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BICATEGORICAL HOMOTOPY PULLBACKS

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Abstract. The homotopy theory of higher categorical structures has become a relevant part of the machinery of algebraic topology and algebraic K-theory, and this paper contains contributions to the study of the relationship between Bénabou's bicategories and the homotopy types of their classifying spaces. Mainly, we state and prove an extension of Quillen's Theorem B by showing, under reasonable necessary conditions, a bicategory-theoretical interpretation of the homotopy-fibre product of the continuous maps induced on classifying spaces by a diagram of bicategories $\mathcal{A} \to \mathcal{B} \leftarrow \mathcal{A}'$. Applications are given for the study of homotopy pullbacks of monoidal categories and of crossed modules.

1. Introduction and summary

If $A \overset{\phi}{\to} B \overset{\phi'}{\leftarrow} A'$ are continuous maps between topological spaces, its *homotopy-fibre product* $A \times^{\mathrm{h}}_{B} A'$ is the subspace of the product $A \times B^I \times A'$, where $I = [0,1]$ and B^I is taken with the compact-open topology, whose points are triples (a, γ, a') with $a \in A$, $a' \in A'$, and $\gamma : \phi a \to \phi' a'$ a path in B joining ϕa and $\phi' a'$, that is $\gamma : I \to B$ is a path starting at $\gamma 0 = \phi a$ and ending at $\gamma 1 = \phi' a'$. In particular, the *homotopy-fibre* of a continuous map $\phi : A \to B$ over a base point $b \in B$ is $\mathrm{Fib}(\phi, b) = A \times^{\mathrm{h}}_{B} \{b\}$, the homotopy-fibre product of ϕ and the constant inclusion map $\{b\} \hookrightarrow B$. That is, $\mathrm{Fib}(\phi, b)$ is the space of pairs (a, γ), where $a \in A$, and $\gamma : \phi a \to b$ is a path in B joining ϕa with the base point b.

If $\mathcal{A} \overset{F}{\to} \mathcal{B} \overset{F'}{\leftarrow} \mathcal{A}'$ are now functors between (small) categories, its *homotopy-fibre product category* is the comma category $F \downarrow F'$ consisting of triples (a, f, a') with $f : Fa \to F'a'$ a morphism in $\mathcal{B}$, in which a morphism from (a_0, f_0, a'_0) to (a_1, f_1, a'_1) is a pair of morphisms $u : a_0 \to a_1$ in $\mathcal{A}$ and $u' : a'_0 \to a'_1$ in A' such that $F'u' \circ f_0 = f_1 \circ Fu$. In particular, the *homotopy-fibre category* $F \downarrow b$ of a functor $F : \mathcal{A} \to \mathcal{B}$, relative to an object $b \in \mathrm{Ob}\mathcal{B}$, is the homotopy-fibre product category of F and the constant functor $\{b\} \hookrightarrow \mathcal{B}$. These naive categorical emulations of the topological constructions are, however, subtle. Let $\mathrm{B} : \mathbf{Cat} \to \mathbf{Top}$ be the classifying space functor. The homotopy-fibre product category $F \downarrow F'$ comes with a canonical map from its classifying space to the homotopy-fibre product space of the induced maps $\mathrm{B}F : \mathrm{B}\mathcal{A} \to \mathrm{B}\mathcal{B}$ and $\mathrm{B}F' : \mathrm{B}\mathcal{A}' \to \mathrm{B}\mathcal{B}$, and

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Barwick and Kan [3, Theorem 3.5] [4, Theorem 8.2] have proven that *this canonical map* $\mathrm{B}(F \downarrow F') \to \mathrm{B}\mathcal{A} \times^{\mathrm{h}}_{\mathrm{B}\mathcal{B}} \mathrm{B}\mathcal{A}'$ *is a homotopy equivalence whenever the maps* $\mathrm{B}(F \downarrow b_0) \to \mathrm{B}(F \downarrow b_1)$*, induced by the different morphisms* $b_0 \to b_1$ *of* $\mathcal{B}$*, are homotopy equivalences.* This result extends the well-known Quillen's Theorem B, which asserts that *under such an hypothesis, the canonical maps* $\mathrm{B}(F \downarrow b) \to \mathrm{Fib}(\mathrm{B}F, \mathrm{B}b)$ *are homotopy equivalences.* Actually, the result by Barwick and Kan is a consequence of a Theorem B by Cisinski [22, Théorèm 6.4.15][1]. Let us stress that Theorem B and its consequent Theorem A have been fundamental for higher algebraic K-theory since the early 1970s, when Quillen [42] published his seminal paper, and they are now two of the most important theorems in the foundation of homotopy theory.

Similar categorical lax limit constructions have been used to describe homotopy pullbacks in many settings of enriched categories, where a homotopy theory has been established (see Grandis [30], for instance). Here, we focus on bicategories. Like categories, small Bénabou bicategories [5] and, in particular, 2-categories and Mac Lane's monoidal categories, are closely related to topological spaces through the classifying space construction, as shown by Carrasco, Cegarra, and Garzón in [16]. This assigns to each bicategory $\mathcal{B}$ a CW-complex $\mathrm{B}\mathcal{B}$, whose cells give a natural geometric meaning to the cells of the bicategory. By this assignment, for example, bigroupoids correspond to homotopy 2-types, that is, to CW-complexes whose n^{th} homotopy groups at any base point vanish for $n \geq 3$ (see Duskin [26, Theorem 8.6]), and homotopy regular monoidal categories to delooping spaces of the classifying spaces of the underlying categories (see Jardine [35, Propositions 3.5 and 3.8]). Further, we should mention that the category of (strict) 2-categories and 2-functors has a Thomason-type Quillen model structure, as was first announced by Worytkiewicz, Hess, Parent and Tonks in [50, Theorem 4.5.1] and fully proved by Ara and Maltsiniotis in [1, Théorème 6.27], such that the classifying space functor $\mathcal{B} \mapsto \mathrm{B}\mathcal{B}$ is a equivalence of homotopy theories between 2-categories and topological spaces.

In the preparatory Section 2 of this paper, for any diagram $\mathcal{A} \xrightarrow{F} \mathcal{B} \xleftarrow{F'} \mathcal{A}'$, where $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{A}'$ are bicategories, F is a lax functor, and F' is an oplax functor (for instance, if F and F' are both homomorphisms), we present a *homotopy-fibre product bicategory* $F \downarrow F'$, whose 0-cells, or objects, are triples (a, f, a') with $f : Fa \to F'a'$ a 1-cell in $\mathcal{B}$ as in the case when F and F' are functors between categories. But now, a 1-cell from (a_0, f_0, a'_0) to (a_1, f_1, a'_1) is a triple (u, β, u') consisting of 1-cells $u : a_0 \to a_1$ in $\mathcal{A}$ and $u' : a'_0 \to a'_1$ in $\mathcal{A}'$, together with a 2-cell $\beta : F'u' \circ f_0 \Rightarrow f_1 \circ Fu$ in $\mathcal{B}$. And $F \downarrow F'$ has 2-cells $(\alpha, \alpha') : (u, \beta, u') \Rightarrow (v, \gamma, v')$, which are given by 2-cells $\alpha : u \Rightarrow v$ in $\mathcal{A}$ and $\alpha' : u' \Rightarrow v'$ in $\mathcal{A}'$ such that $(1_{f_1} \circ F\alpha) \cdot \beta = (\gamma \circ F'\alpha') \circ 1_{f_0}$. In particular, for any object $b \in \mathcal{B}$, we have the *homotopy-fibre bicategories* $F \downarrow b$ and $b \downarrow F'$, in terms of which we state and prove our main results of the paper. These are exposed in Section 3, and they can be summarized as follows (see Theorem 3.1 and Corollary 3.6):

• *For any diagram of bicategories* $\mathcal{A} \xrightarrow{F} \mathcal{B} \xleftarrow{F'} \mathcal{A}'$*, where* F *is a lax functor and* F' *is an oplax functor, there is a canonical map* $\mathrm{B}(F \downarrow F') \to \mathrm{B}\mathcal{A} \times^{\mathrm{h}}_{\mathrm{B}\mathcal{B}} \mathrm{B}\mathcal{A}'$*, from the classifying*

[1]We thank the referee for pointing out this fact.

space of the homotopy-fibre product bicategory to the homotopy-fibre product space of the induced maps $\mathrm{B}F : \mathrm{B}\mathcal{A} \to \mathrm{B}\mathcal{B}$ *and* $\mathrm{B}F' : \mathrm{B}\mathcal{A}' \to \mathrm{B}\mathcal{B}$.

• *For a given lax functor* $F : \mathcal{A} \to \mathcal{B}$, *the following properties are equivalent:*

- For any oplax functor $F' : \mathcal{A}' \to \mathcal{B}$, *the map* $\mathrm{B}(F\downarrow F') \to \mathrm{B}\mathcal{A} \times^{\mathrm{h}}_{\mathrm{B}\mathcal{B}} \mathrm{B}\mathcal{A}'$ *is a homotopy equivalence.*

- For any 1-cell $b_0 \to b_1$ *of* $\mathcal{B}$, *the map* $\mathrm{B}(F\downarrow b_0) \to \mathrm{B}(F\downarrow b_1)$ *is a homotopy equivalence.*

- For any 0-cell b *of* $\mathcal{B}$, *the map* $\mathrm{B}(F\downarrow b) \to \mathrm{Fib}(\mathrm{B}F, \mathrm{B}b)$ *is a homotopy equivalence.*

• *For a given oplax functor* $F' : \mathcal{A}' \to \mathcal{B}$, *the following properties are equivalent:*

-For any lax functor $F : \mathcal{A} \to \mathcal{B}$, *the map* $\mathrm{B}(F\downarrow F') \to \mathrm{B}\mathcal{A} \times^{\mathrm{h}}_{\mathrm{B}\mathcal{B}} \mathrm{B}\mathcal{A}'$ *is a homotopy equivalence.*

- For any 1-cell $b_0 \to b_1$ *of* $\mathcal{B}$, *the map* $\mathrm{B}(b_1 \downarrow F') \to \mathrm{B}(b_0 \downarrow F)$ *is a homotopy equivalence.*

- For any 0-cell b *of* $\mathcal{B}$, *the map* $\mathrm{B}(b\downarrow F') \to \mathrm{Fib}(\mathrm{B}F', \mathrm{B}b)$ *is a homotopy equivalence.*

Let us remark that, if the map $\mathrm{B}(F\downarrow F') \to \mathrm{B}\mathcal{A} \times^{\mathrm{h}}_{\mathrm{B}\mathcal{B}} \mathrm{B}\mathcal{A}'$ is a homotopy equivalence, then, by Dyer and Roitberg [28], there are Mayer-Vietoris type long exact sequences on homotopy groups

$$\cdots \to \pi_{n+1}\mathrm{B}\mathcal{B} \to \pi_n\mathrm{B}(F\downarrow F') \to \pi_n\mathrm{B}\mathcal{A} \times \pi_n\mathrm{B}\mathcal{A}' \to \pi_n\mathrm{B}\mathcal{B} \to \cdots .$$

The above results include the aforementioned results by Barwick and Kan, but also the extension of Quillen's Theorems A and B to lax functors between bicategories stated by Calvo, Cegarra, and Heredia in [15, Theorem 5.4], as well as the generalized Theorem A for lax functors from categories into 2-categories by del Hoyo in [24, Theorem 6.4] (see Corollaries 3.6 and 3.7). Related to this, an interesting relative Theorem A for lax functors between 2-categories has recently been proven by Chiche in [21, Théorème 4.1].

We also study conditions on a bicategory $\mathcal{B}$ in order to ensure that the space $\mathrm{B}(F\downarrow F')$ is always homotopy equivalent to the homotopy-fibre product of the induced maps $\mathrm{B}F : \mathrm{B}\mathcal{A} \to \mathrm{B}\mathcal{B}$ and $\mathrm{B}F' : \mathrm{B}\mathcal{A}' \to \mathrm{B}\mathcal{B}$. Thus, in Theorem 3.8, we prove

• *For a bicategory* $\mathcal{B}$, *the following properties are equivalent:*

- For any diagram $\mathcal{A} \xrightarrow{F} \mathcal{B} \xleftarrow{F'} \mathcal{A}'$, *where* F *is a lax functor and* F' *is an oplax functor, the map* $\mathrm{B}(F\downarrow F') \to \mathrm{B}\mathcal{A} \times^{\mathrm{h}}_{\mathrm{B}\mathcal{B}} \mathrm{B}\mathcal{A}'$ *is a homotopy equivalence*

- For any object b *and* 1*-cell* $b_0 \to b_1$ *in* $\mathcal{B}$, *the induced map* $\mathrm{B}\mathcal{B}(b, b_0) \to \mathrm{B}\mathcal{B}(b, b_1)$ *is a homotopy equivalence.*

- For any object b *and* 1*-cell* $b_0 \to b_1$ *in* $\mathcal{B}$, *the induced map* $\mathrm{B}\mathcal{B}(b_1, b) \to \mathrm{B}\mathcal{B}(b_0, b)$ *is a homotopy equivalence.*

- For any two objects $b, b' \in \mathcal{B}$, *the canonical map*

$$\mathrm{B}\mathcal{B}(b, b') \to \{\gamma : I \to \mathrm{B}\mathcal{B} \mid \gamma(0) = \mathrm{B}b, \gamma(1) = \mathrm{B}b'\} \subseteq \mathrm{B}\mathcal{B}^I$$

is a homotopy equivalence.

For a bicategory $\mathcal{B}$ satisfying the conditions above, we conclude the existence of a canonical homotopy equivalence

$$\mathrm{B}\mathcal{B}(b, b) \simeq \Omega(\mathrm{B}\mathcal{B}, \mathrm{B}b)$$

between the loop space of the classifying space of the bicategory with base point Bb and the classifying space of the category of endomorphisms of b in $\mathcal{B}$ (see Corollary 3.9). This result for $\mathcal{B}$ a 2-category should be attributed to Tillmann [48, Lemma 3.3], but it has been independently proven by both the first author [18, Example 4.4] and by Del Hoyo [24, Theorem 8.5].

Since any monoidal category can be regarded as a bicategory with only one 0-cell, our results are applicable to them. Thus, any diagram of monoidal functors and monoidal categories, $(\mathcal{N},\otimes) \xrightarrow{F} (\mathcal{M},\otimes) \xleftarrow{F'} (\mathcal{N}',\otimes)$, gives rise to a *homotopy-fibre product bicategory* $F \overset{\otimes}{\downarrow} F'$, whose 0-cells are the objects $m \in \mathcal{M}$, whose 1-cells $(n, f, n') : m_0 \to m_1$ consist of objects $n \in \mathcal{N}$ and $n' \in \mathcal{N}'$, and a morphism $f : F'n' \otimes m_0 \to m_1 \otimes Fn$ in $\mathcal{M}$, and whose 2-cells $(u, u') : (n, f, n') \Rightarrow (\bar{n}, \bar{f}, \bar{n}')$ are given by a pair of morphisms, $u : n \to \bar{n}$ in $\mathcal{N}$ and $u' : n' \to \bar{n}'$ in $\mathcal{N}'$, such that $(1 \otimes Fu) \cdot f = \bar{f} \cdot (F'u' \otimes 1)$. In particular, for any monoidal functor F as above, we have the *homotopy-fibre bicategory* $F \overset{\otimes}{\downarrow} \mathrm{I}$, where $\mathrm{I} : ([0],\otimes) \to (\mathcal{M},\otimes)$ denotes the monoidal functor from the trivial (one-arrow) monoidal category $[0]$ to $\mathcal{M}$ that carries its unique object 0 to the unit object I of the monoidal category $\mathcal{M}$. Then, our main conclusions concerning monoidal categories, which are presented throughout Section 4, are summarized as follows (see Theorems 4.2, 4.3, and 4.4).

- *The following properties on a monoidal functor $F : (\mathcal{N},\otimes) \to (\mathcal{M},\otimes)$ are equivalent:*

- For any monoidal functor $F' : (\mathcal{N}',\otimes) \to (\mathcal{M},\otimes)$, the canonical map

$$\mathrm{B}(F \overset{\otimes}{\downarrow} F') \to \mathrm{B}(\mathcal{N},\otimes) \times^{\mathrm{h}}_{\mathrm{B}(\mathcal{M},\otimes)} \mathrm{B}(\mathcal{N}',\otimes)$$

is a homotopy equivalence.

- For any object $m \in \mathcal{M}$, the homomorphism $m\otimes - : F \overset{\otimes}{\downarrow} \mathrm{I} \to F \overset{\otimes}{\downarrow} \mathrm{I}$ induces a homotopy autoequivalence on $\mathrm{B}(F \overset{\otimes}{\downarrow} \mathrm{I})$.

- The canonical map $\mathrm{B}(F \overset{\otimes}{\downarrow} \mathrm{I}) \to \mathrm{Fib}(\mathrm{B}F, \mathrm{BI})$ *is a homotopy equivalence.*

- *The following properties on a monoidal category $(\mathcal{M},\otimes)$ are equivalent:*

- For any diagram of monoidal functors $(\mathcal{N},\otimes) \xrightarrow{F} (\mathcal{M},\otimes) \xleftarrow{F'} (\mathcal{N}',\otimes)$, the canonical map $\mathrm{B}(F \overset{\otimes}{\downarrow} F') \to \mathrm{B}(\mathcal{N},\otimes) \times^{\mathrm{h}}_{\mathrm{B}(\mathcal{M},\otimes)} \mathrm{B}(\mathcal{N}',\otimes)$ *is a homotopy equivalence.*

- For any object $m \in \mathcal{M}$, the functor $m \otimes - : \mathcal{M} \to \mathcal{M}$ induces a homotopy autoequivalence on the classifying space $\mathrm{B}\mathcal{M}$.

- For any object $m \in \mathcal{M}$, the functor $- \otimes m : \mathcal{M} \to \mathcal{M}$ induces a homotopy autoequivalence on the classifying space $\mathrm{B}\mathcal{M}$.

- The canonical map from the classifying space of the underlying category into the loop space of the classifying space of the monoidal category is a homotopy equivalence, $\mathrm{B}\mathcal{M} \simeq \Omega\mathrm{B}(\mathcal{M},\otimes)$.

The equivalence between the two last statements in the first result above might be considered as a version of Quillen's Theorem B for monoidal functors. A monoidal

version of Theorem A follows: *If the homotopy-fibre bicategory of a monoidal functor* $F : (\mathcal{N}, \otimes) \to (\mathcal{M}, \otimes)$ *is contractible, that is,* $\mathrm{B}(F \overset{\otimes}{\downarrow} \mathrm{I}) \simeq \mathrm{pt}$*, then the induced map* $\mathrm{B}F : \mathrm{B}(\mathcal{N}, \otimes) \to \mathrm{B}(\mathcal{M}, \otimes)$ *is a homotopy equivalence.* The equivalence of the three last statements in the second one are essentially due to Stasheff [45].

Thanks to the equivalence between the category of crossed modules and the category of 2-groupoids, by Brown and Higgins [10, Theorem 4.1], our results on bicategories also find application in the setting of crossed modules, what we do in Section 5. Briefly, for any diagram of crossed modules $(\mathcal{G}, \mathcal{P}, \partial) \xrightarrow{(\varphi, F)} (\mathcal{H}, \mathcal{Q}, \partial) \xleftarrow{(\varphi', F')} (\mathcal{G}', \mathcal{P}', \partial)$, we construct its *homotopy-fibre product crossed module* $(\varphi, F)\downarrow(\varphi', F')$, and we prove as the main result here (see Theorem 5.4) the following:

- *There is a canonical homotopy equivalence*

$$\mathrm{B}\big((\varphi, F)\downarrow(\varphi', F')\big) \simeq \mathrm{B}(\mathcal{G}, \mathcal{P}, \partial) \times^{\mathrm{h}}_{\mathrm{B}(\mathcal{H}, \mathcal{Q}, \partial)} \mathrm{B}(\mathcal{G}', \mathcal{P}', \partial)$$

between the classifying space of the homotopy-fibre product crossed module and the homotopy-fibre product space of the induced maps $\mathrm{B}(\varphi, F) : \mathrm{B}(\mathcal{G}, \mathcal{P}, \partial) \to \mathrm{B}(\mathcal{H}, \mathcal{Q}, \partial)$ *and* $\mathrm{B}(\varphi', F') : \mathrm{B}(\mathcal{G}', \mathcal{P}', \partial) \to \mathrm{B}(\mathcal{H}, \mathcal{Q}, \partial)$*.*

(Here, $(\mathcal{G}, \mathcal{P}, \partial) \mapsto \mathrm{B}(\mathcal{G}, \mathcal{P}, \partial)$ denotes the classifying space of crossed modules functor by Brown and Higgins [12].) Recalling that the category of crossed complexes has a closed model structure, as shown by Brown and Golasinki in [9], we also prove that the constructed homotopy-fibre product crossed module $(\varphi, F) \downarrow (\varphi', F')$ is compatible with the construction of homotopy pullbacks in this model category. More precisely, in Theorem 5.6, we prove that

- *If one of the morphisms* (φ, F) *or* (φ', F') *is a fibration, then the canonical morphism*

$$(\mathcal{G}, \mathcal{P}, \partial) \times_{(\mathcal{H}, \mathcal{Q}, \partial)} (\mathcal{G}', \mathcal{P}', \partial) \to (\varphi, F)\downarrow(\varphi', F'),$$

from the pullback crossed module to the homotopy-fibre product crossed module induces a homotopy equivalence on classifying spaces.

The paper also includes some new results concerning classifying spaces of bicategories, which are needed here to obtain the aforementioned results on homotopy-fibre products. On the one hand, although in [16, §4] it was proven that the classifying space construction is a functor from the category of bicategories and homomorphisms to the category **Top** of spaces, in this paper we need to extend that fact as given below (see Lemma 2.9).

- *The assignment* $\mathcal{B} \mapsto \mathrm{B}\mathcal{B}$ *is the function on objects of two functors*

$$\mathbf{Lax} \xrightarrow{\mathrm{B}} \mathbf{Top} \xleftarrow{\mathrm{B}} \mathbf{opLax},$$

where **Lax** *is the category of bicategories and lax functors, and* **opLax** *the category of bicategories and oplax functors.*

On the other hand, we also need to work with Duskin and Street's *geometric nerves* of bicategories [26, 46]. That is, with the simplicial sets $\Delta^{\mathrm{u}}\mathcal{B}$, $\Delta\mathcal{B}$, $\nabla_{\mathrm{u}}\mathcal{B}$, and $\nabla\mathcal{B}$, whose respective p-simplices are the normal lax, lax, normal oplax, and oplax functors from the

category $[p] = \{0 < \cdots < p\}$ into the bicategory $\mathcal{B}$. Although in [16, Theorem 6.1] the existence of homotopy equivalences

$$|\Delta^{\mathrm{u}}\mathcal{B}| \simeq |\Delta\mathcal{B}| \simeq \mathrm{B}\mathcal{B} \simeq |\nabla\mathcal{B}| \simeq |\nabla_{\mathrm{u}}\mathcal{B}|$$

was proved, their natural behaviour is not studied. Then, in Lemma 2.10 we state the following:

• *For any bicategory $\mathcal{B}$, the homotopy equivalence $|\Delta^{\mathrm{u}}\mathcal{B}| \simeq |\Delta\mathcal{B}|$ is natural on normal lax functors, the homotopy equivalence $|\Delta\mathcal{B}| \simeq \mathrm{B}\mathcal{B}$ is homotopy natural on lax functors, the homotopy equivalence $\mathrm{B}\mathcal{B} \simeq |\nabla\mathcal{B}|$ is homotopy natural on oplax functors, and the homotopy equivalence $|\nabla\mathcal{B}| \simeq |\nabla_{\mathrm{u}}\mathcal{B}|$ is natural on normal oplax functors.*

The proofs of these results are quite long and technical. Therefore, to avoid hampering the flow of the paper, we have put most of them into an appendix, comprising Section 6.

2. Preparation: The constructions involved

This section aims to make this paper as self-contained as possible; therefore, at the same time as fixing notations and terminology, we also review some necessary aspects and results about homotopy pullbacks of topological spaces, comma bicategories, and classifying spaces of small bicategories that are used throughout the paper. However, some results, mainly those in Lemmas 2.3, 2.9, and 2.10, are actually new. For a detailed study of the definition of homotopy pullback of continuous maps we refer the reader to Mather's original paper [40] and to the more recent approach by Doeraene [25]. For a general background on simplicial sets and homotopy pullbacks in model categories, we recommend the books by Goerss and Jardine [29] and Hirschhorn [33]. For a complete description of bicategories, lax functors, and lax transformations, we refer the reader to the papers by Bénabou [5, 6] and Street [46].

2.1. Homotopy pullbacks. Throughout this paper, *all topological spaces have the homotopy type of CW-complexes*, so that a continuous map is a homotopy equivalence if and only if it is a weak homotopy equivalence.

If $X \xrightarrow{f} B \xleftarrow{g} Y$ are continuous maps, recall that its *homotopy-fibre product* is the space

$$X \times^{\mathrm{h}}_{B} Y = X \times_B B^I \times_B Y$$

consisting of triples (x, γ, y) with x a point of X, y a point of Y, and $\gamma : I \to B$ a path of B joining $f(x)$ and $g(y)$. This space occurs in the so-called *standard homotopy pullback* of f and g, that is, the homotopy commutative square

$$\begin{array}{ccc} X \times^{\mathrm{h}}_{B} Y & \xrightarrow{f'} & Y \\ {\scriptstyle g'}\downarrow & \overset{F}{\Rightarrow} & \downarrow{\scriptstyle g} \\ X & \xrightarrow{f} & B \end{array}$$

where f' and g' are the evident projection maps, and $F : (X \times^{\mathrm{h}}_B Y) \times I \to B$ is the homotopy from fg' to gf' given by $F(x, \gamma, y, t) = \gamma(t)$. In particular, for any continuous map $g : Y \to B$ and any point $b \in B$, we have the standard homotopy pullback

$$\begin{array}{ccc} \mathrm{Fib}(g,b) & \longrightarrow & Y \\ \downarrow & \overset{F}{\Rightarrow} & \downarrow g \\ \mathrm{pt} & \xrightarrow{\ b\ } & B, \end{array}$$

where $\mathrm{Fib}(g, b) = \mathrm{pt} \times^{\mathrm{h}}_B Y$ is the *homotopy-fibre* of g over b. (We use pt to denote a one-point space.) For any $y \in g^{-1}(b)$, one has the exact homotopy sequence

$$\cdots \to \pi_{n+1}(B, b) \to \pi_n(\mathrm{Fib}(g, b), (\mathrm{Ct}_b, y)) \to \pi_n(Y, y) \to \pi_n(B, b) \to \cdots,$$

from which g is a homotopy equivalence if and only if all its homotopy fibres are contractible.

More generally, following Mather's definition in [40], a homotopy commutative square

$$\begin{array}{ccc} Z & \xrightarrow{f'} & Y \\ {\scriptstyle g'}\downarrow & \overset{H}{\Rightarrow} & \downarrow g \\ X & \xrightarrow{\ f\ } & B, \end{array} \tag{1}$$

where $H : fg' \Rightarrow gf'$ is a homotopy, is called a *homotopy pullback* whenever the induced *whisker* map below is a homotopy equivalence.

$$w : Z \to X \times^{\mathrm{h}}_B Y, \quad z \mapsto (g'(z), H|_{z\times I}, f'(z)) \tag{2}$$

Throughout the paper, we use only basic well-known properties of homotopy pullbacks. For instance, the *homotopy-fibre characterization of homotopy pullback squares*: The homotopy commutative square (1) is a homotopy pullback if and only if, for any point $x \in X$, the composite square

$$\begin{array}{ccccc} \mathrm{Fib}(g', x) & \longrightarrow & Z & \xrightarrow{f'} & Y \\ \downarrow & \Rightarrow & {\scriptstyle g'}\downarrow & \Rightarrow & \downarrow g \\ \mathrm{pt} & \xrightarrow{\ x\ } & X & \xrightarrow{\ f\ } & B \end{array}$$

is a homotopy pullback. That is, if and only if the induced whisker maps on homotopy fibres are homotopy equivalences, $w : \mathrm{Fib}(g', x) \xrightarrow{\sim} \mathrm{Fib}(g, f(x))$; or the *two out of three property* of homotopy pullbacks: Let

$$\begin{array}{ccccc} \bullet & \longrightarrow & \bullet & \longrightarrow & \bullet \\ \downarrow & \Rightarrow & \downarrow & \Rightarrow & \downarrow \\ X' & \longrightarrow & X & \longrightarrow & \bullet \end{array}$$

be a diagram of homotopy commutative squares. If the right square is a homotopy pullback, then the left square is a homotopy pullback if and only if the composite square is as well. If $\pi_0 X' \to \pi_0 X$ is onto and the left and composite squares are homotopy pullbacks, then the right-hand square is a homotopy pullback.

Many other properties are easily deduced from the above ones. For example, the square (1) is a homotopy pullback whenever both maps g and g' are homotopy equivalences. If the square is a homotopy pullback and the map g is a homotopy equivalence, then so is g'. If the square is a homotopy pullback, g' is a homotopy equivalence, and the map $\pi_0 X \to \pi_0 B$ is surjective, then g is also a homotopy equivalence.

Hereafter, any (strictly) commutative square of spaces

$$\begin{array}{ccc} Z & \xrightarrow{f'} & Y \\ {\scriptstyle g'}\downarrow & & \downarrow{\scriptstyle g} \\ X & \xrightarrow{f} & B, \end{array} \tag{3}$$

will be considered equipped with the static homotopy $Z \times I \to B$, $(z,t) \mapsto fg'(z) = gf'(z)$.

2.2. Remark. *A commutative square of spaces, as above, is a homotopy pullback if and only if it is a homotopy pullback in terms of the ordinary Quillen model structure for spaces.* To see that, simply observe that, given the commutative square (3), the whisker map (2) is the composite $Z \to X \times_B Y \xrightarrow{w} X \times^{\mathrm{h}}_B Y$, where $Z \to X \times_B Y$ is the canonical map $z \mapsto (g'(z), f'(z))$ into the topological fibre product. If f or g is a Serre fibration, the map $X \times_B Y \to X \times^{\mathrm{h}}_B Y$ is a homotopy equivalence, and therefore $Z \to X \times^{\mathrm{h}}_B Y$ is a homotopy equivalence if and only if $Z \to X \times_B Y$ is.

