbb{C}\mathbf{P}^n)$, it is easier to calculate the alternating sum of the numbers of cells than to proceed as above. The next example is of a non-closed cellular stratified space, where it is difficult to calculate the Euler characteristic from the numbers of cells.

5.11. Example. For the spheres S^n and S^m, let $X = S^n \times S^m - \{*\}$. This is a cellular stratified space consisting of the cell structure

$$\varphi_j : D_j = \mathrm{Int}(D^j) \longrightarrow S^j - \{*\},$$

which is the restriction of the canonical projection $D^j \to S^j$ collapsing ∂D^j to a single point for $j = n, m$, and

$$\varphi_{n+m} = \varphi_n \times \varphi_m : D_{n+m} = (D^n \times D^m) - (\partial D^n \times \partial D^m) \longrightarrow S^n \times S^m - \{*\},$$

where we regard $D^n \times D^m$ as D^{n+m}. The boundary of D_{n+m} is

$$\begin{aligned}\partial D_{n+m} &= \partial(D^n \times D^m) - (\partial D^n \times \partial D^m) \\ &= (\partial D^n \times D^m) \cup (D^n \times \partial D^m) - (\partial D^n \times \partial D^m) \\ &= \left(\partial D^n \times \mathrm{Int}(D^m)\right) \coprod \left(\mathrm{Int}(D^n) \times \partial D^m\right).\end{aligned}$$

A normal stratification on ∂D^{n+m} is induced by the canonical cell decomposition on ∂I^{n+m} and $\partial I^{n+m} \cong \partial D^{n+m}$, where I is the interval $[0,1]$. The cylindrical structure is given by the inclusions

$$b_{ij} : S^{i-1} \times D_j = S^{i-1} \times \mathrm{Int}(D^j) \hookrightarrow \partial D_{n+m}$$

for $(i,j) = (n,m), (m,n)$. Then the cylindrical face category $C(X)$ has the form:

$$\bullet \xrightarrow{S^{n-1}} \bullet \xleftarrow{S^{m-1}} \bullet.$$

By a family of polyhedral replacements $\{\alpha_j : I^j \cong D^j\}_{j=n,m,m+n}$, the space X is a locally polyhedral cellular stratified space. The similarity matrix

$$\xi = \begin{pmatrix} 1 & 0 & \chi(S^{n-1}) \\ 0 & 1 & \chi(S^{m-1}) \\ 0 & 0 & 1 \end{pmatrix}$$

of $C(X)$ has an inverse matrix

$$\xi^{-1} = \begin{pmatrix} 1 & 0 & -\chi(S^{n-1}) \\ 0 & 1 & -\chi(S^{m-1}) \\ 0 & 0 & 1 \end{pmatrix}.$$

Theorem 5.9 shows that the Euler characteristic of X is

$$\chi(X) = \chi(C(X)) = 3 - \chi(S^{n-1}) - \chi(S^{m-1}) = \begin{cases} -1 & \text{if both } n \text{ and } m \text{ are odd,} \\ 3 & \text{if both } n \text{ and } m \text{ are even,} \\ 1 & \text{otherwise.} \end{cases}$$

Indeed, the space $X = S^n \times S^m - \{*\}$ is homotopy equivalent to $BC(X) = S^{n-1} \vee S^{m-1}$. The Euler characteristic $\chi(X)$ can also be obtained by calculating $H_*(S^{n-1} \vee S^{m-1})$.

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