In [20, Proposition 5.4 and Corollary 5.5], Chachólski, Pitsch, and Scherer characterize continuous maps that always produce homotopy pullback squares when one pulls back with them. Along similar lines, we prove the needed lemma below for maps induced on geometric realizations by simplicial maps. More precisely, we characterize those simplicial maps $g : Y \to B$ such that, for any simplicial map $f : X \to B$, the pullback square of simplicial sets

$$\begin{array}{ccc} X \times_B Y & \xrightarrow{f'} & Y \\ {\scriptstyle g'}\downarrow & & \downarrow{\scriptstyle g} \\ X & \xrightarrow{f} & B \end{array} \tag{4}$$

induces, by taking geometric realizations, a homotopy pullback square of spaces. To do so, recall the canonical homotopy colimit decomposition of a simplicial map, which allows the source of the map to be written as the homotopy colimit of its fibres over the simplices of the target: for a simplicial set B, we can consider its category of simplices $\Delta \downarrow B$ whose objects are the simplicial maps $\Delta[n] \to B$ and whose morphisms are the obvious commutative triangles. For a simplicial map $g : Y \to B$, we can then associate a functor from $\Delta \downarrow B$ to the category of spaces by mapping a simplex $x : \Delta[n] \to B$ to the geometric

realization $|g^{-1}(x)|$ of the simplicial set $g^{-1}(x)$ defined by the pullback square

$$\begin{array}{ccc} g^{-1}(x) & \longrightarrow & Y \\ \downarrow & & \downarrow{\scriptstyle g} \\ \Delta[n] & \xrightarrow{x} & B. \end{array}$$

By [29, Lemma IV.5.2], in the induced commutative diagram of spaces,

$$\begin{array}{ccc} \operatorname*{hocolim}_{x:\Delta[n]\to B} |g^{-1}(x)| & \xrightarrow{\sim} & |Y| \\ \downarrow & (a) & \downarrow{\scriptstyle |g|} \\ \operatorname*{hocolim}_{x:\Delta[n]\to B} |\Delta[n]| & \xrightarrow{\sim} & |B| \end{array}$$

the horizontal maps are both homotopy equivalences.

2.3. Lemma. *For any given simplicial map $g : Y \to B$, the following statements are equivalent:*

(i) For any simplex of B, $x : \Delta[n] \to B$, and for any simplicial map $\sigma : \Delta[m] \to \Delta[n]$, the induced map $|g^{-1}(x\sigma)| \to |g^{-1}(x)|$ is a homotopy equivalence.

(ii) For any simplex $x : \Delta[n] \to B$, the induced pullback square of spaces

$$\begin{array}{ccc} |g^{-1}(x)| & \longrightarrow & |Y| \\ \downarrow & & \downarrow{\scriptstyle |g|} \\ |\Delta[n]| & \xrightarrow{|x|} & |B| \end{array}$$

is a homotopy pullback.

(iii) For any simplicial map $f : X \to B$, the pullback square of spaces

$$\begin{array}{ccc} |X \times_B Y| & \xrightarrow{|f'|} & |Y| \\ {\scriptstyle |g'|}\downarrow & & \downarrow{\scriptstyle |g|} \\ |X| & \xrightarrow{|f|} & |B|, \end{array}$$

induced by (4)*, is a homotopy pullback.*

Proof. $(i) \Rightarrow (ii)$: Let $x : \Delta[n] \to B$ be any simplex of B. We have the diagram

$$\begin{array}{ccccc} |g^{-1}(x)| & \longrightarrow & \operatorname*{hocolim}_{x:\Delta[n]\to B} |g^{-1}(x)| & \xrightarrow{\sim} & |Y| \\ \downarrow & (b) & \downarrow & (a) & \downarrow{\scriptstyle |g|} \\ |\Delta[n]| & \xrightarrow{|x|} & \operatorname*{hocolim}_{x:\Delta[n]\to B} |\Delta[n]| & \xrightarrow{\sim} & |B| \\ \wr\downarrow & (c) & \downarrow\wr & & \\ \mathrm{pt} & \xrightarrow{x} & \operatorname*{hocolim}_{x:\Delta[n]\to B} \mathrm{pt}, & & \end{array}$$

where $\operatorname*{hocolim}_{x:\Delta[n]\to B} \mathrm{pt} = \mathrm{B}(\Delta\downarrow B)$ is the classifying space of the simplex category. Since, by Quillen's Lemma [42, page 14], the composite square $(b)+(c)$ is a homotopy pullback, it follows that (b) is a homotopy pullback. Therefore, the composite $(b)+(a)$ is as well.

$(ii) \Rightarrow (i)$: For any simplicial map $\sigma : \Delta[m] \to \Delta[n]$ and any simplex $x : \Delta[n] \to B$, the right side and the large square in the diagram of spaces

$$\begin{array}{ccccc} |g^{-1}(x\sigma)| & \longrightarrow & |g^{-1}(x)| & \longrightarrow & |Y| \\ \downarrow & & \downarrow & & \downarrow{\scriptstyle |g|} \\ |\Delta[m]| & \xrightarrow{|\sigma|} & |\Delta[n]| & \xrightarrow{|x|} & |B| \end{array}$$

are both homotopy pullback, and therefore so is the left-hand one. As $|\Delta[m]|$ and $|\Delta[n]|$ are both contractible, the map $|\sigma|$ is a homotopy equivalence, and therefore the map $|g^{-1}(x\sigma)| \to |g^{-1}(x)|$ is a homotopy equivalence.

$(i) \Rightarrow (iii)$: Suppose we have the pullback square of simplicial sets (4). Then, for any simplex $x : \Delta[n] \to X$ of X, we have a natural isomorphism of fibres $g'^{-1}(x) \cong g^{-1}(fx)$, and it follows that the map g' also satisfies the same condition (i) as g does. Then, by the already proven part $(i) \Leftrightarrow (ii)$, we know that, for any vertex $x : \Delta[0] \to X$, both the left side and the composite square in the diagram

$$\begin{array}{ccccc} |g'^{-1}(x)| \cong |g^{-1}(fx)| & \longrightarrow & |X \times_B Y| & \longrightarrow & |Y| \\ \downarrow & & {\scriptstyle |g'|}\downarrow & & \downarrow{\scriptstyle |g|} \\ \mathrm{pt} = |\Delta[0]| & \xrightarrow{|x|} & |X| & \xrightarrow{|f|} & |B| \end{array}$$

are homotopy pullbacks. Therefore, from the diagram on whisker maps

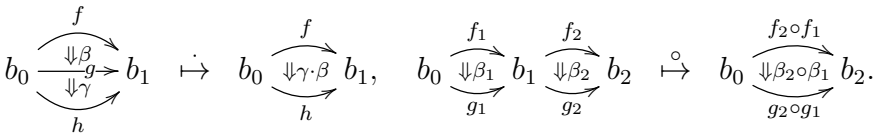
$$\begin{array}{ccc} |g'^{-1}(x)| & \xrightarrow{\sim} & \mathrm{Fib}(|g'|, |x|) \\ {\scriptstyle \wr}\downarrow & & \downarrow{\scriptstyle w} \\ |g^{-1}(fx)| & \xrightarrow{\sim} & \mathrm{Fib}(|g|, |fx|), \end{array}$$

we conclude that the map $\mathrm{Fib}(|g'|, |x|) \to \mathrm{Fib}(|g|, |fx|)$ is a homotopy equivalence. Since the homotopy fibres of any map over points connected by a path are homotopy equivalent, and any point of $|X|$ is path-connected with a 0-cell $|x|$ defined by some 0-simplex $x : \Delta[0] \to X$ as above, the result follows from the homotopy fibre characterization.

$(iii) \Rightarrow (ii)$: This is obvious. ∎

2.4. Some bicategorical conventions. For bicategories, we use the same conventions and notations as Carrasco, Cegarra, and Garzón in [16, §2.4] and [17, §2.1]. Given any bicategory $\mathcal{B}$, its set of *objects* or 0-*cells* is denoted by $\mathrm{Ob}\mathcal{B}$. For each ordered pair of objects (b_0, b_1) of $\mathcal{B}$, $\mathcal{B}(b_0, b_1)$ denotes its hom-category whose objects $f : b_0 \to b_1$ are called the 1-*cells* in $\mathcal{B}$ with source b_0 and target b_1, and whose morphisms $\beta : f \Rightarrow g$ are called 2-*cells* of $\mathcal{B}$. The composition in each hom-category $\mathcal{B}(b_0, b_1)$, that is, the *vertical*

composition of 2-cells, is denoted by the symbol " $\cdot$ ", while the symbol " $\circ$ " is used to denote the *horizontal composition* functors:



Identities are denoted as $1_f : f \Rightarrow f$, for any 1-cell f, and $1_b : b \to b$, for any 0-cell b. The *associativity constraints* of the bicategory are denoted by

$$\boldsymbol{a}_{f_3,f_2,f_1} : (f_3 \circ f_2) \circ f_1 \cong f_3 \circ (f_2 \circ f_1),$$

which are natural in $(f_3, f_2, f_1) \in \mathcal{B}(b_2, b_3) \times \mathcal{B}(b_1, b_2) \times \mathcal{B}(b_0, b_1)$. The *left and right unity constraints* are denoted by $\boldsymbol{l}_f : 1_{b_1} \circ f \cong f$ and $\boldsymbol{r}_f : f \circ 1_{b_0} \cong f$. These are natural in $f \in \mathcal{B}(b_0, b_1)$. These constraint 2-cells must satisfy the well-known pentagon and triangle coherence conditions.

A bicategory in which all the constraints are identities is a 2-*category*. It is the same as a category enriched in the category **Cat** of small categories. As each category $\mathcal{B}$ can be considered as a 2-category in which all deformations are identities, that is, in which each category $\mathcal{B}(b_0, b_1)$ is discrete, several times throughout the paper, categories are considered as special bicategories.

A *lax functor* is written as a pair $F = (F, \widehat{F}) : \mathcal{B} \to \mathcal{C}$, since we will generally denote its structure constraints by

$$\widehat{F}_{f_2,f_1} : Ff_2 \circ Ff_1 \Rightarrow F(f_2 \circ f_1), \quad \widehat{F}_b : 1_{Fb} \Rightarrow F1_b,$$

for each pair of composable 1-cells, and each object of $\mathcal{B}$. Recall that the structure 2-cells $\widehat{F}_{f_2,f_1}$ are natural in $(f_2, f_1) \in \mathcal{B}(b_1, b_2) \times \mathcal{B}(b_0, b_1)$ and they satisfy the usual coherence conditions. Replacing the constraint 2-cells above by $\widehat{F}_{f_2,f_1} : F(f_2 \circ f_1) \Rightarrow Ff_2 \circ Ff_1$ and $\widehat{F}_b : F(1_b) \Rightarrow 1_{Fb}$, we have the notion of *oplax functor* $F = (F, \widehat{F}) : \mathcal{B} \to \mathcal{C}$. Any lax or oplax functor F is termed a *pseudo-functor* or *homomorphism* whenever all the structure constraints $\widehat{F}_{f_2,f_1}$ and $\widehat{F}_b$ are invertible. When these 2-cells are all identities, then F is called a 2-*functor*. If all the unit constraints $\widehat{F}_b$ are identities, then the lax or oplax functor is qualified as (strictly) *unitary* or *normal*.

If $F, F' : \mathcal{B} \to \mathcal{C}$ are lax functors, then a *lax transformation* $\alpha = (\alpha, \widehat{\alpha}) : F \Rightarrow F'$ consists of morphisms $\alpha b : Fb \to F'b$, $b \in \mathrm{Ob}\mathcal{B}$, and 2-cells

$$\begin{array}{ccc} Fb_0 & \xrightarrow{Ff} & Fb_1 \\ {\scriptstyle \alpha b_0}\downarrow & \overset{\widehat{\alpha}_f}{\Rightarrow} & \downarrow{\scriptstyle \alpha b_1} \\ F'b_0 & \xrightarrow[F'f]{} & F'b_1 \end{array}$$

which are natural on the 1-cells $f : b_0 \to b_1$ of $\mathcal{B}$, subject to the usual coherence axioms. Replacing the structure deformation above by $\widehat{\alpha}_f : \alpha b_1 \circ Ff \Rightarrow F'f \circ \alpha b_0$, we have the

notion of *oplax transformation* $\alpha : F \Rightarrow F'$. Any lax or oplax transformation α is termed a *pseudo-transformation* whenever all the naturality 2-cells $\widehat{\alpha}_f$ are invertible. Similarly, we have the notions of lax, oplax, and pseudo transformation between oplax functors.

2.5. Homotopy-fiber product bicategories. We present a bicategorical comma construction in some detail, since it is fundamental for the results of this paper. However, we are not claiming much originality since variations of the quite ubiquitous 'comma category' construction have been considered (just to define "homotopy pullbacks") in many general frameworks of enriched categories (where a homotopy theory has been established); see for instance Grandis [30].

Let $\mathcal{A} \xrightarrow{F} \mathcal{B} \xleftarrow{F'} \mathcal{A}'$ be a diagram where $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{A}'$ are bicategories, F is a lax functor, and F' is an oplax functor. The "*homotopy-fiber product bicategory*"

$$F \downarrow F' \tag{5}$$

is defined as follows:

• *The* 0*-cells* of $F \downarrow F'$ are triples (a, f, a') with a a 0-cell of $\mathcal{A}$, a' a 0-cell of $\mathcal{A}'$, and $f : Fa \to F'a'$ a 1-cell in $\mathcal{B}$.

• *A* 1*-cell* $(u, \beta, u') : (a_0, f_0, a'_0) \to (a_1, f_1, a'_1)$ of $F \downarrow F'$ consists of a 1-cell $u : a_0 \to a_1$ in $\mathcal{A}$, a 1-cell $u' : a'_0 \to a'_1$ in $\mathcal{A}'$, and 2-cell $\beta : F'u' \circ f_0 \Rightarrow f_1 \circ Fu$ in $\mathcal{B}$,

$$\begin{array}{ccc} Fa_0 & \xrightarrow{Fu} & Fa_1 \\ {\scriptstyle f_0}\downarrow & \overset{\beta}{\Rightarrow} & \downarrow{\scriptstyle f_1} \\ F'a'_0 & \xrightarrow{F'u'} & F'a'_1. \end{array}$$

• *A* 2*-cell* in $F \downarrow F'$, $(a_0, f_0, a'_0) \overset{(u,\beta,u')}{\underset{(\bar{u},\bar{\beta},\bar{u}')}{\Downarrow(\alpha,\alpha')}} (a_1, f_1, a'_1)$, is given by a 2-cell $\alpha : u \Rightarrow \bar{u}$ in $\mathcal{A}$ and a 2-cell $\alpha' : u' \Rightarrow \bar{u}'$ in $\mathcal{A}'$ such that the diagram below commutes.

$$\begin{array}{ccc} F'u' \circ f_0 & \xRightarrow{F'\alpha' \circ 1} & F'\bar{u}' \circ f_0 \\ {\scriptstyle \beta}\Downarrow & & \Downarrow{\scriptstyle \bar{\beta}} \\ f_1 \circ Fu & \xRightarrow{1 \circ F\alpha} & f_1 \circ F\bar{u} \end{array}$$

• *The vertical composition of* 2*-cells* in $F \downarrow F'$ is induced by the vertical composition laws in $\mathcal{A}$ and $\mathcal{A}'$, thus $(\bar{\alpha}, \bar{\alpha}') \cdot (\alpha, \alpha') = (\bar{\alpha} \cdot \alpha, \bar{\alpha}' \cdot \alpha')$. The identity at a 1-cell is given by $1_{(u,\beta,u')} = (1_u, 1_{u'})$.

• *The horizontal composition of two* 1*-cells* in $F \downarrow F'$,

$$(a_0, f_0, a'_0) \xrightarrow{(u_1,\beta_1,u'_1)} (a_1, f_1, a'_1) \xrightarrow{(u_2,\beta_2,u'_2)} (a_2, f_2, a'_2)\ , \tag{6}$$

is the 1-cell $(u_2, \beta_2, u_2') \circ (u_1, \beta_1, u_1') = (u_2 \circ u_1, \beta_2 \circledcirc \beta_1, u_2' \circ u_1')$, where $\beta_2 \circledcirc \beta_1$ is the 2-cell pasted of the diagram in $\mathcal{B}$

$$\beta_2 \circledcirc \beta_1 = \quad \begin{array}{ccccc} & & \overset{F(u_2\circ u_1)}{\frown} & & \\ Fa_0 & \xrightarrow{Fu_1} & Fa_1 & \xrightarrow{Fu_2} & Fa_2 \\ {\scriptstyle f_0}\downarrow & \overset{\beta_1}{\Rightarrow} & {\scriptstyle f_1}\downarrow & \overset{\beta_2}{\Rightarrow} & \downarrow{\scriptstyle f_2} \\ F'a_0' & \xrightarrow{F'u_1'} & F'a_1' & \xrightarrow{F'u_2'} & F'a_2', \\ & & \underset{F'(u_2'\circ u_1')}{\smile} & & \end{array} \tag{7}$$

that is, $\beta_2 \circledcirc \beta_1 = \Big(F'(u_2' \circ u_1') \circ f_0 \overset{\widehat{F}'\circ 1}{\Longrightarrow} (F'u_2' \circ F'u_1') \circ f_0 \overset{\boldsymbol{a}}{\Longrightarrow} F'u_2' \circ (F'u_1' \circ f_0) \overset{1\circ\beta_1}{\Longrightarrow}$

$F'u_2' \circ (f_1 \circ Fu_1) \overset{\boldsymbol{a}^{-1}}{\Longrightarrow} (F'u_2' \circ f_1) \circ Fu_1 \overset{\beta_2\circ 1}{\Longrightarrow} (f_2 \circ Fu_2) \circ Fu_1 \overset{\boldsymbol{a}}{\Longrightarrow}$

$f_2 \circ (Fu_2 \circ Fu_1) \overset{1\circ\widehat{F}}{\Longrightarrow} f_2 \circ F(u_2 \circ u_1) \Big)$.

• *The horizontal composition of* 2*-cells* in $F \downarrow F'$ is given by composing horizontally the 2-cells in $\mathcal{A}$ and $\mathcal{A}'$, thus $(\alpha_2, \alpha_2') \circ (\alpha_1, \alpha_1') = (\alpha_2 \circ \alpha_1, \alpha_2' \circ \alpha_1')$.

• *The identity* 1*-cell* in $F \downarrow F'$, at an object (a, f, a'), is $(1_a, \mathring{1}_{(a,f,a')}, 1_{a'})$, where $\mathring{1}_{(a,f,a')}$ is the 2-cell in $\mathcal{B}$ obtained by pasting the diagram

$$\mathring{1}_{(a,f,a')} = \quad \begin{array}{ccc} & \overset{F1_a}{\frown} & \\ Fa & \xrightarrow{1_{Fa}} & Fa \\ {\scriptstyle f}\downarrow & \boldsymbol{r}^{-1}\cdot\boldsymbol{l} \cong & \downarrow{\scriptstyle f} \\ F'a' & \xrightarrow[1_{F'a'}]{} & F'a', \\ & \underset{F'1_{a'}}{\smile} & \end{array}$$

that is, $\mathring{1}_{(a,f,a')} = \Big(F'1_{a'} \circ f \overset{\widehat{F}'\circ 1}{\Longrightarrow} 1_{F'a'} \circ f \overset{\boldsymbol{l}}{\Longrightarrow} f \overset{\boldsymbol{r}^{-1}}{\Longrightarrow} f \circ 1_{Fa} \overset{1\circ\widehat{F}}{\Longrightarrow} f \circ F1_a \Big)$.

• *The associativity, right and left unit constraints* of the bicategory $F \downarrow F'$ are provided by those of $\mathcal{A}$ and $\mathcal{A}'$ by the formulas

$$\boldsymbol{a}_{(u_3,\beta_3,u_3'),(u_2,\beta_2,u_2'),(u_1,\beta_1,u_1')} = (\boldsymbol{a}_{u_3,u_2,u_1}, \boldsymbol{a}_{u_3',u_2',u_1'}), \quad \boldsymbol{l}_{(u,\beta,u')} = (\boldsymbol{l}_u, \boldsymbol{l}_{u'}), \quad \boldsymbol{r}_{(u,\beta,u')} = (\boldsymbol{r}_u, \boldsymbol{r}_{u'}).$$

2.5.1. The main square. There is a (non-commutative!) square, which is of fundamental interest for the discussions below:

$$\begin{array}{ccc} F \downarrow F' & \xrightarrow{P'} & \mathcal{A}' \\ {\scriptstyle P}\downarrow & & \downarrow{\scriptstyle F'} \\ \mathcal{A} & \xrightarrow{F} & \mathcal{B} \end{array} \tag{8}$$

where P and P' are projection 2-functors, which act on cells of $F\downarrow F'$ by

$$a_0 \underset{\bar{u}}{\overset{u}{\Downarrow\alpha}} a_1 \quad \overset{P}{\longleftarrow} \quad (a_0, f_0, a_0') \underset{(\bar{u},\bar{\beta},\bar{u}')}{\overset{(u,\beta,u')}{\Downarrow(\alpha,\alpha')}} (a_1, f_1, a_1') \quad \overset{P'}{\longmapsto} \quad a_0' \underset{\bar{u}'}{\overset{u'}{\Downarrow\alpha'}} a_1'. \tag{9}$$

2.5.2. Two pullback squares. We consider here three particular cases of the construction (5):

- For any lax functor $F : \mathcal{A} \to \mathcal{B}$, the bicategory $F\downarrow\mathcal{B} := F\downarrow 1_{\mathcal{B}}$.

- For any oplax functor $F' : \mathcal{A}' \to \mathcal{B}$, the bicategory $\mathcal{B}\downarrow F' := 1_{\mathcal{B}}\downarrow F'$.

- For any bicategory $\mathcal{B}$, the bicategory $\mathcal{B}\downarrow\mathcal{B} := 1_{\mathcal{B}}\downarrow 1_{\mathcal{B}}$.

There are commutative squares

$$\begin{array}{ccc} F\downarrow F' & \xrightarrow{\bar{F}} & \mathcal{B}\downarrow F' \\ {\scriptstyle P}\downarrow & & \downarrow{\scriptstyle P} \\ \mathcal{A} & \xrightarrow{F} & \mathcal{B}, \end{array} \qquad \begin{array}{ccc} F\downarrow F' & \xrightarrow{P'} & \mathcal{A}' \\ {\scriptstyle \bar{F}'}\downarrow & & \downarrow{\scriptstyle F'} \\ F\downarrow\mathcal{B} & \xrightarrow{P'} & \mathcal{B}, \end{array} \tag{10}$$

where the first one is in the category of bicategories and lax functors, and the second one in the category of oplax functors. The lax functor $\bar{F} : F\downarrow F' \to \mathcal{B}\downarrow F'$ in the first square is given on cells by applying F to the first components

$$(a_0, f_0, a_0') \underset{(\bar{u},\bar{\beta},\bar{u}')}{\overset{(u,\beta,u')}{\Downarrow(\alpha,\alpha')}} (a_1, f_1, a_1') \quad \overset{\bar{F}}{\mapsto} \quad (Fa_0, f_0, a_0') \underset{(F\bar{u},\bar{\beta},\bar{u}')}{\overset{(Fu,\beta,u')}{\Downarrow(F\alpha,\alpha')}} (Fa_1, f_1, a_1'),$$

while the oplax functor $\bar{F}' : F\downarrow F' \to F\downarrow\mathcal{B}$ in the second one acts on cells through the application of F' to the last components

$$(a_0, f_0, a_0') \underset{(\bar{u},\bar{\beta},\bar{u}')}{\overset{(u,\beta,u')}{\Downarrow(\alpha,\alpha')}} (a_1, f_1, a_1') \quad \overset{\bar{F}'}{\mapsto} \quad (a_0, f_0, F'a_0') \underset{(\bar{u},\bar{\beta},F'\bar{u}')}{\overset{(u,\beta,F'u')}{\Downarrow(\alpha,F'\alpha')}} (a_1, f_1, F'a_1').$$

At any pair of composable 1-cells in $F\downarrow F'$ as in (6), their respective structure constraints for the composition are the 2-cells

$$(\widehat{F}_{u_2,u_1}, 1_{u_2'\circ u_1'}) : \bar{F}(u_2,\beta_2,u_2') \circ \bar{F}(u_1,\beta_1,u_1') \Rightarrow \bar{F}\big((u_2,\beta_2,u_2')\circ(u_1,\beta_1,u_1')\big),$$
$$(1_{u_2\circ u_1}, \widehat{F}'_{u_2',u_1'}) : \bar{F}'\big((u_2,\beta_2,u_2')\circ(u_1,\beta_1,u_1')\big) \Rightarrow \bar{F}'(u_2,\beta_2,u_2') \circ \bar{F}'(u_1,\beta_1,u_1'),$$

and, at any object (a, f, a') of $F\downarrow F'$, their respective constraints for the identity are

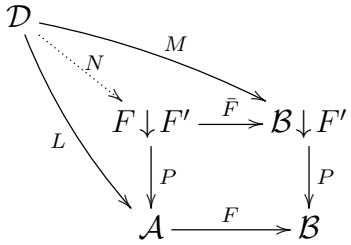
$$(\widehat{F}_a, 1_{1_{a'}}) : 1_{\bar{F}(a,f,a')} \Rightarrow \bar{F}1_{(a,f,a')}, \quad (1_{1_a}, \widehat{F}'_{a'}) : \bar{F}'1_{(a,f,a')} \Rightarrow 1_{\bar{F}'(a,f,a')}.$$

Although neither the category of bicategories and lax functors nor the category of bicategories and oplax functors have pullbacks in general, the following fact holds.

2.6. Lemma. (*i*) *The first square in* (10) *is a pullback in the category of bicategories and lax functors.*

(*ii*) *The second square in* (10) *is a pullback in the category of bicategories and oplax functors.*

Proof. (*i*) Any pair of lax functors $L : \mathcal{D} \to \mathcal{A}$ and $M : \mathcal{D} \to \mathcal{B}{\downarrow}F'$ such that $FL = PM$ determines a unique oplax functor $N : \mathcal{D} \to F{\downarrow}F'$



such that $PN = L$ and $\bar{F}N = M$, which is defined as follows: The lax functor M carries any object $d \in \mathrm{Ob}\mathcal{D}$ to an object of $\mathcal{B} \downarrow F$ which is necessarily written in the form $M(d) = (FL(d), f(d), a'(d))$, with $a'(d)$ an object of $\mathcal{A}'$ and $f(d) : FL(d) \to F'a'(d)$ a 1-cell in $\mathcal{B}$. Similarly, for any 1-cell $h : d_0 \to d_1$ in $\mathcal{D}$, we have $M(h) = (FL(h), \beta(h), u'(h))$ for some 1-cell $u'(h) : a'(d_0) \to a'(d_1)$ in $\mathcal{A}'$ and some 2-cell

$$\begin{array}{ccc} FL(d_0) & \xrightarrow{FL(h)} & FL(d_1) \\ {\scriptstyle f(d_0)}\downarrow & \overset{\beta(h)}{\Rightarrow} & \downarrow{\scriptstyle f(d_1)} \\ F'a'(d_0) & \xrightarrow[F'u'(h)]{} & F'a'(d_1), \end{array}$$

in $\mathcal{B}$, and for any 2-cell $\gamma : h_0 \Rightarrow h_1$ in $\mathcal{D}$, we have $M(\gamma) = (FL(\gamma), \alpha'(\gamma))$, for some 2-cell $\alpha'(\gamma) : u'(h_0) \Rightarrow u'(h_1)$ in $\mathcal{A}'$. Also, for any object d and any pair of composable 1-cells $h_1 : d_0 \to d_1$ and $h_2 : d_1 \to d_2$ in $\mathcal{D}$, the attached structure 2-cells of M can be respectively written in a similar form as

$$\widehat{M}_d = (F(\widehat{L}_d) \cdot \widehat{F}_{L(d)}, \widehat{\alpha}'_d) : 1_{M(d)} \Rightarrow M(1_d),$$
$$\widehat{M}_{h_2,h_1} = (F(\widehat{L}_{h_2,h_1}) \cdot \widehat{F}_{L(h_2),L(h_1)}, \widehat{\alpha}'_{h_2,h_1}) : M(h_2) \circ M(h_1) \Rightarrow M(h_2 \circ h_1),$$

for some 2-cells $\widehat{\alpha}'_{h_2,h_1}$ and $\widehat{\alpha}'_d$ in $\mathcal{A}'$. Then, the claimed $N : \mathcal{D} \to F{\downarrow}F'$ is the lax functor which acts on cells by

$$d_0 \underset{\bar{h}}{\overset{h}{\Downarrow\gamma}} d_1 \quad \overset{N}{\mapsto} \quad (L(d_0), f(d_0), a'(d_0)) \underset{(L(\bar{h}),\beta(\bar{h}),u'(\bar{h}))}{\overset{(L(h),\beta(h),u'(h))}{\Downarrow(L(\gamma),\alpha'(\gamma))}} (L(d_1), f(d_1), a'(d_1))$$

and its respective structure 2-cells, for any object d and any pair of composable 1-cells $h_1 : d_0 \to d_1$ and $h_2 : d_1 \to d_2$ in $\mathcal{D}$, are

$$\widehat{N}_d = (\widehat{L}_d, \widehat{\alpha}'_d) : 1_{N(d)} \Rightarrow N(1_d), \quad \widehat{N}_{h_2,h_1} = (\widehat{L}_{h_2,h_1}, \widehat{\alpha}'_{h_2,h_1}) : N(h_2) \circ N(h_1) \Rightarrow N(h_2 \circ h_1).$$

The proof of (*ii*) is parallel to that given above for part (*i*), and it is left to the reader. ∎

2.7. The homotopy-fibre bicategories. For any 0-cell $b \in \mathcal{B}$, we also denote by $b : [0] \to \mathcal{B}$ the normal homomorphism such that $b(0) = b$, and whose structure isomorphism is $\boldsymbol{l} : 1_b \otimes 1_b \cong 1_b$. Then, we have the bicategories

- $F{\downarrow}b$, for any lax functor $F : \mathcal{A} \to \mathcal{B}$.

- $b{\downarrow}F'$, for any oplax functor $F' : \mathcal{A}' \to \mathcal{B}$.

- $b{\downarrow}\mathcal{B} := b\downarrow 1_{\mathcal{B}}$, and $\mathcal{B}{\downarrow}b := 1_{\mathcal{B}}{\downarrow}b$.

Given F and F' as above, any 1-cell $p : b_0 \to b_1$ in $\mathcal{B}$ determines 2-functors

$$p_* : F{\downarrow}b_0 \to F{\downarrow}b_1, \quad p^* : b_1{\downarrow}F' \to b_0{\downarrow}F', \tag{11}$$

respectively given on cells by

$$(a_0, f_0) \underset{(\bar{u},\bar{\beta})}{\overset{(u,\beta)}{\Downarrow\alpha}} (a_1, f_1) \;\overset{p_*}{\longmapsto}\; (a_0, p\circ f_0) \underset{(\bar{u},p\odot\bar{\beta})}{\overset{(u,p\odot\beta)}{\Downarrow\alpha}} (a_1, p\circ f_1),$$

$$(f_0, a'_0) \underset{(\bar{\beta},\bar{u}')}{\overset{(\beta,u')}{\Downarrow\alpha'}} (f_1, a'_1) \;\overset{p^*}{\longmapsto}\; (f_0\circ p, a'_0) \underset{(\bar{\beta}\odot p,\bar{u}')}{\overset{(\beta\odot p,u')}{\Downarrow\alpha'}} (f_1\circ p, a'_1),$$

where, for any $(u,\beta) : (a_0, f_0) \to (a_1, f_1)$ in $F{\downarrow}b_0$ and $(\beta, u') : (f_0, a'_0) \to (f_1, a'_1)$ in $b_1{\downarrow}F'$, the 2-cells $p \odot \beta$ and $\beta \odot p$ are respectively obtained by pasting the diagrams

$$p\odot\beta : \begin{array}{ccc} Fa_0 & \xrightarrow{Fu} & Fa_1 \\ {\scriptstyle f_0}\downarrow & \overset{\beta}{\Rightarrow} & \downarrow{\scriptstyle f_1} \\ b_0 & \xrightarrow{1} & b_0 \\ {\scriptstyle p}\downarrow & \cong & \downarrow{\scriptstyle p} \\ b_1 & \xrightarrow{1} & b_1 \end{array} \qquad \beta\odot p : \begin{array}{ccc} b_0 & \xrightarrow{1} & b_0 \\ {\scriptstyle p}\downarrow & \cong & \downarrow{\scriptstyle p} \\ b_1 & \xrightarrow{1} & b_1 \\ {\scriptstyle f_0}\downarrow & \overset{\beta}{\Rightarrow} & \downarrow{\scriptstyle f_1} \\ F'a'_0 & \xrightarrow{F'u'} & F'a'_1 \end{array}$$

that is,

$$p\odot\beta = \Big(1\circ(p\circ f_0) \overset{\boldsymbol{l}}{\Longrightarrow} p\circ f_0 \overset{1\circ\boldsymbol{l}^{-1}}{\Longrightarrow} p\circ(1\circ f_0) \overset{1\circ\beta}{\Longrightarrow} p\circ(f_1\circ Fu) \overset{\boldsymbol{a}^{-1}}{\Longrightarrow} (p\circ f_1)\circ Fu\Big),$$

$$\beta\odot p = \Big(F'u'\circ(f_0\circ p) \overset{\boldsymbol{a}^{-1}}{\Longrightarrow} (F'u'\circ f_0)\circ p \overset{\beta\circ 1}{\Longrightarrow} (f_1\circ 1)\circ p \overset{\boldsymbol{r}\circ 1}{\Longrightarrow} f_1\circ p \overset{\boldsymbol{r}^{-1}}{\Longrightarrow} (f_1\circ p)\circ 1\Big).$$

2.8. Classifying spaces of bicategories. Briefly, let us recall from [16, Definition 3.1] that the *classifying space* $\mathrm{B}\mathcal{B}$ of a (small) bicategory $\mathcal{B}$ is defined as the geometric realization of the *Grothendieck nerve* or *pseudo-simplicial nerve* of the bicategory, that is, the pseudo-functor from Δ^{op} to the 2-category **Cat** of small categories

$$\underline{\mathrm{N}}\mathcal{B} : \Delta^{op} \to \mathbf{Cat}, \quad [p] \mapsto \coprod_{(b_0,\dots,b_p)} \mathcal{B}(b_{p-1}, b_p) \times \mathcal{B}(b_{p-2}, b_{p-1}) \times \cdots \times \mathcal{B}(b_0, b_1), \tag{12}$$

whose face and degeneracy functors are defined in the standard way by using the horizontal composition and identity morphisms of the bicategory, and the natural isomorphisms $d_i d_j \cong d_{j-1} d_i$, etc., being given from the associativity and unit constraints of the bicategory (see Theorem 6.1 in the Appendix, for more details). Thus,

$$\mathrm{B}\mathcal{B} = \mathrm{B} \textstyle\int_{\Delta} \underline{\mathrm{N}}\mathcal{B}$$

is the classifying space of the category $\int_{\Delta} \underline{\mathrm{N}}\mathcal{B}$ obtained by the Grothendieck construction [31] on the pseudofunctor $\underline{\mathrm{N}}\mathcal{B}$. In other words, $\mathrm{B}\mathcal{B} = |\underline{\mathrm{N}} \int_{\Delta} \underline{\mathrm{N}}\mathcal{B}|$ is the geometric realization of the simplicial set nerve of the category $\int_{\Delta} \underline{\mathrm{N}}\mathcal{B}$. When $\mathcal{B}$ is a 2-category, then $\mathrm{B}\mathcal{B}$ is homotopy equivalent to Segal's classifying space [44] of the topological category obtained from $\mathcal{B}$ by replacing the hom-categories $\mathcal{B}(x, y)$ by their classifying spaces $\mathrm{B}\mathcal{B}(x, y)$, see [16, Remark 3.2].

In [16, §4], it is proven that the classifying space construction, $\mathcal{B} \mapsto \mathrm{B}\mathcal{B}$, is a functor $\mathrm{B} : \mathbf{Hom} \to \mathbf{Top}$, from the category of bicategories and homomorphisms to the category **Top** of spaces (actually of CW-complexes). In this paper, we need the extension of this fact stated in part (i) of the lemma below.

2.9. Lemma. (i) *The assignment $\mathcal{B} \mapsto \mathrm{B}\mathcal{B}$ is the function on objects of two functors into the category of spaces*

$$\mathbf{Lax} \xrightarrow{\mathrm{B}} \mathbf{Top} \xleftarrow{\mathrm{B}} \mathbf{opLax}\,,$$

where **Lax** *(resp.* **opLax***) is the category of bicategories with lax (resp. oplax) functors between them as morphisms.*

(ii) *If $F, G : \mathcal{B} \to \mathcal{C}$ are two lax or oplax functors between bicategories, then any lax or oplax transformation between them $\alpha : F \Rightarrow G$ determines a homotopy, $\mathrm{B}\alpha : \mathrm{B}F \Rightarrow \mathrm{B}G : \mathrm{B}\mathcal{B} \to \mathrm{B}\mathcal{C}$, between the induced maps on classifying spaces.*

Proof. It is given in the Appendix, Corollaries 6.4 and 6.7. ∎

Other possibilities for defining $\mathrm{B}\mathcal{B}$ come from the *geometric nerves* of the bicategory, first defined by Street [46] and studied, among others, by Duskin [26], Gurski [32] and Carrasco, Cegarra, and Garzón [16]; that is, the simplicial sets

$$\begin{aligned}
&\Delta^{\mathrm{u}}\mathcal{B} : \Delta^{op} \to \mathbf{Set}, && [p] \mapsto \mathrm{NorLax}([p], \mathcal{B}),\\
&\Delta\mathcal{B} : \Delta^{op} \to \mathbf{Set}, && [p] \mapsto \mathrm{Lax}([p], \mathcal{B}),\\
&\nabla_{\mathrm{u}}\mathcal{B} : \Delta^{op} \to \mathbf{Set}, && [p] \mapsto \mathrm{NorOpLax}([p], \mathcal{B}),\\
&\nabla\mathcal{B} : \Delta^{op} \to \mathbf{Set}, && [p] \mapsto \mathrm{OpLax}([p], \mathcal{B}),
\end{aligned} \tag{13}$$

whose respective p-simplices are the normal lax, lax, normal oplax, and oplax functors from the category $[p]$ into the bicategory $\mathcal{B}$. In the Homotopy Invariance Theorem [16, Theorem 6.1] the existence of homotopy equivalences

$$|\Delta^{\mathrm{u}}\mathcal{B}| \simeq |\Delta\mathcal{B}| \simeq \mathrm{B}\mathcal{B} \simeq |\nabla\mathcal{B}| \simeq |\nabla_{\mathrm{u}}\mathcal{B}|, \tag{14}$$

it is proven, but their natural behaviour is not studied. Since, to establish the results in this paper, we need to know that all the homotopy equivalences above are homotopy natural, we state the following

2.10. Lemma. *For any bicategory $\mathcal{B}$, the first homotopy equivalence in* (14) *is natural on normal lax functors, the second one is homotopy natural on lax functors, the third one is homotopy natural on oplax functors, and the fourth one is natural on normal oplax functors.*

Proof. By [16, Theorem 6.2], the homotopy equivalence $|\Delta^{\mathrm{u}}\mathcal{B}| \simeq |\Delta\mathcal{B}|$ is induced on geometric realizations by the inclusion map $\Delta^{\mathrm{u}}\mathcal{B} \hookrightarrow \Delta\mathcal{B}$. Therefore, it is clearly natural on normal lax functors between bicategories. Similarly, the homotopy equivalence $|\nabla_{\mathrm{u}}\mathcal{B}| \simeq |\nabla\mathcal{B}|$ is natural on normal oplax functors. The proof for the other two is more complicated and is given in the Appendix, Corollary 6.6. ∎

3. Inducing homotopy pullbacks on classifying spaces

Quillen's Theorem B [42] provides a sufficient condition on a functor between small categories $F : \mathcal{A} \to \mathcal{B}$ for the classifying space $\mathrm{B}(F\downarrow b)$ to be a homotopy-fibre over the 0-cell $|b| \in \mathrm{B}\mathcal{B}$ of the induced map $\mathrm{B}F : \mathrm{B}\mathcal{A} \to \mathrm{B}\mathcal{B}$, for each object $b \in \mathrm{Ob}\mathcal{B}$. The condition is that the maps $\mathrm{B}p_* : \mathrm{B}(F\downarrow b) \to \mathrm{B}(F\downarrow b')$ are homotopy equivalences for every morphism $p : b \to b'$ in the category $\mathcal{B}$. That condition was referred to by Dwyer, Kan, and Smith in [27, §6] by saying that "the functor F has the property B" (see also Barwick, and Kan in [3, 4]), and by Cisinski in [22, 6.4.1] by saying that "the functor F is locally constant". To state our theorem below, we shall adapt that terminology to the bicategorical setting, and we will say that

- (B_l) *a lax functor* between bicategories $F : \mathcal{A} \to \mathcal{B}$ *has the property* B_l if, for any 1-cell $p : b_0 \to b_1$ in $\mathcal{B}$, the 2-functor $p_* : F\downarrow b_0 \to F\downarrow b_1$ in (11) induces a homotopy equivalence on classifying spaces, $\mathrm{B}(F\downarrow b_0) \simeq \mathrm{B}(F\downarrow b_1)$.

- (B_o) *an oplax functor* between bicategories $F' : \mathcal{A}' \to \mathcal{B}$ *has the property* B_o if, for any 1-cell $p : b_0 \to b_1$ in $\mathcal{B}$, the 2-functor $p^* : b_1\downarrow F' \to b_0\downarrow F'$ in (11) induces a homotopy equivalence on classifying spaces, $\mathrm{B}(b_1\downarrow F') \simeq \mathrm{B}(b_0\downarrow F')$.

The main result in this paper can be summarized as follows:

3.1. THEOREM. *Let* $\mathcal{A} \xrightarrow{F} \mathcal{B} \xleftarrow{F'} \mathcal{A}'$ *be a diagram of bicategories, where* F *is a lax functor and* F' *is an oplax functor (for instance, if* F *and* F' *are any two homomorphisms).*

(*i*) *There is a homotopy* $\mathrm{B}F\,\mathrm{B}P \Rightarrow \mathrm{B}F'\,\mathrm{B}P'$*, so that the square below, which is induced by* (8) *on classifying spaces, is homotopy commutative.*

$$\begin{array}{ccc} \mathrm{B}(F\downarrow F') & \xrightarrow{\mathrm{B}P'} & \mathrm{B}\mathcal{A}' \\ {\scriptstyle \mathrm{B}P}\downarrow & \Rightarrow & \downarrow{\scriptstyle \mathrm{B}F'} \\ \mathrm{B}\mathcal{A} & \xrightarrow{\mathrm{B}F} & \mathrm{B}\mathcal{B} \end{array} \tag{15}$$

(*ii*) *Suppose that* F *has the property* B_l *or* F' *has the property* B_o*. Then, the square* (15) *is a homotopy pullback.*

Therefore, by Dyer and Roitberg [28], for each $a \in \mathrm{Ob}\mathcal{A}$ *and* $a' \in \mathrm{Ob}\mathcal{A}'$ *such that* $Fa = F'a'$ *there is an induced Mayer-Vietoris type long exact sequence on homotopy groups based at the* 0*-cells* $\mathrm{B}a$ *of* $\mathrm{B}\mathcal{A}$*,* $\mathrm{B}Fa$ *of* $\mathrm{B}\mathcal{B}$*,* $\mathrm{B}a'$ *of* $\mathrm{B}\mathcal{A}'$*, and* $\mathrm{B}(a, 1, a')$ *of* $\mathrm{B}(F\downarrow F')$*,*

$$\cdots \to \pi_{n+1}\mathrm{B}\mathcal{B} \to \pi_n\mathrm{B}(F\downarrow F') \to \pi_n\mathrm{B}\mathcal{A} \times \pi_n\mathrm{B}\mathcal{A}' \to \pi_n\mathrm{B}\mathcal{B} \to \cdots$$

$$\cdots \to \pi_1\mathrm{B}(F\downarrow F') \to \pi_1\mathrm{B}\mathcal{A} \times \pi_1\mathrm{B}\mathcal{A}' \to \pi_1\mathrm{B}\mathcal{B} \to \pi_0\mathrm{B}(F\downarrow F') \to \pi_0(\mathrm{B}\mathcal{A} \times \mathrm{B}\mathcal{A}').$$

(*iii*) *If the square* (15) *is a homotopy pullback for every* $F' = b : [0] \to \mathcal{B}$*,* $b \in \mathrm{Ob}\mathcal{B}$*, then* F *has the property* B_l*. Similarly, if the square* (15) *is a homotopy pullback for any* $F = b : [0] \to \mathcal{B}$*,* $b \in \mathrm{Ob}\mathcal{B}$*, then* F' *has the property* B_o*.*

The remainder of this section is devoted to the proof of this theorem. We shall start by recalling from [15, Lemma 5.2] the following lemma.

3.2. LEMMA. *For any object* b *of a bicategory* $\mathcal{B}$*, the classifying spaces of the comma bicategories* $\mathcal{B}{\downarrow}b$ *and* $b{\downarrow}\mathcal{B}$ *are contractible, that is,* $\mathrm{B}(\mathcal{B}{\downarrow}b) \simeq \mathrm{pt} \simeq \mathrm{B}(b{\downarrow}\mathcal{B})$*.*

We also need the auxiliary result below. To state it, we use that, for any given diagram $F : \mathcal{A} \to \mathcal{B} \leftarrow \mathcal{A}' : F'$, with F a lax functor and F' an oplax functor, and for each objects a of $\mathcal{A}$ and a' of $\mathcal{A}'$, there are normal homomorphisms

$$Fa\downarrow F' \xrightarrow{J} F\downarrow F' \xleftarrow{J'} F\downarrow F'a', \tag{16}$$

where J acts on cells by

$$(f_0, a'_0) \underset{(\bar\beta,\bar u')}{\overset{(\beta,u')}{\Downarrow \alpha'}} (f_1, a'_1) \;\overset{J}{\longmapsto}\; (a, f_0, a'_0) \underset{(1_a,\imath(\bar\beta,\bar u'),\bar u')}{\overset{(1_a,\imath(\beta,u'),u')}{\Downarrow (1,\alpha')}} (a, f_1, a'_1),$$

where, for any 1-cell $(\beta, u') : (f_0, a'_0) \to (f_1, a'_1)$ in $Fa\downarrow F'$, the 2-cell $\imath(\beta, u')$ is defined as the composite

$$\imath(\beta, u') = \big(F'u' \circ f_0 \overset{\beta}{\Longrightarrow} f_1 \circ 1_{Fa} \overset{1\circ\widehat{F}_a}{\Longrightarrow} f_1 \circ F1_a\big),$$

and whose constraints, at pairs of 1-cells $(f_0, a'_0) \xrightarrow{(\beta_1, u'_1)} (f_1, a'_1) \xrightarrow{(\beta_2, u'_2)} (f_2, a'_2)$ in $Fa{\downarrow}F'$, are the 2-cells of $F{\downarrow}F'$

$$(\boldsymbol{l}_{1_a}, 1_{u'_2 \circ u'_1}) : (1_a \circ 1_a, \imath(\beta_2, u'_2) \circledcirc \imath(\beta_1, u'_1), u'_2 \circ u'_1) \cong (1_a, \imath(\beta_2 \circledcirc \beta_1, u'_2 \circ u'_1), u'_2 \circ u'_1).$$

Similarly, J' acts by

$$(a_0, f_0) \underset{(\bar{u},\bar{\beta})}{\overset{(u,\beta)}{\Downarrow \alpha}} (a_1, f_1) \;\overset{J'}{\longmapsto}\; (a_0, f_0, a') \underset{(\bar{u},\imath'(\bar{u},\bar{\beta}),1_{a'})}{\overset{(u,\imath'(u,\beta),1_{a'})}{\Downarrow (\alpha,1)}} (a_1, f_1, a'),$$

where, for any 1-cell $(u, \beta) : (a_0, f_0) \to (a_1, f_1)$ in $F{\downarrow}F'a'$, the 2-cell $\imath'(u, \beta)$ is defined as the composites

$$\imath'(u, \beta) = \big(F'1_{a'} \circ f_0 \overset{\widehat{F}'\circ 1}{\Longrightarrow} 1_{F'a'} \circ f_0 \overset{\beta}{\Longrightarrow} f_1 \circ Fu\big),$$

and whose constraints, at pairs of 1-cells $(a_0, f_0) \xrightarrow{(u_1, \beta_1)} (a_1, f_1) \xrightarrow{(u_2, \beta_2)} (a_2, f_2)$ in $F{\downarrow}F'a'$, are the 2-cells of $F{\downarrow}F'$

$$(1_{u_2 \circ u_1}, \boldsymbol{l}_{1_{a'}}) : (u_2 \circ u_1, \imath'(u_2, \beta_2) \circledcirc \imath'(u_1, \beta_1), 1_{a'} \circ 1_{a'}) \cong (u_2 \circ u_1, \imath'(u_2 \circ u_1, \beta_2 \circledcirc \beta_1), 1_{a'}).$$

3.3. Lemma. *Let $\mathcal{A} \xrightarrow{F} \mathcal{B} \xleftarrow{F'} \mathcal{A}'$ be any diagram of bicategories, where F is a lax functor and F' is an oplax functor.*

(i) If $\mathcal{A}$ is a category with an initial object 0, then the homomorphism J in (16) *induces a homotopy equivalence on classifying spaces,* $\mathrm{B}(F0{\downarrow}F') \simeq \mathrm{B}(F{\downarrow}F')$.

(ii) If $\mathcal{A}'$ is a category with a terminal object 1, then the homomorphism J' in (16) *induces a homotopy equivalence on classifying spaces,* $\mathrm{B}(F{\downarrow}F'1) \simeq \mathrm{B}(F{\downarrow}F')$.

Proof. We only prove (i) since the proof of (ii) is parallel. Let $\langle a\rangle : 0 \to a$ be the unique morphism in $\mathcal{A}$ from the initial object to a. There is a 2-functor $L : F{\downarrow}F' \to F0{\downarrow}F'$ given on cells by

$$(a_0, f_0, a'_0) \underset{(u,\bar{\beta},\bar{u}')}{\overset{(u,\beta,u')}{\Downarrow (1,\alpha')}} (a_1, f_1, a'_1) \;\overset{L}{\mapsto}\; (f_0 \circ F\langle a_0\rangle, a'_0) \underset{(\ell(u,\bar{\beta},\bar{u}'),\bar{u}')}{\overset{(\ell(u,\beta,u'),u')}{\Downarrow \alpha'}} (f_1 \circ F\langle a_1\rangle, a'_1),$$

where, for any 1-cell $(u, \beta, u') : (a_0, f_0, a'_0) \to (a_1, f_1, a'_1)$ of $F{\downarrow}F'$, $\ell(u, \beta, u')$ is the 2-cell of $\mathcal{B}$ obtained by pasting the diagram

$$\ell(u,\beta,u') : \quad \begin{array}{ccc} F0 & \xrightarrow{1} & F0 \\ {\scriptstyle F\langle a_0\rangle}\downarrow & \overset{r^{-1}\cdot\widehat{F}}{\cong} & \downarrow{\scriptstyle F\langle a_1\rangle} \\ Fa_0 & \xrightarrow{Fu} & Fa_1 \\ {\scriptstyle f_0}\downarrow & \overset{\beta}{\Rightarrow} & \downarrow{\scriptstyle f_1} \\ F'a'_0 & \xrightarrow[F'u']{} & F'a'_1 \end{array}$$

that is,

$$\ell(u,\beta,u') = \Big(F'u' \circ (f_0 \circ F\langle a_0\rangle) \overset{\boldsymbol{a}^{-1}}{\Longrightarrow} (F'u' \circ f_0) \circ F\langle a_0\rangle \overset{\beta\circ 1}{\Longrightarrow} (f_1 \circ Fu) \circ F\langle a_0\rangle \overset{\boldsymbol{a}}{\Longrightarrow} f_1 \circ (Fu \circ F\langle a_0\rangle) \overset{1\circ\widehat{F}}{\Longrightarrow} f_1 \circ F(u \circ \langle a_0\rangle) = f_1 \circ F\langle a_1\rangle \overset{\boldsymbol{r}^{-1}}{\Longrightarrow} (f_1 \circ F\langle a_1\rangle) \circ 1 \Big).$$

In addition, there are two pseudo-transformations

$$1_{F_0\downarrow F'} \Rightarrow LJ, \quad JL \Rightarrow 1_{F\downarrow F'}.$$

The first one has as a component, at any object (f, a') of $F_0\downarrow F'$, the 1-cell

$$(\eta(f,a'), 1_{a'}) : (f,a') \to (f \circ F1_0, a'),$$
$$\eta(f,a') = \big(F'1_{a'} \circ f \overset{\widehat{F'}\circ 1}{\Longrightarrow} 1_{F'a'} \circ f \overset{\boldsymbol{l}}{\Longrightarrow} f \overset{\boldsymbol{r}^{-1}}{\Longrightarrow} f \circ 1_{F_0} \overset{1\circ\widehat{F}}{\Longrightarrow} f \circ F1_0 \overset{\boldsymbol{r}^{-1}}{\Longrightarrow} (f \circ F1_0) \circ 1_{F_0} \big)$$

while its naturality component, at any 1-cell $(\beta, u') : (f_0, a'_0) \to (f_1, a'_1)$ of $F_0\downarrow F'$, is given by the canonical isomorphism $\boldsymbol{l}^{-1} \cdot \boldsymbol{r} : u' \circ 1_{a'_0} \cong 1_{a'_1} \circ u'$,

$$\begin{array}{ccc} (f_0, a'_0) & \xrightarrow{\quad (\beta,u') \quad} & (f_1, a'_1) \\ {\scriptstyle (\eta,1)}\downarrow & \boldsymbol{l}^{-1}\overset{\cong}{\cdot}\boldsymbol{r} & \downarrow{\scriptstyle (\eta,1)} \\ (f_0 \circ F1_0, a'_0) & \xrightarrow[(\ell(1_0,\imath(\beta,u'),u'),u')]{} & (f_1 \circ F1_0, a'_1). \end{array}$$

As for the pseudo-transformation $JL \Rightarrow 1_{F\downarrow F'}$, it associates to an object (a, f, a') in $F\downarrow F'$ the 1-cell

$$(\langle a\rangle, \epsilon(a,f,a'), 1_{a'}) : (0, f \circ F\langle a\rangle, a') \to (a,f,a')$$
$$\epsilon(a,f,a') = \big(F'1_{a'} \circ (f \circ F\langle a\rangle) \overset{\widehat{F'}\circ 1}{\Longrightarrow} 1_{F'a'} \circ (f \circ F\langle a\rangle) \overset{\boldsymbol{l}}{\Longrightarrow} f \circ F\langle a\rangle \big)$$

while its naturality component, at a 1-cell $(u, \beta, u') : (a_0, f_0, a'_0) \to (a_1, f_1, a'_1)$ of $F\downarrow F'$, is

$$\begin{array}{ccc} (0, f_0 \circ F\langle a_0\rangle, a'_0) & \xrightarrow{\quad (1_0,\imath(\ell(u,\beta,u'),u'),u') \quad} & (0, f_1 \circ F\langle a_1\rangle, a'_1) \\ {\scriptstyle (\langle a_0\rangle,\epsilon,1)}\downarrow & (1,\boldsymbol{l}^{-1}\overset{\cong}{\cdot}\boldsymbol{r}) & \downarrow{\scriptstyle (\langle a_1\rangle,\epsilon,1)} \\ (a_0, f_0, a'_0) & \xrightarrow[(u,\beta,u')]{} & (a_1, f_1, a'_1). \end{array}$$

Therefore, by Lemma 2.9, there are homotopies $\mathrm{B}J\,\mathrm{B}L \Rightarrow 1_{\mathrm{B}(F\downarrow F')}$ and $1_{\mathrm{B}(F_0\downarrow F')} \Rightarrow \mathrm{B}L\,\mathrm{B}J$ making $\mathrm{B}J$ a homotopy equivalence. ∎

As we will see below, the following result is the key for proving Theorem 3.1.

3.4. Lemma. (*i*) *If an oplax functor $F' : \mathcal{A}' \to \mathcal{B}$ has the property* B_o*, then, for any lax functor $F : \mathcal{A} \to \mathcal{B}$, the commutative square*

$$\begin{array}{ccc} \mathrm{B}(F\downarrow F') & \xrightarrow{\mathrm{B}\bar{F}} & \mathrm{B}(\mathcal{B}\downarrow F') \\ {\scriptstyle \mathrm{B}P}\downarrow & & \downarrow{\scriptstyle \mathrm{B}P} \\ \mathrm{B}\mathcal{A} & \xrightarrow{\mathrm{B}F} & \mathrm{B}\mathcal{B}, \end{array} \tag{17}$$

induced by the first square in (10) *on classifying spaces, is a homotopy pullback.*

(*ii*) *If a lax functor $F : \mathcal{A} \to \mathcal{B}$ has the property* B_l*, then, for any oplax functor $F' : \mathcal{A}' \to \mathcal{B}$, the commutative square*

$$\begin{array}{ccc} \mathrm{B}(F\downarrow F') & \xrightarrow{\mathrm{B}P'} & \mathrm{B}\mathcal{A}' \\ {\scriptstyle \mathrm{B}\bar{F}'}\downarrow & & \downarrow{\scriptstyle \mathrm{B}F'} \\ \mathrm{B}(F\downarrow \mathcal{B}) & \xrightarrow{\mathrm{B}P'} & \mathrm{B}\mathcal{B}, \end{array} \tag{18}$$

induced by the second square in (10) *on classifying spaces, is a homotopy pullback.*

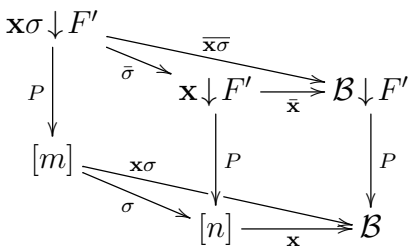
Proof. Suppose that $F' : \mathcal{A}' \to \mathcal{B}$ is any given oplax functor having the property B_o. We will prove that the simplicial map $\Delta P : \Delta(\mathcal{B}\downarrow F') \to \Delta\mathcal{B}$, induced on geometric nerves by the projection 2-functor $P : \mathcal{B}\downarrow F' \to \mathcal{B}$ in (9), satisfies the condition (*i*) of Lemma 2.3. To do so, let $\mathbf{x} : [n] \to \mathcal{B}$ be any geometric n-simplex of $\mathcal{B}$. Thanks to Lemma 2.6 (*i*), the square

$$\begin{array}{ccc} \mathbf{x}\downarrow F' & \xrightarrow{\bar{\mathbf{x}}} & \mathcal{B}\downarrow F' \\ {\scriptstyle P}\downarrow & & \downarrow{\scriptstyle P} \\ [n] & \xrightarrow{\mathbf{x}} & \mathcal{B} \end{array}$$

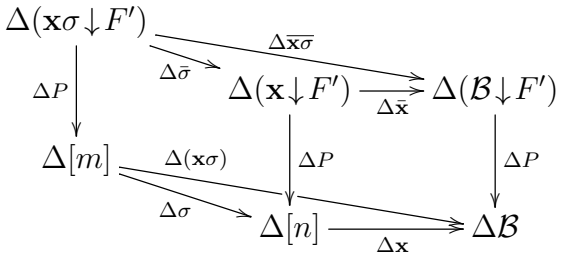
is a pullback in the category of bicategories and lax functors, whence the square induced by taking geometric nerves

$$\begin{array}{ccc} \Delta(\mathbf{x}\downarrow F') & \xrightarrow{\Delta\bar{\mathbf{x}}} & \Delta(\mathcal{B}\downarrow F') \\ {\scriptstyle \Delta P}\downarrow & & \downarrow{\scriptstyle \Delta P} \\ \Delta[n] & \xrightarrow{\Delta\mathbf{x}} & \Delta\mathcal{B} \end{array}$$

is a pullback in the category of simplicial sets. Therefore, $\Delta P^{-1}(\Delta\mathbf{x}) \cong \Delta(\mathbf{x}\downarrow F')$. Furthermore, for any map $\sigma : [m] \to [n]$ in the simplicial category, the diagram of lax functors



is commutative, whence the induced diagram of simplicial maps



is also commutative. Consequently, the diagram below commutes.

$$\begin{array}{ccc} \Delta P^{-1}(\Delta\mathbf{x}\Delta\sigma) & \longrightarrow & \Delta P^{-1}(\Delta\mathbf{x}) \\ \cong\downarrow & & \downarrow\cong \\ \Delta(\mathbf{x}\sigma\downarrow F') & \xrightarrow{\Delta\bar{\sigma}} & \Delta(\mathbf{x}\downarrow F') \end{array}$$

Therefore, it suffices to prove that the lax functor $\bar{\sigma} : \mathbf{x}\sigma \downarrow F' \to \mathbf{x} \downarrow F'$ induces a homotopy equivalence on classifying spaces, $\mathrm{B}(\mathbf{x}\sigma \downarrow F') \simeq \mathrm{B}(\mathbf{x} \downarrow F')$. But note that we have the diagram

$$\begin{array}{ccc} \mathbf{x}\sigma 0\downarrow F' & \xrightarrow{J} & \mathbf{x}\sigma\downarrow F' \\ {\scriptstyle \mathbf{x}(0,\sigma 0)^*}\downarrow & \overset{\theta}{\Rightarrow} & \downarrow{\scriptstyle \bar{\sigma}} \\ \mathbf{x}0\downarrow F' & \xrightarrow{J} & \mathbf{x}\downarrow F' \end{array}$$

where the homomorphisms J are given as in (16), and θ is the pseudo-transformation that assigns to every object (f, a') of $\mathbf{x}\sigma 0\downarrow F'$ the 1-cell of $\mathbf{x}\downarrow F'$

$$\big((0,\sigma 0), \theta(f,a'), 1_{a'}\big) : (0, f\circ\mathbf{x}(0,\sigma 0), a') \to (\sigma 0, f, a'),$$

where the 2-cell of $\mathcal{B}$

$$\begin{array}{ccc} \mathbf{x}0 & \xrightarrow{\mathbf{x}(0,\sigma 0)} & \mathbf{x}\sigma 0 \\ {\scriptstyle f\circ\mathbf{x}(0,\sigma 0)}\downarrow & \overset{\theta(f,a')}{\Rightarrow} & \downarrow{\scriptstyle f} \\ F'a' & \xrightarrow[F'1_{a'}]{} & F'a' \end{array}$$

is the composite $\theta(f,a') = \big(F'1_{a'}\circ(f\circ\mathbf{x}(0,\sigma 0)) \overset{\widehat{F'}\circ 1}{\Longrightarrow} 1_{F'a'}\circ(f\circ\mathbf{x}(0,\sigma 0)) \overset{\boldsymbol{l}}{\Longrightarrow} f\circ\mathbf{x}(0,\sigma 0)\big)$, and its naturality component at any 1-cell $(\beta, u') : (f_0, a'_0) \to (f_1, a'_1)$

$$\begin{array}{ccc} (0, f_0\circ\mathbf{x}(0,\sigma 0), a'_0) & \xrightarrow{((0,0),\imath(\beta\circledcirc\mathbf{x}(0,\sigma 0),u'),u')} & (0, f_1\circ\mathbf{x}(0,\sigma 0), a'_1) \\ {\scriptstyle ((0,\sigma 0),\theta(f_0,a'_0),1_{a'_0})}\downarrow & \overset{(1,\boldsymbol{l}^{-1}_{\cong}\cdot\boldsymbol{r})}{} & \downarrow{\scriptstyle ((0,\sigma 0),\theta(f_1,a'_1),1_{a'_1})} \\ (\sigma 0, f_0, a'_0) & \xrightarrow[((\sigma 0,\sigma 0),\imath(\beta,u'),u')]{} & (\sigma 0, f_1, a'_1) \end{array}$$

is given by the canonical isomorphism $\boldsymbol{l}^{-1}\cdot\boldsymbol{r} : u'\circ 1_{a'_1} \cong 1_{a'_2}\circ u'$ in $\mathcal{A}'$. Therefore, by Lemma 2.9, the induced square on classifying spaces

$$\begin{array}{ccc} \mathrm{B}(\mathbf{x}\sigma 0{\downarrow}F') & \xrightarrow{\mathrm{B}J} & \mathrm{B}(\mathbf{x}\sigma{\downarrow}F') \\ {\scriptstyle \mathrm{B}\mathbf{x}(0,\sigma 0)^*}\downarrow & \overset{\mathrm{B}\theta}{\Rightarrow} & \downarrow{\scriptstyle \mathrm{B}\bar{\sigma}} \\ \mathrm{B}(\mathbf{x}0{\downarrow}F') & \xrightarrow{\mathrm{B}J} & \mathrm{B}(\mathbf{x}{\downarrow}F') \end{array}$$

is homotopy commutative. Moreover, by Lemma 3.3(i), both maps BJ in the square are homotopy equivalences and, since the oplax functor F' has the property B_o, the map $\mathrm{B}\mathbf{x}(0,\sigma 0)^* : \mathrm{B}(\mathbf{x}\sigma 0{\downarrow}F') \to \mathrm{B}(\mathbf{x}0{\downarrow}F')$ is also a homotopy equivalence. It follows that the remaining map in the square has the same property, that is, the map $\mathrm{B}\bar{\sigma} : \mathrm{B}(\mathbf{x}\sigma{\downarrow}F') \simeq \mathrm{B}(\mathbf{x}{\downarrow}F')$ is a homotopy equivalence, as required.

Suppose now that $F : \mathcal{A} \to \mathcal{B}$ is any lax functor. Again, by Lemma 2.6(i), the first square in (10) is a pullback in the category of bicategories and lax functors, whence the square induced by taking geometric nerves

$$\begin{array}{ccc} \Delta(F{\downarrow}F') & \xrightarrow{\Delta\bar{F}} & \Delta(\mathcal{B}{\downarrow}F') \\ {\scriptstyle \Delta P}\downarrow & & \downarrow{\scriptstyle \Delta P} \\ \Delta\mathcal{A} & \xrightarrow{\Delta F} & \Delta\mathcal{B} \end{array} \tag{19}$$

is a pullback in the category of simplicial sets. By what has been already proven above, it follows from Lemma 2.3 (iii) that the commutative square

$$\begin{array}{ccc} |\Delta(F{\downarrow}F')| & \xrightarrow{|\Delta\bar{F}|} & |\Delta(\mathcal{B}{\downarrow}F')| \\ {\scriptstyle |P|}\downarrow & & \downarrow{\scriptstyle |P|} \\ |\Delta\mathcal{A}| & \xrightarrow{|\Delta F|} & |\Delta\mathcal{B}| \end{array} \qquad \overset{(14)}{\simeq} \qquad \begin{array}{ccc} \mathrm{B}(F{\downarrow}F') & \xrightarrow{\mathrm{B}\bar{F}} & \mathrm{B}(\mathcal{B}{\downarrow}F') \\ {\scriptstyle \mathrm{B}P}\downarrow & & \downarrow{\scriptstyle \mathrm{B}P} \\ \mathrm{B}\mathcal{A} & \xrightarrow{\mathrm{B}F} & \mathrm{B}\mathcal{B} \end{array}$$

is a homotopy pullback. This completes the proof of part (i) of the lemma.

The proof of part (ii) follows similar lines, but using the geometric nerve functor ∇ instead of Δ as above. Thus, for example, given any lax functor $F : \mathcal{A} \to \mathcal{B}$ having the property B_l, we start by proving that the simplicial map $\nabla P' : \nabla(F\downarrow\mathcal{B}) \to \nabla\mathcal{B}$ satisfies the condition (i) in Lemma 2.3, which we do by first getting natural simplicial isomorphisms $\nabla P'^{-1}(\nabla\mathbf{x}') \cong \nabla(F\downarrow\mathbf{x}')$, for the different oplax functors $\mathbf{x}' : [n] \to \mathcal{B}$ (i.e., the simplices of $\nabla\mathcal{B}$), and then by proving that any simplicial map $\sigma : [m] \to [n]$ induces a homotopy equivalence $\mathrm{B}(F\downarrow\mathbf{x}'\sigma) \simeq \mathrm{B}(F\downarrow\mathbf{x}')$. Here, we need to use the homomorphisms $J' : F\downarrow\mathbf{x}'n \to F\downarrow\mathbf{x}'$ in (16), which induce homotopy equivalences on classifying spaces by Lemma 3.3 (ii), and the existence of a pseudo-transformation $\theta' : \bar{\sigma}\,J' \Rightarrow J'\,\mathbf{x}'(\sigma m, n)_*$, which assigns to every object (a, f) of $F\downarrow\mathbf{x}'\sigma m$ the 1-cell $\big(1_a, \theta'(a,f), (\sigma m, n)\big) : (a, f, \sigma m) \to (a, \mathbf{x}'(\sigma m, n)\circ f, n)$, where

$$\theta'(a,f) = \big(\mathbf{x}'(\sigma m,n)\circ f \overset{\boldsymbol{r}^{-1}}{\Longrightarrow} (\mathbf{x}'(\sigma m,n)\circ f)\circ 1_{Fa} \overset{1\circ\widehat{F}}{\Longrightarrow} (\mathbf{x}'(\sigma m,n)\circ f)\circ F1_a\big).$$

Using Lemma 2.6 (ii) therefore, we deduce that, for any lax functor $F' : \mathcal{A}' \to \mathcal{B}$, the square

$$\begin{array}{ccc} \nabla(F\downarrow F') & \xrightarrow{\nabla P'} & \nabla\mathcal{A}' \\ {\scriptstyle \nabla\bar{F}'}\downarrow & & \downarrow{\scriptstyle \nabla F'} \\ \nabla(F\downarrow\mathcal{B}) & \xrightarrow{\nabla P'} & \nabla\mathcal{B}, \end{array}$$

is a pullback in the category of simplicial sets which, by Lemma 2.3, induces a homotopy pullback square on geometric realizations. It follows that (18) is a homotopy pullback. ∎

With the corollary below we will be ready to complete the proof of Theorem 3.1.

3.5. Corollary. (i) *For any lax functor $F : \mathcal{A} \to \mathcal{B}$, the projection 2-functor $P : F\downarrow \mathcal{B} \to \mathcal{A}$ induces a homotopy equivalence on classifying spaces,* $\mathrm{B}(F\downarrow\mathcal{B}) \simeq \mathrm{B}\mathcal{A}$.

(ii) *For any oplax functor $F' : \mathcal{A}' \to \mathcal{B}$, the projection 2-functor $P' : \mathcal{B}\downarrow F' \to \mathcal{A}'$ induces a homotopy equivalence on classifying spaces,* $\mathrm{B}(\mathcal{B}\downarrow F') \simeq \mathrm{B}\mathcal{A}'$.

Proof. Once again we limit ourselves to proving (i). Let $F : \mathcal{A} \to \mathcal{B}$ be a lax functor.

The identity homomorphism $1_{\mathcal{B}} : \mathcal{B} \to \mathcal{B}$ has the property B_o since, for any object $b \in \mathrm{Ob}\mathcal{B}$, the classifying space of the comma bicategory $b\downarrow\mathcal{B}$ is contractible, by Lemma 3.2. Therefore, Lemma 3.4 (i) applies to the case when $F' = 1_{\mathcal{B}}$, and tells us that the induced commutative square

$$\begin{array}{ccc} \mathrm{B}(F\downarrow\mathcal{B}) & \xrightarrow{\mathrm{B}\bar{F}} & \mathrm{B}(\mathcal{B}\downarrow\mathcal{B}) \\ {\scriptstyle \mathrm{B}P}\downarrow & & \downarrow{\scriptstyle \mathrm{B}P} \\ \mathrm{B}\mathcal{A} & \xrightarrow{\mathrm{B}F} & \mathrm{B}\mathcal{B}, \end{array}$$

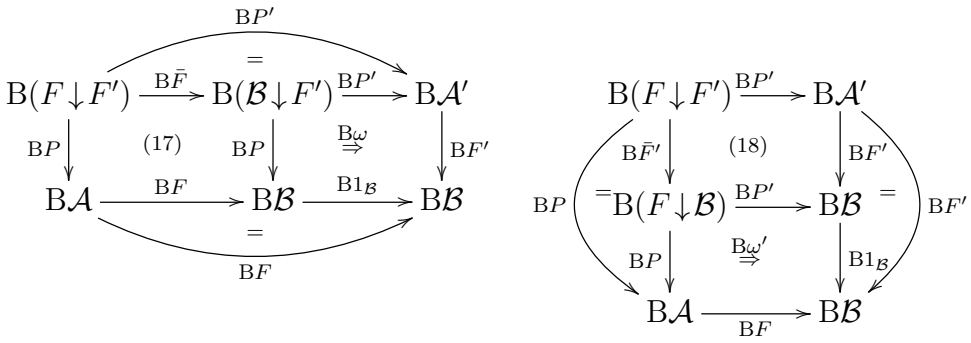
is a homotopy pullback. So, it is enough to prove that the map $\mathrm{B}P : \mathrm{B}(\mathcal{B}\downarrow\mathcal{B}) \to \mathrm{B}\mathcal{B}$ is a homotopy equivalence. To do so, let b be any object of $\mathcal{B}$, and let us particularize the square above to the case where $F = b : [0] \to \mathcal{B}$. Then, we find the commutative homotopy pullback square

$$\begin{array}{ccc} \mathrm{B}(b\downarrow\mathcal{B}) & \xrightarrow{\mathrm{B}\bar{b}} & \mathrm{B}(\mathcal{B}\downarrow\mathcal{B}) \\ {\scriptstyle \mathrm{B}P}\downarrow & & \downarrow{\scriptstyle \mathrm{B}P} \\ \mathrm{pt} & \xrightarrow{\mathrm{B}b} & \mathrm{B}\mathcal{B}, \end{array}$$

where, by Lemma 3.2, the left vertical map is a homotopy equivalence. This tells us that the different homotopy fibres of the map $\mathrm{B}P : \mathrm{B}(\mathcal{B}\downarrow\mathcal{B}) \to \mathrm{B}\mathcal{B}$ over the 0-cells of $\mathrm{B}\mathcal{B}$ are all contractible, and consequently $\mathrm{B}P$ is actually a homotopy equivalence. ∎

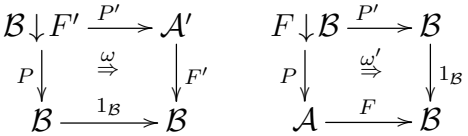
We can now complete the proof of Theorem 3.1:

For any diagram $\mathcal{A} \xrightarrow{F} \mathcal{B} \xleftarrow{F'} \mathcal{A}'$, where F is a lax functor and F' is an oplax functor, the square (15) occurs as the outside region in both of the following two diagrams:



(20)

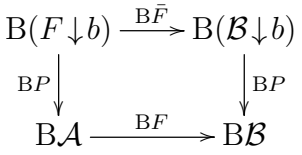
where the inner squares with the homotopies labelled Bω and Bω' are the particular cases of the squares (15) obtained when $F = 1_{\mathcal{B}}$ and when $F' = 1_{\mathcal{B}}$, respectively. The homotopies are respectively induced, by Lemma 2.9, by the lax transformations



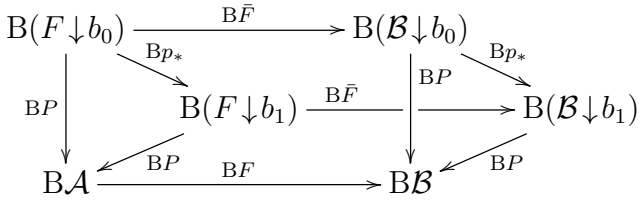
which are defined as follows: The lax transformation ω associates to any object (b, f, a') of $\mathcal{B}\downarrow F'$ the 1-cell $f : b \to F'a'$, and its naturality component at any 1-cell $(p, \beta, u') : (b_0, f_0, a'_0) \to (b_1, f_1, a'_1)$ is the 2-cell $\beta : F'u' \circ f_0 \Rightarrow f_1 \circ p$. Similarly, ω' associates to any object (a, f, b) of $F\downarrow\mathcal{B}$ the 1-cell $f : Fa \to b$, and its naturality component at any 1-cell $(u, \beta, p) : (a_0, f_0, b_0) \to (a_1, f_1, b_1)$ is $\beta : p \circ f_0 \Rightarrow f_1 \circ Fu$. Since, by Corollary 3.5, both maps $\mathrm{B}P' : \mathrm{B}(\mathcal{B}\downarrow F') \to \mathrm{B}\mathcal{A}'$ and $\mathrm{B}P : \mathrm{B}(F\downarrow\mathcal{B}) \to \mathrm{B}\mathcal{A}$ are homotopy equivalences, both squares are homotopy pullbacks. The other inner squares are those referred to therein.

The above implies the part (i) of Theorem 3.1 and, furthermore, it follows that the square (15) is a homotopy pullback whenever one of the inner squares (17) or (18) is a homotopy pullback. Therefore, Lemma 3.4 implies part (ii).

For proving part (iii), suppose a lax functor $F : \mathcal{A} \to \mathcal{B}$ is given such that the square (15) is a homotopy pullback for any $F' = b : [0] \to \mathcal{B}$, $b \in \mathrm{Ob}\mathcal{B}$. It follows from the diagram on the left in (20) that the inner square (17)



is a homotopy pullback for any object $b \in \mathcal{B}$. Then, if $p : b_0 \to b_1$ is any 1-cell of $\mathcal{B}$, since we have the commutative diagram



we deduce that the square

$$\begin{array}{ccc} \mathrm{B}(F\downarrow b_0) & \xrightarrow{\mathrm{B}\bar{F}} & \mathrm{B}(\mathcal{B}\downarrow b_0) \\ {\scriptstyle \mathrm{B}p_*}\downarrow & & \downarrow{\scriptstyle \mathrm{B}p_*} \\ \mathrm{B}(F\downarrow b_1) & \xrightarrow{\mathrm{B}\bar{F}} & \mathrm{B}(\mathcal{B}\downarrow b_1) \end{array}$$

is also a homotopy pullback. Therefore, as $\mathrm{B}(\mathcal{B}\downarrow b_0) \simeq \mathrm{pt} \simeq \mathrm{B}(\mathcal{B}\downarrow b_1)$, by Lemma 3.2, we conclude that the induced map $\mathrm{B}p_* : \mathrm{B}(F \downarrow b_0) \simeq \mathrm{B}(F \downarrow b_1)$ is a homotopy equivalence. That is, the lax functor F has the property B_l. ∎

As a corollary, we obtain the following theorem, which is just the well-known Quillen's Theorem B [42] when the lax or oplax functor F in the hypothesis is an ordinary functor between small categories. The generalization of Theorem B to lax functors between bicategories was originally stated and proven by Calvo, Cegarra, and Heredia in [15, Theorem 5.4], and it also generalizes a similar result by the first author in [18, Theorem 3.2] for the case when F is a 2-functor between 2-categories.

3.6. Corollary. (*i*) *If a lax functor $F : \mathcal{A} \to \mathcal{B}$ has the property B_l then, for every object $b \in \mathcal{B}$, there is an induced homotopy fibre sequence*

$$\mathrm{B}(F\downarrow b) \xrightarrow{\mathrm{B}P} \mathrm{B}\mathcal{A} \xrightarrow{\mathrm{B}F} \mathrm{B}\mathcal{B}.$$

(*ii*) *If an oplax functor $F' : \mathcal{A}' \to \mathcal{B}$ has the property B_o then, for every object $b \in \mathcal{B}$, there is an induced homotopy fibre sequence*

$$\mathrm{B}(b\downarrow F') \xrightarrow{\mathrm{B}P'} \mathrm{B}\mathcal{A}' \xrightarrow{\mathrm{B}F'} \mathrm{B}\mathcal{B}.$$

Proof. It follows from Theorem 3.1, by taking $F' = b : [0] \to \mathcal{B}$ to obtain part (*i*) and $F = b : [0] \to \mathcal{B}$ for part (*ii*). ∎

By the above result in [15, 18], the bicategories $F\downarrow b$ and $b\downarrow F'$ are called *homotopy-fibre bicategories*. The following consequence was proven in [15, Theorem 5.6], and it shows a generalization of Quillen's Theorem A [42].

3.7. Corollary. *(i) Let $F : \mathcal{A} \to \mathcal{B}$ be a lax functor such that the classifying spaces of its homotopy-fibre categories are contractible, that is,* $\mathrm{B}(F\downarrow b) \simeq \mathrm{pt}$ *for every object* $b \in \mathcal{B}$. *Then, the induced map on classifying spaces* $\mathrm{B}F : \mathrm{B}\mathcal{A} \to \mathrm{B}\mathcal{B}$ *is a homotopy equivalence.*

(ii) Let $F' : \mathcal{A}' \to \mathcal{B}$ be an oplax functor such that the classifying spaces of its homotopy-fibre categories are contractible, that is, $\mathrm{B}(b\downarrow F') \simeq \mathrm{pt}$ *for every object* $b \in \mathcal{B}$. *Then, the induced map on classifying spaces* $\mathrm{B}F' : \mathrm{B}\mathcal{A}' \to \mathrm{B}\mathcal{B}$ *is a homotopy equivalence.*

Particular cases of the above results have also been stated by Bullejos and Cegarra in [13, Theorem 1.2], for the case when $F : \mathcal{A} \to \mathcal{B}$ is any 2-functor between 2-categories, and by del Hoyo in [24, Theorem 6.4], for the case when F is a lax functor from a category $\mathcal{A}$ to a 2-category $\mathcal{B}$. In [21, Théorème 6.5], Chiche proved a relative Theorem A for lax functors between 2-categories, which also specializes by giving the particular case of Corollary 3.7 when F is any lax functor between 2-categories.

Next we study conditions on a bicategory $\mathcal{B}$ in order for the square (15) to always be a homotopy pullback. We use that, for any two objects b, b' of a bicategory $\mathcal{B}$, there is a diagram

$$\begin{array}{ccc} \mathcal{B}(b,b') & \longrightarrow & [0] \\ \downarrow & \overset{\gamma}{\Rightarrow} & \downarrow{\scriptstyle b'} \\ [0] & \xrightarrow{\ b\ } & \mathcal{B}, \end{array} \tag{21}$$

in which γ is the lax transformation defined by $\gamma f = f$, for any 1-cell $f : b \to b'$ in $\mathcal{B}$, and whose naturality component at a 2-cell $\beta : f_0 \Rightarrow f_1$, for any $f_0, f_1 : b \to b'$, is the composite 2-cell $\widehat{\gamma}_\beta = \big(1_{b'} \circ f_0 \overset{\boldsymbol{l}}{\cong} f_0 \overset{\beta}{\Rightarrow} f_1 \overset{\boldsymbol{r}^{-1}}{\cong} f_1 \circ 1_b\big)$.

3.8. Theorem. *The following properties of a bicategory $\mathcal{B}$ are equivalent:*

(*i*) *For any diagram of bicategories $\mathcal{A} \xrightarrow{F} \mathcal{B} \xleftarrow{F'} \mathcal{A}'$, where F is a lax functor and F' is an oplax functor, the induced square* (15)

$$\begin{array}{ccc} \mathrm{B}(F\downarrow F') & \xrightarrow{\mathrm{B}P'} & \mathrm{B}\mathcal{A}' \\ {\scriptstyle \mathrm{B}P}\downarrow & \Rightarrow & \downarrow{\scriptstyle \mathrm{B}F'} \\ \mathrm{B}\mathcal{A} & \xrightarrow{\mathrm{B}F} & \mathrm{B}\mathcal{B} \end{array}$$

is a homotopy pullback.

(*ii*) *Any lax functor $F : \mathcal{A} \to \mathcal{B}$ has the property* B_l.

(*iii*) *Any oplax functor $F' : \mathcal{A}' \to \mathcal{B}$ has the property* B_o.

(*iv*) *For any object b and 1-cell $p : b_0 \to b_1$ in $\mathcal{B}$, the functor $p_* : \mathcal{B}(b, b_0) \to \mathcal{B}(b, b_1)$ induces a homotopy equivalence on classifying spaces,* $\mathrm{B}\mathcal{B}(b, b_0) \simeq \mathrm{B}\mathcal{B}(b, b_1)$.

(*v*) *For any object b and 1-cell $p : b_0 \to b_1$ in $\mathcal{B}$, the functor $p^* : \mathcal{B}(b_1, b) \to \mathcal{B}(b_0, b)$ induces a homotopy equivalence on classifying spaces,* $\mathrm{B}\mathcal{B}(b_1, b) \simeq \mathrm{B}\mathcal{B}(b_0, b)$.

(*vi*) *For any two objects* $b, b' \in \mathcal{B}$*, the homotopy commutative square*

$$\begin{array}{ccc} \mathrm{B}\mathcal{B}(b,b') & \longrightarrow & \mathrm{pt} \\ \downarrow & \overset{\mathrm{B}\gamma}{\Rightarrow} & \downarrow{\scriptstyle \mathrm{B}b'} \\ \mathrm{pt} & \xrightarrow{\mathrm{B}b} & \mathrm{B}\mathcal{B}, \end{array}$$

induced by (21)*, is a homotopy pullback. That is, the whisker map*

$$\mathrm{B}\mathcal{B}(b,b') \to \{\gamma : I \to \mathrm{B}\mathcal{B} \mid \gamma(0) = \mathrm{B}b, \gamma(1) = \mathrm{B}b'\} \subseteq \mathrm{B}\mathcal{B}^I$$

is a homotopy equivalence.

Proof. The implications (*i*) $\Leftrightarrow$ (*ii*) $\Leftrightarrow$ (*iii*) are all direct consequences of Theorem 3.1. For the remaining implications, let us take into account that, for any objects $b, b' \in \mathcal{B}$ there is quite an obvious isomorphism of categories $b \downarrow b' \cong \mathcal{B}(b, b')$. With this identification in mind, we see that the homomorphism $b : [0] \to \mathcal{B}$ has the property B_l (resp. B_o) if and only if, for any 1-cell $p : b_0 \to b_1$ in $\mathcal{B}$, the functor $p_* : \mathcal{B}(b, b_0) \to \mathcal{B}(b, b_1)$ (resp. $p^* : \mathcal{B}(b_1, b) \to \mathcal{B}(b_0, b)$) induces a homotopy equivalence on classifying spaces. Therefore, the implications (*ii*) $\Rightarrow$ (*iv*) and (*iii*) $\Rightarrow$ (*v*) are clear.

Furthermore, we see that the square in (*vi*) identifies the square

$$\begin{array}{ccc} \mathrm{B}(b{\downarrow}b') & \xrightarrow{\mathrm{B}P'} & \mathrm{B}[0] \\ {\scriptstyle \mathrm{B}P}\downarrow & \Rightarrow & \downarrow{\scriptstyle \mathrm{B}b'} \\ \mathrm{B}[0] & \xrightarrow{\mathrm{B}b} & \mathrm{B}\mathcal{B}. \end{array}$$

Then, for b fixed, it follows from Theorem 3.1 that the square in (*vi*) is a homotopy pullback for any b' if and only if $b : [0] \to \mathcal{B}$ has the property B_l, that is, the equivalence of statements (*vi*) $\Leftrightarrow$ (*iv*) holds.

Finally, to complete the proof, we are going to prove that (*iv*) $\Rightarrow$ (*iii*) and we shall leave it to the reader the proof that (*v*) $\Rightarrow$ (*ii*) since it is parallel. By hypothesis, for any object $b \in \mathrm{Ob}\mathcal{B}$, the normal homomorphism $b : [0] \to \mathcal{B}$ has the property B_l. Then, by Theorem 3.1 (*ii*), for any oplax functor $F' : \mathcal{A}' \to \mathcal{B}$ the square

$$\begin{array}{ccc} \mathrm{B}(b{\downarrow}F') & \xrightarrow{\mathrm{B}P'} & \mathrm{B}\mathcal{A}' \\ {\scriptstyle \mathrm{B}P}\downarrow & \Rightarrow & \downarrow{\scriptstyle \mathrm{B}F'} \\ \mathrm{B}[0] & \xrightarrow{\mathrm{B}b} & \mathrm{B}\mathcal{B} \end{array}$$

is a homotopy pullback for any object $b \in \mathcal{B}$. Therefore, by Theorem 3.1 (*iii*), F' has the property B_o. ∎

We can state that

(B) *a bicategory $\mathcal{B}$ has the property* B if it has the properties in Theorem *3.8*.

For example, *bigroupoids*, that is, bicategories whose 1-cells are invertible up to a 2-cell, and whose 2-cells are strictly invertible, have the property B: If $\mathcal{B}$ is any bigroupoid, for any object b and 1-cell $p : b_0 \to b_1$ in $\mathcal{B}$, the functor $p^* : \mathcal{B}(b_1, b) \to \mathcal{B}(b_0, b)$ is actually an equivalence of categories and, therefore, induces a homotopy equivalence on classifying spaces $\mathrm{B}p^* : \mathrm{B}\mathcal{B}(b_1, b) \simeq \mathrm{B}\mathcal{B}(b_0, b)$. Recall that, by the correspondence $\mathcal{B} \mapsto \mathrm{B}\mathcal{B}$, bigroupoids correspond to homotopy 2-types, that is, CW-complexes whose n^{th} homotopy groups at any base point vanish for $n \geq 3$ (see Duskin [26, Theorem 8.6]).

3.9. Corollary. *If a bicategory $\mathcal{B}$ has the property* B*, then, for any object $b \in \mathcal{B}$, there is a homotopy equivalence*

$$\Omega(\mathrm{B}\mathcal{B}, \mathrm{B}b) \simeq \mathrm{B}\mathcal{B}(b, b) \tag{22}$$

between the loop space of the classifying space of the bicategory with base point Bb *and the classifying space of the category of endomorphisms of b in $\mathcal{B}$.*

The above homotopy equivalence is already known when the bicategory is strict, that is, when $\mathcal{B}$ is a 2-category. It appears as a main result in the paper by Del Hoyo [24, Theorem 8.5], and it was also stated at the same time by the first author in [18, Example 4.4]. Indeed, that homotopy equivalence (22), for the case when $\mathcal{B}$ is a 2-category, can be deduced from a result by Tillmann about simplicial categories in [48, Lemma 3.3].

3.10. Homotopy pullbacks of 2-categories. As we recalled in the introduction, the category **2-Cat** of (strict) 2-categories and 2-functors has a Thomason-type Quillen model structure [1, Théorème 6.27], such that the classifying space functor is an equivalence of homotopy theories between 2-categories and topological spaces [1, Corollaire 6.31]. Thus, in this model category, a 2-functor $F : \mathcal{A} \to \mathcal{B}$ is a weak equivalence if and only if the induced map on classifying spaces $\mathrm{B}F : \mathrm{B}\mathcal{A} \to \mathrm{B}\mathcal{B}$ is a homotopy equivalence, and a commutative square in **2-Cat**

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{G'} & \mathcal{A}' \\ {\scriptstyle G}\downarrow & & \downarrow{\scriptstyle F'} \\ \mathcal{A} & \xrightarrow{F} & \mathcal{B} \end{array} \tag{23}$$

is a homotopy pullback if and only if the induced on classifying spaces

$$\begin{array}{ccc} \mathrm{B}\mathcal{C} & \xrightarrow{\mathrm{B}G'} & \mathrm{B}\mathcal{A}' \\ {\scriptstyle \mathrm{B}G}\downarrow & & \downarrow{\scriptstyle \mathrm{B}F'} \\ \mathrm{B}\mathcal{A} & \xrightarrow{\mathrm{B}F} & \mathrm{B}\mathcal{B} \end{array} \tag{24}$$

is a homotopy pullback of spaces (see Remark 2.2). This is, for example, the case when F or F' in (23) is a fibration, $\mathcal{C} = \mathcal{A} \times_{\mathcal{B}} \mathcal{A}'$ is the pullback 2-category, and G and and G' are the respective projection 2-functors.

Our result in Theorem 3.1 has a natural interpretation in this setting as below. Observe that, when F and F' are 2-functors between 2-categories as above, the homotopy-fibre product bicategory $F \downarrow F'$ as well as the homotopy-fibre bicategories $F \downarrow b$ are actually 2-categories.

3.11. Proposition. *Let* (23) *be a commutative square in* **2-Cat**. *Suppose the* 2*-functor* F *has property* B_l. *Then the square is a homotopy pullback in the model category* **2-Cat** *(with the 'Thomason' model structure) if and only if the canonical* 2*-functor*

$$V : \mathcal{C} \to F \downarrow F'$$

$$c \overset{h}{\underset{h'}{\Downarrow \alpha}} c' \quad \overset{V}{\longmapsto} \quad (Gc, 1, G'c) \overset{(Gh,1,G'h)}{\underset{(Gh',1,G'h')}{\Downarrow (G\alpha, G'\alpha)}} (Gc', 1, G'c'),$$

is a weak equivalence of 2*-categories.*

Proof. The square (24) is the composite of the induced squares

$$\begin{array}{ccccc} \mathrm{B}C & \xrightarrow{\mathrm{B}V} & \mathrm{B}(F \downarrow F') & \xrightarrow{\mathrm{B}P'} & \mathrm{B}\mathcal{A}' \\ {\scriptstyle \mathrm{B}G}\downarrow & & {\scriptstyle \mathrm{B}P}\downarrow & \Rightarrow & \downarrow{\scriptstyle \mathrm{B}F'} \\ \mathrm{B}\mathcal{A} & = & \mathrm{B}\mathcal{A} & \xrightarrow{\mathrm{B}F} & \mathrm{B}\mathcal{B}, \end{array} \tag{25}$$

where the right square is (15). In effect, one easily verifies that $PV = G$ and $P'V = G'$. Further, the homotopy $\mathrm{B}F\,\mathrm{B}P \Rightarrow \mathrm{B}F'\,\mathrm{B}P'$ is induced by the lax transformation $\omega : FP \Rightarrow F'P'$ with $\omega(a, f, a') = f : Fa \to Fa'$, and whose naturality component at any 1-cell $(u, \beta, u') : (a_0, f_0, a_0') \to (a_1, f_1, a_1')$ is the 2-cell $\beta : F'u' \circ f_0 \Rightarrow f_1 \circ Fu$. The composite $\omega V : FPV = FG \Rightarrow F'G'V = F'G'$ is then the identity transformation, whence $\mathrm{B}\omega\mathrm{B}V$ is the static homotopy on $FG = F'G'$.

Suppose now that F has property B_l, so that the right square in (25) is a homotopy pullback of spaces, by Theorem 3.1. It follows that the composite square (24) is a homotopy pullback if and only if the left square in (25) is as well. As the later is a homotopy pullback if and only if the map $\mathrm{B}V$ is a homotopy equivalence, the proposition follows. ∎

3.12. The case when both functors are lax. For a diagram $\mathcal{A} \xrightarrow{F} \mathcal{B} \xleftarrow{G} \mathcal{C}$, where both F and G are lax functors, the comma bicategory $F \downarrow G$ is not defined (unless G is a homomorphism). However, we can obtain a bicategorical model for the homotopy pullback of the induced maps $\mathrm{B}\mathcal{A} \xrightarrow{\mathrm{B}F} \mathrm{B}\mathcal{B} \xleftarrow{\mathrm{B}G} \mathrm{B}\mathcal{C}$ as follows: Let

$$F \downarrow_2 G := F \downarrow P'$$

be the comma bicategory defined as in (5) by the diagram $\mathcal{A} \xrightarrow{F} \mathcal{B} \xleftarrow{P'} G \downarrow \mathcal{B}$, where P' is the projection 2-functor (9) (the notation is taken from Dwyer, Kan, and Smith in

[27] and Barwick and Kan in [3, 4]). Thus, $F\downarrow_2 G$ has 0-cells tuples (a, f, b, g, c), where $Fa \xrightarrow{f} b \xleftarrow{g} Gc$ are 1-cells of $\mathcal{B}$. A 1-cell

$$(u, \beta, p, \beta', v) : (a_0, f_0, b_0, g_0, c_0) \to (a_1, f_1, b_1, g_1, c_1)$$

in $F\downarrow_2 G$ consists of 1-cells $u : a_0 \to a_1$, $p : b_0 \to b_1$, and $v : c_0 \to c_1$, in $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$, respectively, together with 2-cells β and β' of $\mathcal{B}$ as in the diagram

$$\begin{array}{ccccc} Fa_0 & \xrightarrow{f_0} & b_0 & \xleftarrow{g_0} & Gc_0 \\ {\scriptstyle Fu}\downarrow & \overset{\beta}{\Leftarrow} & {\scriptstyle p}\downarrow & \overset{\beta'}{\Rightarrow} & \downarrow{\scriptstyle Gv} \\ Fa_1 & \xrightarrow[f_1]{} & b_1 & \xleftarrow[g_1]{} & Gc_1, \end{array}$$

and a 2-cell

$$(a_0, f_0, b_0, g_0, c_0) \overset{(u,\beta,p,\beta',v)}{\underset{(\bar{u},\bar{\beta},\bar{p},\bar{\beta}',\bar{v})}{\Downarrow(\alpha,\delta,\rho)}} (a_1, f_1, b_1, g_1, c_1),$$

is given by 2-cells $\alpha : u \Rightarrow \bar{u}$ in $\mathcal{A}$, $\delta : p \Rightarrow \bar{p}$ in $\mathcal{B}$, and $\rho : v \Rightarrow \bar{v}$ in $\mathcal{C}$, such that the diagrams below commute.

$$\begin{array}{ccc} p \circ f_0 & \xRightarrow{\delta\circ 1} & \bar{p} \circ f_0 \\ {\scriptstyle\beta}\Downarrow & & \Downarrow{\scriptstyle\bar{\beta}} \\ f_1 \circ Fu & \xRightarrow{1\circ F\alpha} & f_1 \circ F\bar{u} \end{array} \qquad \begin{array}{ccc} p \circ g_0 & \xRightarrow{\delta\circ 1} & \bar{p} \circ g_0 \\ {\scriptstyle\beta'}\Downarrow & & \Downarrow{\scriptstyle\bar{\beta}'} \\ g_1 \circ Gv & \xRightarrow{1\circ G\rho} & g_1 \circ G\bar{v} \end{array}$$

There is a (non-commutative) square

$$\begin{array}{ccc} F\downarrow_2 G & \xrightarrow{Q} & \mathcal{C} \\ {\scriptstyle P}\downarrow & & \downarrow{\scriptstyle G} \\ \mathcal{A} & \xrightarrow{F} & \mathcal{B} \end{array} \tag{26}$$

where P and Q are projection 2-functors, which act on cells of $F\downarrow_2 G$ by

$$a_0 \overset{u}{\underset{\bar{u}}{\Downarrow\alpha}} a_1 \overset{P}{\longmapsto} (a_0, f_0, b_0, g_0, c_0) \overset{(u,\beta,p,\beta',v)}{\underset{(\bar{u},\bar{\beta},\bar{p},\bar{\beta}',\bar{v})}{\Downarrow(\alpha,\delta,\rho)}} (a_1, f_1, b_1, g_1, c_1) \overset{Q}{\longmapsto} c_0 \overset{v}{\underset{\bar{v}}{\Downarrow\rho}} c_1,$$

and we have the result given below.

3.13. THEOREM. *Let* $\mathcal{A} \xrightarrow{F} \mathcal{B} \xleftarrow{G} \mathcal{C}$ *be a diagram where* F *and* G *are lax functors.*

(*i*) *There is a homotopy* $\mathrm{B}F\,\mathrm{B}P \Rightarrow \mathrm{B}G\,\mathrm{B}Q$ *so that the square below, which is induced by* (26) *on classifying spaces, is homotopy commutative.*

$$\begin{array}{ccc} \mathrm{B}(F\downarrow_2 G) & \xrightarrow{\mathrm{B}Q} & \mathrm{B}\mathcal{C} \\ {\scriptstyle \mathrm{B}P}\downarrow & \Rightarrow & \downarrow{\scriptstyle \mathrm{B}G} \\ \mathrm{B}\mathcal{A} & \xrightarrow{\mathrm{B}F} & \mathrm{B}\mathcal{B} \end{array}$$

(*ii*) *The square above is a homotopy pullback whenever* F *or* G *has property* B_l.

PROOF. The part (*i*) follows from Theorem 3.1 (*i*) and the definition of $F\downarrow_2 G$. For the part (*ii*), since $F\downarrow_2 G \cong G\downarrow_2 F$, it is enough, by symmetry, to prove the theorem when F has the property B_l. In this case, we have the homotopy commutative diagram

$$\begin{array}{ccccc} & & \overset{\mathrm{B}Q}{\frown} & & \\ \mathrm{B}(F\downarrow_2 G) & \xrightarrow{\mathrm{B}P'} & \mathrm{B}(G\downarrow\mathcal{B}) & \xrightarrow{\mathrm{B}P} & \mathrm{B}\mathcal{C} \\ {\scriptstyle \mathrm{B}P}\downarrow & (15) & {\scriptstyle \mathrm{B}P'}\downarrow & (15) & \downarrow{\scriptstyle \mathrm{B}G} \\ \mathrm{B}\mathcal{A} & \xrightarrow{\mathrm{B}F} & \mathrm{B}\mathcal{B} & \xrightarrow{\mathrm{B}1_{\mathcal{B}}} & \mathrm{B}\mathcal{B} \\ & & \underset{\mathrm{B}F}{\smile} & & \end{array}$$

where, by Theorem 3.1, the inner squares (15) are both homotopy pullback. Then, the outside square is also a homotopy pullback, as claimed. ∎

4. Homotopy pullbacks of monoidal categories.

Recall [43, 38] that a *monoidal category* $(\mathcal{M},\otimes) = (\mathcal{M},\otimes,\mathrm{I},\boldsymbol{a},\boldsymbol{l},\boldsymbol{r})$ consists of a category $\mathcal{M}$ equipped with a tensor product $\otimes : \mathcal{M}\times\mathcal{M}\to\mathcal{M}$, a unit object I, and natural and coherent isomorphisms $\boldsymbol{a} : (m_3\otimes m_2)\otimes m_1 \cong m_3\otimes(m_2\otimes m_1)$, $\boldsymbol{l} : \mathrm{I}\otimes m \cong m$, and $\boldsymbol{r} : m\otimes\mathrm{I}\cong m$. Any monoidal category $(\mathcal{M},\otimes)$ can be viewed as a bicategory $\Sigma\mathcal{M}$ with only one object, say $*$, the objects m of $\mathcal{M}$ as 1-cells $m : *\to *$, and the morphisms of $\mathcal{M}$ as 2-cells. Thus, $\Sigma\mathcal{M}(*,*) = \mathcal{M}$, and the horizontal composition of cells is given by the tensor functor. The identity at the object is $1_* = \mathrm{I}$, the unit object of the monoidal category, and the associativity, left unit and right unit constraints for $\Sigma\mathcal{M}$ are precisely those of the monoidal category, that is, $\boldsymbol{a}$, $\boldsymbol{l}$, and $\boldsymbol{r}$, respectively. Furthermore, a monoidal functor $F = (F,\widehat{F}) : (\mathcal{N},\otimes)\to(\mathcal{M},\otimes)$ amounts precisely to a homomorphism $\Sigma F : \Sigma\mathcal{N}\to\Sigma\mathcal{M}$.

For any monoidal category $(\mathcal{M},\otimes)$, the Grothendieck nerve (12) of the bicategory $\Sigma\mathcal{M}$ is exactly the pseudo-simplicial category that the monoidal category defines by the reduced bar construction (see Jardine [35, Corollary 1.7]), whose category of p-simplices is $\mathcal{M}^p$, the p-fold power of the underlying category $\mathcal{M}$. Therefore, the *classifying space of*

the monoidal category $\mathrm{B}(\mathcal{M},\otimes)$ [35, §3] is the same as the classifying space $\mathrm{B}\Sigma\mathcal{M}$ of the one-object bicategory it defines [14], and thus the bicategorical results obtained above are applicable to monoidal functors between monoidal categories. This, briefly, can be done as follows:

Given any diagram $(\mathcal{N},\otimes)\xrightarrow{F}(\mathcal{M},\otimes)\xleftarrow{F'}(\mathcal{N}',\otimes)$, where F and F' are monoidal functors between monoidal categories, the "*homotopy- fibre product bicategory*"

$$F\overset{\otimes}{\downarrow}F' \tag{27}$$

(the notation $\overset{\otimes}{\downarrow}$ is to avoid confusion with the comma category $F\downarrow F'$ of the underlying functors) has as 0-cells the objects $m\in\mathcal{M}$. A 1-cell $(n,f,n'):m_0\to m_1$ of $F\overset{\otimes}{\downarrow}F'$ consists of objects $n\in\mathcal{N}$ and $n'\in\mathcal{N}'$, and a morphism $f:F'n'\otimes m_0\to m_1\otimes Fn$ in $\mathcal{B}$. A 2-cell in $F\overset{\otimes}{\downarrow}F'$,

$$m_0 \;\overset{(n,f,n')}{\underset{(\bar n,\bar f,\bar n')}{\Downarrow(u,u')}}\; m_1,$$

is given by a pair of morphisms, $u:n\to\bar n$ in $\mathcal{N}$ and $u':n'\to\bar n'$ in $\mathcal{N}'$, such that the diagram below commutes.

$$\begin{array}{ccc} F'n'\otimes m_0 & \xrightarrow{F'u'\otimes 1} & F'\bar n'\otimes m_0 \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle \bar f} \\ m_1\otimes Fn & \xrightarrow{1\otimes Fu} & m_1\circ F\bar n \end{array}$$

The vertical composition of 2-cells is given by the composition of morphisms in $\mathcal{N}$ and $\mathcal{N}'$. The horizontal composition of the 1-cells $m_0\xrightarrow{(n_1,f_1,n_1')}m_1\xrightarrow{(n_2,f_2,n_2')}m_2$ is the 1-cell

$$(n_2\otimes n_1,\, f_2\circledcirc f_1,\, n_2'\otimes n_1'):m_0\to m_2,$$

$$\begin{aligned} f_2\circledcirc f_1 = \big(F'(n_2'\otimes n_1')\otimes m_0 &\overset{\widehat{F'^{-1}}\otimes 1}{\cong} (F'n_2'\otimes F'n_1')\otimes m_0 \overset{\boldsymbol{a}}{\cong} F'n_2'\otimes(F'n_1'\otimes m_0)\xrightarrow{1\otimes f_1} \\ F'n_2'\otimes(m_1\otimes Fn_1) &\overset{\boldsymbol{a}^{-1}}{\cong} (F'n_2'\otimes m_1)\otimes Fn_1 \xrightarrow{f_2\otimes 1}(m_2\otimes Fn_2)\otimes Fn_1\overset{\boldsymbol{a}}{\cong} \\ m_2\otimes(Fn_2\circ Fn_1) &\overset{1\otimes\widehat{F}}{\cong} m_2\otimes F(n_2\otimes n_1)\big), \end{aligned}$$

and the horizontal composition of 2-cells is given by the tensor product of morphisms in $\mathcal{N}$ and $\mathcal{N}'$. The identity 1-cell, at any 0-cell m, is $(\mathrm{I},\mathring{1}_m,\mathrm{I}):m\to m$, where

$$\mathring{1}_m=\big(F'\mathrm{I}\otimes m\overset{\widehat{F'^{-1}}\otimes 1}{\cong}\mathrm{I}\otimes m\overset{\boldsymbol{l}}{\cong}m\overset{\boldsymbol{r}^{-1}}{\cong}m\otimes\mathrm{I}\overset{1\otimes\widehat{F}}{\cong}m\otimes F\mathrm{I}\big).$$

The associativity, right, and left unit constraints of the bicategory $F\overset{\otimes}{\downarrow}F'$ are provided by those of $\mathcal{N}$ and $\mathcal{N}'$ by the formulas

$$\boldsymbol{a}_{(n_3,f_3,n_3'),(n_2,f_2,n_2'),(n_1,f_1,n_1')}=(\boldsymbol{a}_{n_3,n_2,n_1},\boldsymbol{a}_{n_3',n_2',n_1'}),\ \boldsymbol{l}_{(n,f,n')}=(\boldsymbol{l}_n,\boldsymbol{l}_{n'}),\ \boldsymbol{r}_{(n,f,n')}=(\boldsymbol{r}_n,\boldsymbol{r}_{n'}).$$

4.1. Remark. Let us stress that $F \overset{\otimes}{\downarrow} F'$ is not a monoidal category but a genuine bicategory, since it generally has more than one object.

In particular, for any monoidal functor $F : (\mathcal{N}, \otimes) \to (\mathcal{M}, \otimes)$, we have the *homotopy-fibre bicategories* (cf. [13])

$$F \overset{\otimes}{\downarrow} \mathrm{I}, \quad \mathrm{I} \overset{\otimes}{\downarrow} F \tag{28}$$

where we denote by $\mathrm{I} : ([0], \otimes) \to (\mathcal{M}, \otimes)$ the monoidal functor that carries 0 to the unit object I, and whose structure isomorphism is $\boldsymbol{l}_{\mathrm{I}} = \boldsymbol{r}_{\mathrm{I}} : \mathrm{I} \otimes \mathrm{I} \cong \mathrm{I}$. Every object $m \in \mathcal{M}$ determines 2-endofunctors

$$m \otimes - : F \overset{\otimes}{\downarrow} \mathrm{I} \to F \overset{\otimes}{\downarrow} \mathrm{I}, \quad - \otimes m : \mathrm{I} \overset{\otimes}{\downarrow} F \to \mathrm{I} \overset{\otimes}{\downarrow} F,$$

respectively given on cells by

$$m_0 \underset{(\bar{n},\bar{f})}{\overset{(n,f)}{\Downarrow u}} m_1 \overset{m\otimes -}{\longmapsto} m \otimes m_0 \underset{(\bar{n},m\odot\bar{f})}{\overset{(n,m\odot f)}{\Downarrow u}} m \otimes m_1, \quad m_0 \underset{(\bar{g},\bar{n}')}{\overset{(g,n')}{\Downarrow u'}} m_1 \overset{-\otimes m}{\longmapsto} m_0 \otimes m \underset{(\bar{g}\odot m,\bar{n}')}{\overset{(g\odot m,n')}{\Downarrow u'}} m_1 \otimes m,$$

where, for any $(n, f) : m_0 \to m_1$ in $F \overset{\otimes}{\downarrow} \mathrm{I}$ and $(g, n) : m_0 \to m_1$ in $\mathrm{I} \overset{\otimes}{\downarrow} F$,

$$m \odot f = \big(\mathrm{I} \otimes (m \otimes m_0) \overset{\boldsymbol{l}}{\cong} m \otimes m_0 \overset{1\otimes \boldsymbol{l}^{-1}}{\cong} m \otimes (\mathrm{I} \otimes m_0) \xrightarrow{1\otimes f} m \otimes (m_1 \otimes Fn) \overset{\boldsymbol{a}^{-1}}{\cong} (m \otimes m_1) \otimes Fn\big),$$

$$g \odot m = \big(Fn \otimes (m_0 \otimes m) \overset{\boldsymbol{a}^{-1}}{\cong} (Fn \otimes m_0) \otimes m \xrightarrow{g\otimes 1} (m_1 \otimes \mathrm{I}) \otimes m \overset{\boldsymbol{r}\otimes 1}{\cong} m_1 \otimes m \overset{\boldsymbol{r}^{-1}}{\cong} (m_1 \otimes m) \otimes \mathrm{I}\big).$$

We state that

(B_l) *the monoidal functor F has the property* B_l if, for any object $m \in \mathcal{M}$, the induced map $\mathrm{B}(m \otimes -) : \mathrm{B}(F \overset{\otimes}{\downarrow} \mathrm{I}) \to \mathrm{B}(F \overset{\otimes}{\downarrow} \mathrm{I})$ is a homotopy autoequivalence.

(B_o) *the monoidal functor F has the property* B_o if, for any object $m \in \mathcal{M}$, the induced map $\mathrm{B}(- \otimes m) : \mathrm{B}(\mathrm{I} \overset{\otimes}{\downarrow} F) \to \mathrm{B}(\mathrm{I} \overset{\otimes}{\downarrow} F)$ is a homotopy autoequivalence.

Our main result here is a direct consequence of Theorem 3.1, after taking into account the identifications $\mathrm{B}(\mathcal{M}, \otimes) = \mathrm{B}\Sigma\mathcal{M}$, $\mathrm{B}F = \mathrm{B}\Sigma F$, $F \overset{\otimes}{\downarrow} F' = \Sigma F \downarrow \Sigma F'$, $F \overset{\otimes}{\downarrow} \mathrm{I} = \Sigma F \downarrow *$, and $\mathrm{I} \overset{\otimes}{\downarrow} F = * \downarrow \Sigma F$, and the fact that a monoidal functor has the property B_l or B_o if and only if the homomorphism ΣF has that property. This result is as given below.

4.2. Theorem. (*i*) *Suppose* $(\mathcal{N}, \otimes) \xrightarrow{F} (\mathcal{M}, \otimes) \xleftarrow{F'} (\mathcal{N}', \otimes)$ *are monoidal functors between monoidal categories, such that F has the property* B_l *or F' has the property* B_o. *Then, there is an induced homotopy pullback square*

$$\begin{array}{ccc} \mathrm{B}(F \overset{\otimes}{\downarrow} F') & \xrightarrow{\mathrm{B}P'} & \mathrm{B}(\mathcal{N}', \otimes) \\ {\scriptstyle \mathrm{B}P}\downarrow & \Rightarrow & \downarrow{\scriptstyle \mathrm{B}F'} \\ \mathrm{B}(\mathcal{N}, \otimes) & \xrightarrow{\mathrm{B}F} & \mathrm{B}(\mathcal{M}, \otimes). \end{array} \tag{29}$$

Therefore, there is an induced Mayer-Vietoris type long exact sequence on homotopy groups, based at the 0*-cells* $\mathrm{B}*$ *of* $\mathrm{B}(\mathcal{M},\otimes)$, $\mathrm{B}(\mathcal{N},\otimes)$, *and* $\mathrm{B}(\mathcal{N}',\otimes)$ *respectively, and the* 0*-cell* $\mathrm{BI} \in \mathrm{B}(F \overset{\otimes}{\downarrow} F')$,

$$\cdots \to \pi_{n+1}\mathrm{B}(\mathcal{M},\otimes) \to \pi_n \mathrm{B}(F \overset{\otimes}{\downarrow} F') \to \pi_n \mathrm{B}(\mathcal{N},\otimes) \times \pi_n \mathrm{B}(\mathcal{N}',\otimes) \to \pi_n \mathrm{B}(\mathcal{M},\otimes) \to$$

$$\cdots \to \pi_1 \mathrm{B}(F \overset{\otimes}{\downarrow} F') \to \pi_1 \mathrm{B}(\mathcal{N},\otimes) \times \pi_1 \mathrm{B}(\mathcal{N}',\otimes) \to \pi_1 \mathrm{B}(\mathcal{M},\otimes) \to \pi_0 \mathrm{B}(F \overset{\otimes}{\downarrow} F') \to 0.$$

(*ii*) *Given a monoidal functor* $F : (\mathcal{N},\otimes) \to (\mathcal{M},\otimes)$, *if the square* (29) *is a homotopy pullback for every monoidal functor* $F' : (\mathcal{N}',\otimes) \to (\mathcal{M},\otimes)$, *then* F *has the property* B_l. *Similarly, if* F' *is a monoidal functor such that the square* (29) *is a homotopy pullback for any monoidal functor* F, *as above, then* F' *has the property* B_o.

Similarly, from Corollaries 3.6 and 3.7, we get the following extensions of Quillen's Theorems A and B to monoidal functors:

4.3. Theorem. *Let* $F : (\mathcal{N},\otimes) \to (\mathcal{M},\otimes)$ *be any monoidal functor.*

(*i*) *If* F *has the property* B_l, *then there is an induced homotopy fibre sequence*

$$\mathrm{B}(F \overset{\otimes}{\downarrow} \mathrm{I}) \longrightarrow \mathrm{B}(\mathcal{N},\otimes) \longrightarrow \mathrm{B}(\mathcal{M},\otimes).$$

(*ii*) *If* F *has the property* B_o, *then there is an induced homotopy fibre sequence*

$$\mathrm{B}(\mathrm{I} \overset{\otimes}{\downarrow} F) \longrightarrow \mathrm{B}(\mathcal{N},\otimes) \longrightarrow \mathrm{B}(\mathcal{M},\otimes).$$

(*iii*) *If the classifying space of any of the two homotopy-fibre bicategories of* F *is contractible, that is, if* $\mathrm{B}(F \overset{\otimes}{\downarrow} \mathrm{I}) \simeq \mathrm{pt}$ *or* $\mathrm{B}(\mathrm{I} \overset{\otimes}{\downarrow} F) \simeq \mathrm{pt}$, *then the induced map on classifying spaces* $\mathrm{B}F : \mathrm{B}(\mathcal{N},\otimes) \simeq \mathrm{B}(\mathcal{M},\otimes)$ *is a homotopy equivalence.*

For the last statement in the following theorem, let us note that there is a diagram of bicategories

$$\begin{array}{ccc} \mathcal{M} & \longrightarrow & [0] \\ \downarrow & \overset{\gamma}{\Rightarrow} & \downarrow{\scriptstyle *} \\ [0] & \xrightarrow{*} & \Sigma\mathcal{M} \end{array} \tag{30}$$

in which γ is the lax transformation defined by $\gamma m = m : * \to *$, for any object $m \in \mathcal{M}$, and whose naturality component at a morphism $f : m_0 \to m_1$, is the composite 2-cell $\widehat{\gamma}_f = \big(\mathrm{I} \otimes m_0 \overset{\boldsymbol{l}}{\cong} m_0 \overset{f}{\Rightarrow} m_1 \overset{\boldsymbol{r}^{-1}}{\cong} m_1 \circ \mathrm{I}\big)$. Then, we have an induced homotopy commutative square on classifying spaces

$$\begin{array}{ccc} \mathrm{B}\mathcal{M} & \longrightarrow & \mathrm{pt} \\ \downarrow & \overset{\mathrm{B}\gamma}{\Rightarrow} & \downarrow{\scriptstyle *} \\ \mathrm{pt} & \xrightarrow{*} & \mathrm{B}(\mathcal{M},\otimes) \end{array}$$

and a corresponding whisker map

$$\mathrm{B}\mathcal{M} \to \Omega(\mathrm{B}(\mathcal{M}, \otimes), *). \tag{31}$$

Theorem 3.8 particularizes by giving

4.4. Theorem. *The following properties of a monoidal category* $(\mathcal{M}, \otimes)$ *are equivalent:*

(*i*) *For any diagram of monoidal functors* $(\mathcal{N}, \otimes) \xrightarrow{F} (\mathcal{M}, \otimes) \xleftarrow{F'} (\mathcal{N}', \otimes)$ *, the induced square* (29)

$$\begin{array}{ccc} \mathrm{B}(F \overset{\otimes}{\downarrow} F') & \xrightarrow{\mathrm{B}P'} & \mathrm{B}(\mathcal{N}', \otimes) \\ {\scriptstyle \mathrm{B}P}\downarrow & \Rightarrow & \downarrow{\scriptstyle \mathrm{B}F'} \\ \mathrm{B}(\mathcal{N}, \otimes) & \xrightarrow{\mathrm{B}F} & \mathrm{B}(\mathcal{M}, \otimes). \end{array}$$

is a homotopy pullback.

(*ii*) *Any monoidal functor* $F : (\mathcal{N}, \otimes) \to (\mathcal{M}, \otimes)$ *has property* B_l.

(*iii*) *Any monoidal functor* $F : (\mathcal{N}, \otimes) \to (\mathcal{M}, \otimes)$ *has property* B_o.

(*iv*) *For any object* $m \in \mathcal{M}$*, the functor* $m \otimes - : \mathcal{M} \to \mathcal{M}$ *induces a homotopy autoequivalence on the classifying space* $\mathrm{B}\mathcal{M}$.

(*v*) *For any object* $m \in \mathcal{M}$*, the functor* $- \otimes m : \mathcal{M} \to \mathcal{M}$ *induces a homotopy autoequivalence on the classifying space* $\mathrm{B}\mathcal{M}$.

(*vi*) *The whisker map* (31) *is a homotopy equivalence*

$$\mathrm{B}\mathcal{M} \simeq \Omega(\mathrm{B}(\mathcal{M}, \otimes), *)$$

between the classifying space of the underlying category and the loop space of the classifying space of the monoidal category.

The implications (*iv*) $\Rightarrow$ (*vi*) and (*v*) $\Rightarrow$ (*vi*) in the above theorem are essentially due to Stasheff [45], but several other proofs can be found in the literature (see Jardine [35, Propositions 3.5 and 3.8], for example). When the equivalent properties in Theorem 4.4 hold, we say that *the monoidal category is homotopy regular*. For example, *regular monoidal categories* (as termed by Saavedra [43, Chap. I, (0.1.3)]), that is, monoidal categories $(\mathcal{M}, \otimes)$ where, for every object $m \in \mathcal{M}$, the functor $m \otimes - : \mathcal{M} \to \mathcal{M}$ is an autoequivalence of the underlying category $\mathcal{M}$, and, in particular, *categorical groups* (so named by Joyal and Street in [36, Definition 3.1] and also termed *Gr-categories* by Breen in [8, §2, 2.1]), that is, monoidal categories whose objects are invertible up to an isomorphism, and whose morphisms are all invertible, are homotopy regular.

5. Homotopy pullbacks of crossed modules

Thanks to the equivalence between the category of crossed modules and the category of 2-groupoids, the results in Section 3 can be applied to crossed modules. To do so in some detail, we shall start by briefly reviewing crossed modules and their classifying spaces.

Recall that, if $\mathcal{P}$ is any (small) groupoid, then the category of (left) *$\mathcal{P}$-groups* has objects the functors $\mathcal{P} \to \mathbf{Gp}$, from $\mathcal{P}$ into the category of groups, and its morphisms, called *$\mathcal{P$-group homomorphisms*, are natural transformations. If $\mathcal{G}$ is a $\mathcal{P}$-group, then, for any arrow $p : a \to b$ in $\mathcal{P}$, we write the associated group homomorphism $\mathcal{G}(a) \to \mathcal{G}(b)$ by $g \mapsto {}^p g$, so that the equalities ${}^1 g = g$, ${}^{(q\circ p)} g = {}^q({}^p g)$, and ${}^p(g \cdot g') = {}^p g \cdot {}^p g'$ hold whenever they make sense. Here, the symbol $\circ$ denotes composition in the groupoid $\mathcal{P}$, whereas $\cdot$ denotes multiplication in $\mathcal{G}$. For instance, the assignment to each object of $\mathcal{P}$ its isotropy group, $a \mapsto \mathrm{Aut}_{\mathcal{P}}(a)$, is the function on objects of a $\mathcal{P}$-group $\mathrm{Aut}_{\mathcal{P}} : \mathcal{P} \to \mathbf{Gp}$ such that ${}^p q = p \circ q \circ p^{-1}$, for any $p : a \to b$ in $\mathcal{P}$ and $q \in \mathrm{Aut}_{\mathcal{P}}(a)$. Then, a *crossed module* (of groupoids) is a triplet

$$(\mathcal{G}, \mathcal{P}, \partial)$$

consisting of a groupoid $\mathcal{P}$, a $\mathcal{P}$-group $\mathcal{G}$, and a $\mathcal{P}$-group homomorphism $\partial : \mathcal{G} \to \mathrm{Aut}_{\mathcal{P}}$, called the *boundary map*, such that the Peiffer identity ${}^{\partial g} g' = g \cdot g' \cdot g^{-1}$ holds, for any $g, g' \in \mathcal{G}(a)$, $a \in \mathrm{Ob}\mathcal{P}$.

When a group P is regarded as a groupoid $\mathcal{P}$ with exactly one object, the above definition by Brown and Higgins [11] recovers the more classic notion of crossed module (G, P, ∂) due to Whitehead and Mac Lane [49, 39], now called *crossed modules of groups*. In fact, if $(\mathcal{G}, \mathcal{P}, \partial)$ is any crossed module, then, for any object a of $\mathcal{P}$, the triplet $(\mathcal{G}(a), \mathrm{Aut}_{\mathcal{P}}(a), \partial_a)$ is precisely a crossed module of groups.

Composition with any given functor $F : \mathcal{P} \to \mathcal{Q}$ defines a functor from the category of $\mathcal{Q}$-groups to the category of $\mathcal{P}$-groups: $(\varphi : \mathcal{G} \to \mathcal{H}) \mapsto (\varphi F : \mathcal{G}F \to \mathcal{H}F)$. For the particular case of the $\mathcal{Q}$-group of automorphisms $\mathrm{Aut}_{\mathcal{Q}}$, we have the $\mathcal{P}$-group homomorphism $F : \mathrm{Aut}_{\mathcal{P}} \to \mathrm{Aut}_{\mathcal{Q}} F$, which, at any $a \in \mathcal{P}$, is given by the map $\mathrm{Aut}_{\mathcal{P}}(a) \to \mathrm{Aut}_{\mathcal{Q}}(Fa)$, $q \mapsto Fq$, defined by the functor F. Then, a *morphism of crossed modules*

$$(\varphi, F) : (\mathcal{G}, \mathcal{P}, \partial) \to (\mathcal{H}, \mathcal{Q}, \partial)$$

consists of a functor $F : \mathcal{P} \to \mathcal{Q}$ together with a $\mathcal{P}$-group homomorphism $\varphi : \mathcal{G} \to \mathcal{H}F$ such that the square below commutes.

$$\begin{array}{ccc} \mathcal{G} & \xrightarrow{\partial} & \mathrm{Aut}_{\mathcal{P}} \\ {\scriptstyle\varphi}\downarrow & & \downarrow{\scriptstyle F} \\ \mathcal{H}F & \xrightarrow{\partial F} & \mathrm{Aut}_{\mathcal{Q}}F. \end{array}$$

The category of crossed modules, where compositions and identities are defined in the natural way, is denoted by **Xmod**. Let us now recall from Brown and Higgins [10, Theorem 4.1] that there is an equivalence between the category of crossed modules and the category of 2-groupoids

$$\beta : \mathbf{Xmod} \xrightarrow{\sim} 2\text{-}\mathbf{Gpd}, \tag{32}$$

which is as follows: Given any crossed module $(\mathcal{G}, \mathcal{P}, \partial)$, $\mathcal{P}$ is the underlying groupoid of the 2-groupoid $\beta(\mathcal{G}, \mathcal{P}, \partial)$, whose 2-cells

$$a_0 \overset{p}{\underset{\bar{p}}{\overset{\Downarrow g}{\rightrightarrows}}} a_1$$

are those elements $g \in \mathcal{G}(a_0)$ such that $\bar{p} \circ \partial g = p$. The vertical and horizontal composition of 2-cells are, respectively, given by

$$a_0 \overset{p}{\underset{\bar{\bar{p}}}{\overset{\Downarrow g}{\underset{\Downarrow \bar{g}}{\rightrightarrows}}}} a_1 \quad \overset{\cdot}{\mapsto} \quad a_0 \overset{p}{\underset{\bar{\bar{p}}}{\overset{\Downarrow \bar{g} \cdot g}{\rightrightarrows}}} a_1 \,, \qquad a_0 \overset{p_1}{\underset{\bar{p}_1}{\overset{\Downarrow g_1}{\rightrightarrows}}} a_1 \overset{p_2}{\underset{\bar{p}_2}{\overset{\Downarrow g_2}{\rightrightarrows}}} a_2 \quad \overset{\circ}{\mapsto} \quad a_0 \overset{p_2 \circ p_1}{\underset{\bar{p}_2 \circ \bar{p}_1}{\overset{\Downarrow \bar{p}_1^{-1} g_2 \cdot g_1}{\rightrightarrows}}} a_2.$$

A morphism of crossed modules $(\varphi, F) : (\mathcal{G}, \mathcal{P}, \partial) \to (\mathcal{H}, \mathcal{Q}, \partial)$ is carried by the equivalence to the 2-functor $\beta(\varphi, F) : \beta(\mathcal{G}, \mathcal{P}, \partial) \to \beta(\mathcal{H}, \mathcal{Q}, \partial)$ acting on cells by

$$a_0 \overset{p}{\underset{\bar{p}}{\overset{\Downarrow g}{\rightrightarrows}}} a_1 \;\mapsto\; Fa_0 \overset{Fp}{\underset{F\bar{p}}{\overset{\Downarrow \varphi g}{\rightrightarrows}}} Fa_1.$$

5.1. Example. A striking example of crossed module is $\Pi(X, A, S) = (\pi_2(X, A), \pi(A, S), \partial)$, which comes associated to any triple (X, A, S), where X is any topological space, $A \subseteq X$ a subspace, and $S \subseteq A$ a set of (base) points. Here, $\pi(A, S)$ is the fundamental groupoid of homotopy classes of paths in A between points in S, $\pi_2(X, A) : \pi(A, S) \to \mathbf{Gp}$ is the functor associating to each $a \in S$ the relative homotopy group $\pi_2(X, A, a)$, and, at any $a \in S$, the boundary map $\partial : \pi_2(X, A, a) \to \pi_1(A, a)$ is the usual boundary homomorphism in the exact sequence of homotopy groups based at a of the pair (X, A):

$$\left[\begin{array}{ccc} a & \overset{u}{\to} & a \\ \| & g & \| \\ a & = & a \end{array}\right] \overset{\partial}{\mapsto} \quad a \xrightarrow{[u]} a.$$

Furthermore, $\pi(A, S)$ is the underlying groupoid of the *Whitehead 2-groupoid* $W(X, A, S)$ presented by Moerdijk and Svensson [41], whose 2-cells

$$\left[\begin{array}{ccc} a & \overset{v}{\to} & b \\ \| & g & \| \\ a & \underset{w}{\to} & b \end{array}\right] : [v] \Rightarrow [w] : a \to b,$$

are equivalence classes of maps $g : I \times I \to X$, from the square $I \times I$ into X, which are constant along the vertical edges with values in S, and map the horizontal edges into A; two such maps are equivalent if they are homotopic by a homotopy that is constant along the vertical edges and deforms the horizontal edges within A.

Both constructions $\Pi(X,A,S)$ and $W(X,A,S)$ correspond to each other by the equivalence of categories (32). More precisely, there is a natural isomorphism

$$\beta\Pi(X,A,S) \cong W(X,A,S), \tag{33}$$

which is the identity on 0- and 1-cells, and carries a 2-cell $[g] : [v] \Rightarrow [w]$ of $\beta\Pi(X,A,S)$ to the 2-cell $1_{[w]} \circ [g] : [v] \Rightarrow [w]$ of $W(X,A,S)$:

$$\left(\left[\begin{array}{ccc} a & \xrightarrow{u} & a \\ \| & {\scriptstyle g(s,t)} & \| \\ a & = & a \end{array} \right] \; : \; [v] \Rightarrow [w] \right) \;\; \mapsto \;\; \left(\left[\begin{array}{ccc} a & \xrightarrow{u} a & \xrightarrow{w} b \\ \| & {\scriptstyle g(2s,t)} & \| {\scriptstyle w(2s-1)} \| \\ a & = a & \xrightarrow[w]{} b \end{array} \right] \; : \; [v] \Rightarrow [w] \right).$$

For a simplicial set K, its *fundamental*, or *homotopy*, *crossed module* $\Pi(K)$ is defined as the crossed module

$$\Pi(K) = \Pi\big(|K|, |K^{(1)}|, |K^{(0)}|\big) \tag{34}$$

constructed in Example 5.1 (here, $K^{(n)}$ denotes the n-skeleton, as usual). The construction $K \mapsto \Pi(K)$ gives rise to a functor $\Pi : \mathbf{SimpSet} \to \mathbf{Xmod}$, from the category of simplicial sets to the category of crossed modules. To go in the other direction, we have the notion of *nerve of a crossed module*, which is actually a special case of the definition of nerve for crossed complexes by Brown and Higgins [12]. Thus, the *nerve* $N(\mathcal{G},\mathcal{P},\partial)$ of a crossed module $(\mathcal{G},\mathcal{P},\partial)$ is the simplicial set

$$N(\mathcal{G},\mathcal{P},\partial) : \Delta^{op} \longrightarrow \mathbf{Set}, \quad [n] \mapsto \mathbf{Xmod}\big(\Pi(\Delta[n]), (\mathcal{G},\mathcal{P},\partial)\big), \tag{35}$$

whose n-simplices are all morphisms of crossed modules $\Pi(\Delta[n]) \to (\mathcal{G},\mathcal{P},\partial)$.

The *classifying space* $\mathrm{B}(\mathcal{G},\mathcal{P},\partial)$ of a crossed module $(\mathcal{G},\mathcal{P},\partial)$ is the geometric realization of its nerve, that is,

$$\mathrm{B}(\mathcal{G},\mathcal{P},\partial) = |N(\mathcal{G},\mathcal{P},\partial)|. \tag{36}$$

By [12, Proposition 2.6], $\mathrm{B}(\mathcal{G},\mathcal{P},\partial)$ is a CW-complex whose 0-cells identify with the objects of the groupoid $\mathcal{P}$ and whose homotopy groups, at any $a \in \mathrm{Ob}\mathcal{P}$, can be algebraically computed as

$$\pi_i\big(\mathrm{B}(\mathcal{G},\mathcal{P},\partial), a\big) = \begin{cases} \text{the set of connected components of } \mathcal{P}, \text{ if } i = 0, \\ \mathrm{Coker}\, \partial : \mathcal{G}(a) \to \mathrm{Aut}_{\mathcal{P}}(a), \text{ if } i = 1, \\ \mathrm{Ker}\, \partial : \mathcal{G}(a) \to \mathrm{Aut}_{\mathcal{P}}(a), \text{ if } i = 2, \\ 0, \text{ if } i \geq 3. \end{cases} \tag{37}$$

Therefore, classifying spaces of crossed modules are homotopy 2-types. Furthermore, it is a consequence of [12, Theorem 4.1] that, for any CW-complex X with $\pi_i(X,a) = 0$ for all $i > 2$ and base 0-cell a, there is a homotopy equivalence $X \simeq \mathrm{B}\Pi(X, X^{(1)}, X^{(0)})$. Therefore, crossed modules are algebraic models for homotopy 2-types.

5.2. Lemma. *For any crossed module $(\mathcal{G},\mathcal{P},\partial)$, there is a homotopy natural homotopy equivalence*

$$\mathrm{B}(\mathcal{G},\mathcal{P},\partial) \simeq \mathrm{B}\beta(\mathcal{G},\mathcal{P},\partial). \tag{38}$$

PROOF. By [12, Theorem 2.4], the functor $\Pi : \mathbf{SimpSet} \to \mathbf{Xmod}$ is left adjoint to the nerve functor $N : \mathbf{Xmod} \to \mathbf{SimpSet}$. Furthermore, in [41, Theorem 2.3] Moerdijk and Svensson show that the *Whitehead 2-groupoid functor* $W : \mathbf{SimpSet} \to 2\text{-}\mathbf{Gpd}$, $K \mapsto W(K) = W\big(|K|, |K^{(1)}|, |K^{(0)}|\big)$ (see Example 5.1) is left adjoint to the unitary geometric nerve functor $\Delta^{\mathrm{u}} : 2\text{-}\mathbf{Gpd} \to \mathbf{SimpSet}$. Since, owing to the isomorphisms (33), there is a natural isomorphism $\beta\,\Pi \cong W$, we conclude that $\Delta^{\mathrm{u}}\beta \cong N$. Therefore, for $(\mathcal{G}, \mathcal{P}, \partial)$ any crossed module, $\mathrm{B}(\mathcal{G}, \mathcal{P}, \partial) = |N(\mathcal{G}, \mathcal{P}, \partial)| \cong |\Delta^{\mathrm{u}}\beta(\mathcal{G}, \mathcal{P}, \partial)| \overset{(14)}{\simeq} \mathrm{B}\beta(\mathcal{G}, \mathcal{P}, \partial)$. ∎

5.3. REMARK. For any crossed module $(\mathcal{G}, \mathcal{P}, \partial)$, the n-simplices of $\Delta^{\mathrm{u}}\beta(\mathcal{G}, \mathcal{P}, \partial)$, that is, the normal lax functors $[n] \to \beta(\mathcal{G}, \mathcal{P}, \partial)$, are precisely systems of data

$$(g, p, a) = \big(g_{i,j,k}, p_{i,j}, a_i\big)_{0 \le i \le j \le k \le n}$$

consisting of objects a_i of $\mathcal{P}$, arrows $p_{i,j} : a_i \to a_j$ of $\mathcal{P}$, with $p_{i,i} = 1$, and elements $g_{i,j,k} \in \mathcal{G}(a_i)$, with $g_{i,i,j} = g_{i,j,j} = 1$, such that the following conditions hold:

$$\begin{aligned} &\partial(g_{i,j,k}) = p_{i,k}^{-1} \circ p_{j,k} \circ p_{i,j} && \text{for } i \le j \le k, \\ &g_{i,j,k}^{-1} \cdot g_{i,k,l}^{-1} \cdot g_{i,j,l} \cdot {}^{p_{i,j}^{-1}} g_{j,k,l} = 1 && \text{for } i \le j \le k \le l. \end{aligned}$$

Thus, the unitary geometric nerve $\Delta^{\mathrm{u}}\beta(\mathcal{G}, \mathcal{P}, \partial)$ coincides with the simplicial set called by Dakin [23, Chapter 5, §3] the *nerve of the crossed module* $(\mathcal{G}, \mathcal{P}, \partial)$ (cf. [12, page 99] and [2, Chapter 1, §11]). ¿From the above explicit description, it is easily proven that the nerve of a crossed module is a Kan complex whose homotopy groups are given as in (37).

Thanks to Lemma 5.2, the bicategorical results obtained in Section 3 are transferable to the setting of crossed modules. To do so, if

$$(\mathcal{G}, \mathcal{P}, \partial) \xrightarrow{(\varphi, F)} (\mathcal{H}, \mathcal{Q}, \partial) \xleftarrow{(\varphi', F')} (\mathcal{G}', \mathcal{P}', \partial)$$

is any diagram in **Xmod**, then its *"homotopy-fiber product crossed module"*

$$(\varphi, F)\!\downarrow\!(\varphi', F') = \big(\mathcal{G}_{\varphi, F\downarrow\varphi', F'}, \mathcal{P}_{\varphi, F\downarrow\varphi', F'}, \partial\big) \tag{39}$$

is constructed as follows:

- *The groupoid* $\mathcal{P}_{\varphi, F\downarrow\varphi', F'}$ has objects the triples (a, q, a'), with $a \in \mathrm{Ob}\mathcal{P}$, $a' \in \mathrm{Ob}\mathcal{P}'$, and $q : Fa \to F'a'$ a morphism in $\mathcal{Q}$. A morphism $(p, h, p') : (a_0, q_0, a_0') \to (a_1, q_1, a_1')$ consists of a morphism $p : a_0 \to a_1$ in $\mathcal{P}$, a morphism $p' : a_0' \to a_1'$ in $\mathcal{P}'$, and an element $h \in \mathcal{H}(Fa_0)$, which measures the lack of commutativity of the square

$$\begin{array}{ccc} Fa_0 & \xrightarrow{q_0} & F'a_0' \\ {\scriptstyle Fp}\downarrow & & \downarrow{\scriptstyle F'p'} \\ Fa_1 & \xrightarrow{q_1} & F'a_1' \end{array}$$

in the sense that the following equation holds: $\partial h = Fp^{-1} \circ q_1^{-1} \circ F'p' \circ q_0$. The composition of two morphisms $(a_0, q_0, a'_0) \xrightarrow{(p_1,h_1,p'_1)} (a_1, q_1, a'_1) \xrightarrow{(p_2,h_2,p'_2)} (a_2, q_2, a'_2)$ is given by the formula

$$(p_2, h_2, p'_2) \circ (p_1, h_1, p'_1) = (p_2 \circ p_1, {}^{Fp_1^{-1}}h_2 \cdot h_1, p'_2 \circ p'_1).$$

For every object (a, q, a'), its identity is $1_{(a,q,a')} = (1_a, 1, 1_{a'})$, and the inverse of any morphism (p, h, p') as above is $(p, h, p')^{-1} = (p^{-1}, {}^{Fp}h^{-1}, p'^{-1})$.

- *The functor* $\mathcal{G}_{\varphi,F\downarrow\varphi',F'} : \mathcal{P}_{\varphi,F\downarrow\varphi',F'} \to \mathbf{Gp}$ is defined on objects by

$$\mathcal{G}_{\varphi,F\downarrow\varphi',F'}(a, q, a') = \mathcal{G}(a) \times \mathcal{G}'(a'),$$

and, for any morphism $(p, h, p') : (a_0, q_0, a'_0) \to (a_1, q_1, a'_1)$, the associated homomorphism is given by ${}^{(p,h,p')}(g, g') = ({}^{p}g, {}^{p'}g')$.

- *The boundary map* $\partial : \mathcal{G}_{\varphi,F\downarrow\varphi',F'} \to \mathrm{Aut}_{\mathcal{P}_{\varphi,F\downarrow\varphi',F'}}$, at any object (a, q, a') of the groupoid $\mathcal{P}_{\varphi,F\downarrow\varphi',F'}$, is given by the formula

$$\partial(g, g') = (\partial g, \varphi g^{-1} \cdot {}^{q^{-1}}\varphi' g', \partial g').$$

For any crossed module $(\mathcal{H}, \mathcal{Q}, \partial)$, we identify any object $b \in \mathcal{Q}$ with the morphism from the trivial crossed module $b : (1, 1, 1) \to (\mathcal{H}, \mathcal{Q}, \partial)$ such that $b(1) = b$, so that, for any morphism of crossed modules $(\varphi, F) : (\mathcal{G}, \mathcal{P}, \partial) \to (\mathcal{H}, \mathcal{Q}, \partial)$, we have defined the *"homotopy-fibre crossed module"*

$$(\varphi, F)\downarrow b.$$

Next, we summarize our results in this setting of crossed modules. The crossed module (39) comes with a (non-commutative) square

$$\begin{array}{ccc} (\varphi, F)\downarrow(\varphi', F') & \xrightarrow{(\pi',P')} & (\mathcal{G}', \mathcal{P}', \partial) \\ {\scriptstyle (\pi,P)}\downarrow & & \downarrow{\scriptstyle (\varphi',F')} \\ (\mathcal{G}, \mathcal{P}, \partial) & \xrightarrow{(\varphi,F)} & (\mathcal{H}, \mathcal{Q}, \partial), \end{array} \tag{40}$$

where

$$(a_0 \xrightarrow{p} a_1) \xleftarrow{P} \big((a_0, q_0, a'_0) \xrightarrow{(p,h,p')} (a_1, q_1, a'_1)\big) \xmapsto{P'} (a'_0 \xrightarrow{p'} a'_1)$$
$$g \xleftarrow{\pi} (g, g') \xmapsto{\pi'} g'$$

5.4. Theorem. *The following statements hold:*

(*i*) *For any morphisms of crossed modules* $(\mathcal{G}, \mathcal{P}, \partial) \xrightarrow{(\varphi,F)} (\mathcal{H}, \mathcal{Q}, \partial) \xleftarrow{(\varphi',F')} (\mathcal{G}', \mathcal{P}', \partial)$, *there is a homotopy* $\mathrm{B}(\varphi, F)\,\mathrm{B}(\pi, P) \Rightarrow \mathrm{B}(\varphi', F')\,\mathrm{B}(\pi', P')$ *making the homotopy commutative square*

$$\begin{array}{ccc} \mathrm{B}((\varphi, F)\downarrow(\varphi', F')) & \xrightarrow{\mathrm{B}(\pi',P')} & \mathrm{B}(\mathcal{G}', \mathcal{P}', \partial) \\ {\scriptstyle \mathrm{B}(\pi,P)}\downarrow & \Rightarrow & \downarrow{\scriptstyle \mathrm{B}(\varphi',F')} \\ \mathrm{B}(\mathcal{G}, \mathcal{P}, \partial) & \xrightarrow{\mathrm{B}(\varphi,F)} & \mathrm{B}(\mathcal{H}, \mathcal{Q}, \partial), \end{array} \tag{41}$$

induced by (40) *on classifying spaces, a homotopy pullback square.*

(*ii*) *For any morphism of crossed modules* $(\varphi, F) : (\mathcal{G}, \mathcal{P}, \partial) \to (\mathcal{H}, \mathcal{Q}, \partial)$ *and every object* $b \in \mathcal{Q}$*, there is an induced homotopy fibre sequence*

$$\mathrm{B}((\varphi, F)\downarrow b) \xrightarrow{\mathrm{B}(\pi,P)} \mathrm{B}(\mathcal{G}, \mathcal{P}, \partial) \xrightarrow{\mathrm{B}(\varphi,F)} \mathrm{B}(\mathcal{H}, \mathcal{Q}, \partial).$$

(*iii*) *A morphism of crossed modules* $(\varphi, F) : (\mathcal{G}, \mathcal{P}, \partial) \to (\mathcal{H}, \mathcal{Q}, \partial)$ *induces a homotopy equivalence on classifying spaces,* $\mathrm{B}(\varphi, F) : \mathrm{B}(\mathcal{G}, \mathcal{P}, \partial) \simeq \mathrm{B}(\mathcal{H}, \mathcal{Q}, \partial)$*, if and only if, for every object* $b \in \mathcal{Q}$*, the space* $\mathrm{B}((\varphi, F)\downarrow b)$ *is contractible.*

(*iv*) *For any crossed module* $(\mathcal{G}, \mathcal{P}, \partial)$ *and object* $a \in \mathcal{P}$*, there is a homotopy equivalence*

$$\mathrm{B}\big((\mathcal{G}, \mathcal{P}, \partial)(a)\big) \simeq \Omega(\mathrm{B}(\mathcal{G}, \mathcal{P}, \partial), a),$$

where $(\mathcal{G}, \mathcal{P}, \partial)(a)$ *is the groupoid whose objects are the automorphisms* $p : a \to a$ *in* $\mathcal{P}$*, and whose arrows* $g : p \to q$ *are those elements* $g \in \mathcal{G}(a)$ *such that* $p = q \circ \partial g$*.*

Proof. (*i*) Let us apply the equivalence of categories (32) to the square of crossed modules (40). Then, by direct comparison, we see that the equation between squares of 2-groupoids

$$\begin{array}{ccc} \beta\big((\varphi, F)\downarrow(\varphi', F')\big) & \xrightarrow{\beta(\pi',P')} & \beta(\mathcal{G}', \mathcal{P}', \partial) \\ {\scriptstyle \beta(\pi,P)}\downarrow & & \downarrow{\scriptstyle \beta(\varphi',F')} \\ \beta(\mathcal{G}, \mathcal{P}, \partial) & \xrightarrow{\beta(\varphi,F)} & \beta(\mathcal{H}, \mathcal{Q}, \partial) \end{array} \quad = \quad \begin{array}{ccc} \big(\beta(\varphi, F)\downarrow\beta(\varphi', F')\big) & \xrightarrow{P'} & \beta(\mathcal{G}', \mathcal{P}', \partial) \\ {\scriptstyle P}\downarrow & & \downarrow{\scriptstyle \beta(\varphi',F')} \\ \beta(\mathcal{G}, \mathcal{P}, \partial) & \xrightarrow{\beta(\varphi,F)} & \beta(\mathcal{H}, \mathcal{Q}, \partial) \end{array}$$

holds, where the square on the right is (8) for the 2-functors $\beta(\varphi, F)$ and $\beta(\varphi', F')$. As any 2-groupoid has property (*iv*) in Theorem 3.8 (see the comment before Corollary 3.9), that theorem gives a homotopy $\mathrm{B}\beta(\varphi, F)\,\mathrm{B}\beta(\pi, P) \Rightarrow \mathrm{B}\beta(\varphi', F')\,\mathrm{B}\beta(\pi', P')$ such that the induced square

$$\begin{array}{ccc} \mathrm{B}\beta\big((\varphi, F)\downarrow(\varphi', F')\big) & \xrightarrow{\mathrm{B}\beta(\pi',P')} & \mathrm{B}\beta(\mathcal{G}', \mathcal{P}', \partial) \\ {\scriptstyle \mathrm{B}\beta(\pi,P)}\downarrow & \Rightarrow & \downarrow{\scriptstyle \mathrm{B}\beta(\varphi',F')} \\ \mathrm{B}\beta(\mathcal{G}, \mathcal{P}, \partial) & \xrightarrow{\mathrm{B}\beta(\varphi,F)} & \mathrm{B}\beta(\mathcal{H}, \mathcal{Q}, \partial) \end{array}$$

is a homotopy pullback. It follows that the square (41) is also a homotopy pullback since, by Lemma 5.2, it is homotopy equivalent to the square above.

The implications $(i) \Rightarrow (ii) \Rightarrow (iii)$ are clear, and (*iv*) follows from Corollary 3.9, as $(\mathcal{G}, \mathcal{P}, \partial)(a) = \beta(\mathcal{G}, \mathcal{P}, \partial)(a, a)$ and $\mathrm{B}(\mathcal{G}, \mathcal{P}, \partial) \simeq \mathrm{B}\beta(\mathcal{G}, \mathcal{P}, \partial)$. ∎

We can easily show how the construction $(\varphi, F)\downarrow(\varphi', F')$ works on basic examples (see below).

5.5. Example. (*i*) *Let* $P \xrightarrow{F} Q \xleftarrow{F'} P'$ *be homomorphisms of groups. These induce homomorphisms of crossed modules of groups* $(1, P, 1) \xrightarrow{(1,F)} (1, Q, 1) \xleftarrow{(1,F')} (1, P', 1)$*, whose*

homotopy-fibre product is $(1,F)\downarrow(1,F') = (1, F\downarrow F', 1)$*, where* $F\downarrow F'$ *is the groupoid having as objects the elements* $q \in Q$ *and as morphisms* $(p,p') : q_0 \to q_1$ *those pairs* $(p,p') \in P \times P'$ *such that* $q_1 \cdot Fp = F'p' \cdot q_0$*. Thus,* (41) *particularizes by giving a homotopy pullback square*

$$\begin{array}{ccc} \mathrm{B}(F\downarrow F') & \longrightarrow & K(P',1) \\ \downarrow & \Rightarrow & \downarrow \\ K(P,1) & \longrightarrow & K(Q,1). \end{array}$$

(*ii*) *Let* $A \xrightarrow{\varphi} B \xleftarrow{\varphi'} A'$ *be homomorphisms of abelian groups. These induce homomorphisms of crossed modules of groups* $(A,1,1) \xrightarrow{(\varphi,1)} (B,1,1) \xleftarrow{(\varphi',1)} (A',1,1)$*, whose homotopy-fibre product is the abelian crossed module of groups* $(\varphi,1)\downarrow(\varphi',1) = (A\times A', B, \partial)$*, where the coboundary map is given by* $\partial(a,a') = \varphi' a' - \varphi a$*. Thus,* (41) *particularizes by giving a homotopy pullback square*

$$\begin{array}{ccc} \mathrm{B}(A\times A', B, \partial) & \longrightarrow & K(A',2) \\ \downarrow & \Rightarrow & \downarrow \\ K(A,2) & \longrightarrow & K(B,2). \end{array}$$

Let us stress that, as Example 5.5(*i*) shows, the homotopy-fiber product crossed module $(\varphi,F)\downarrow(\varphi',F')$ may be a genuine crossed module of groupoids even in the case when both (φ,F) and (φ',F') are morphisms between crossed modules of groups. The reader can find in this fact a good reason to be interested in the study of general crossed modules over groupoids.

To finish, recall that the category of crossed complexes has a closed model structure as described by Brown and Golasinski [9]. In this homotopy structure, a morphism of crossed modules $(\varphi,F) : (\mathcal{G},\mathcal{P},\partial) \to (\mathcal{H},\mathcal{Q},\partial)$ is a weak equivalence if the induced map on classifying spaces $\mathrm{B}(\varphi,F)$ is a homotopy equivalence, and it is a fibration (see Howie [34]) whenever the following conditions hold: (*i*) $F : \mathcal{P} \to \mathcal{Q}$ is a fibration of groupoids, that is, for every object $a \in \mathcal{P}$ and every morphism $q : Fa \to b$ in $\mathcal{Q}$, there is a morphism $p : a \to a'$ in $\mathcal{P}$ such that $Fp = q$, and (*ii*) for any object $a \in \mathcal{P}$, the homomorphism $\varphi : \mathcal{G}(a) \to \mathcal{H}(Fa)$ is surjective. Then, it is natural to ask whether the constructed homotopy-fiber product crossed module $(\varphi,F)\downarrow(\varphi',F')$ is compatible with the homotopy pullback in the model category of crossed complexes. The answer is positive as a consequence of the theorem below, and this fact implies that the classifying space functor $(\mathcal{G},\mathcal{P},\partial) \mapsto \mathrm{B}(\mathcal{G},\mathcal{P},\partial)$ preserves homotopy pullbacks.

5.6. Theorem. *Let* $(\mathcal{G},\mathcal{P},\partial) \xrightarrow{(\varphi,F)} (\mathcal{H},\mathcal{Q},\partial) \xleftarrow{(\varphi',F')} (\mathcal{G}',\mathcal{P}',\partial)$ *be morphisms of crossed modules. If one of them is a fibration, then the canonical morphism*

$$(\mathcal{G},\mathcal{P},\partial) \times_{(\mathcal{H},\mathcal{Q},\partial)} (\mathcal{G}',\mathcal{P}',\partial) \to (\varphi,F)\downarrow(\varphi',F')$$

induces a homotopy equivalence $\mathrm{B}\big((\mathcal{G},\mathcal{P},\partial) \times_{(\mathcal{H},\mathcal{Q},\partial)} (\mathcal{G}',\mathcal{P}',\partial)\big) \simeq \mathrm{B}\big((\varphi,F)\downarrow(\varphi',F')\big)$.

PROOF. Let us observe that the pullback crossed module of (φ, F) and (φ', F') is

$$(\mathcal{G}, \mathcal{P}, \partial) \times_{(\mathcal{H},\mathcal{Q},\partial)} (\mathcal{G}', \mathcal{P}', \partial) = (\mathcal{G} \times_{\mathcal{H}F} \mathcal{G}', \mathcal{P} \times_{\mathcal{Q}} \mathcal{P}', \partial),$$

where $\mathcal{P} \times_{\mathcal{Q}} \mathcal{P}'$ is the pullback groupoid of $F : \mathcal{P} \to \mathcal{Q}$ and $F' : \mathcal{P}' \to \mathcal{Q}$. The functor $\mathcal{G} \times_{F} \mathcal{G}' : \mathcal{P} \times_{\mathcal{Q}} \mathcal{P}' \to \mathbf{Gp}$ is defined on objects by

$$(\mathcal{G} \times_{\mathcal{H}F} \mathcal{G}')(a, a') = \mathcal{G}(a) \times_{\mathcal{H}(Fa)} \mathcal{G}'(a') = \{(g, g') \in \mathcal{G}(a) \times \mathcal{G}'(a') \mid \varphi_a(g) = \varphi'_{a'}(g')\},$$

and the homomorphism associated to any morphism $(p, p') : (a, a') \to (b, b')$ in $\mathcal{P} \times_{\mathcal{Q}} \mathcal{P}'$ is given by ${}^{(p,p')}(g, g') = ({}^{p}g, {}^{p'}g')$. The boundary map $\partial : \mathcal{G} \times_{\mathcal{H}F} \mathcal{G}' \to \mathrm{Aut}_{\mathcal{P}\times_{\mathcal{Q}}\mathcal{P}'}$, at any object of the groupoid $\mathcal{P} \times_{\mathcal{Q}} \mathcal{P}'$, is given by the formula $\partial(g, g') = (\partial g, \partial g')$.

The canonical morphism

$$(\jmath, J) : (\mathcal{G} \times_{\mathcal{H}F} \mathcal{G}', \mathcal{P} \times_{\mathcal{Q}} \mathcal{P}', \partial) \to \left(\mathcal{G}_{\varphi,F\downarrow\varphi',F'}, \mathcal{P}_{\varphi,F\downarrow\varphi',F'}, \partial\right) \tag{42}$$

is as follows: The functor $J : \mathcal{P} \times_{\mathcal{Q}} \mathcal{P}' \to \mathcal{P}_{\varphi,F\downarrow\varphi',F'}$ sends a morphism $(p, p') : (a, a') \to (b, b')$ to the morphism $(p, 1_{\mathcal{H}(Fa)}, p') : (a, 1_{Fa}, a') \to (b, 1_{Fb}, b')$, and the $\mathcal{P} \times_{\mathcal{Q}} \mathcal{P}'$-group homomorphism $\jmath : \mathcal{G} \times_{\mathcal{H}F} \mathcal{G}' \to \mathcal{P}_{\varphi,F\downarrow\varphi',F'} J$ is given at any object $(a, a') \in \mathcal{P} \times_{\mathcal{Q}} \mathcal{P}'$ by the inclusion map $\mathcal{G}(a) \times_{\mathcal{H}(Fa)} \mathcal{G}'(a') \hookrightarrow \mathcal{G}(a) \times \mathcal{G}'(a')$.

Next, we assume that (φ, F) is a fibration. Then, we verify that the canonical morphism (42) induces isomorphisms between the corresponding homotopy groups. Recall from (37) how to compute the homotopy groups of the classifying space of a crossed module.

• *The map $\pi_0(\jmath, J)$ is a bijection.*

Injectivity: Suppose objects $(a, a'), (b, b') \in \mathcal{P} \times_{\mathcal{Q}} \mathcal{P}'$, such that there is a morphism $(p, h, p') : (a, 1_{Fa}, a') \to (b, 1_{Fb}, b')$ in $\mathcal{P}_{\varphi,F\downarrow\varphi',F'}$. Then, as $\varphi : \mathcal{G}(a) \to \mathcal{H}(Fa)$ is surjective, there is $g \in \mathcal{G}(a)$ such that $\varphi(g) = h$, whence $(p \circ \partial g, p') : (a, a') \to (b, b')$ is a morphism in $\mathcal{P} \times_{\mathcal{Q}} \mathcal{P}'$.

Surjectivity: Let (a, q, a') be an object of $\mathcal{P}_{\varphi,F\downarrow\varphi',F'}$. As $F : \mathcal{P} \to \mathcal{Q}$ is a fibration of groupoids, there is a morphism $p : a \to b$ in $\mathcal{P}$ such that $Fp = q$. Then, (b, a') is an object of the groupoid $\mathcal{P} \times_{\mathcal{Q}} \mathcal{P}'$ with $J(b, a') = (b, 1_{Fb}, a')$ in the same connected component of (a, q, a'), since we have the morphism $(p, 1_{\mathcal{H}(Fb)}, 1_{a'}) : (a, q, a') \to (b, 1_{Fb}, a')$.

• *The homomorphisms $\pi_1(\jmath, J)$ are isomorphisms.* Let (a, a') be any object of $\mathcal{P} \times_{\mathcal{Q}} \mathcal{P}'$.

Injectivity: Let $[(p, p')]$ be an element in the kernel of the homomorphism $\pi_1(\jmath, J)$ at (a, a'), that is, such that $[(p, 1_{\mathcal{H}(Fa)}, p')] = [(1_a, 1_{\mathcal{H}(Fa)}, 1_{a'})]$. This means that there is $(g, g') \in \mathcal{G}(a) \times \mathcal{G}'(a')$ with $\partial g = p$, $\partial(g') = p'$ and $\varphi(g)^{-1} \cdot \varphi'(g') = 1$. The last equation says that (g, g') is an element of $\mathcal{G}(a) \times_{\mathcal{H}(Fa)} \mathcal{G}'(a)$ which, by the first two, satisfies that $\partial(g, g') = (p, p')$. Hence, $[(p, p')] = [(1_a, 1_{a'})]$.

Surjectivity: Let $(p, h, p') : (a, 1_{Fa}, a') \to (a, 1_{Fa}, a')$ be an automorphism of $\mathcal{P}_{\varphi,F\downarrow\varphi',F'}$. As $\varphi : \mathcal{G}(a) \to \mathcal{H}(Fa)$ is surjective, there is a $g \in \mathcal{G}(a)$ such that $\varphi(g) = h$. Then, we have

$$\begin{aligned}(p, h, p')^{-1} \circ J(p \circ \partial g, p') &= (p, h, p')^{-1} \circ (p \circ \partial g, 1_{\mathcal{H}(Fa)}, p') \\ &= (p^{-1} \circ p \circ \partial g, h^{-1}, p'^{-1} \circ p') \\ &= (\partial g, \varphi(g)^{-1} \cdot 1_{\mathcal{H}(Fa)}, 1_{a'}) = \partial(g, 1_{\mathcal{G}'(a')})\end{aligned}$$

and therefore $[(p, h, p')] = [J(p \circ \partial g, p')]$.

• *The homomorphisms $\pi_2(\jmath, J)$ are isomorphisms.* At any object $(a, a') \in \mathcal{P} \times_{\mathcal{Q}} \mathcal{P}'$, the homomorphism $\pi_2(\jmath, J)$ is the restriction to the kernels of the boundary maps of the inclusion $\mathcal{G}(a) \times_{\mathcal{H}(Fa)} \mathcal{G}'(a') \hookrightarrow \mathcal{G}(a) \times \mathcal{G}'(a')$. Then, it is clearly injective. To see the surjectivity, let $(g, g') \in \mathcal{G}(a) \times \mathcal{G}'(a')$ with $\partial(g, g') = (1_a, 1_{\mathcal{H}(Fa)}, 1_{a'})$. Then, we have $\partial g = 1_a$, $\partial g' = 1_{a'}$ and $\varphi(g)^{-1} \cdot \varphi'(g') = 1_{\mathcal{H}(Fa)}$. That is, that $(g, g') \in \mathcal{G}(a) \times_{\mathcal{H}(Fa)} \mathcal{G}'(a')$ and $(\partial g, \partial g') = (1_a, 1_{a'})$. ∎

6. Appendix: Proofs of Lemmas 2.9 and 2.10

We shall only address lax functors below, but the discussions are easily dualized in order to obtain the corresponding results for oplax functors.

Our first goal is to accurately determine the functorial behaviour of the Grothendieck nerve construction $\mathcal{B} \mapsto \underline{\mathrm{N}}\mathcal{B}$ (12) on lax functors between bicategories by means of the theorem below. The result in the first part of it is already known: see [16, §3, (21)], where a proof is given using Jardine's Supercoherence Theorem in [35], or [19, Theorem 3.1], where it has been extended to tricategories. However, for the second part, we need a new proof of the existence of the pseudo-simplicial category $\underline{\mathrm{N}}\mathcal{B}$, with a more explicit and detailed construction of it.

6.1. Theorem. *(i) Any bicategory $\mathcal{B}$ defines a normal pseudo-simplicial category, that is, a unitary pseudo-functor from the simplicial category Δ^{op} into the 2-category of small categories,*

$$\underline{\mathrm{N}}\mathcal{B} = (\underline{\mathrm{N}}\mathcal{B}, \widehat{\underline{\mathrm{N}}}\mathcal{B}) : \Delta^{\mathrm{op}} \to \mathbf{Cat},$$

which is called the Grothendieck *or* pseudo-simplicial nerve *of the bicategory, whose category of p-simplices, for $p \geq 0$, is*

$$\underline{\mathrm{N}}\mathcal{B}_p := \coprod_{(x_p,\ldots,x_0) \in \mathrm{Ob}\mathcal{B}^{p+1}} \mathcal{B}(x_{p-1}, x_p) \times \mathcal{B}(x_{p-2}, x_{p-1}) \times \cdots \times \mathcal{B}(x_0, x_1).$$

(ii) Any lax functor between bicategories $F : \mathcal{B} \to \mathcal{B}'$ induces a lax transformation (i.e., a lax simplicial functor)

$$\underline{\mathrm{N}}F = (\underline{\mathrm{N}}F, \widehat{\underline{\mathrm{N}}}F) : \underline{\mathrm{N}}\mathcal{B} \to \underline{\mathrm{N}}\mathcal{B}'.$$

For any pair of composable lax functors $F : \mathcal{B} \to \mathcal{B}'$ and $F' : \mathcal{B}' \to \mathcal{B}''$, the equality $\underline{\mathrm{N}}F' \, \underline{\mathrm{N}}F = \underline{\mathrm{N}}(F'F)$ holds, and, for any bicategory $\mathcal{B}$, $\underline{\mathrm{N}}1_{\mathcal{B}} = 1_{\underline{\mathrm{N}}\mathcal{B}}$.

Before starting with the proof, we shall describe some needed constructions and a few auxiliary facts. Given a category $\mathcal{I}$ and a bicategory $\mathcal{B}$, we denote by

$$\mathbb{L}\mathrm{ax}(\mathcal{I}, \mathcal{B})$$

the category whose objects are lax functors $F : \mathcal{I} \to \mathcal{B}$, and whose morphisms are *relative to object lax transformations*, as termed by Bullejos and Cegarra in [13], but also called

icons by Lack in [37]. That is, for any two lax functors $F, G : \mathcal{I} \to \mathcal{B}$, a morphism $\Phi : F \Rightarrow G$ may exist only if F and G agree on objects, and it is then given by 2-cells in $\mathcal{B}$, $\Phi a : Fa \Rightarrow Ga$, for every arrow $a : i \to j$ in $\mathcal{I}$, such that the diagrams

$$\begin{array}{ccc} Fa \circ Fb & \overset{\widehat{F}_{a,b}}{\Longrightarrow} & F(ab) \\ {\scriptstyle \Phi a \circ \Phi b}\Downarrow & & \Downarrow{\scriptstyle \Phi(ab)} \\ Ga \circ Gb & \underset{\widehat{G}_{a,b}}{\Longrightarrow} & G(ab), \end{array} \qquad \begin{array}{ccc} & 1_{Fi=Gi} & \\ {\scriptstyle \widehat{F}_i}\swarrow & & \searrow{\scriptstyle \widehat{G}_i} \\ F1_i & \xRightarrow{\Phi 1_i} & G1_i, \end{array}$$

commute for each pair of composable arrows $i \xrightarrow{b} j \xrightarrow{a} k$ and each object i. The composition of morphisms $\Phi : F \Rightarrow G$ and $\Psi : G \Rightarrow H$, for $F, G, H : \mathcal{I} \to \mathcal{B}$ lax functors, is $\Psi \cdot \Phi : F \Rightarrow H$, where $(\Psi \cdot \Phi)a = \Psi a \cdot \Phi a : Fa \Rightarrow Ha$, for each arrow $a : i \to j$ in $\mathcal{I}$. The identity morphism of a lax functor $F : \mathcal{I} \to \mathcal{B}$ is $1_F : F \Rightarrow F$, where $(1_F)a = 1_{Fa}$, the identity of Fa in the category $\mathcal{B}(Fi, Fj)$, for each $a : i \to j$ in $\mathcal{I}$.

Let us now replace the category $\mathcal{I}$ above by a (directed) graph $\mathcal{G}$. For any bicategory $\mathcal{B}$, there is a category

$$\mathbb{G}\mathrm{raph}(\mathcal{G}, \mathcal{B}),$$

where an object $f : \mathcal{G} \to \mathcal{B}$ consists of a pair of maps that assign an object fi to each vertex $i \in \mathcal{G}$ and a 1-cell $fa : fi \to fj$ to each edge $a : i \to j$ in $\mathcal{G}$, respectively. A morphism $\phi : f \Rightarrow g$ may exist only if f and g agree on vertices, that is, $fi = gi$ for all $i \in \mathcal{G}$; and then it consists of a map that assigns to each edge $a : i \to j$ in the graph a 2-cell $\phi a : fa \Rightarrow ga$ of $\mathcal{B}$. Compositions in $\mathbb{G}\mathrm{raph}(\mathcal{G}, \mathcal{B})$ are defined in the natural way by the same rules as those stated above for the category $\mathbb{L}\mathrm{ax}(\mathcal{I}, \mathcal{B})$.

6.2. Lemma. *Let $\mathcal{I} = \mathcal{I}(\mathcal{G})$ be the free category generated by a graph $\mathcal{G}$, let $\mathcal{B}$ be a bicategory, and let*

$$R : \mathbb{L}\mathrm{ax}(\mathcal{I}(\mathcal{G}), \mathcal{B}) \to \mathbb{G}\mathrm{raph}(\mathcal{G}, \mathcal{B})$$

be the functor defined by restriction to the basic graph. Then, there is a functor

$$J : \mathbb{G}\mathrm{raph}(\mathcal{G}, \mathcal{B}) \to \mathbb{L}\mathrm{ax}(\mathcal{I}, \mathcal{B}),$$

and a natural transformation

$$\mathrm{v} : JR \Rightarrow 1_{\mathbb{L}\mathrm{ax}(\mathcal{I}, \mathcal{B})}, \tag{43}$$

such that $RJ = 1_{\mathbb{G}\mathrm{raph}(\mathcal{G}, \mathcal{B})}$, $\mathrm{v}J = 1_J$, $R\mathrm{v} = 1_R$. Thus, the functor R is right adjoint to the functor J.

Proof. To describe the functor J, we use the following useful construction: For any list $(x_0, \ldots, x_p)$ of objects in the bicategory $\mathcal{B}$, let

$$\overset{\mathrm{or}}{\circ} : \mathcal{B}(x_{p-1}, x_p) \times \mathcal{B}(x_{p-2}, x_{p-1}) \times \cdots \times \mathcal{B}(x_0, x_1) \longrightarrow \mathcal{B}(x_0, x_p)$$

denote the functor obtained by iterating horizontal composition in the bicategory, which acts on objects and arrows of the product category by the recursive formula

$$\overset{\text{or}}{\circ}(u_p,\dots,u_1)=\begin{cases} u_1 & \text{if } \ p=1,\\ u_p\circ\big(\overset{\text{or}}{\circ}(u_{p-1},\dots,u_1)\big) & \text{if } \ p\geq 2.\end{cases}$$

Then, the homomorphism J takes a graph map, say $f:\mathcal{G}\to\mathcal{B}$, to the unitary pseudo-functor from the free category

$$J(f)=F:\mathcal{I}\to\mathcal{B},$$

such that $Fi=fi$, for any vertex i of $\mathcal{G}$ (= objects of $\mathcal{I}$), and associates to strings $a:a(0)\overset{a_1}{\to}\cdots\overset{a_p}{\to}a(p)$ in $\mathcal{G}$ the 1-cells $Fa=\overset{\text{or}}{\circ}(fa_p,\dots,fa_1):fa(0)\to fa(p)$. The structure 2-cells $\widehat{F}_{a,b}:Fa\circ Fb\Rightarrow F(ab)$, for any pair of strings in the graph, $a=a_p\cdots a_1$ as above and $b=b_q\cdots b_1$ with $b(q)=a(0)$, are canonically obtained from the associativity constraints in the bicategory: first by taking $\widehat{F}_{a_1,b}=1_{F(a_1b)}$ when $p=1$ and then, recursively for $p>1$, defining $\widehat{F}_{a,b}$ as the composite

$$\widehat{F}_{a,b}:\ Fa\circ Fb\overset{\boldsymbol{a}}{\Longrightarrow}Fa_p\circ(Fa'\circ Fb)\overset{1\circ\widehat{F}_{a',b}}{\Longrightarrow}F(ab),$$

where $a'=a_{p-1}\cdots a_1$ (whence $Fa=Fa_p\circ Fa'$). The coherence conditions for F are easily verified by using the coherence and naturality of the associativity constraint $\boldsymbol{a}$ of the bicategory.

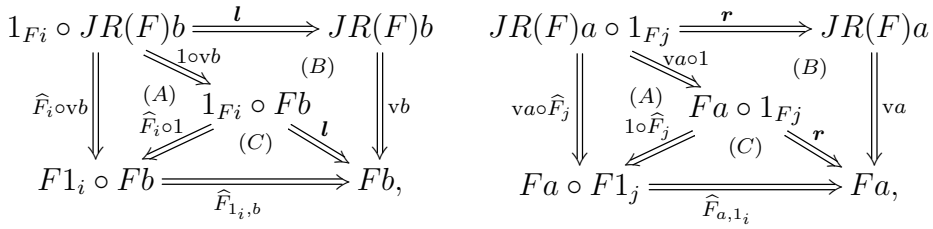
Any morphism $\phi:f\Rightarrow g$ in $\mathbb{G}\text{raph}(\mathcal{G},\mathcal{B})$ is taken by J to the morphism $J(\phi):F\Rightarrow G$ of $\mathbb{L}\text{ax}(\mathcal{I},\mathcal{B})$, consisting of the 2-cells in the bicategory $\overset{\text{or}}{\circ}(\phi a_p,\dots,\phi a_1):Fa\Rightarrow Ga$, attached to the strings of adjacent edges in the graph $a=a_p\cdots a_1$. The coherence conditions of $J(\phi)$ are consequence of the naturality of the associativity constraint $\boldsymbol{a}$ of the bicategory. If $\phi:f\Rightarrow g$ and $\psi:g\Rightarrow h$ are 1-cells in $\mathbb{G}\text{raph}(\mathcal{G},\mathcal{B})$, then $J(\psi)\cdot J(\phi)=J(\psi\cdot\phi)$ follows from the functoriality of the composition $\circ$, and so J is a functor.

The lax transformation v is defined as follows: The component of this lax transformation at a lax functor $F:\mathcal{I}\to\mathcal{B}$, $\text{v}:JR(F)\Rightarrow F$, is defined on identities by $\text{v}1_i=\widehat{F}_i:1_{Fi}\Rightarrow F1_i$, for any vertex i of $\mathcal{G}$, and it associates to each string of adjacent edges in the graph $a=a_p\cdots a_1$ the 2-cell $\text{v}a:\overset{\text{or}}{\circ}(Fa_p,\dots,Fa_1)\Rightarrow Fa$, which is given by taking $\text{v}a_1=1_{Fa_1}$ if $p=1$, and then, recursively for $p>1$, by taking va as the composite

$$\text{v}a=\ \Big(\overset{\text{or}}{\circ}(Fa_p,\dots,Fa_1)\overset{1\circ\text{v}a'}{\Longrightarrow}Fa_p\circ Fa'\overset{\widehat{F}_{a_p,a'}}{\Longrightarrow}Fa\Big),$$

where $a'=a_{p-1}\cdots a_1$. The naturality condition $\widehat{F}_{a,b}\circ(\text{v}a\circ\text{v}b)=\text{v}(ab)\circ\widehat{JR(F)}_{a,b}$, for any pair of composable morphisms in $\mathcal{I}$, can be checked as follows: when $a=1_i$ or $b=1_j$

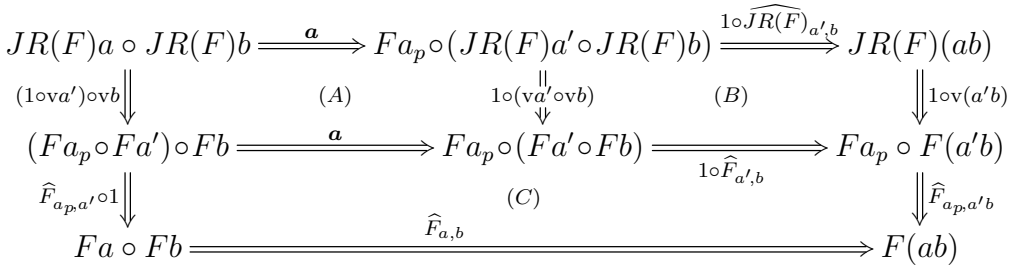
are identities, then it is a consequence of the commutativity of the diagrams



where the regions labelled with (A) commute by the functoriality of $\circ$, those with (B) by the naturality of $\boldsymbol{l}$ and $\boldsymbol{r}$, and those with (C) by the coherence of F. Now, for arbitrary strings a and b in the graph with $b(q) = a(0)$, we study the coherence recursively on the length of a. The case when $p = 1$ is the obvious commutative diagram

$$\begin{array}{ccc} Fa_1 \circ JR(F)b & \xRightarrow{1} & JR(F)(a_1 b) \\ {\scriptstyle 1\circ \mathrm{v}b}\Downarrow & & \Downarrow{\scriptstyle \mathrm{v}(a_1 b)=\widehat{F}_{a_1,b}\cdot(1\circ \mathrm{v}b)} \\ Fa_1 \circ Fb & \xRightarrow[\widehat{F}_{a_1,b}]{} & F(a_1 b), \end{array}$$

and then, for $p > 1$, the result is a consequence of the diagram



where (A) commutes by the naturality of $\boldsymbol{a}$, (B) by induction, and (C) by the coherence of F.

To verify the equalities $RJ = 1$, $\mathrm{v}J = 1$, and $R\mathrm{v} = 1$ is straightforward. ∎

Let $\mathcal{I} = \mathcal{I}(\mathcal{G})$ again be the free category generated by a graph $\mathcal{G}$, as in Lemma 6.2 above, and suppose now that $F : \mathcal{B} \to \mathcal{B}'$ is a lax functor. Then, the square

$$\begin{array}{ccc} \mathbb{L}\mathrm{ax}(\mathcal{I}, \mathcal{B}) & \xrightarrow{R} & \mathbb{G}\mathrm{raph}(\mathcal{G}, \mathcal{B}) \\ {\scriptstyle F_*}\downarrow & & \downarrow{\scriptstyle F_*} \\ \mathbb{L}\mathrm{ax}(\mathcal{I}, \mathcal{B}') & \xrightarrow{R'} & \mathbb{G}\mathrm{raph}(\mathcal{G}, \mathcal{B}') \end{array} \tag{44}$$

commutes and, since $RJ = 1$, we have the equalities

$$R'F_*JR = F_*RJR = F_*R = R'F_*. \tag{45}$$

Furthermore, the naturality of $\mathrm{v} : JR \Rightarrow 1$ and $\mathrm{v}' : J'R' \Rightarrow 1$ means that the square

$$\begin{array}{ccc} J'R'F_*JR & \xRightarrow{\mathrm{v}'F_*JR} & F_*JR \\ {\scriptstyle J'R'F_*\mathrm{v}}\Downarrow & & \Downarrow{\scriptstyle F_*\mathrm{v}} \\ J'R'F_* & \xRightarrow{\mathrm{v}'F_*} & F_* \end{array}$$

commutes. As $R\mathrm{v} = 1_R$ and then $J'R'F_*\mathrm{v} = J'F_*R\mathrm{v} = J'F_*1_R = 1_{J'R'F_*}$, we have the equality

$$F_*\mathrm{v} \circ \mathrm{v}'F_*JR = \mathrm{v}'F_*. \tag{46}$$

6.3. Proof of Theorem 6.1. (i): Let us note that, for any integer $p \geq 0$, the category $[p]$ is free on the graph

$$\mathcal{G}_p = (0 \to 1 \cdots \to p).$$

Then, for any given bicategory $\mathcal{B}$, the existence of an adjunction

$$J_p \dashv R_p : \underline{\mathrm{N}}\mathcal{B}_p = \mathbb{G}\mathrm{raph}(\mathcal{G}_p, \mathcal{B}) \rightleftarrows \mathbb{L}\mathrm{ax}([p], \mathcal{B}) \tag{47}$$

follows from Lemma 6.2, where R_p is the functor defined by restricting to the basic graph $\mathcal{G}_p$ of the category $[p]$, where $R_pJ_p = 1$, whose unity is the identity, and whose counit $\mathrm{v}_p : J_pR_p \Rightarrow 1$ satisfies the equalities $\mathrm{v}_pJ_p = 1$ and $R_p\mathrm{v}_p = 1$.

If $a : [q] \to [p]$ is any map in the simplicial category, then the associated functor $\underline{\mathrm{N}}\mathcal{B}_a : \underline{\mathrm{N}}\mathcal{B}_p \to \underline{\mathrm{N}}\mathcal{B}_q$ is the composite

$$\begin{array}{ccc} \underline{\mathrm{N}}\mathcal{B}_p & \overset{\underline{\mathrm{N}}\mathcal{B}_a}{\dashrightarrow} & \underline{\mathrm{N}}\mathcal{B}_q \\ {\scriptstyle J_p}\downarrow & & \uparrow{\scriptstyle R_q} \\ \mathbb{L}\mathrm{ax}([p], \mathcal{B}) & \xrightarrow{a^*} & \mathbb{L}\mathrm{ax}([q], \mathcal{B}). \end{array}$$

Thus, $\underline{\mathrm{N}}\mathcal{B}_a$ maps the component category of $\underline{\mathrm{N}}\mathcal{B}_p$ at $(x_p, \ldots, x_0)$ into the component at $(x_{a(q)}, \ldots, x_{a(0)})$ of $\underline{\mathrm{N}}\mathcal{B}_q$, and it acts both on objects and morphisms of $\underline{\mathrm{N}}\mathcal{B}_p$ by the formula $\underline{\mathrm{N}}\mathcal{B}_p(u_p, \ldots, u_1) = (v_q, \ldots, v_1)$, where, for $0 \leq k < q$,

$$v_{k+1} = \begin{cases} \overset{\mathrm{or}}{\circ}(u_{a(k+1)}, \ldots, u_{a(k)+1}) & \textit{if} \quad a(k) < a(k+1), \\ 1 & \textit{if} \quad a(k) = a(k+1), \end{cases}$$

whence, in particular, the usual formulas below for the face and degeneracy functors.

$$d_i(u_p, \ldots, u_1) = \begin{cases} (u_p, \ldots, u_2) & \text{if} \quad i = 0, \\ (u_p, \ldots, u_{i+1} \circ u_i, \ldots, u_1) & \text{if} \quad 0 < i < p, \\ (u_{p-1}, \ldots, u_1) & \text{if} \quad i = p, \end{cases}$$

$$s_i(u_p, \ldots, u_1) = (u_p, \ldots, u_{i+1}, 1, u_i, \ldots, u_0).$$

The structure natural transformation

$$\underline{\mathrm{N}}\mathcal{B}_p \;\overset{\underline{\mathrm{N}}\mathcal{B}_b\ \underline{\mathrm{N}}\mathcal{B}_a}{\underset{\underline{\mathrm{N}}\mathcal{B}_{ab}}{\Downarrow \widehat{\underline{\mathrm{N}}}\mathcal{B}_{a,b}}}\; \underline{\mathrm{N}}\mathcal{B}_n, \tag{48}$$

for each pair of composable maps $[n] \xrightarrow{b} [q] \xrightarrow{a} [p]$ in Δ, is

$$\underline{\mathrm{N}}\mathcal{B}_b\ \underline{\mathrm{N}}\mathcal{B}_a = R_n b^* J_q R_q a^* J_p \xRightarrow{\widehat{\underline{\mathrm{N}}}\mathcal{B}_{a,b} = R_n b^* \mathrm{v}_q a^* J_p} R_n b^* a^* J_p = R_n (ab)^* J_p = \underline{\mathrm{N}}\mathcal{B}_{ab}.$$

Let us stress that, in spite of the natural transformation v in (43) not being invertible, the natural transformation $\widehat{\underline{\mathrm{N}}}\mathcal{B}_{a,b}$ in (48) is invertible since, for any $\mathbf{x} \in \underline{\mathrm{N}}\mathcal{B}_p$, the lax functor $a^* J_p \mathbf{x}$ is actually a homomorphism and therefore $\mathrm{v}_q a^* J_p \mathbf{x}$ is an isomorphism. Consequently, we only need to prove that these constraints $\widehat{\underline{\mathrm{N}}}\mathcal{B}_{a,b}$ verify the coherence conditions for lax functors:

If $a = 1_{[p]}$, then $\widehat{\underline{\mathrm{N}}}\mathcal{B}_{1,b} = R_n b^* \mathrm{v}_p J_p = R_n b^* 1_{J_p} = 1_{\underline{\mathrm{N}}\mathcal{B}_b}$. Similarly, $\widehat{\underline{\mathrm{N}}}\mathcal{B}_{a,1} = 1_{\underline{\mathrm{N}}\mathcal{B}_a}$. Furthermore, for every triplet of composable arrows $[m] \xrightarrow{c} [n] \xrightarrow{b} [q] \xrightarrow{a} [p]$, the diagram

$$\begin{array}{ccc}
\underline{\mathrm{N}}\mathcal{B}_c\ \underline{\mathrm{N}}\mathcal{B}_b\ \underline{\mathrm{N}}\mathcal{B}_a & \xRightarrow{\underline{\mathrm{N}}\mathcal{B}_c\ \widehat{\underline{\mathrm{N}}}\mathcal{B}_{a,b}} & \underline{\mathrm{N}}\mathcal{B}_c\ \underline{\mathrm{N}}\mathcal{B}_{ab} \\
{\scriptstyle \widehat{\underline{\mathrm{N}}}\mathcal{B}_{b,c}\ \underline{\mathrm{N}}\mathcal{B}_a}\Downarrow & & \Downarrow{\scriptstyle \widehat{\underline{\mathrm{N}}}\mathcal{B}_{ab,c}} \\
\underline{\mathrm{N}}\mathcal{B}_{bc}\ \underline{\mathrm{N}}\mathcal{B}_a & \xRightarrow{\widehat{\underline{\mathrm{N}}}\mathcal{B}_{a,bc}} & \underline{\mathrm{N}}\mathcal{B}_{abc},
\end{array}$$

is commutative since it is obtained by applying the functors $R_m c^*$ on the left, and $a^* J_p$ on the right, to the diagram

$$\begin{array}{ccc}
J_n R_n b^* J_q R_q & \xRightarrow{J_n R_n b^* \mathrm{v}_q} & J_n R_n b^* \\
{\scriptstyle \mathrm{v}_n b^* J_q R_q}\Downarrow & & \Downarrow{\scriptstyle \mathrm{v}_n b^*} \\
b^* J_q R_q & \xRightarrow[b^* \mathrm{v}_q]{} & b^*,
\end{array} \tag{49}$$

which commutes by the naturality of v_n.

(*ii*): Suppose now that $F : \mathcal{B} \to \mathcal{B}'$ is a lax functor. Then, at any integer $p \geq 0$, the functor $\underline{\mathrm{N}}F_p : \underline{\mathrm{N}}\mathcal{B}_p \to \underline{\mathrm{N}}\mathcal{B}'_p$ is the composite

$$\begin{array}{ccc}
\underline{\mathrm{N}}\mathcal{B}_p & \overset{\underline{\mathrm{N}}F_p}{\dashrightarrow} & \underline{\mathrm{N}}\mathcal{B}'_p \\
{\scriptstyle J_p}\downarrow & & \uparrow{\scriptstyle R'_p} \\
\mathbb{L}\mathrm{ax}([p], \mathcal{B}) & \xrightarrow{F_*} & \mathbb{L}\mathrm{ax}([p], \mathcal{B}'),
\end{array}$$

which is explicitly given both on objects and arrows by the simple formula $\underline{\mathrm{N}}F_p(u_p,\dots,u_1) = (Fu_p,\dots,Fu_1)$. The structure natural transformation

$$\begin{array}{ccc} \underline{\mathrm{N}}\mathcal{B}_p & \xrightarrow{\underline{\mathrm{N}}\mathcal{B}_a} & \underline{\mathrm{N}}\mathcal{B}_q \\ {\scriptstyle \underline{\mathrm{N}}F_p}\downarrow & \overset{\widehat{\underline{\mathrm{N}}}F_a}{\Rightarrow} & \downarrow{\scriptstyle \underline{\mathrm{N}}F_q} \\ \underline{\mathrm{N}}\mathcal{B}'_p & \xrightarrow[\underline{\mathrm{N}}\mathcal{B}'_a]{} & \underline{\mathrm{N}}\mathcal{B}'_q, \end{array}$$

at each map $a:[q]\to[p]$ in Δ, is

$$\begin{aligned} \underline{\mathrm{N}}\mathcal{B}'_a\,\underline{\mathrm{N}}F_p = R'_q a^* J'_p R'_p F_* J_p \xRightarrow{\widehat{\underline{\mathrm{N}}}F_a = R'_q a^* \mathrm{v}'_p F_* J_p} & R'_q a^* F_* J_p = R'_q F_* a^* J_p \overset{(45)}{=} R'_q F_* J_q R_q a^* J_p \\ & = \underline{\mathrm{N}}F_q\,\underline{\mathrm{N}}\mathcal{B}_a. \end{aligned}$$

This family of natural transformations $\widehat{\underline{\mathrm{N}}}F_a$ verifies the coherence conditions for lax transformations: If $a = 1_{[p]}$, then $\widehat{\underline{\mathrm{N}}}F_1 = R'_p \mathrm{v}'_p F_* J_p = 1_{R'_p F_* J_p} = 1_{\widehat{\underline{\mathrm{N}}}F_p}$. Suppose that $b:[n]\to[q]$ is any other map of Δ, then the coherence diagram

$$\begin{array}{ccccc} \underline{\mathrm{N}}\mathcal{B}'_b\,\underline{\mathrm{N}}\mathcal{B}'_a\,\underline{\mathrm{N}}F_p & \xRightarrow{\underline{\mathrm{N}}\mathcal{B}'_b\,\widehat{\underline{\mathrm{N}}}F_a} & \underline{\mathrm{N}}\mathcal{B}'_b\,\underline{\mathrm{N}}F_q\,\underline{\mathrm{N}}\mathcal{B}_a & \xRightarrow{\widehat{\underline{\mathrm{N}}}F_b\,\underline{\mathrm{N}}\mathcal{B}_a} & \underline{\mathrm{N}}F_n\,\underline{\mathrm{N}}\mathcal{B}_b\,\underline{\mathrm{N}}\mathcal{B}_a \\ {\scriptstyle \widehat{\underline{\mathrm{N}}}\mathcal{B}'_{a,b}\,\underline{\mathrm{N}}F_p}\Downarrow & & & & \Downarrow{\scriptstyle \underline{\mathrm{N}}F_n\,\widehat{\underline{\mathrm{N}}}\mathcal{B}_{a,b}} \\ \underline{\mathrm{N}}\mathcal{B}'_{ab}\,\underline{\mathrm{N}}F_p & & \xRightarrow{\widehat{\underline{\mathrm{N}}}F_{ab}} & & \underline{\mathrm{N}}F_n\,\underline{\mathrm{N}}\mathcal{B}_{ab} \end{array}$$

commutes, since

$$\begin{aligned} (\underline{\mathrm{N}}F_n\,\widehat{\underline{\mathrm{N}}}\mathcal{B}_{a,b}) \circ (\widehat{\underline{\mathrm{N}}}F_b\,\underline{\mathrm{N}}\mathcal{B}_a) \circ (\underline{\mathrm{N}}\mathcal{B}'_b\,\widehat{\underline{\mathrm{N}}}F_a) & \\ = (R'_n F_* J_n R_n b^* \mathrm{v}_q a^* J_p) \circ (R'_n b^* \mathrm{v}'_q F_* J_q R_q a^* J_p) \circ (R'_n b^* J'_q R'_q a^* \mathrm{v}'_p F_* J_p) & \\ \overset{(45)}{=} (R'_n b^* F_* \mathrm{v}_q a^* J_p) \circ (R'_n b^* \mathrm{v}'_q F_* J_q R_q a^* J_p) \circ (R'_n b^* J'_q R'_q a^* \mathrm{v}'_p F_* J_p) & \\ \overset{(46)}{=} (R'_n b^* \mathrm{v}'_q a^* F_* J_p) \circ (R'_n b^* J'_q R'_q a^* \mathrm{v}'_p F_* J_p) & \\ \overset{(49)}{=} (R'_n b^* a^* \mathrm{v}'_p F_* J_p) \circ (R'_n b^* \mathrm{v}'_q a^* J'_p R'_p F_* J_p) = \widehat{\underline{\mathrm{N}}}F_{ab} \circ (\widehat{\underline{\mathrm{N}}}\mathcal{B}'_{a,b}\,\underline{\mathrm{N}}F_p). & \end{aligned}$$

To finish, let $F:\mathcal{B}\to\mathcal{B}'$ and $F':\mathcal{B}'\to\mathcal{B}''$ be lax functors. Then, $\underline{\mathrm{N}}F'\,\underline{\mathrm{N}}F = \underline{\mathrm{N}}(F'F)$ and $\underline{\mathrm{N}}1_{\mathcal{B}} = 1_{\underline{\mathrm{N}}\mathcal{B}}$ since, at any $[p]$ and $a:[q]\to[p]$ in Δ, we have

$$\begin{aligned} \underline{\mathrm{N}}F'_p\,\underline{\mathrm{N}}F_p = R'' F'_* J'_p R'_p F_* J_p \overset{(45)}{=} R''_p F'_* F_* J_p = R''_p (F'F)_* J_p = \underline{\mathrm{N}}(F'F)_p, & \\ \widehat{\underline{\mathrm{N}}F'\underline{\mathrm{N}}}F_a = \underline{\mathrm{N}}F'_q\,\widehat{\underline{\mathrm{N}}}F_a \circ \widehat{\underline{\mathrm{N}}}F'_a\,\underline{\mathrm{N}}F_p = (R''_q F'_* J'_q R'_q a^* \mathrm{v}'_p F_* J_p) \circ (R''_q a^* \mathrm{v}''_p F'_* J'_p R'_p F_* J_p) & \\ \overset{(45)}{=} (R''_q a^* F'_* \mathrm{v}'_p F_* J_p) \circ (R''_q a^* \mathrm{v}''_p F'_* J'_p R'_p F_* J_p) \overset{(46)}{=} R''_q a^* \mathrm{v}''_p F'_* F_* J_p = \widehat{\underline{\mathrm{N}}}(F'F)_a, & \\ \underline{\mathrm{N}}1_p = R_p 1_* J_p \overset{(44)}{=} R_p J_p = 1_{\underline{\mathrm{N}}\mathcal{B}_p}, & \\ \widehat{\underline{\mathrm{N}}}1_a = R_q a^* \mathrm{v}_p 1_* J_p = R_q a^* \mathrm{v}_p J_p = R_q a^* 1_{J_p} = 1_{R_q a^* J_p} = 1_{\underline{\mathrm{N}}\mathcal{B}_a}. & \end{aligned}$$

This completes the proof of Theorem 6.1 and lets us prepare to prove the first part of Lemma 2.9.

6.4. Corollary. *The assignment $\mathcal{B} \mapsto \mathrm{B}\mathcal{B}$ is the function on objects of a functor*

$$\mathrm{B} : \mathbf{Lax} \to \mathbf{Top}.$$

Proof. By Theorem 6.1, any lax functor $F : \mathcal{B} \to \mathcal{B}'$ gives rise to a lax simplicial functor $\underline{\mathrm{N}}F : \underline{\mathrm{N}}\mathcal{B} \to \underline{\mathrm{N}}\mathcal{B}'$, hence to a functor $\int_\Delta \underline{\mathrm{N}}F : \int_\Delta \underline{\mathrm{N}}\mathcal{B} \to \int_\Delta \underline{\mathrm{N}}\mathcal{B}'$ and then to a cellular map $\mathrm{B}F : \mathrm{B}\mathcal{B} \to \mathrm{B}\mathcal{B}'$. For $F = 1_\mathcal{B}$, we have $\int_\Delta \underline{\mathrm{N}}1_\mathcal{B} = \int_\Delta 1_{\underline{\mathrm{N}}\mathcal{B}} = 1_{\int_\Delta \underline{\mathrm{N}}\mathcal{B}}$, whence $\mathrm{B}1_\mathcal{B} = 1_{\mathrm{B}\mathcal{B}}$. For any other lax functor $F' : \mathcal{B}' \to \mathcal{B}''$, the equality $\underline{\mathrm{N}}F'\,\underline{\mathrm{N}}F = \underline{\mathrm{N}}(F'F)$ gives that $\int_\Delta \underline{\mathrm{N}}(F'F) = \int_\Delta \underline{\mathrm{N}}F'\underline{\mathrm{N}}F = \int_\Delta \underline{\mathrm{N}}F' \int_\Delta \underline{\mathrm{N}}F$, whence $\mathrm{B}(F'F) = \mathrm{B}F'\,\mathrm{B}F$. ∎

In [16, Definition 5.2], Carrasco, Cegarra, and Garzón defined the *categorical geometric nerve* of a bicategory $\mathcal{B}$ as the simplicial category

$$\underline{\Delta}\mathcal{B} : \Delta^{op} \to \mathbf{Cat}, \quad [p] \mapsto \mathbb{L}\mathrm{ax}([p], \mathcal{B}),$$

whose category of p-simplices is the category of lax functors $\mathbf{x} : [p] \to \mathcal{B}$, with relative to objects lax transformations (i.e., icons) between them as arrows. The proposition below shows how $\underline{\Delta}\mathcal{B}$ relates with the Grothendieck nerve $\underline{\mathrm{N}}\mathcal{B}$.

6.5. Proposition. *For any bicategory $\mathcal{B}$, there is a lax simplicial functor*

$$R = (R, \widehat{R}) : \underline{\Delta}\mathcal{B} \to \underline{\mathrm{N}}\mathcal{B} \tag{50}$$

inducing a homotopy equivalence

$$\mathrm{B}\textstyle\int_\Delta R : \mathrm{B}\int_\Delta \underline{\Delta}\mathcal{B} \xrightarrow{\sim} \mathrm{B}\int_\Delta \underline{\mathrm{N}}\mathcal{B} = \mathrm{B}\mathcal{B}, \tag{51}$$

which is natural in $\mathcal{B}$ on lax functors. That is, for any lax functor $F : \mathcal{B} \to \mathcal{B}'$, the square of spaces below commutes.

$$\begin{array}{ccc} \mathrm{B}\int_\Delta \underline{\Delta}\mathcal{B} & \xrightarrow{\mathrm{B}\int_\Delta R} & \mathrm{B}\mathcal{B} \\ {\scriptstyle \mathrm{B}\int_\Delta \underline{\Delta}F}\downarrow & & \downarrow{\scriptstyle \mathrm{B}F} \\ \mathrm{B}\int_\Delta \underline{\Delta}\mathcal{B}' & \xrightarrow{\mathrm{B}\int_\Delta R'} & \mathrm{B}\mathcal{B}' \end{array} \tag{52}$$

Proof. At any object $[p]$ of the simplicial category, R is given by the functor in (47)

$$R_p : \underline{\Delta}\mathcal{B}_p = \mathbb{L}\mathrm{ax}([p], \mathcal{B}) \longrightarrow \mathbb{G}\mathrm{raph}(\mathcal{G}_p, \mathcal{B}) = \underline{\mathrm{N}}\mathcal{B}_p,$$

and, at any map $a : [q] \to [p]$, the natural transformation

$$\begin{array}{ccc} \underline{\Delta}\mathcal{B}_p & \xrightarrow{a^*} & \underline{\Delta}\mathcal{B}_q \\ {\scriptstyle R_p}\downarrow & \overset{\widehat{R}_a}{\Rightarrow} & \downarrow{\scriptstyle R_q} \\ \underline{\mathrm{N}}\mathcal{B}_p & \xrightarrow[\underline{\mathrm{N}}\mathcal{B}_a]{} & \underline{\mathrm{N}}\mathcal{B}_q, \end{array}$$

is defined by $\underline{\mathrm{N}}\mathcal{B}_a R_p = R_q a^* J_p R_p \xrightarrow{\widehat{R}_a = R_q a^* \mathrm{v}_p} R_q a^*$. When $a = 1_{[p]}$, clearly $\widehat{R}_{1_{[p]}} = R_p \mathrm{v}_p = 1_{R_p}$ and, for any $b : [n] \to [q]$, the commutativity coherence condition

$$\begin{array}{ccc} \underline{\mathrm{N}}\mathcal{B}_b\, \underline{\mathrm{N}}\mathcal{B}_a\, R_p & \xRightarrow{\underline{\mathrm{N}}\mathcal{B}_b \widehat{R}_a} & \underline{\mathrm{N}}\mathcal{B}_b\, R_q\, a^* \\ {\scriptstyle \widehat{\mathrm{N}}\mathcal{B}_{a,b} R_p}\Downarrow & & \Downarrow{\scriptstyle \widehat{R}_b a^*} \\ \underline{\mathrm{N}}\mathcal{B}_{ab}\, R_p & \xRightarrow{\widehat{R}_{ab}} & R_n b^* a^* = R_n (ab)^*, \end{array}$$

holds since, by (49), $R_n b^* a^* \mathrm{v}_p \circ R_n b^* \mathrm{v}_q a^* J_p R_p = R_n b^* \mathrm{v}_q a^* \circ R_n b^* J_q R_q a^* \mathrm{v}_p$.

By [42, Corollary 1], every functor $R_p : \underline{\Delta}\mathcal{B}_p \to \underline{\mathrm{N}}\mathcal{B}_p$ induces a homotopy equivalence on classifying spaces $\mathrm{B}R_p : \mathrm{B}\underline{\Delta}\mathcal{B}_p \xrightarrow{\sim} \mathrm{B}\underline{\mathrm{N}}\mathcal{B}_p$ since it has the functor J_p in (47) as a left adjoint. Then, the induced map in (51) is actually a homotopy equivalence by [47, Corollary 3.3.1].

Now let $F : \mathcal{B} \to \mathcal{B}'$ be any lax functor. Then, the square

$$\begin{array}{ccc} \underline{\Delta}\mathcal{B} & \xrightarrow{R} & \underline{\mathrm{N}}\mathcal{B} \\ {\scriptstyle \underline{\Delta}F}\downarrow & & \downarrow{\scriptstyle \underline{\mathrm{N}}F} \\ \underline{\Delta}\mathcal{B}' & \xrightarrow{R'} & \underline{\mathrm{N}}\mathcal{B}' \end{array}$$

commutes since, for any integer $p \geq 0$ and $a : [q] \to [p]$, we have

$$\begin{aligned} \underline{\mathrm{N}}F_p\, R_p &= R'_p F_* J_p R_p \overset{(45)}{=} R'_p F_* = R'_p\, \underline{\Delta}F_p, \\ \widehat{\underline{\mathrm{N}}F R}_a &= \underline{\mathrm{N}}F_q \widehat{R}_a \circ \widehat{\mathrm{N}}F_a R_p = R'_q F_* J_q R_q a^* \mathrm{v}_p \circ R'_q a^* \mathrm{v}'_p F_* J_p R_p \\ &\overset{(45)}{=} R'_q a^* F_* \mathrm{v}_p \circ R'_q a^* \mathrm{v}'_p F_* J_p R_p \overset{(46)}{=} R'_q a^* \mathrm{v}'_p F_* = \widehat{R}_a F_* = \widehat{R' \underline{\Delta} F}_a. \end{aligned}$$

Hence, the commutativity of the square (52) follows:

$$\begin{aligned} \mathrm{B}F\ \mathrm{B}\textstyle\int_\Delta R = \mathrm{B}\int_\Delta \underline{\mathrm{N}}F\ \mathrm{B}\int_\Delta R = \mathrm{B}(\int_\Delta \underline{\mathrm{N}}F\ \int_\Delta R) = \mathrm{B}\int_\Delta (\underline{\mathrm{N}}F\, R) \\ = \mathrm{B}\int_\Delta (R'\, \underline{\Delta}F) = \mathrm{B}(\int_\Delta R' \int_\Delta \underline{\Delta}F) = \mathrm{B}\int_\Delta R'\ \mathrm{B}\int_\Delta \underline{\Delta}F. \end{aligned}$$

■

We are now ready to complete the proof of Lemmas 2.9 and 2.10.

6.6. Corollary. *For any bicategory $\mathcal{B}$, there is a homotopy equivalence*

$$\kappa : |\Delta\mathcal{B}| \xrightarrow{\sim} \mathrm{B}\mathcal{B}, \tag{53}$$

which is homotopy natural on lax functors. That is, for any lax functor $F : \mathcal{B} \to \mathcal{B}'$, there is a homotopy $\kappa'\, |\Delta F| \Rightarrow \mathrm{B}F\, \kappa$,

$$\begin{array}{ccc} |\Delta\mathcal{B}| & \xrightarrow{\kappa} & \mathrm{B}\mathcal{B} \\ {\scriptstyle |\Delta F|}\downarrow & \Rightarrow & \downarrow{\scriptstyle \mathrm{B}F} \\ |\Delta\mathcal{B}'| & \xrightarrow{\kappa'} & \mathrm{B}\mathcal{B}'. \end{array} \tag{54}$$

PROOF. Let $\mathrm{N}\underline{\Delta}\mathcal{B} : \Delta^{op} \to \mathbf{SimplSet}$ be the bisimplicial set obtained from the simplicial category $\underline{\Delta}\mathcal{B} : \Delta^{op} \to \mathbf{Cat}$ with the nerve of categories functor $\mathrm{N} : \mathbf{Cat} \to \mathbf{SimplSet}$.

As $\Delta\mathcal{B}$ is the simplicial set of objects of the simplicial category $\underline{\Delta}\mathcal{B}$, if we regard $\Delta\mathcal{B}$ as a discrete simplicial category (i.e., with only identities as arrows), we have a simplicial category inclusion map $\Delta\mathcal{B} \hookrightarrow \underline{\Delta}\mathcal{B}$, whence a bisimplicial inclusion map $\mathrm{N}\Delta\mathcal{B} \hookrightarrow \mathrm{N}\underline{\Delta}\mathcal{B}$, where $\mathrm{N}\Delta\mathcal{B}$ is the bisimplicial set that is constant the simplicial set $\Delta\mathcal{B}$ in the vertical direction. Then, we have an induced simplicial set map on diagonals $i : \Delta\mathcal{B} \to \operatorname{diag}\mathrm{N}\underline{\Delta}\mathcal{B}$. This map is clearly natural in $\mathcal{B}$ on lax functors and, by [16, Theorem 6.2], it induces a homotopy equivalence on geometric realizations. Furthermore, a result by Bousfield and Kan [7, Chap. XII, 4.3] and Thomason's Homotopy Colimit Theorem [47] give us the existence of simplicial maps $\mu : \operatorname{hocolim}\mathrm{N}\underline{\Delta}\mathcal{B} \to \operatorname{diag}\mathrm{N}\underline{\Delta}\mathcal{B}$ and $\eta : \operatorname{hocolim}\mathrm{N}\underline{\Delta}\mathcal{B} \to \mathrm{N}\int_{\Delta}\underline{\Delta}\mathcal{B}$, which are natural on lax functors and both induce homotopy equivalences on geometric realizations.

We then have a chain of homotopy equivalences between spaces

$$|\Delta\mathcal{B}| \xrightarrow{|i|} |\operatorname{diag}\mathrm{N}\underline{\Delta}\mathcal{B}| \xleftarrow{|\mu|} |\operatorname{hocolim}\mathrm{N}\underline{\Delta}\mathcal{B}| \xrightarrow{|\eta|} \mathrm{B}\textstyle\int_{\Delta}\underline{\Delta}\mathcal{B} \xrightarrow{\mathrm{B}\int_{\Delta}R} \mathrm{B}\mathcal{B},$$

where the last one on the right is the homotopy equivalence (50), all of them natural on lax functors $F : \mathcal{B} \to \mathcal{B}'$. Therefore, taking $|\mu|^{\bullet} : |\operatorname{diag}\mathrm{N}\underline{\Delta}\mathcal{B}| \to |\operatorname{hocolim}\mathrm{N}\underline{\Delta}\mathcal{B}|$ to be any homotopy inverse map of $|\mu|$, we have a homotopy equivalence

$$\kappa = \mathrm{B}\textstyle\int_{\Delta}R \cdot |\eta| \cdot |\mu|^{\bullet} \cdot |i| : |\Delta\mathcal{B}| \xrightarrow{\sim} \mathrm{B}\mathcal{B},$$

which is homotopy natural on lax functors, as required. ∎

6.7. COROLLARY. *If $F, F' : \mathcal{B} \to \mathcal{B}'$ are two lax functors between bicategories, then any lax or oplax transformation between them $\alpha : F \Rightarrow F'$ determines a homotopy, $\mathrm{B}\alpha : \mathrm{B}F \Rightarrow \mathrm{B}F' : \mathrm{B}\mathcal{B} \to \mathrm{B}\mathcal{B}'$, between the induced maps on classifying spaces.*

PROOF. In the proof of [16, Proposition 7.1 (ii)] it is proven that any $\alpha : F \Rightarrow G$ gives rise to a homotopy $H(\alpha) : |\Delta F| \Rightarrow |\Delta F'| : |\Delta\mathcal{B}| \to |\Delta\mathcal{B}'|$. Then, a homotopy $\mathrm{B}\alpha : \mathrm{B}F \Rightarrow \mathrm{B}F'$ is obtained as the composite of the homotopies

$$\mathrm{B}F \Longrightarrow \mathrm{B}F\kappa\kappa^{\bullet} \overset{(54)}{\Longrightarrow} \kappa'|\Delta F|\kappa^{\bullet} \overset{\kappa' H(\alpha)\kappa^{\bullet}}{\Longrightarrow} \kappa'|\Delta F'|\kappa^{\bullet} \overset{(54)}{\Longrightarrow} \mathrm{B}F'\kappa\kappa^{\bullet} \Longrightarrow \mathrm{B}F',$$

where $\kappa^{\bullet}$ is a homotopy inverse of the homotopy equivalence $\kappa : |\Delta\mathcal{B}| \to \mathrm{B}\mathcal{B}$ in (53). ∎

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